A tripos surd

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1. *Introduction*

By the final third of the nineteenth century, the Mathematical Tripos examination had come to demand 44+ hours of gruelling, pressure-packed unrelenting high-level problem solving spread out over eight days [1]. This titanic tournament dwarfs the olympiads of today although it is legitimately regarded as their almost legendary progenitor. Many of the problems set in this 'mega-olympiad' were degree-level research questions and hundreds of papers have been written because of them.

Every year some tripos questions treated approximations, and our interest was caught by the following striking fourth-root surd approximation taken from the Tripos of 1886:*

> If $M = N^4 + x$, and x is small compared with N, then a good approximation for $\sqrt[4]{M}$ is:
>
> $$\frac{51}{56}N + \frac{5}{56}\frac{M}{N^3} + \frac{27Nx}{14(7M + 5N^4)}. \tag{1}$$
>
> Show that when $N = 10$, $x = 1$, this approximation is accurate to 16 places of decimals.

Making the numerical substitutions, we find

$$\sqrt[4]{10001} - \frac{1920160001}{192011200} = -5.695655\ldots \times 10^{-18},$$

which shows that the approximation is in excess and just misses being accurate to 17 decimal places!

Two questions naturally arise:

(a) In general, how large is the error:

$$E(x) = \sqrt[4]{M} - \left\{\frac{51}{56}N + \frac{5}{56}\frac{M}{N^3} + \frac{27Nx}{14(7M + 5N^4)}\right\}? \tag{2}$$

(b) How did the author discover the formula (1)?

Answering these questions will lead us to some interesting and subtle mathematics.

* We have not seen the examination paper itself. This quoted version is taken from Hardy [2, p. 431]. The problem is also quoted by Chrystal [3, p. 220] in a slightly different way. . . He writes p instead of M and leaves out 'compared with p,' and writes 'approximately' instead of 'a good approximation for $\sqrt[4]{M}$' is. . . It is a pity that there is no on-line source for the problem statements in the old tripos examinations listed by year.

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2. *The Accuracy of the Surd Approximation*

We will prove two theorems.

Theorem 1: If $\frac{7}{12}\frac{|x|}{N^4} \neq -1$, the error, $E(x)$, is given by

$$E(x) = -\frac{77}{2048}\left\{\frac{1}{(1 + X)^{15/4}} - \frac{28}{33}\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}}\right\}\frac{x^4}{N^{15}}, \tag{3}$$

where X is between 0 and $\frac{x}{N^4}$. Moreover, if

$$-\frac{20}{77} < \frac{x}{N^4} < 0.05336\ldots ,$$

the tripos surd approximation (1) *overestimates* the true value of $\sqrt[4]{M}$.

We write the error term with a minus sign to emphasize that the approximation overestimates the true value of the surd.

If we apply (3) to the original statement, and note that $0 < X < \frac{1}{10000}$, we see that

$$|E| \leqslant \frac{77}{2048}\left\{\frac{1}{(1 + 0)^{15/4}} - \frac{28}{33}\frac{1}{1 + \frac{7}{12}\frac{1}{10000}}\right\}\frac{1}{10^{15}} = 5.698475\ldots \times 10^{-18},$$

which coincides with the true error up to 10^{-20} inclusive.

The form of the error term (3) shows why the surd approximation is more accurate than the Taylor polynomial alone − see (4) − namely, by subtracting the term $\frac{28}{33} \div \left(1 + \frac{7}{12}\frac{x}{N^4}\right)$ from the error $1/(1 + X)^{15/4}$ in the Taylor polynomial, the approximation restores part of the true sum lost by truncation.

Although the formula (3) for the error is exact, the presence of the unknown quantity X can make it inconvenient in applications. Moreover, in order to transform (3) into an inequality bounding $E(x)$, one needs information on the size of x. Unfortunately, the original problem statement only says '… x is small compared with N…' which is not quantitatively precise. Moreover, x can be positive or negative, which complicates the analysis. We will prove the following estimates.

Theorem 2: If M differs from N^4 by less than p% of either, then $\sqrt[4]{M}$ differs from the tripos surd approximation by less than

$$\frac{77}{2048}\left\{\left(1 + \frac{p}{100}\right)^{15/4} - \frac{28}{33}\right\}\left(\frac{p}{100 + p}\right)^4 \cdot N$$

if x is negative, and by less than

$$\frac{77}{2048}\left\{1 - \frac{28}{33}\frac{1}{1 + \frac{7}{12}\frac{p}{100}}\right\}\left(\frac{p}{100}\right)^4 \cdot N$$

if x is positive.

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Thus, if $p = 1$, then $\sqrt[4]{M}$ differs from the tripos surd approximation by less than $\dfrac{N}{17000000000}$ if the difference is positive, and by less than $\dfrac{N}{14600000000}$ if it is negative. The proofs of the two theorems are based on the standard Maclaurin expansion of $\sqrt[4]{1 + x/N^4}$ with the Lagrange form of the remainder, which we state as a separate lemma.

Lemma 3: If $1 + \dfrac{x}{N^4} \geqslant 0$, there exists a number X between 0 and $\dfrac{x}{N^4}$ such that the following expansion is valid:

$$\sqrt[4]{1 + \frac{x}{N^4}} = 1 + \frac{1}{4}\frac{x}{N^4} - \frac{3}{32}\frac{x^2}{N^8} + \frac{7}{128}\frac{x^3}{N^{12}} - \frac{77}{2048}\left(\frac{1}{1+X}\right)^{15/4}\frac{x^4}{N^{16}}. \tag{4}$$

Proof of Theorem 1: Let $S(x)$ be the surd approximation:

$$S(x) = \frac{51}{56}N + \frac{5}{56}\frac{M}{N^3} + \frac{27Nx}{14(7M + 5N^4)}. \tag{5}$$

Then, using $M = N^4 + x$, we obtain

$$\begin{aligned} S(x) &= N\left\{\frac{51}{56} + \frac{5}{56}\frac{M}{N^4} + \frac{27x}{14(7M + 5N^4)}\right\} \\ &= N\left\{1 + \frac{5}{56}\frac{x}{N^4} + \frac{27x}{14(7[N^4 + x] + 5N^4)}\right\} \\ &= N\left\{1 + \frac{5}{56}\frac{x}{N^4} + \frac{9}{56}\frac{x}{N^4}\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}}\right\}. \end{aligned}$$

Now we use the identity

$$\frac{1}{1 + \frac{7}{12}y} \equiv 1 - \frac{7}{12}y + \frac{7^2}{12^2}y^2 - \frac{\frac{7^3}{12^3}y^3}{1 + \frac{7}{12}y}, \qquad \frac{7}{12}y \neq -1$$

with $y = \frac{x}{N^4}$. Then

$$\begin{aligned} \frac{9y}{56}\frac{1}{1 + \frac{7}{12}y} &\equiv \frac{9y}{56}\left(1 - \frac{7}{12}y + \frac{7^2}{12^2}y^2 - \frac{7^3}{12^3}\frac{y^3}{1 + \frac{7}{12}y}\right) \\ &\equiv \frac{9}{56}y - \frac{3}{32}y^2 + \frac{7}{128}y^3 - \frac{49}{1536}\frac{y^3}{1 + \frac{7}{12}y} \end{aligned}$$

and this leads to

$$S(x) = N\left\{1 + \frac{1}{4}\frac{x}{N^4} - \frac{3}{32}\frac{x^2}{N^8} + \frac{7}{128}\frac{x^3}{N^{12}} - \frac{49}{1536}\frac{x^4}{N^{16}}\frac{1}{1 + \frac{7}{12}y}\right\},$$

where we have used the assumption $\dfrac{7}{12}\dfrac{|x|}{N^4} \neq -1$. But (4) and $M = N^4 + x$ show us that

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$$\sqrt[4]{M} \equiv \sqrt[4]{N^4 + x} = \left\{1 + \frac{1}{4}\frac{x}{N^4} - \frac{3}{32}\frac{x^2}{N^8} + \frac{7}{128}\frac{x^3}{N^{12}} - \frac{77}{2048}\left(\frac{1}{1 + X}\right)^{15/4}\frac{x^4}{N^{16}}\right\}.$$

Subtracting, we obtain

$$E(x) = \sqrt[4]{M} - S(x) = -\frac{77}{2048}\left\{\frac{1}{(1 + X)^{15/4}} - \frac{28}{33}\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}}\right\}\frac{x^4}{N^{15}}$$

for some X between 0 and $\frac{x}{N^4}$. This completes the proof of the formula for $E(x)$.

The proof that $S(x)$ overestimates $\sqrt[4]{1 + \frac{x}{N^4}}$ is more troublesome because of the uncertainty of the value of X. (The value of X is not 'uncertain'; we do not know the value, but it is not a random variable.) Analytically, we have to prove that for certain positive and negative values of x the following inequality is valid:

$$\frac{1}{(1 + X)^{15/4}} - \frac{28}{33}\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}} > 0.$$

If $x > 0$, then writing u for $\frac{x}{N^4}$ we require

$$(1 + u)^{15/4} \leqslant \frac{33}{28}\left(1 + \frac{7}{12}u\right) = \frac{33}{28} + \frac{11}{16}u$$

which holds for $u < 0.05336\ldots$. If $x < 0$ then the desired inequality holds if

$$1 \leqslant \frac{33}{28}\left(1 + \frac{7}{12}u\right),$$

which holds for $u > -\frac{20}{77} = -0.259\ldots$. Therefore,

$$-\frac{20}{77} < \frac{x}{N^4} < 0.05336\ldots \Rightarrow \frac{1}{(1 + X)^{15/4}} - \frac{33}{28}\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}} > 0.$$

Doubtless these bounds can be improved, but now we have a quantitative formulation of '... x is small compared with N'. We note that the example in the original statement has $\frac{x}{N^4} = \frac{1}{10000} = 0.0001 < 0.05336\ldots$ which fulfils our inequality with plenty to spare.

Proof of Theorem 2: Suppose that x is positive and $0 < X < \frac{x}{N^4}$. Then, by assumption,

$$0 < X < \frac{x}{N^4} < \frac{p}{100} \Rightarrow 1 < 1 + X < 1 + \frac{x}{N^4} < 1 + \frac{p}{100}$$
$$\Rightarrow 1 + \frac{7}{12}\frac{x}{N^4} < 1 + \frac{7}{12}\frac{p}{100}$$
$$\Rightarrow -\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}} < \frac{1}{1 + \frac{7}{12}\frac{p}{100}}$$

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$$\Rightarrow |E(x)| = \left| -\frac{77}{2048}\left\{\frac{1}{(1+X)^{15/4}} - \frac{28}{33}\frac{1}{1+\frac{7}{12}\frac{x}{N^4}}\right\}\right| N$$

$$< \frac{77}{2048}\left\{1 - \frac{28}{33}\frac{1}{1+\frac{7}{12}\frac{p}{100}}\right\}\left(\frac{p}{100}\right)^4 N,$$

since $\dfrac{1}{(1+X)^{15/4}} < 1$.

Suppose that x is negative and that $\frac{x}{N^4} < X < 0$. Then $M = N^4 - |x|$ and

$$|x| < \frac{p}{100}\cdot M \Rightarrow \frac{|x|}{N^4} < \frac{p}{100}\left(1 - \frac{|x|}{N^4}\right)$$

$$\Rightarrow \frac{|x|}{N^4} < \frac{p}{100 + p} \tag{6}$$

$$\Rightarrow 1 - \frac{p}{100 + p} < 1 + \frac{x}{N^4} < 1 + X < 1$$

$$\Rightarrow \frac{1}{1 + X} < 1 + \frac{p}{100}$$

$$\Rightarrow \frac{1}{(1 + X)^{15/4}} < \left(1 + \frac{p}{100}\right)^{15/4}. \tag{7}$$

Moreover,

$$-\frac{1}{1 + \frac{7}{12}\frac{x}{N^4}} < -1.$$

Therefore the formula (3) and the inequalities (6) and (7) above allow us to conclude that

$$|E(x)| < \frac{77}{2048}\left\{\left(1 + \frac{p}{100}\right)^{15/4} - \frac{28}{33}\right\}\left(\frac{p}{100 + p}\right)^4 N.$$

This completes the proof.

3. *Discovering the approximation*

We seek an approximation, $s(x)$, of the form

$$\sqrt[4]{M} = \sqrt[4]{N^4 + x} \approx AN + B\frac{M}{N^3} + \frac{CNx}{DM + EN^4}. \tag{8}$$

where the coefficients A, B, C, D, E are to be determined so that the approximation is as accurate as possible. This means that it coincides with the Maclaurin expansion to as high a power as possible.

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Expanding the right-hand side of $s(x)$, i.e. of (8), into powers of $\frac{x}{N^4}$, we obtain

$$s(x) = N\left\{A + B + \left(B + \frac{C}{D+E}\right)\frac{x}{N^4} - \frac{CD}{(D+E)^2}\frac{x^2}{N^8} + \frac{CD^2}{(D+E)^3}\frac{x^3}{N^{12}} - \dots\right\}.$$

Comparing this with (4) we obtain the following system of equations:

$$A + B = 1,$$
$$B + \frac{C}{D + E} = \frac{1}{4},$$
$$-\frac{CD}{(D + E)^2} = -\frac{3}{32},$$
$$\frac{CD^2}{(D + E)^3} = \frac{7}{128}.$$

Then, writing F for $D + E$ the last two equations give us $D = 7$. We then obtain, in order:

$$\frac{C}{F} = \frac{9}{56}, \qquad B = \frac{5}{56}, \qquad A = \frac{51}{56},$$

and now using $E = \frac{7}{12}D$, also

$$\frac{C}{D} = \frac{27}{98}, \qquad \frac{C}{E} = \frac{27}{70}.$$

Substituting the values of $\frac{C}{D}$ and $\frac{C}{E}$ into the fraction $\dfrac{CNx}{DM + EN^4}$ in $s(x)$, the common factors D in the numerator and denominator cancel and it collapses to the fraction $\dfrac{27Nx}{14(7M + 5N^4)}$ in $S(x)$. This shows us that $s(x) \equiv S(x)$ and that $s(x)$ is uniquely determined.

It is interesting to note that we used four equations with five unknowns to obtain the tripos surd approximation. If we equate the coefficient of $\frac{x^4}{N^{15}}$ in the Maclaurin expansion, namely $-\frac{77}{2048}$, with the corresponding coefficient in $s(x)$, namely $-\dfrac{CD}{(D + E)^4}$, so as to obtain a fifth equation for the five unknowns, then the third, fourth, and new fifth equations give the inconsistent result $\dfrac{D}{F} = \dfrac{7}{12}$ and $\dfrac{D}{F} = \dfrac{11}{16}$. Therefore, $S(x)$ is the best possible and unique approximation of the given form.

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References

1. W. W. Rouse Ball, *A history of the study of mathematics at Cambridge*, Cambridge University Press (1889).
2. G. H. Hardy, *A course of pure mathematics*, (10th edn.), Cambridge University Press (1958).
3. G. Chrystal, *Textbook of algebra* (Part II), Dover (1961).

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