A $p(x)$-Laplacian extension of the Díaz-Saa inequality and some applications

Peter Takáč

Institut für Mathematik, Universität Rostock Ulmenstraße 69,
Haus 3 D-18055 Rostock, Germany (peter.takac@uni-rostock.de)

Jacques Giacomoni

LMAP (UMR 5142) Université de Pau et des Pays de l'Adour Avenue
de l'Université, F-64013 Pau cedex, France
(jacques.giacomoni@univ-pau.fr)

The main result of this work is a new extension of the well-known inequality by Díaz
and Saa which, in our case, involves an anisotropic operator, such as the
$p(x)$-Laplacian, $\Delta_{p(x)}u \equiv \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. Our present extension of this
inequality enables us to establish several new results on the uniqueness of solutions
and comparison principles for some anisotropic quasilinear elliptic equations. Our
proofs take advantage of certain convexity properties of the energy functional
associated with the $p(x)$-Laplacian.

1. Introduction

This work is concerned with an extension of a well-known inequality due to
Díaz and Saa [**5**] to certain quasilinear elliptic operators that are pointwise $p(x)$-
homogeneous, but anisotropic, in general, such as the $p(x)$-Laplacian $\Delta_{p(x)}u \equiv$
$\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ with a variable exponent $p(x) \in (1, \infty)$. Such operators have
been studied extensively, for example, in Diening *et al.* [**6**], and in Rădulescu
and Repovš [**18**]. Interesting applications to a model of *electrorheological fluids*
are presented in [**6**, §14.4, pp. 470–481] and the monograph by Růžička [**19**].
However, to our best knowledge, the Díaz and Saa inequality [**5**] still has not
been extended from the original case of a constant exponent $p(x) \equiv p = \mathrm{const} \in$
$(1, \infty)$ to a *variable* exponent $p(x)$. This inequality turns out to be equiva-
lent with the convexity of a $p(x)$-power type energy functional, as suggested in
Brézis and Oswald [**3**] for $p(x) \equiv p = 2$, and generalized in Fleckinger *et al.* [**12**]

© 2019 The Royal Society of Edinburgh

to any constant $p(x) \equiv p \in (1, \infty)$. In applications to quasilinear elliptic operators (with p constant, $1 < p < \infty$), this equivalence played a decisive role in the works by Girg and Takáč [**13**, § 4.1, pp. 289–292] and Takáč, Tello, and Ulm [**22**, lemma 2.4, p. 79].

To be more specific, the functional in question, $\mathcal{W} : W \to \mathbb{R}_+$, is defined by

$$\mathcal{W}(v) \equiv \mathcal{W}_{p(x),r}(v) \stackrel{\text{def}}{=} \int_\Omega \frac{r}{p(x)} \left| \nabla(|v(x)|^{1/r}) \right|^{p(x)} \mathrm{d}x \qquad (1.1)$$

for every function $v \in L^{p(x)/r}(\Omega)$ such that $|v|^{1/r} \in W_0^{1,p(x)}(\Omega)$; the set of all such functions $v : \Omega \to \mathbb{R}$ is denoted by $W \equiv W_{p(x),r}$. Here, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) whose boundary is a compact manifold (with smoothness to be specified later if $N \geqslant 2$), $r \in [1, \infty)$ is a given constant, and $p \in L^\infty(\Omega)$ is an essentially bounded function satisfying $p(x) > 1$ and $p(x) \geqslant r$ for almost all $x \in \Omega$ (whose smoothness will be specified later, as well). We will show in § 2 that this functional is *convex* on the cone

$$\overset{\bullet}{V} \stackrel{\text{def}}{=} \{v : \Omega \to (0, \infty) : v \in W\} \subset W \qquad (1.2)$$

of all positive functions $v \in W$. The convexity of the restriction $\mathcal{W} : \overset{\bullet}{V} \to \mathbb{R}_+$ to $\overset{\bullet}{V}$ is well-known to be equivalent with the monotonicity of its (set-valued) subdifferential $\partial \mathcal{W}(v)$ at $v \in \overset{\bullet}{V}$ that is a nonempty set only for certain elements $v \in \overset{\bullet}{V}$ which might not be easy to determine; cf. Girg and Takáč [**13**, § 4.1, pp. 289–292]. To avoid this problem, we restrict ourselves only to certain directional derivatives of $\mathcal{W}$ which exist in the classical sense.

We consider two functions $v_1, v_2 \in \overset{\bullet}{V}$ such that $v_1/v_2, v_2/v_1 \in L^\infty(\Omega)$. Consequently, also $v \stackrel{\text{def}}{=} (1 - \theta)v_1 + \theta v_2 \in \overset{\bullet}{V}$ is valid for all $\theta \in (-\delta, 1 + \delta)$, where $\delta \in (0, 1)$ is small enough. The function

$$\theta \mapsto \Phi(\theta) \stackrel{\text{def}}{=} \mathcal{W}(v) = \mathcal{W}((1 - \theta)v_1 + \theta v_2) : (-\delta, 1 + \delta) \to \mathbb{R}_+$$

is convex and differentiable with the derivative

$$\Phi'(\theta) = \int_\Omega \left| \nabla(|v(x)|^{1/r}) \right|^{p(x)-2} \nabla(|v(x)|^{1/r}) \cdot \nabla \left(\frac{v_2 - v_1}{v^{1-(1/r)}} \right) \mathrm{d}x; \qquad (1.3)$$

see theorem 2.2 below. The monotonicity of the derivative $\theta \mapsto \Phi'(\theta) : (-\delta, 1 + \delta) \to \mathbb{R}$ yields $\Phi'(1) - \Phi'(0) \geqslant 0$, that is,

$$\left\langle -\frac{\Delta_{p(x)}(v_1(x)^{1/r})}{v_1(x)^{(r-1)/r}} + \frac{\Delta_{p(x)}(v_2(x)^{1/r})}{v_2(x)^{(r-1)/r}}, \quad v_1 - v_2 \right\rangle$$
$$= \int_\Omega \left(-\frac{\Delta_{p(x)}(v_1(x)^{1/r})}{v_1(x)^{(r-1)/r}} + \frac{\Delta_{p(x)}(v_2(x)^{1/r})}{v_2(x)^{(r-1)/r}} \right) (v_1 - v_2) \, \mathrm{d}x \geqslant 0, \qquad (1.4)$$

provided the integration by parts in equation (1.3) can be justified. In this case, we may substitute $w_i = v_i^{1/r} > 0$ in $W_0^{1,p(x)}(\Omega)$; $i = 1, 2$, to derive the following

extension of the *Díaz and Saa inequality* (theorem 2.5 below):

$$\int_{\Omega} \left(-\frac{\Delta_{p(x)} w_1(x)}{w_1(x)^{r-1}} + \frac{\Delta_{p(x)} w_2(x)}{w_2(x)^{r-1}} \right) (w_1^r - w_2^r) \, \mathrm{d}x \geqslant 0 \qquad (1.5)$$

for all pairs $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both w_1/w_2, $w_2/w_1 \in L^\infty(\Omega)$. The special case $p(x) \equiv r = \mathrm{const} \in (1, \infty)$ yields the classical Díaz and Saa inequality established in [**5**].

To verify inequality (1.5), in §2 below (theorem 2.5), we slightly modify the method used in [**12**, **13**, **22**]. Our proof inequality (1.5) is based on the convexity of the restriction of the functional $\mathcal{W}$ to the cone $\overset{\bullet}{V} \subset W$. Finally, in §3, we present a few applications of our main results to some nonlinear boundary value problems with the Dirichlet $p(x)$-Laplacian $\Delta_{p(x)}$ and the power-type nonlinearity $|u(x)|^{q(x)-2}u(x)$, where the following (uniform) *'subhomogeneity' condition* is assumed:

$$1 < q(x) \leqslant r \leqslant p(x) \quad \text{for almost every } x \in \Omega, \qquad (1.6)$$

with a suitable (uniform separation) constant $r \in (1, \infty)$. This condition is related to abstract subhomogeneity conditions introduced in the well-known monograph by Krasnosel'skiĭ and Zabreĭko [**14**] in several different abstract settings in ordered Banach spaces.

2. Main results and their proofs

It is easy to see that the set $\overset{\bullet}{V}$ defined in equation (1.2) is a *convex cone*, that is,
(i) $\lambda \in (0, \infty)$, $f \in \overset{\bullet}{V} \Rightarrow \lambda f \in \overset{\bullet}{V}$; and (ii) $f, g \in \overset{\bullet}{V} \Rightarrow f + g \in \overset{\bullet}{V}$.

DEFINITION 2.1. A functional $\mathcal{W} : \overset{\bullet}{V} \to \mathbb{R}$ will be called ***ray-strictly convex*** (***strictly convex***, respectively) if it satisfies

$$\mathcal{W}\left((1-\theta)v_1 + \theta v_2\right) \leqslant (1-\theta) \cdot \mathcal{W}(v_1) + \theta \cdot \mathcal{W}(v_2) \qquad (2.1)$$

$$\text{for all } v_1, v_2 \in \overset{\bullet}{V} \quad \text{and for all } \theta \in (0,1),$$

where the inequality is strict ($<$) unless $v_2/v_1 \equiv \mathrm{const} > 0$ is a constant (for $\mathcal{W}$ 'strictly convex' always strict if $v_1 \neq v_2$).

We assume that $\Omega \subset \mathbb{R}^N$ is either a bounded open interval in $\mathbb{R}^1$ ($N = 1$) or else a bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$) whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Additional hypotheses on the smoothness of the boundary $\partial\Omega$ (such as interior sphere condition at $\partial\Omega$) will be added later in the applications (§3).

For the sake of simplicity, we assume that $r \in [1, \infty)$ is a given constant and $p : \Omega \to (1, \infty)$ is a continuous function, such that

$$1 < p_- \overset{\text{def}}{=} \inf_\Omega p(x) \leqslant p_+ \overset{\text{def}}{=} \sup_\Omega p(x) < \infty \quad \text{and} \quad 1 \leqslant r \leqslant p_-. \tag{2.2}$$

We assume that the function A of $(x, \xi) \in \Omega \times \mathbb{R}^N$ extends to a continuous and nonnegative function $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$, and it verifies the following hypothesis: For every fixed $x \in \Omega$, the function $A(x, \cdot) : \mathbb{R}^N \to \mathbb{R}_+$ is *positively $p(x)$-homogeneous*, that is,

$$A(x, t\xi) = |t|^{p(x)} A(x, \xi) \quad \text{for all } t \in \mathbb{R}, \ \xi \in \mathbb{R}^N. \tag{2.3}$$

It is evident from equation (2.3) that

$$A(x, \xi) = A\left(x, \frac{\xi}{|\xi|}\right) \cdot |\xi|^{p(x)}, \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{\mathbf{0}\},$$

determines the growth of $A(x, \xi)$ with respect to $\xi \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, for any fixed $x \in \Omega$. Let $\mathbb{S}^{N-1} \overset{\text{def}}{=} \{\xi \in \mathbb{R}^N : |\xi| = 1\}$ denote the unit sphere in $\mathbb{R}^N$ centred at the origin $\mathbf{0} \in \mathbb{R}^N$. We remark that the 'coefficient' $A(x, \xi/|\xi|)$ in the last equation above is bounded from above by a positive constant, thanks to $A : \overline{\Omega} \times \mathbb{S}^{N-1} \to \mathbb{R}_+$ being continuous on the compact set $\overline{\Omega} \times \mathbb{S}^{N-1} \subset \mathbb{R}^N \times \mathbb{R}^N$.

The simplest example of A is, of course, $A(x, \xi) = |\xi|^{p(x)}$ for $(x, \xi) \in \Omega \times \mathbb{R}^N$, in which case $A(x, \xi) = 1$ for all $(x, \xi) \in \Omega \times \mathbb{S}^{N-1}$. This case leads to the functional $\mathcal{W}$ defined in equation (1.1).

The next theorem is our main result on the functional $\mathcal{W}_A : W \to \mathbb{R}_+$ defined by

$$\mathcal{W}_A(v) \equiv \mathcal{W}_{A,p(x),r}(v) \overset{\text{def}}{=} \int_\Omega \frac{r}{p(x)} A\left(x, \nabla(|v(x)|^{1/r})\right) \mathrm{d}x \tag{2.4}$$

$$= \int_\Omega \frac{r}{p(x)} A\left(x, \frac{\nabla(|v|^{1/r})}{|\nabla(|v|^{1/r})|}\right) \cdot \left|\nabla(|v|^{1/r})\right|^{p(x)} \mathrm{d}x$$

for every function $v \in W$; see equation (1.1) in §1.

THEOREM 2.2 (Convexity). *Let $r \in [1, \infty)$ and $p : \Omega \to (1, \infty)$ satisfy (2.2). Assume that $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$ is continuous and satisfies the $p(x)$-homogeneity hypothesis (2.3). In addition, assume that the function*

$$\xi \mapsto \mathfrak{N}(x, \xi) \overset{\text{def}}{=} A(x, \xi)^{r/p(x)} : \mathbb{R}^N \to \mathbb{R}_+ \tag{2.5}$$

*is strictly convex for every $x \in \Omega$. Then the restriction of the functional $\mathcal{W}_A : W \to \mathbb{R}_+$ to the convex cone $\overset{\bullet}{V}$ is **ray-strictly convex** on $\overset{\bullet}{V}$.*

*Furthermore, if $p(x) \not\equiv r$ in Ω, that is, if $r = p_- \equiv p(x) \equiv p_+$ does not hold in Ω, then $\mathcal{W}_A$ is even **strictly convex** on $\overset{\bullet}{V}$.*

REMARK 2.3. (i) In the classical setting with $p(x) \equiv p \in (1, \infty)$ being a constant and $r = 1$ (cf. Takáč [21]), $\mathfrak{N}(x, \xi) \equiv \mathfrak{N}(\xi) = |\xi| = \|\xi\|_{\ell_2}$ ($\xi \in \mathbb{R}^N$) is the standard

Euclidean norm in $\mathbb{R}^N$. Hence, the functional $\mathcal{W}_A : W \to \mathbb{R}_+$ defined in (1.1) and (2.4) takes the form

$$\mathcal{W}_A(v) \equiv \mathcal{W}_{A,p,1}(v) = \frac{1}{p} \int_\Omega |\nabla v(x)|^p \, \mathrm{d}x = \frac{1}{p} \|v\|^p_{W_0^{1,p}(\Omega)}$$

for every $v \in W = W_0^{1,p}(\Omega)$, thanks to $|\nabla |v|| = |\nabla v|$ a.e. in Ω.

(ii) In fact, Part (i) was the motivation for expressing the function $\xi \mapsto A(x,\xi) = \mathfrak{N}(x,\xi)^{p(x)/r} : \mathbb{R}^N \to \mathbb{R}_+$ as a power ($p(x)/r \geqslant 1$) of the (strictly convex) function $\xi \mapsto \mathfrak{N}(x,\xi) : \mathbb{R}^N \to \mathbb{R}_+$ that may be taken to be a strictly convex norm on $\mathbb{R}^N$ depending on $x \in \Omega$; cf. Takáč, Tello, and Ulm [**22**, remark 2.1, p. 78].

(iii) We note that (in theorem 2.2 above) the function $\xi \mapsto A(x,\xi) = \mathfrak{N}(x,\xi)^{p(x)/r} : \mathbb{R}^N \to \mathbb{R}_+$ is strictly convex for each fixed $x \in \Omega$, thanks to the power function $t \mapsto t^{p(x)/r} : \mathbb{R}_+ \to \mathbb{R}_+$ being strictly monotone increasing and convex. Consequently, $A(x,\xi) > A(x,\mathbf{0}) = 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, and $A : \overline{\Omega} \times \mathbb{S}^{N-1} \to \mathbb{R}_+$ is bounded below and above on the compact set $\overline{\Omega} \times \mathbb{S}^{N-1} \subset \mathbb{R}^N \times \mathbb{R}^N$ by some positive constants; hence, the 'coefficient' $A(x,\xi/|\xi|)$ in the integrand in equation (2.4), if $\xi = \nabla(|v|^{1/r}) \neq \mathbf{0}$, is bounded from below and above by some positive constants $c_1, c_2 \in (0, \infty)$,

$$0 < c_1 \leqslant A\left(x, \tfrac{\xi}{|\xi|}\right) \leqslant c_2 < \infty.$$

This shows that also the ratio of the functionals in (2.4) and (1.1) is bounded from below and above by the same positive constants as above, that is,

$$c_1 \cdot \mathcal{W}(v) \leqslant \mathcal{W}_A(v) \leqslant c_2 \cdot \mathcal{W}(v) \quad \text{for every } v \in \overset{\bullet}{V}.$$

Proof of theorem 2.2. Recalling definition 2.1, let us consider any $v_1, v_2 \in \overset{\bullet}{V}$ and $\theta \in (0,1)$. Let us denote $v = (1-\theta)v_1 + \theta v_2$; hence, $v \in \overset{\bullet}{V}$. We obtain easily

$$\nabla(v_i(x)^{1/r}) = \frac{v_i^{1/r}}{r} \frac{\nabla v_i}{v_i} \quad \text{for } i = 1, 2, \quad \text{and}$$

$$\nabla(v(x)^{1/r}) = \frac{1}{r} \frac{(1-\theta)\nabla v_1 + \theta \nabla v_2}{[(1-\theta)v_1 + \theta v_2]^{1-(1/r)}}$$

$$= \frac{v^{1/r}}{r} \frac{(1-\theta)\nabla v_1 + \theta \nabla v_2}{v}$$

$$= \frac{v^{1/r}}{r} \left[(1-\theta) \frac{v_1}{v} \cdot \frac{\nabla v_1}{v_1} + \theta \frac{v_2}{v} \cdot \frac{\nabla v_2}{v_2} \right],$$

with the convex combination of positive coefficients $(1-\theta)\,v_1/v$ and $\theta\,v_2/v$,

$$(1-\theta)\frac{v_1}{v} + \theta \frac{v_2}{v} = 1.$$

Now let $x \in \Omega$ be fixed. Since $\xi \mapsto \mathfrak{N}(x, \xi)$ is strictly convex, by our hypothesis, we may apply the identities from above to conclude that

$$
\mathfrak{N}\left(x, (1-\theta)\, \frac{v_1}{v} \cdot \frac{\nabla v_1}{v_1} + \theta\, \frac{v_2}{v} \cdot \frac{\nabla v_2}{v_2}\right)
$$
$$
\leqslant (1-\theta)\, \frac{v_1}{v} \cdot \mathfrak{N}\left(x, \frac{\nabla v_1}{v_1}\right) + \theta\, \frac{v_2}{v} \cdot \mathfrak{N}\left(x, \frac{\nabla v_2}{v_2}\right).
\tag{2.6}
$$

The equality holds if and only if

$$
\frac{\nabla v_1(x)}{v_1(x)} = \frac{\nabla v_2(x)}{v_2(x)}, \quad \text{which is equivalent to} \quad \nabla\left(\frac{v_2(x)}{v_1(x)}\right) = 0.
\tag{2.7}
$$

Notice that the homogeneity conditions (2.3) and (2.5) yield

$$
\mathfrak{N}(x, t\xi) = |t|^r\, \mathfrak{N}(x, \xi) \quad \text{for all } t \in \mathbb{R},\ \xi \in \mathbb{R}^N.
\tag{2.8}
$$

Consequently, we multiply inequality (2.6) by v/r^r to obtain the following equivalent inequality,

$$
\mathfrak{N}\left(x, \nabla(v(x)^{1/r})\right) = \frac{v}{r^r} \cdot \mathfrak{N}\left(x, (1-\theta)\, \frac{v_1}{v} \cdot \frac{\nabla v_1}{v_1} + \theta\, \frac{v_2}{v} \cdot \frac{\nabla v_2}{v_2}\right)
$$
$$
\leqslant (1-\theta)\, \frac{v_1}{r^r} \cdot \mathfrak{N}\left(x, \frac{\nabla v_1}{v_1}\right) + \theta\, \frac{v_2}{r^r} \cdot \mathfrak{N}\left(x, \frac{\nabla v_2}{v_2}\right)
\tag{2.9}
$$
$$
= (1-\theta) \cdot \mathfrak{N}\left(x, \nabla(v_1(x)^{1/r})\right) + \theta \cdot \mathfrak{N}\left(x, \nabla(v_2(x)^{1/r})\right).
$$

Finally, by remark 2.3, we conclude that inequality (2.9) entails

$$
A\left(x, \nabla(v(x)^{1/r})\right) \leqslant (1-\theta) \cdot A\left(x, \nabla(v_1(x)^{1/r})\right) + \theta \cdot A\left(x, \nabla(v_2(x)^{1/r})\right).
\tag{2.10}
$$

We multiply the last inequality, (2.10), by $r/p(x)$, then integrate the product over Ω to derive the convexity of the restriction of the functional $\mathcal{W}_A$ to the convex cone $\overset{\bullet}{V} \subset W$.

To derive that $\mathcal{W}_A$ is even ray-strictly convex on $\overset{\bullet}{V}$, let us consider any pair $v_1, v_2 \in \overset{\bullet}{V}$ with $v_1 \not\equiv v_2$ in Ω. We observe that the equality in the convexity inequality (2.1) forces both conditions, (2.7) and $p(x)/r = 1$, to hold simultaneously at almost every point $x \in \Omega$. These conditions are then equivalent with $v_2/v_1 \equiv \text{const}$ ($\neq 1$) in Ω and $p(x) \equiv r$ in Ω. Thus, if $p(x) \not\equiv r$ in Ω, then $\mathcal{W}_A$ is even strictly convex on $\overset{\bullet}{V}$. $\qquad\square$

Our second theorem is concerned with the extension of the Díaz and Saa *inequality* as formulated inequality (1.5). Here, we need to assume a more specific form of the function $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$. Besides the homogeneity hypothesis (2.5), we assume that A and its partial gradient $\partial_\xi A \equiv (\partial A/\partial \xi_i)_{i=1}^N$ with respect to $\xi \in \mathbb{R}^N$ satisfy the following structural hypothesis, upon the substitution $\mathbf{a}(x, \xi) \overset{\text{def}}{=} (1/p(x))\, \partial_\xi A(x, \xi)$ with $a_i = (1/p(x))\,(\partial A/\partial \xi_i)$:

HYPOTHESIS 2.4. (**A**)   Given any fixed $x \in \Omega$, the function $A(x, \cdot) : \mathbb{R}^N \to \mathbb{R}_+$ verifies the *positive $p(x)$-homogeneity* hypothesis (2.3). Furthermore, we assume that $A \in C(\overline{\Omega} \times \mathbb{R}^N) \cap C^1(\Omega \times \mathbb{R}^N)$ and its partial gradient $\partial_\xi A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies $1/p\,(\partial A/\partial \xi_i) = a_i \in C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ for all $i = 1, 2, \ldots, N$, together with the following *ellipticity* and *growth conditions*: There exist some constants $\gamma, \Gamma \in (0, \infty)$ such that

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \xi_j}(x, \xi) \cdot \eta_i \eta_j \geqslant \gamma \cdot |\xi|^{p(x)-2} \cdot |\eta|^2, \tag{2.11}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial \xi_j}(x, \xi) \right| \leqslant \Gamma \cdot |\xi|^{p(x)-2}, \tag{2.12}$$

for all $x \in \Omega$, all $\xi \in \mathbb{R}^N \setminus \{0\}$, and all $\eta \in \mathbb{R}^N$.

Owing to the homogeneity hypothesis (2.5), it suffices to assume that the inequalities in (2.11) and (2.12) hold for all $\xi \in \mathbb{S}^{N-1}$ only.

THEOREM 2.5 (The Díaz and Saa inequality). *Let $r \in [1, \infty)$ and $p : \Omega \to (1, \infty)$ satisfy* (2.2). *Assume that $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$ satisfies Hypothesis* (**A**) *and, in addition, the function $\xi \mapsto \mathfrak{N}(x, \xi) = A(x, \xi)^{r/p(x)} : \mathbb{R}^N \to \mathbb{R}_+$ is strictly convex for every $x \in \Omega$. Then the following inequality*

$$\int_\Omega \left(-\frac{\operatorname{div} \mathbf{a}(x, \nabla w_1(x))}{w_1(x)^{r-1}} + \frac{\operatorname{div} \mathbf{a}(x, \nabla w_2(x))}{w_2(x)^{r-1}} \right) (w_1^r - w_2^r) \, \mathrm{d}x \geqslant 0 \tag{2.13}$$

holds (in the sense of distributions) for all pairs $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both $w_1/w_2, w_2/w_1 \in L^\infty(\Omega)$. Moreover, if the equality (=) in (2.13) *occurs, then we have the following two statements:*

(a) $w_2/w_1 \equiv \operatorname{const} > 0$ *in* Ω.

(b)   *If also $p(x) \not\equiv r$ in Ω, then even $w_1 \equiv w_2$ holds in Ω.*

REMARK 2.6. The distributional inequality (2.13) has to be interpreted in the following way:

$$\int_\Omega \mathbf{a}(x, \nabla w_1(x)) \cdot \nabla \left(w_1 - \frac{w_2^r}{w_1^{r-1}} \right) \mathrm{d}x$$
$$\geqslant \int_\Omega \mathbf{a}(x, \nabla w_2(x)) \cdot \nabla \left(\frac{w_1^r}{w_2^{r-1}} - w_2 \right) \mathrm{d}x \tag{2.14}$$

for all pairs $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both $w_1/w_2, w_2/w_1 \in L^\infty(\Omega)$. Both integrals above are defined as Lebesgue integrals, thanks to the inequalities in (2.11) and (2.12) combined with the following standard

identities,

$$\nabla\left(\frac{w_2^r}{w_1^{r-1}}\right) = r\left(\frac{w_2}{w_1}\right)^{r-1}\nabla w_2 - (r-1)\left(\frac{w_2}{w_1}\right)^r\nabla w_1,$$

$$\nabla\left(\frac{w_1^r}{w_2^{r-1}}\right) = r\left(\frac{w_1}{w_2}\right)^{r-1}\nabla w_1 - (r-1)\left(\frac{w_1}{w_2}\right)^r\nabla w_2,$$

where w_1/w_2, $w_2/w_1 \in L^\infty(\Omega)$ and all gradients belong to $L^{p(x)}(\Omega)$, whence also w_2^r/w_1^{r-1}, $w_1^r/w_2^{r-1} \in W_0^{1,p(x)}(\Omega)$.

Proof of theorem 2.5. Recalling definition 2.1, let us consider any pair w_1, $w_2 \in W_0^{1,p(x)}(\Omega)$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both w_1/w_2, $w_2/w_1 \in L^\infty(\Omega)$. Consequently, there is a number $\delta \in (0,1)$, sufficiently small, such that

$$v \stackrel{\text{def}}{=} (1-\theta)w_1^r + \theta w_2^r \in \overset{\bullet}{V} \quad \text{and} \quad v^{1/r} \in W_0^{1,p(x)}(\Omega) \ \text{ for all } \ \theta \in (-\delta, 1+\delta).$$

The function

$$\theta \mapsto \Phi(\theta) \stackrel{\text{def}}{=} \mathcal{W}(v) = \mathcal{W}_A\left((1-\theta)w_1^r + \theta w_2^r\right) : (-\delta, 1+\delta) \to \mathbb{R}_+$$

is convex and differentiable with the derivative

$$\Phi'(\theta) = \int_\Omega \mathbf{a}(x, \nabla(v(x)^{1/r})) \cdot \nabla\left(\frac{w_2^r - w_1^r}{v^{1-(1/r)}}\right) \mathrm{d}x. \tag{2.15}$$

To provide a rigorous proof of the convexity claim, one has to consider two arbitrary points θ_1, $\theta_2 \in \mathbb{R}$, such that $-\delta < \theta_1 < \theta_2 < 1+\delta$, and all their convex combinations $\theta = (1-t)\theta_1 + t\theta_2 \in (-\delta, 1+\delta)$ with $t \in [0,1]$. For $0 \leqslant \theta_1 < \theta_2 \leqslant 1$ the convexity is known, by theorem 2.2. However, if at least one of the following inequalities holds, $-\delta < \theta_1 < 0$ and/or $1 < \theta_2 < 1+\delta$, the convexity inequality $\Phi(\theta) \leqslant (1-t)\Phi(\theta_1) + t\Phi(\theta_2)$ still remains to be verified. Of course, the number $\delta > 0$ needs to be taken small enough. We leave this easy exercise to the reader.

The monotonicity of the derivative $\theta \mapsto \Phi'(\theta) : (-\delta, 1+\delta) \to \mathbb{R}$ yields $\Phi'(0) \leqslant \Phi'(1)$, which is equivalent with inequality (2.14), thanks to $v = w_1^r$ if $\theta = 0$, and $v = w_2^r$ if $\theta = 1$. It is now easy to see that inequality (2.13) is a distributional interpretation of (2.14) after integration by parts.

Finally, let us assume that the equality ($=$) in (2.13) is valid. This forces $\Phi'(0) = \Phi'(1)$ above; hence, $\Phi'(\theta) = \Phi'(0)$ for all $\theta \in [0,1]$, by the monotonicity of $\Phi' : [0,1] \to \mathbb{R}$. It follows that $\Phi : [0,1] \to \mathbb{R}$ must be linear, that is, $\Phi(\theta) = (1-\theta)\Phi(0) + \theta\Phi(1) \in \mathbb{R}$ for all $\theta \in [0,1]$. Recalling our definition of Φ above and theorem 2.2, we conclude that $w_2/w_1 \equiv \text{const} > 0$ in Ω. This proves statement (a).

To verify statement (b), suppose that the constant above $w_2/w_1 \equiv \text{const} \neq 1$ in Ω. Then the equality in both inequalities, (2.9) and (2.10), is possible only if $p(x) \equiv r$ in Ω. Statement (b) follows. $\qquad\square$

Our third (and last) theorem is a *weak comparison principle* for positive solutions $u \in W_0^{1,p(x)}(\Omega)$ of the following (uniformly) *'subhomogeneous'* Dirichlet boundary

value problem:

$$\begin{cases} -\operatorname{div}\mathbf{a}(x, \nabla u(x)) = f(x)\, u(x)^{r-1} & \text{for } x \in \Omega; \qquad u > 0 \quad \text{a.e. in } \Omega; \\ \qquad\qquad u = 0 \quad \text{for } x \in \partial\Omega. \end{cases} \tag{2.16}$$

Here, $f \in L^\infty(\Omega)$ is a given nonnegative function, $f \geqslant 0$ a.e. in Ω.

THEOREM 2.7 (Weak comparison principle). *Let all* $r \in [1, \infty)$, *p* : $\Omega \to (1, \infty)$, $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$, *and the function* $\xi \mapsto \mathfrak{N}(x, \xi) = A(x, \xi)^{r/p(x)} : \mathbb{R}^N \to \mathbb{R}_+$ *satisfy the same hypotheses as in theorem* 2.5 *above. In addition, assume that* $p(x) \not\equiv r$ *in* Ω, *that is,* $p(x) > r$ *on a subset of* Ω *with positive Lebesgue measure.*

Finally, let $u_i \in W_0^{1,p(x)}(\Omega)$ *be a positive solution of the Dirichlet boundary value problem* (2.16) *(in the sense of distributions) with* $f = f_i \in L^\infty(\Omega)$ *for* $i = 1, 2$, *respectively, where* $0 \leqslant f_1 \leqslant f_2$ *a.e. in* Ω. *If* $u_1/u_2, u_2/u_1 \in L^\infty(\Omega)$ *then we have also* $u_1 \leqslant u_2$ *a.e. in* Ω.

REMARK 2.8. Conditions $u_1/u_2, u_2/u_1 \in L^\infty(\Omega)$ imposed on the solutions u_1 and u_2 are easy to verify for $f_1 \not\equiv 0$ (hence, also $f_2 \not\equiv 0$) in Ω, by the regularity results in Fan and Zhao [**10**, theorem 4.1, p. 312] and Fan [**7**, theorem 1.2, p. 400] combined with the Hopf boundary point lemma from Zhang [**23**, theorems 1.1 and 1.2, p. 26]; see our proof of theorem 3.4 below.

It will be obvious from our proof of theorem 2.7 below that the following simple generalization of this theorem to weak sub- and supersolutions is a direct consequence of the proof. (We leave the details concerning only the last two inequalities of the proof to an interested reader.)

THEOREM 2.9 (Weak comparison principle for sub- and supersolutions). *Let all* $r \in [1, \infty)$, *p* : $\Omega \to (1, \infty)$, $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$, *and the function* $\xi \mapsto \mathfrak{N}(x, \xi) = A(x, \xi)^{r/p(x)} : \mathbb{R}^N \to \mathbb{R}_+$ *satisfy the same hypotheses as in theorem* 2.5 *above. In addition, assume that* $p(x) \not\equiv r$ *in* Ω, *that is,* $p(x) > r$ *on a subset of* Ω *with positive Lebesgue measure.*

Finally, let $u_i \in W_0^{1,p(x)}(\Omega)$ *(i = 1, 2) be a pair of positive functions satisfying* $u_1/u_2, u_2/u_1 \in L^\infty(\Omega)$ *together with the following inequalities (in the sense of distributions) with* $f_i \in L^\infty(\Omega)$ *for* $i = 1, 2$, *respectively, where* $0 \leqslant f_1 \leqslant f_2$ *a.e. in* Ω:

$$\begin{cases} -\operatorname{div}\mathbf{a}(x, \nabla u_1(x)) \leqslant f_1(x)\, u_1(x)^{r-1} & \text{for } x \in \Omega; \quad u_1 > 0 \quad \text{a.e. in } \Omega; \\ \qquad\qquad u_1 = 0 \quad \text{for } x \in \partial\Omega. \end{cases} \tag{2.17}$$

$$\begin{cases} -\operatorname{div}\mathbf{a}(x, \nabla u_2(x)) \geqslant f_2(x)\, u_2(x)^{r-1} & \text{for } x \in \Omega; \quad u_2 > 0 \quad \text{a.e. in } \Omega; \\ \qquad\qquad u_2 = 0 \quad \text{for } x \in \partial\Omega. \end{cases} \tag{2.18}$$

Then also $u_1 \leqslant u_2$ *a.e. in* Ω *holds.*

We quote a well-known fact from the theory of distributions that any nonnegative distribution in $\mathcal{D}'(\Omega)$ may be identified with a nonnegative Radon measure on Ω.

This result shows that the left-hand side of both inequalities (2.17) and (2.18) must be a Radon measure on Ω.

Proof of theorem 2.7. We proceed in analogy with the proof of theorem 2.5 above. We set $w_i = u_i$; $i = 1, 2$, and define

$$v \equiv v(\theta) \stackrel{\text{def}}{=} u_2^r + \theta(u_1^r - u_2^r)^+ \quad \text{for all } \theta \in (-\delta, 1 + \delta),$$

where $\delta \in (0, 1)$ is a sufficiently small number, such that $v \in \overset{\bullet}{V}$ for every $\theta \in (-\delta, 1 + \delta)$. As usual, the symbol $\xi^+ = \max\{\xi, 0\} \geqslant 0$ stands for the positive part of a real number $\xi \in \mathbb{R}$. Hence, we have also $v^{1/r} \in W_0^{1,p(x)}(\Omega)$. Notice that

$$v = \begin{cases} u_2^r & \text{if } u_1 \leqslant u_2, \\ u_2^r + \theta(u_1^r - u_2^r) & \text{if } u_1 > u_2. \end{cases}$$

On the contrary to our claim $u_1 \leqslant u_2$ a.e. in Ω, let us assume that $(u_1^r - u_2^r)^+ > 0$ holds on a subset $\Omega_+ = \{x \in \Omega : u_1(x) > u_2(x)\} \subset \Omega$ of positive Lebesgue measure.

By theorem 2.2, thanks to our hypothesis $p(x) \not\equiv r$ in Ω, the function

$$\theta \mapsto \Phi(\theta) \stackrel{\text{def}}{=} \mathcal{W}(v) = \mathcal{W}_A\left(u_2^r + \theta(u_1^r - u_2^r)^+\right) : (-\delta, 1 + \delta) \to \mathbb{R}_+$$

is strictly convex and differentiable with the derivative

$$\Phi'(\theta) = \int_\Omega \mathbf{a}(x, \nabla(v(x)^{1/r})) \cdot \nabla\left(\frac{(u_1^r - u_2^r)^+}{v^{1 - \frac{1}{r}}}\right) \mathrm{d}x. \tag{2.19}$$

The strict convexity of $\Phi : (-\delta, 1 + \delta) \to \mathbb{R}$ and the monotonicity of its derivative Φ' yield $\Phi'(0) < \Phi'(1)$, which is equivalent with

$$\int_\Omega \mathbf{a}(x, \nabla u_2(x)) \cdot \nabla\left(\frac{(u_1^r - u_2^r)^+}{u_2^{r-1}}\right) \mathrm{d}x$$

$$< \int_\Omega \mathbf{a}\left(x, \nabla\left[(u_2^r + (u_1^r - u_2^r)^+)^{1/r}\right]\right) \cdot \nabla\left(\frac{(u_1^r - u_2^r)^+}{(u_2^r + (u_1^r - u_2^r)^+)^{1-(1/r)}}\right) \mathrm{d}x.$$

By remark 2.6, the last inequality has the following distributional interpretation,

$$-\int_\Omega \frac{\operatorname{div} \mathbf{a}(x, \nabla u_2(x))}{u_2(x)^{r-1}} (u_1^r - u_2^r)^+ \, \mathrm{d}x$$

$$< -\int_\Omega \frac{\operatorname{div} \mathbf{a}\left(x, \nabla\left[(u_2^r + (u_1^r - u_2^r)^+)^{1/r}\right]\right)}{(u_2^r + (u_1^r - u_2^r)^+)^{1-\frac{1}{r}}} (u_1^r - u_2^r)^+ \, \mathrm{d}x. \tag{2.20}$$

As it is well-known from the theory of Sobolev spaces of type $W_0^{1,p(x)}(\Omega)$, both integrands above vanish almost everywhere in the Lebesgue measurable set $\Omega_- =$

$\{x \in \Omega : u_1(x) \leqslant u_2(x)\} \subset \Omega$. Consequently, inequality (2.20) reads

$$\int_{\Omega_+} \left(-\frac{\text{div}\,\mathbf{a}(x, \nabla u_2(x))}{u_2(x)^{r-1}} + \frac{\text{div}\,\mathbf{a}(x, \nabla u_1(x))}{u_1(x)^{r-1}} \right) (u_1^r - u_2^r)\,\mathrm{d}x < 0. \qquad (2.21)$$

By our hypotheses, we have

$$-\frac{\text{div}\,\mathbf{a}(x, \nabla u_2(x))}{u_2(x)^{r-1}} + \frac{\text{div}\,\mathbf{a}(x, \nabla u_1(x))}{u_1(x)^{r-1}} = f_2(x) - f_1(x) \geqslant 0 \quad \text{for a.e. } x \in \Omega.$$

Since also $u_1^r - u_2^r > 0$ a.e. in Ω_+, inequality (2.21) leads to a contradiction.

Thus, we have proved that the set Ω_+ must have Lebesgue measure equal to zero. $\qquad\square$

3. Applications to differential equations

In this section, we give two applications of theorems 2.2 and 2.5. Throughout this section, we impose the following hypotheses on Ω and $p(x)$:

HYPOTHESIS 3.1 ($\mathbf{\Omega}$). *If $N = 1$ then Ω is a bounded open interval in $\mathbb{R}^1$. If $N \geqslant 2$ then Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, and Ω satisfies also the interior sphere condition at every point of $\partial\Omega$.*

It is clear that for $N \geqslant 2$, Hypothesis ($\mathbf{\Omega}$) is satisfied if, for instance, $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. We write $\overline{\Omega} = \Omega \cup \partial\Omega$ for the closure of Ω in $\mathbb{R}^N$.

HYPOTHESIS 3.2 ($\mathbf{p}$). *We assume that $p : \overline{\Omega} \to (1, \infty)$ is α_1-Hölder-continuous, that is, $p \in C^{0,\alpha_1}(\overline{\Omega})$ for some $\alpha_1 \in (0, 1)$, and p satisfies (2.2) with a given constant $r \in [1, \infty)$, that is,*

$$1 < p_- \stackrel{\text{def}}{=} \inf_\Omega p(x) \leqslant p_+ \stackrel{\text{def}}{=} \sup_\Omega p(x) < \infty \quad \text{and} \quad 1 \leqslant r \leqslant p_-.$$

Our first application is the following nonlinear Dirichlet boundary value problem taken from Diening *et al.* [**6**, equation (13.3.2), p. 418],

$$\begin{cases} -\Delta_{p(x)}u = f(x, u) & \text{in } \Omega; \\ \qquad u = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ in } \Omega. \end{cases} \qquad (3.1)$$

We impose the following hypotheses on the function f:

(**f1**) $f : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative continuous function such that $f(x, 0) = 0$ for all $x \in \Omega$.

(**f2**) The function $s \longmapsto f(x, s)/s^{r-1} : (0, \infty) \to \mathbb{R}_+$ is strictly monotone decreasing for every $x \in \Omega$.

(**f3**) The following two limits are uniform with respect to $x \in \Omega$:

$$\frac{f(x,s)}{s^{r-1}} \longrightarrow +\infty \quad \text{as } s \to 0+ \quad \text{and} \quad \frac{f(x,s)}{s^{r-1}} \longrightarrow 0 \quad \text{as } s \to +\infty.$$

Equivalently, we require

$$\frac{1}{s^{r-1}} \cdot \inf_{x \in \Omega} f(x,s) \longrightarrow +\infty \quad \text{as } s \to 0+ \quad \text{and}$$

$$\frac{1}{s^{r-1}} \cdot \sup_{x \in \Omega} f(x,s) \longrightarrow 0 \quad \text{as } s \to +\infty.$$

A typical example of the function f satisfying all Hypotheses (**f1**)–(**f3**), with $f(x,s) = h(x)\, s^{q(x)-1}$ for $x \in \overline{\Omega}$ and $s \in \mathbb{R}_+$, is given below in example 3.6. Here, $h \in C(\overline{\Omega})$ is a positive function and $q \in C(\overline{\Omega})$ satisfies $1 \leqslant q(x) \leqslant q_+ \stackrel{\text{def}}{=} \sup_\Omega q(x) < r = p_-$ for every $x \in \overline{\Omega}$. In fact, we may choose any number $r \in (q_+,\, p_-]$ while requiring $q_+ < p_-$. As a consequence, in this example we must have $1 \leqslant q_+ < r \leqslant p_-$ whence $r > 1$.

We remark that Hypothesis (**f3**) implies the following asymptotic behaviour of the function $s \mapsto f(x,s) : (0,\infty) \to \mathbb{R}_+$ as $s \to 0+$: Given any $\varepsilon > 0$, there is a constant $s_\varepsilon \in (0,\infty)$ such that

$$f(x,s) \geqslant \frac{1}{\varepsilon}\, s^{r-1} \quad \text{holds for all } (x,s) \in \overline{\Omega} \times [0, s_\varepsilon]. \tag{3.2}$$

In contrast, Hypotheses (**f1**) and (**f3**) limit the asymptotic behaviour of $f(x,\cdot)$ as $s \to +\infty$ as follows: Given any $\varepsilon > 0$, there is a constant $C_\varepsilon \in (0,\infty)$ such that

$$0 \leqslant f(x,s) \leqslant \varepsilon\, s^{r-1} + C_\varepsilon \quad \text{holds for all } (x,s) \in \overline{\Omega} \times \mathbb{R}_+. \tag{3.3}$$

We define the notion of a *nonnegative* weak solution to problem (3.1) as follows:

DEFINITION 3.3. A nonnegative function $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is called a **nonnegative weak solution** of problem (3.1) if, for every test function $\phi \in W_0^{1,p(x)}(\Omega)$, the following equation holds,

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla \phi(x)\, \mathrm{d}x = \int_\Omega f(x, u(x))\, \phi(x)\, \mathrm{d}x. \tag{3.4}$$

If u satisfies also $u > 0$ throughout Ω, we call u a **positive weak solution**.

Problem (3.1) has already been treated in Fan and Zhang [**9**] where the existence of a weak solution in $W_0^{1,p(x)}(\Omega)$ is proved; see also Diening *et al.* [**6**, theorem 13.3.3, p. 418]. Of course, the trivial solution $u \equiv 0$ in Ω is a nonnegative weak solution to problem (3.1).

The following theorem describes the solvability of the boundary value problem (3.1) for positive weak solutions.

THEOREM 3.4. *Under the* Hypotheses (**Ω**), (**p**), *and* (**f1**)–(**f3**), *problem* (3.1) *possesses a unique nonnegative and nontrivial weak solution* $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

This solution belongs to the class $C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, \alpha)$, and satisfies also the Hopf maximum principle,

$$u(x) > 0 \ \text{ for all } \ x \in \Omega \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}}(x) < 0 \ \text{ for all } \ x \in \partial\Omega. \qquad (3.5)$$

Of course, $u = 0$ on the boundary $\partial\Omega$. Hence, u is also a positive weak solution.

As usual, the symbol $\boldsymbol{\nu}(x) \in \mathbb{R}^N$ stands for the *unit outward normal* to the boundary $\partial\Omega$ at the point $x \in \partial\Omega$.

Proof. We extend the domain of f to all of $\overline{\Omega} \times \mathbb{R}$ by setting $f(x, s) = 0$ for $(x, s) \in \overline{\Omega} \times (-\infty, 0)$. We define the potential F for the function f as follows:

$$F(x, u) \stackrel{\text{def}}{=} \int_0^u f(x, s)\,\mathrm{d}s = \begin{cases} \int_0^u f(x, s)\,\mathrm{d}s & \text{if } 0 \leqslant u < \infty; \\ 0 & \text{if } -\infty < u < 0, \end{cases} \qquad (3.6)$$

for $(x, u) \in \overline{\Omega} \times \mathbb{R}$. Hence, $f(x, s) = (\partial F / \partial u)(x, s)$ for $(x, s) \in \overline{\Omega} \times \mathbb{R}$. Clearly, for each fixed $x \in \overline{\Omega}$, $F(x, \cdot) : \mathbb{R} \to \mathbb{R}_+$ is a monotone increasing function, owing to $(\partial F / \partial u)(x, s) = f(x, s) \geqslant 0$.

Next, we obtain a nonnegative weak solution to problem (3.1) from a global minimizer of the energy functional $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) \equiv \mathcal{E}_{p(x)}(u) \stackrel{\text{def}}{=} \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x, u(x))\,\mathrm{d}x \qquad (3.7)$$

for every function $u \in W_0^{1,p(x)}(\Omega)$. This functional is well-defined, by the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^r(\Omega)$, which is even compact, and the estimate in (3.3). The reader is referred to the monograph by Diening *et al.* [**6**, §§ 8.3 and 8.4] for Sobolev embeddings and their compactness. Furthermore, $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is coercive thanks to inequality (3.3) and $r \leqslant p_-$, that is,

$$\|u\|_{W_0^{1,p(x)}(\Omega)} \stackrel{\text{def}}{=} \|\nabla u\|_{L^{p(x)}(\Omega)} \longrightarrow +\infty \quad \implies \quad \mathcal{E}(u) \to +\infty. \qquad (3.8)$$

It is also weakly lower semicontinuous, by [**6**, § 13.2, pp. 412–417]. Thus, by a basic result from the calculus of variations (Struwe [**20**, theorem 1.2, p. 4]), $\mathcal{E}$ possesses a global minimizer $u_0 \in W_0^{1,p(x)}(\Omega)$. Since also $|u_0| \in W_0^{1,p(x)}(\Omega)$ with the Sobolev gradient $\nabla |u_0| = \nabla u_0$ almost everywhere in the set $\Omega^+ = \{x \in \Omega : u_0(x) \geqslant 0\}$, and $\nabla |u_0| = -\nabla u_0$ almost everywhere in $\Omega^- = \{x \in \Omega : u_0(x) \leqslant 0\}$, we have $|\nabla|u_0|| = |\nabla u_0|$ a.e. in $\Omega = \Omega^+ \cup \Omega^-$. From this equality, combined with $F(x, u) > 0$ for $u > 0$ and $F(x, u) = 0$ for $u \leqslant 0$, we deduce that $\mathcal{E}(|u_0|) \leqslant \mathcal{E}(u_0)$ which shows that also $|u_0|$ is a a global minimizer for $\mathcal{E}$ on $W_0^{1,p(x)}(\Omega)$. This means that $\mathcal{E}(u_0) \leqslant \mathcal{E}(|u_0|) \leqslant$

$\mathcal{E}(u_0)$ which forces

$$\mathcal{E}(u_0) = \int_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x - \int_\Omega F(x, u_0(x)) \, \mathrm{d}x$$

$$= \mathcal{E}(|u_0|) = \int_\Omega \frac{1}{p(x)} \big| \nabla |u_0(x)| \big|^{p(x)} \, \mathrm{d}x - \int_\Omega F(x, |u_0(x)|) \, \mathrm{d}x.$$

The arguments above yield

$$\int_{\Omega^-} F(x, |u_0(x)|) \, \mathrm{d}x = \int_{\Omega^-} F(x, u_0(x)) \, \mathrm{d}x = 0.$$

Hence, we get $u_0(x) = 0$ for a.e. $x \in \Omega^-$. We have proved that $u_0 \geqslant 0$ a.e. in Ω.

We now exclude the possibility that $u_0 \equiv 0$ in Ω, that is, $u_0 = 0$ a.e. in Ω. Since $\mathcal{E}(0) = 0$, we only need to find a function $u_1 \in W_0^{1,p(x)}(\Omega)$ such that $\mathcal{E}(u_1) < 0$. Then $\mathcal{E}(u_0) \leqslant \mathcal{E}(u_1) < 0$ prevents the case $u_0 \equiv 0$ in Ω with $\mathcal{E}(u_0) = 0$. To this end, choose $\phi \in C_c^1(\Omega)$ to be an arbitrary nonnegative C^1-function with compact support in Ω, $\phi \not\equiv 0$ in Ω. For $0 < t \leqslant 1$ we estimate

$$\mathcal{E}(t\phi) = \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \phi(x)|^{p(x)} \, \mathrm{d}x - \int_\Omega F(x, t\phi(x)) \, \mathrm{d}x \qquad (3.9)$$

$$\leqslant \frac{t^{p_-}}{p_-} \int_\Omega |\nabla \phi(x)|^{p(x)} \, \mathrm{d}x - \int_\Omega F(x, t\phi(x)) \, \mathrm{d}x.$$

In order to estimate the last integral, from inequality (3.2) we deduce that, given any $\varepsilon > 0$, there is a constant $t_\varepsilon \in (0, 1]$ such that

$$F(x, t\phi(x)) \geqslant \frac{1}{r\varepsilon} [t\, \phi(x)]^r \quad \text{holds for all } (x, t) \in \overline{\Omega} \times [0, t_\varepsilon]. \qquad (3.10)$$

We apply this estimate to inequality (3.9) and recall that $1 < r \leqslant p_-$, thus arriving at

$$\mathcal{E}(t\phi) \leqslant \frac{t^r}{r} \int_\Omega |\nabla \phi(x)|^{p(x)} \, \mathrm{d}x - \frac{t^r}{r\varepsilon} \int_\Omega \phi(x)^r \, \mathrm{d}x$$

$$= -\frac{t^r}{r} \left(\frac{1}{\varepsilon} \int_\Omega \phi(x)^r \, \mathrm{d}x - \int_\Omega |\nabla \phi(x)|^{p(x)} \, \mathrm{d}x \right)$$

for all $t \in [0, t_\varepsilon]$. Choosing $\varepsilon > 0$ small enough, we conclude that $\mathcal{E}(t\phi) < 0$ whenever $0 < t \leqslant t_\varepsilon$. In addition to $u_0 \geqslant 0$ a.e. in Ω, we have proved also $u_0 \not\equiv 0$ in Ω.

Since $u_0 \in W_0^{1,p(x)}(\Omega)$ is a global minimizer for the functional $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, it is also a critical point for $\mathcal{E}$ and, hence, a nonnegative weak solution to problem (3.1) provided $u_0 \in L^\infty(\Omega)$.

Now let $u \in W_0^{1,p(x)}(\Omega)$ be any nonnegative critical point for $\mathcal{E}$, $u \not\equiv 0$ in Ω. This means that u is a weak solution to problem (3.1) in the sense of Fan and Zhao [**10**, definition 4.1, p. 311]. We may apply their regularity result [**10**, theorem 4.1, p. 312] (and its proof) to conclude that $u \in L^\infty(\Omega)$. This means that u is a nonnegative weak solution to problem (3.1) also in the sense of our definition 3.3

above. Moreover, we get $u \in C^{0,\beta'}(\Omega)$ for some $\beta' \in (0, \alpha)$, by [**10**, theorem 4.2, p. 315]. Furthermore, thanks to our Hypothesis (**p**) on p, that is, $p \in C^{0,\alpha_1}(\overline{\Omega})$ for some $\alpha_1 \in (0, 1)$, we may apply a stronger regularity result due to Fan [**7**, theorem 1.2, p. 400] to obtain $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$. Finally, we apply the strong maximum principle and the Hopf boundary point lemma, respectively, from Zhang [**23**, theorems 1.1 and 1.2, p. 26] to conclude that both inequalities claimed in (3.5) are valid.

Clearly, the global minimizer $u_0 \in W_0^{1,p(x)}(\Omega)$ for the functional $\mathcal{E}$ obtained above enjoys analogous regularity and positivity properties as does u. As a simple consequence, both ratios u/u_0 and u_0/u are continuous positive functions over the domain Ω and can be extended to positive continuous functions over the closure $\overline{\Omega}$, by l'Hospital's rule,

$$\lim_{x \to x_0} \frac{u(x)}{u_0(x)} = \lim_{t \to 0+} \frac{u(x_0 - t\boldsymbol{\nu}(x_0))}{u_0(x_0 - t\boldsymbol{\nu}(x_0))} = \frac{\partial u}{\partial \boldsymbol{\nu}}(x_0) \bigg/ \frac{\partial u_0}{\partial \boldsymbol{\nu}}(x_0) > 0, \qquad (3.11)$$

where $x_0 \in \partial\Omega$ is an arbitrary boundary point and $x \in \Omega$ ranges inside Ω near x_0, for example, $x = x_0 - t\boldsymbol{\nu}(x_0)$ with $t > 0$ small enough. The last ratio, $x_0 \mapsto (\partial u/\partial \boldsymbol{\nu})(x_0)/(\partial u_0/\partial \boldsymbol{\nu})(x_0)$, being positive and continuous over the compact boundary $\partial\Omega$, we conclude that both ratios, u/u_0 and u_0/u, can be extended to positive continuous functions over the closure $\overline{\Omega}$. Consequently, both ratios are bounded. We apply our theorem 2.5 (the Díaz and Saa inequality) to arrive at the uniqueness of a nonnegative and nontrivial weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ to problem (3.1), that is, $u = u_0$, as follows. Setting $w_1 = u$ and $w_2 = u_0$ in theorem 2.5, the left-hand side inequality (2.13) becomes

$$\int_\Omega \left(\frac{-\Delta_{p(x)}u}{u(x)^{r-1}} - \frac{-\Delta_{p(x)}u_0}{u_0(x)^{r-1}} \right) (u(x)^r - u_0(x)^r) \, \mathrm{d}x$$
$$= \int_\Omega \left(\frac{f(x, u(x))}{u(x)^{r-1}} - \frac{f(x, u_0(x))}{u_0(x)^{r-1}} \right) (u(x)^r - u_0(x)^r) \, \mathrm{d}x \leqslant 0, \qquad (3.12)$$

since the function $s \mapsto f(x, s)/s^{r-1} : (0, \infty) \to \mathbb{R}_+$ is strictly monotone decreasing for every $x \in \Omega$, by Hypothesis (**f2**). However, inequality (2.13), precisely the opposite inequality '$\geqslant$' must be valid. We conclude that the equality in (3.12) above must hold. That is possible only if $u(x) = u_0(x)$ at almost every point $x \in \Omega$, by Hypothesis (**f2**), that is, $u \equiv u_0$ in Ω, by the regularity derived above.

Our proof of theorem 3.4 is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

REMARK 3.5. In our proof of theorem 3.4 above, we have proved that any nonnegative critical point $u \in W_0^{1,p(x)}(\Omega)$ for the energy functional $\mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by equation (3.7) must be bounded, that is, $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Hence, u is a nonnegative weak solution to problem (3.1) in the sense of our definition 3.3 above. The decisive argument here is the regularity result in Fan and Zhao [**10**, theorem 4.1, p. 312].

EXAMPLE 3.6. (a) A typical example of the function f satisfying all conditions in theorem 3.4 is $f(x, s) = h(x) s^{q(x)-1}$, with a positive function $h \in C(\overline{\Omega})$ and

$q \in C(\overline{\Omega})$ such that

$$1 \leqslant q(x) \leqslant q_+ \stackrel{\text{def}}{=} \sup_{\Omega} q(x) < r = p_- \quad \text{for every } x \in \overline{\Omega}.$$

Consequently, f satisfies Hypotheses (**f1**)–(**f3**) with $r = p_-$.

(b) Our condition on r, that is, $q(x) < r = p_-$ for every $x \in \overline{\Omega}$, is trivially sharp in the following sense: If $q(x) \equiv r \equiv p(x)$ is a constant in $\overline{\Omega}$, $1 < r < \infty$, and $h(x) \equiv \lambda_{1,r}(\Omega)$ is the first eigenvalue of the 'positive' r-Laplacian $-\Delta_r u = -\text{div}(|\nabla u|^{r-2}\nabla u)$ in Ω with zero Dirichlet boundary conditions, and $\varphi_{1,r} \in W_0^{1,r}(\Omega)$ denotes the associated first eigenfunction normalized by $\varphi_{1,r}(x) > 0$ in Ω and $\int_\Omega \varphi_{1,r}(x)^r \, \mathrm{d}x = 1$, then any nonnegative multiple $t\varphi_{1,r}$ ($t \in \mathbb{R}_+$) is a nonnegative weak solution to problem (3.1); hence, this problem admits an infinite number of solutions.

In contrast, if $0 \leqslant h(x) < \lambda_{1,r}(\Omega)$ holds for all $x \in \Omega$, then the variational characterization of the first eigenvalue $\lambda_{1,r}(\Omega)$ by the Rayleigh quotient

$$\lambda_{1,r}(\Omega) = \min_{u \neq 0} \frac{\int_\Omega |\nabla u(x)|^r \, \mathrm{d}x}{\int_\Omega |u(x)|^r \, \mathrm{d}x}$$

with every minimizer $\varphi \in W_0^{1,r}(\Omega)$ taking the form $\varphi = t\varphi_{1,r}$ for some $t \in \mathbb{R} \setminus \{0\}$, leaves only the trivial zero solution to problem (3.1) with $f(x, s) = h(x) s^{r-1}$.

(c) Furthermore, if $1 \leqslant q(x) \leqslant r = p_- = \inf_{x \in \Omega} p(x)$ holds for every $x \in \Omega$, with $p(x) > r$ for all $x \in \Omega_0$ in a subset $\Omega_0 \subset \Omega$ of positive Lebesgue measure, then problem (3.1) possesses at most one nonnegative and nontrivial weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Any such weak solution u belongs to the class $C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, \alpha)$, and satisfies also the Hopf maximum principle (3.5). Of course, $u = 0$ on the boundary $\partial\Omega$. Hence, u is also a positive weak solution. This claim follows easily from the fact that the reaction function $f(x, s) = h(x) s^{q(x)-1}$ has the following properties:

(**f1'**) $f : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative continuous function satisfying $f(x, 0) = 0$ for all $x \in \Omega$ with $q(x) > 1$.

(**f2'**) The function

$$s \longmapsto \frac{f(x,s)}{s^{r-1}} = \frac{h(x)}{s^{r-q(x)}} \; : \; (0, \infty) \to \mathbb{R}_+$$

is strictly monotone decreasing for every $x \in \Omega_1 = \{x \in \Omega : q(x) < r\}$, while being $= h(x)$ for all $x \in \Omega \setminus \Omega_1 = \{x \in \Omega : q(x) = r\}$.

Recall that $h > 0$ in all of Ω. Consequently, if there were two distinct positive weak solutions, say, u_0 and u as in our proof of theorem 3.4 above, then inequality (3.12) would force $u(x) = u_0(x)$ for every $x \in \Omega_1$. Moreover, by theorem 2.5, Part (b), we have even $u(x) = u_0(x)$ for every $x \in \Omega$, thanks to $p(x) > r$ for all $x \in \Omega_0$.

(d) Similarly to case (c) above, if $1 \leqslant q(x) \leqslant r = p_- \equiv p(x)$ holds for every $x \in \Omega$, with $p(x) \equiv r \in (1, \infty)$ being a constant and $q(x) < r$ for all $x \in \Omega_1$ in a subset $\Omega_1 \subset \Omega$ of positive Lebesgue measure, then problem (3.1) possesses at most one nonnegative and nontrivial weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. The reasoning

for this is similar as in case (c): First, we may take $\Omega_1 = \{x \in \Omega : q(x) < r\}$; its Lebesgue measure is > 0. Again, inequality (3.12) forces $u(x) = u_0(x)$ for every $x \in \Omega_1$. Moreover, by theorem 2.5, Part (a), we have even $u(x) = c \cdot u_0(x)$ for almost every $x \in \Omega$, where $c \in (0, \infty)$ is a constant. But $\Omega_1 \subset \Omega$ has positive Lebesgue measure which yields $c = 1$. The uniqueness result follows.

(e) Finally, in Mihăilescu and Rădulescu [15], the nonuniqueness of weak solutions is established in case $1 < \min_{x \in \overline{\Omega}} q(x) < p_- < \max_{x \in \overline{\Omega}} q(x)$ (see also some other related results in case $p(x) = q(x)$ in Fan, Zhang, and Zhao [11]).

REMARK 3.7.

(i) Theorem 3.4 solves the open problem raised in Fan [8] (see remark 2.3 on p. 1443) and improves the uniqueness results given for problem (3.1) in [8].

(ii) The uniqueness property does not hold for solutions with changing sign, even if p is a constant. For more details, we refer to examples exhibiting two distinct critical points of the energy functional $\mathcal{E}_\lambda$, with $\lambda > 0$, defined on $W_0^{1,p}(\Omega)$ by

$$\mathcal{E}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda}{p} \int_\Omega |u|^p \, \mathrm{d}x - \int_\Omega f(x) \, u \, \mathrm{d}x.$$

Such examples were constructed in del Pino, Elgueta, and Manásevich [17, equation (5.26) on p. 12] for $2 < p < \infty$ and in Fleckinger, Hernández, and Takáč [12, example 2 on p. 148] for $1 < p < 2$.

(iii) The easiest problem of type (3.1), with the right-hand side $f(x, u) \equiv f(x)$ being independent from the unknown function $u = u(x)$, $f \in L^\infty(\Omega)$, can be treated in a similar way as in theorem 3.4; one has to take $r = 1$ in the proof, particularly in theorem 2.5 when applying it to an anlogue of inequality (3.12). Then this inequality is actually an equality with the right-hand side $= 0$. The uniqueness of a weak solution to problem

$$- \Delta_{p(x)} u = f(x) \ \text{in} \ \Omega; \quad u = 0 \ \text{on} \ \partial\Omega, \quad u > 0 \ \text{in} \ \Omega,$$

then follows by theorem 2.5, Part (a). But this uniqueness result is valid for any (possibly sign-changing) weak solution $u \in W_0^{1,p(x)}(\Omega)$, by a classical argument that takes advantage of the strict convexity of the functional $\mathcal{W}_A \equiv \mathcal{W}_{A,p(x),1}$ on $W_0^{1,p(x)}(\Omega)$ $(r = 1)$; see, for example, [6, theorem 13.3.3, p. 418] or [9, theorem 4.3, p. 1848].

Our second example is the following simple generalization of problem (3.1):

$$\begin{cases} -\Delta_{p(x)} u + g(x, u) = f(x, u) & \text{in} \ \Omega; \\ u = 0 & \text{on} \ \partial\Omega, \quad u > 0 \ \text{in} \ \Omega. \end{cases} \tag{3.13}$$

Here, we have a new monotone nonlinear operator on the left-hand side, $-\Delta_{p(x)} u + g(x, u)$, whose homogeneity properties with respect to the function u are similar to those of $-\Delta_{p(x)} u = -\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$. We recall that $p : \overline{\Omega} \to (1, \infty)$ is a continuous function, such that it satisfies Hypothesis (**p**) together with inequalities

(2.2), where $r \in \mathbb{R}$ is a given constant, $1 < r \leqslant p_-$. The function $f : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy all Hypotheses (**f1**)–(**f3**).

We impose the following hypotheses on the function g:

(**g1**) $g : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative continuous function such that $g(x, 0) = 0$ for all $x \in \Omega$ and $g(x, s) > 0$ for all $(x, s) \in \Omega \times (0, \infty)$.

(**g2**)   The function $s \longmapsto g(x,s)/s^{r-1} : (0, \infty) \to \mathbb{R}_+$ is monotone increasing for every $x \in \Omega$, but *not* necessarily *strictly* monotone increasing.

(**g3**)   The following limit is uniform with respect to $x \in \Omega$:

$$\limsup_{s \to +\infty} \frac{g(x,s)}{s^{m(x)-1}} \leqslant C \equiv \text{const} < \infty \quad \text{for all } x \in \Omega,$$

where $m : \overline{\Omega} \to \mathbb{R}_+$ is some suitable continuous function that satisfies $1 < m(x) < p^*(x)$, where

$$p^*(x) \overset{\text{def}}{=} \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N; \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

The authors in [**6**, §8.3, pp. 265–272] call $p^*(x) \in [1, +\infty]$ the *Sobolev conjugate exponent* and prove the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ for $p_+ = \sup_\Omega p(x) < N$ ([**6**, §8.3, theorem 8.3.1, p. 265]) under the additional regularity hypothesis on $p(x)$ requiring $p \in \mathcal{P}^{\log}(\Omega)$, cf. [**6**, §4.1, definition 4.1.4, p. 101], that is, $1/p(x)$ is globally log-Hölder-continuous in Ω. This additional hypothesis (log-Hölder continuity) is always satisfied in our situation, provided $p : \overline{\Omega} \to (1, \infty)$ is a continuous function that obeys our hypotheses above, that is, p satisfies Hypothesis (**p**) together with inequalities (2.2), where we now assume also $p_+ < N$, in addition to $1 < r \leqslant p_-$.

It is worth noticing that Hypotheses (**g1**) and (**g2**) imply

(**g2'**)   Also $s \longmapsto g(x,s) : \mathbb{R}_+ \to \mathbb{R}_+$ is a *strictly* monotone increasing function for every $x \in \Omega$.

Moreover, Hypotheses (**g1**) and (**g2**) combined entail

$$g(x,s) \leqslant C_0 \, s^{r-1} \quad \text{holds for all } (x,s) \in \overline{\Omega} \times [0, s_0]. \tag{3.14}$$

Here, $s_0 \in [1, \infty)$ is an arbitrary number and

$$C_0 = C_0(s_0) = \frac{\sup_{x \in \Omega} g(x, s_0)}{s_0^{r-1}} < \infty$$

is a positive constant depending solely on s_0.

We remark that Hypotheses (**g1**) and (**g3**) limit the asymptotic behaviour of $g(x, \cdot)$ as $s \to +\infty$ as follows: Given any $\varepsilon > 0$, there is a constant $C'_\varepsilon \in (0, \infty)$

such that

$$0 \leqslant g(x,s) \leqslant (C+\varepsilon)\, s^{m(x)-1} + C'_{\varepsilon} \quad \text{holds for all } (x,s) \in \overline{\Omega} \times \mathbb{R}_+. \tag{3.15}$$

In analogy with our definition 3.3 adapted to problem (3.1), we define the notion of a *nonnegative* weak solution to problem (3.13) as follows:

DEFINITION 3.8. A nonnegative function $u \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$ is called a ***nonnegative weak solution*** of problem (3.13) if, for every test function $\phi \in W_0^{1,p(x)}(\Omega)$, the following equation holds,

$$\int_{\Omega} |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla \phi(x)\, \mathrm{d}x + \int_{\Omega} g(x, u(x))\, \phi(x)\, \mathrm{d}x$$
$$= \int_{\Omega} f(x, u(x))\, \phi(x)\, \mathrm{d}x. \tag{3.16}$$

If u satisfies also $u > 0$ throughout Ω, we call u a ***positive weak solution***.

Problem (3.13) fits into a more general class of variational problems treated in Fan and Zhao [**10**, equation 4.1, p. 310]. However, the authors are interested only in some standard regularity properties of weak solutions, like (local and global) boundedness and Hölder continuity ([**10**, § 4, pp. 310–317]).

We now generalize the existence and uniqueness result in theorem 3.4 to the boundary value problem (3.13) for positive weak solutions.

THEOREM 3.9. *Under the Hypotheses* (**Ω**)*,* (**p**)*,* (**f1**)–(**f3**)*, and* (**g1**)–(**g3**)*, problem* (3.13) *possesses a unique nonnegative and nontrivial weak solution* $u \in W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)$*. This solution belongs to the class* $C^{1,\beta}(\overline{\Omega})$*, for some* $\beta \in (0, \alpha)$*, and satisfies also the Hopf maximum principle* (3.5)*,*

$$u(x) > 0 \ \ for \ all \ x \in \Omega \quad and \quad \frac{\partial u}{\partial \boldsymbol{\nu}}(x) < 0 \ \ for \ all \ x \in \partial\Omega.$$

Of course, $u = 0$ *on the boundary* $\partial\Omega$*. Hence,* u *is also a positive weak solution.*

Proof. First, let us recall that the potential F for the function f has been defined in equation (3.6). We define the potential G for the function g in a similar way: First, we extend the domain of g to all of $\overline{\Omega} \times \mathbb{R}$ by setting $g(x,s) = 0$ for $(x,s) \in \overline{\Omega} \times (-\infty, 0)$. Then we define the potential G for the function g by

$$G(x,u) \stackrel{\text{def}}{=} \int_0^u g(x,s)\,\mathrm{d}s = \begin{cases} \int_0^u g(x,s)\,\mathrm{d}s & \text{if } 0 \leqslant u < \infty; \\ 0 & \text{if } -\infty < u < 0, \end{cases} \tag{3.17}$$

for $(x, u) \in \overline{\Omega} \times \mathbb{R}$. Hence, $g(x,s) = (\partial G/\partial u)(x,s)$ for $(x,s) \in \overline{\Omega} \times \mathbb{R}$. Clearly, for each fixed $x \in \overline{\Omega}$, $G(x, \cdot) : \mathbb{R} \to \mathbb{R}_+$ is a monotone increasing function, owing to $(\partial G/\partial u)(x,s) = g(x,s) \geqslant 0$.

Again, we obtain a nonnegative weak solution to problem (3.13) from a global minimizer of the energy functional $\hat{\mathcal{E}} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$
\begin{aligned}
\hat{\mathcal{E}}(u) \equiv \hat{\mathcal{E}}_{p(x)}(u) &\overset{\text{def}}{=} \int_\Omega \frac{1}{p(x)} \, |\nabla u(x)|^{p(x)} \, \mathrm{d}x \\
&+ \int_\Omega G(x, u(x)) \, \mathrm{d}x - \int_\Omega F(x, u(x)) \, \mathrm{d}x
\end{aligned}
\tag{3.18}
$$

for every function $u \in W_0^{1,p(x)}(\Omega)$. By our proof of theorem 3.4, the first and last summands on the right-hand side of equation (3.18) are well-defined. The same is true of the second summand, thanks to inequalities (3.14) and (3.15) supplemented by the Sobolev embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ for $p_+ < N$ ([6, §8.3, theorem 8.3.1, p. 265]).

By the standard properties of the 'smaller' functional $\mathcal{E}(u) = \hat{\mathcal{E}}(u) - \int_\Omega G(x, u(x)) \, \mathrm{d}x$ defined in equation (3.7), that have been verified in the proof of theorem 3.4 above, also our present functional $\hat{\mathcal{E}} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is coercive thanks to inequality (3.3) and $r \leqslant p_-$, that is, it satisfies an analogue of (3.8),

$$
\|u\|_{W_0^{1,p(x)}(\Omega)} \overset{\text{def}}{=} \|\nabla u\|_{L^{p(x)}(\Omega)} \longrightarrow +\infty \implies \hat{\mathcal{E}}(u) \to +\infty.
$$

It is also weakly lower semicontinuous, by [6, §13.2, pp. 412–417]. Consequently, a basic result from the calculus of variations yields the existence of a global minimizer $u_0 \in W_0^{1,p(x)}(\Omega)$ for $\mathcal{E}$. We claim that $u_0 \geqslant 0$ a.e. in Ω. Clearly, also its positive part, $u_0^+ \overset{\text{def}}{=} \max\{u_0, 0\} \geqslant 0$, is in $W_0^{1,p(x)}(\Omega)$ and thus satisfies $\hat{\mathcal{E}}(u_0^+) \geqslant \hat{\mathcal{E}}(u_0)$. Denoting $\Omega^+ = \{x \in \Omega : u_0(x) \geqslant 0\}$ and $\Omega^- = \{x \in \Omega : u_0(x) \leqslant 0\}$, we calculate

$$
\begin{aligned}
\hat{\mathcal{E}}(u_0) = \int_\Omega \frac{1}{p(x)} \, |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x \\
+ \int_\Omega G(x, u_0(x)) \, \mathrm{d}x - \int_\Omega F(x, u_0(x)) \, \mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{E}}(u_0) &= \int_{\Omega^+} \frac{1}{p(x)} \, |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x + \int_{\Omega^-} \frac{1}{p(x)} \, |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x \\
&\quad + \int_{\Omega^+} G(x, u_0(x)) \, \mathrm{d}x - \int_{\Omega^+} F(x, u_0(x)) \, \mathrm{d}x \\
&= \hat{\mathcal{E}}(u_0^+) + \int_{\Omega^-} \frac{1}{p(x)} \, |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x \\
&\geqslant \hat{\mathcal{E}}(u_0) + \int_{\Omega^-} \frac{1}{p(x)} \, |\nabla u_0(x)|^{p(x)} \, \mathrm{d}x \geqslant \hat{\mathcal{E}}(u_0).
\end{aligned}
$$

These inequalities force $\nabla u_0^-(x) = -\nabla u_0(x) \equiv 0$ a.e. in Ω^-, whence $u_0^-(x) \equiv 0$ a.e. in Ω^-. We have proved $u_0 \geqslant 0$ a.e. in Ω as claimed.

In order to exclude the possibility that $u_0 \equiv 0$ in Ω, we now construct a function $u_1 \in W_0^{1,p(x)}(\Omega)$ such that $\hat{\mathcal{E}}(u_1) < 0 = \hat{\mathcal{E}}(0)$. First, we take an arbitrary nonnegative C^1-function $\phi \in C_c^1(\Omega)$ with compact support in Ω, $\phi \not\equiv 0$ in Ω. For $0 < t \leqslant 1$,

we estimate

$$\hat{\mathcal{E}}(t\phi) = \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla\phi(x)|^{p(x)} \, \mathrm{d}x + \int_\Omega G(x, t\phi(x)) \, \mathrm{d}x - \int_\Omega F(x, t\phi(x)) \, \mathrm{d}x \quad (3.19)$$

$$\leqslant \frac{t^{p_-}}{p_-} \int_\Omega |\nabla\phi(x)|^{p(x)} \, \mathrm{d}x + \int_\Omega G(x, t\phi(x)) \, \mathrm{d}x - \int_\Omega F(x, t\phi(x)) \, \mathrm{d}x.$$

We estimate the difference of the last two integrals as follows. We combine inequalities (3.2) and (3.14) to deduce that, given any $\varepsilon > 0$ small enough, $\varepsilon < 1/C_0$, there is a constant $t'_\varepsilon \in (0, 1]$ such that

$$F(x, t\phi(x)) - G(x, t\phi(x)) \geqslant \frac{1}{r\varepsilon} [t\,\phi(x)]^r - \frac{C_0}{r} [t\,\phi(x)]^r$$

$$= \frac{\varepsilon^{-1} - C_0}{r} [t\,\phi(x)]^r \qquad (3.20)$$

holds for all $(x, t) \in \overline{\Omega} \times [0, t'_\varepsilon]$.

Applying this estimate to inequality (3.19), we arrive at

$$\hat{\mathcal{E}}(t\phi) \leqslant \frac{t^r}{r} \int_\Omega |\nabla\phi(x)|^{p(x)} \, \mathrm{d}x - \frac{t^r}{r} (\varepsilon^{-1} - C_0) \int_\Omega \phi(x)^r \, \mathrm{d}x$$

$$= -\frac{t^r}{r} \left((\varepsilon^{-1} - C_0) \int_\Omega \phi(x)^r \, \mathrm{d}x - \int_\Omega |\nabla\phi(x)|^{p(x)} \, \mathrm{d}x \right)$$

for all $t \in [0, t'_\varepsilon]$. Choosing $\varepsilon > 0$ small enough, we conclude that $\hat{\mathcal{E}}(t\phi) < 0$ whenever $0 < t \leqslant t'_\varepsilon$. In addition to $u_0 \geqslant 0$ a.e. in Ω, we have proved also $u_0 \not\equiv 0$ in Ω, thanks to $\hat{\mathcal{E}}(u_0) \leqslant \hat{\mathcal{E}}(t\phi) < 0 = \hat{\mathcal{E}}(0)$.

Since $u_0 \in W_0^{1,p(x)}(\Omega)$ is a global minimizer for the functional $\hat{\mathcal{E}} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, it is also a critical point for $\hat{\mathcal{E}}$ and, hence, a nonnegative weak solution to problem (3.13) provided $u_0 \in L^\infty(\Omega)$.

Now let $u \in W_0^{1,p(x)}(\Omega)$ be any nonnegative critical point for $\hat{\mathcal{E}}$, $u \not\equiv 0$ in Ω. This means that u is a weak solution to problem (3.13) in the sense of Fan and Zhao [**10**, definition 4.1, p. 311]. We may apply their regularity result [**10**, theorem 4.1, p. 312] (and its proof) to conclude that $u \in L^\infty(\Omega)$. This means that u is a nonnegative weak solution to problem (3.1) also in the sense of our definition 3.3 above. By another result in [**10**, theorem 4.4, p. 317], u is even Hölder-continuous in $\overline{\Omega}$, $u \in C^{0,\beta'}(\overline{\Omega})$ for some $\beta' \in (0, \alpha)$.

The regularity property $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, \alpha)$ and l'Hospital's rule (3.11) are obtained by the same arguments as in the proof of theorem 3.4 above. In particular, the continuity and boundedness of both ratios, u/u_0 and u_0/u, in the closure $\overline{\Omega}$ follows. Thus, it remains to apply our theorem 2.5 (the Díaz and Saa inequality) to arrive at the uniqueness of a nonnegative and nontrivial weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ to problem (3.13), that is, $u = u_0$.

Setting $w_1 = u$ and $w_2 = u_0$ in theorem 2.5, the left-hand side of inequality (2.13) becomes

$$\int_\Omega \left(\frac{-\Delta_{p(x)}u}{u(x)^{r-1}} - \frac{-\Delta_{p(x)}u_0}{u_0(x)^{r-1}} \right) (u(x)^r - u_0(x)^r) \, \mathrm{d}x$$

$$= \int_\Omega \left[\left(\frac{f(x, u(x))}{u(x)^{r-1}} - \frac{f(x, u_0(x))}{u_0(x)^{r-1}} \right) \right.$$
$$\left. - \left(\frac{g(x, u(x))}{u(x)^{r-1}} - \frac{g(x, u_0(x))}{u_0(x)^{r-1}} \right) \right] (u(x)^r - u_0(x)^r) \, \mathrm{d}x \leqslant 0,$$

as the function $s \mapsto [f(x, s) - g(x, s)]/s^{r-1} : (0, \infty) \to \mathbb{R}_+$ is strictly monotone decreasing for every $x \in \Omega$, by Hypotheses (**f2**) and (**g2**). Since the opposite inequality '$\geqslant$' must be valid, by inequality (2.13), we conclude that the equality above must hold. This forces $u(x) = u_0(x)$ at almost every point $x \in \Omega$, by Hypotheses (**f2**) and (**g2**), that is, $u \equiv u_0$ in Ω.

The proof of theorem 3.9 is finished. $\qquad\square$

Theorem 3.9 has the following interesting special case.

COROLLARY 3.10. *Assume that* $p \in C^{0,\alpha_1}(\overline{\Omega})$ *for some* $\alpha_1 \in (0, 1)$ *and the constant* $r \in [1, \infty)$ *satisfy* Hypothesis (**p**) *together with* $r < p_-$, *that is,*

$$1 \leqslant r < p_- \stackrel{\text{def}}{=} \inf_\Omega p(x) \leqslant p_+ \stackrel{\text{def}}{=} \sup_\Omega p(x) < \infty.$$

Let $h, \ell \in C(\overline{\Omega})$ *and* $q, Q \in C(\overline{\Omega})$ *be two pairs of strictly positive functions such that*

$$\begin{cases} 1 \leqslant q_- \stackrel{\text{def}}{=} \inf_\Omega q(x) \leqslant q_+ \stackrel{\text{def}}{=} \sup_\Omega q(x) \\ < r < p_- \stackrel{\text{def}}{=} \inf_\Omega p(x) \leqslant p_+ \stackrel{\text{def}}{=} \sup_\Omega p(x) < \infty, \end{cases} \tag{3.21}$$

$$r \leqslant Q_- \stackrel{\text{def}}{=} \inf_\Omega Q(x) \leqslant Q_+ \stackrel{\text{def}}{=} \sup_\Omega Q(x) < \infty. \tag{3.22}$$

Let $f, g : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ *be defined by* $f(x, s) = h(x) s^{q(x)-1}$ *and* $g(x, s) = \ell(x) s^{Q(x)-1}$ *for* $(x, s) \in \overline{\Omega} \times \mathbb{R}_+$. *Then the conclusion of theorem 3.9 for problem* (3.13) *taking the following special form,*

$$\begin{cases} -\Delta_{p(x)}u + \ell(x) u^{Q(x)-1} = h(x) u^{q(x)-1} & in \ \Omega; \\ \qquad\qquad\qquad u = 0 & on \ \partial\Omega, \quad u > 0 \ in \ \Omega, \end{cases} \tag{3.23}$$

is valid.

Proof. It is a matter of easy, direct calculations that functions f and g satisfy all Hypotheses (**f1**)–(**f3**) and all Hypotheses (**g1**)–(**g3**), respectively. Notice that $1 \leqslant q(x) < r \leqslant \min\{p(x), Q(x)\}$ holds for all $x \in \overline{\Omega}$. $\qquad\square$

Our last application concerns a nonlocal boundary value problem of *Kirchhoff's type* involving local and nonlocal nonlinearities treated for example in Chen, Kuo, and Wu [**4**]. This problem is motivated by the stationary (elliptic) case of an evolutionary hyperbolic equation that arises in the study of string or membrane vibrations, where $u = u(x, t)$ stands for the displacement at $x \in \Omega$ and time

$t \in \mathbb{R}_+$, cf. [**4**, equation (1.2), p. 1877]. The mathematical model for the stationary displacement $u = u(x)$ at $x \in \Omega$ takes the following form,

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} \, \mathrm{d}x\right) \Delta_{p(x)} u = f(x, u) & \text{in } \Omega; \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ in } \Omega. \end{cases} \tag{3.24}$$

In the original physics problem, $p(x) \equiv 2$ is constant.

In addition to our Hypotheses (**f1**)–(**f3**) imposed on the function f at the beginning of this section, we impose the following hypotheses on the function M:

(**M1**) $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative continuous function with $M(0) > 0$.

(**M2**) $M : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing, but *not* necessarily *strictly* monotone increasing.

(**M3**) $M : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded, that is, the monotone limit $M(s) \nearrow M(+\infty) < \infty$ as $s \nearrow +\infty$ is finite.

As a consequence of Hypothesis (**M2**) we obtain also

$$M(0)\, t \leqslant \hat{M}(t) \stackrel{\text{def}}{=} \int_0^t M(s)\mathrm{d}s \leqslant M(+\infty)\, t \quad \text{for every } t \in \mathbb{R}_+. \tag{3.25}$$

Clearly, $\hat{M} : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone increasing and convex (possibly *not* strictly convex). Recalling the potential F introduced in equation (3.6), we observe that problem (3.24) corresponds to the Euler equation for a critical point $u \in W_0^{1,p(x)}(\Omega)$ of the energy functional $\mathcal{J} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}_+$ defined by

$$\mathcal{J}(u) \equiv \mathcal{J}_{p(x),f}(u) \stackrel{\text{def}}{=} \hat{M}\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} \, \mathrm{d}x\right) - \int_\Omega F(x, u(x)) \, \mathrm{d}x \tag{3.26}$$

for every function $u \in W_0^{1,p(x)}(\Omega)$. This functional is well-defined, by the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^r(\Omega)$, which is even compact, and by the estimate in (3.3). The reader is referred to the monograph by Diening *et al.* [**6**, §§ 8.3 and 8.4] for Sobolev embeddings and their compactness. It is coercive thanks to inequalities (3.3), (3.25), and $r \leqslant p_-$, that is, it satisfies an analogue of (3.8),

$$\|u\|_{W_0^{1,p(x)}(\Omega)} \stackrel{\text{def}}{=} \|\nabla u\|_{L^{p(x)}(\Omega)} \longrightarrow +\infty \quad \Longrightarrow \quad \mathcal{J}(u) \to +\infty.$$

Furthermore, it is easy to see that $\mathcal{J}$ is Gâteaux-differentiable on $W_0^{1,p(x)}(\Omega)$ with the Gâteaux derivative

$$\begin{aligned} [\mathcal{J}'(u)](x) &= \hat{M}'\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} \, \mathrm{d}x\right) \cdot \left[-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)\right] \\ &\quad - f(x, u) \\ &= M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} \, \mathrm{d}x\right) \cdot (-\Delta_{p(x)}u) - f(x, u), \quad x \in \Omega, \end{aligned} \tag{3.27}$$

which may be interpreted as a distribution over Ω, that is, it belongs to the locally convex space $\mathcal{D}'(\Omega)$ of all distributions over Ω which is the dual space of $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$.

We have the following analogue of theorems 3.4 and 3.9 for positive weak solutions to the boundary value problem (3.24):

THEOREM 3.11. *Under the Hypotheses (Ω), ($\mathbf{p}$), ($\mathbf{f1}$)–($\mathbf{f3}$), and ($\mathbf{M1}$)–($\mathbf{M3}$), the nonlocal Kirchhoff problem (3.24) possesses a unique nonnegative and nontrivial weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. This solution belongs to the class $C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, \alpha)$, and satisfies also the Hopf maximum principle (3.5),*

$$u(x) > 0 \ \text{for all} \ x \in \Omega \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}}(x) < 0 \ \text{for all} \ x \in \partial\Omega.$$

Of course, $u = 0$ on the boundary $\partial\Omega$. Hence, u is also a positive weak solution.

Proof. Although we could generalize the Díaz and Saa inequality (1.5) (proved in theorem 2.5) to the class of nonlocal quasilinear elliptic operators as suggested in the Kirchhoff problem (3.24), we prefer to give a direct proof of our theorem which, however, follows very closely the same ideas as does our proof of theorem 2.5.

We begin with the following trivial observation; we use the same notation as does our convexity result in theorem 2.5:

The composition functional $\hat{M} \circ \mathcal{W}_A : W \to \mathbb{R}_+$ is given by

$$\left[\hat{M} \circ \mathcal{W}_A\right](v) \equiv \hat{M}\left(\mathcal{W}_{A,p(x),r}(v)\right) \tag{3.28}$$

$$= \hat{M}\left(\int_\Omega \frac{r}{p(x)} A\left(x, \frac{\nabla(|v|^{1/r})}{|\nabla(|v|^{1/r})|}\right) \cdot \left|\nabla(|v|^{1/r})\right|^{p(x)} \mathrm{d}x\right)$$

for every function $v \in W$; see equations (1.1) (§1), (2.4), and (3.25). In particular, concerning the Kirchhoff problem (3.24), we take $A(x, \xi) = |\xi|^{p(x)}$ for $(x, \xi) \in \Omega \times \mathbb{R}^N$, in which case $A(x, \xi) = 1$ for all $(x, \xi) \in \Omega \times \mathbb{S}^{N-1}$.

By our convexity result in theorem 2.2, the restriction of the functional $\mathcal{W}_A : W \to \mathbb{R}_+$ to the convex cone $\overset{\bullet}{V}$ is ray-strictly convex on $\overset{\bullet}{V}$. Recall from above that $\hat{M} : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone increasing and convex. Consequently, an easy exercise in elementary analysis reveals that also the composition functional $\hat{M} \circ \mathcal{W}_A : W \to \mathbb{R}_+$ must be ray-strictly convex on $\overset{\bullet}{V}$. By our Hypotheses ($\mathbf{f1}$) and ($\mathbf{f2}$) on f, for every fixed $x \in \Omega$, also the function $t \mapsto -F(x, t^{1/r}) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly convex, owing to the partial derivative

$$t \mapsto \frac{\partial}{\partial t} F(x, t^{1/r}) = \frac{1}{r} t^{-1+(1/r)} \cdot \frac{\partial F}{\partial s}(x, t^{1/r}) = \frac{1}{r} \cdot \frac{f(x, t^{1/r})}{(t^{1/r})^{r-1}} : (0, \infty) \to \mathbb{R}_+$$

being strictly monotone decreasing on $(0, \infty)$. From these two convexity results, we deduce that also the functional

$$v \longmapsto \hat{\mathcal{J}}(v) \stackrel{\text{def}}{=} \mathcal{J}\left(|v|^{1/r}\right) = \left[\hat{M} \circ \mathcal{W}_A\right]\left(|v|^{1/r}\right) - \int_\Omega F\left(x, |v(x)|^{1/r}\right) \, \mathrm{d}x \quad (3.29)$$

$$= \hat{M}\left(\int_\Omega \frac{r}{p(x)} \cdot \left|\nabla(|v|^{1/r})\right|^{p(x)} \mathrm{d}x\right) - \int_\Omega F\left(x, |v(x)|^{1/r}\right) \mathrm{d}x$$

must be strictly convex on $\overset{\bullet}{V}$.

Now we are ready to prove the uniqueness claim in our theorem: On the contrary, let us assume that u_1, $u_2 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ are two different nonnegative and nontrivial weak solutions to problem (3.24) that satisfy also the Hopf maximum principle (3.5). In particular, we have $\mathcal{J}'(u_1) = \mathcal{J}'(u_2) = 0$ in $\mathcal{D}'(\Omega)$. Setting $v_1 = u_1^r$ and $v_2 = u_2^r$, we get also the Gâteaux derivatives $\hat{\mathcal{J}}'(v_1) = \hat{\mathcal{J}}'(v_2) = 0$ as distributions in $\mathcal{D}'(\Omega)$. Moreover, we have v_1, $v_2 \in \overset{\bullet}{V}$ and v_1/v_2, $v_2/v_1 \in L^\infty(\Omega)$. Consequently, also $v \stackrel{\text{def}}{=} (1-\theta)v_1 + \theta v_2 \in \overset{\bullet}{V}$ is valid for all $\theta \in (-\delta, 1+\delta)$, where $\delta \in (0, 1)$ is small enough. The function

$$\theta \mapsto \Phi(\theta) \stackrel{\text{def}}{=} \hat{\mathcal{J}}(v) = \hat{\mathcal{J}}\left((1-\theta)v_1 + \theta v_2\right) : (-\delta, 1+\delta) \to \mathbb{R}_+$$

is strictly convex and differentiable with the derivative

$$\Phi'(\theta) = M\left(\int_\Omega \frac{r}{p(x)} \cdot \left|\nabla(|v|^{1/r})\right|^{p(x)} \mathrm{d}x\right)$$

$$\times \int_\Omega \left|\nabla(|v(x)|^{1/r})\right|^{p(x)-2} \nabla(|v(x)|^{1/r}) \cdot \nabla\left(\frac{v_2 - v_1}{v^{1-(1/r)}}\right) \mathrm{d}x \quad (3.30)$$

$$- \frac{1}{r} \int_\Omega f(x, |v(x)|^{1/r}) \cdot \frac{v_2 - v_1}{v^{1-(1/r)}} \, \mathrm{d}x.$$

The monotonicity of the derivative $\theta \mapsto \Phi'(\theta) : (-\delta, 1+\delta) \to \mathbb{R}$ yields

$$0 \leqslant \Phi'(t) - \Phi'(0) \leqslant \Phi'(1) - \Phi'(0) \quad \text{for every } t \in [0, 1].$$

But $\hat{\mathcal{J}}'(v_1) = \hat{\mathcal{J}}'(v_2) = 0$ in $\mathcal{D}'(\Omega)$ forces $\Phi'(0) = \Phi'(1) = 0$ whence $\Phi'(t) = 0$ for every $t \in [0, 1]$. We conclude that $\Phi(t) = \Phi(0)$ for every $t \in [0, 1]$ which contradicts the strict convexity of Φ on $[0, 1]$.

The uniqueness part of our theorem is proved.

To verify the existence part, we apply analogous arguments as in our proofs of theorems 3.4 and 3.9. Recalling that the energy functional $\mathcal{J}$ defined in equation (3.26) is coercive and weakly lower semicontinuous on $W_0^{1,p(x)}(\Omega)$, by [6, § 13.2, pp. 412–417], we conclude that it possesses a global minimizer $u_0 \in W_0^{1,p(x)}(\Omega)$, by [20, theorem 1.2, p. 4]. To verify $u_0 \geqslant 0$ a.e. in Ω, we first observe that also its positive part, $u_0^+ \stackrel{\text{def}}{=} \max\{u_0, 0\} \geqslant 0$, is in $W_0^{1,p(x)}(\Omega)$ and thus satisfies $\mathcal{J}(u_0^+) \geqslant \mathcal{J}(u_0)$.

Recalling $\Omega^+ = \{x \in \Omega : u_0(x) \geqslant 0\}$ and $\Omega^- = \{x \in \Omega : u_0(x) \leqslant 0\}$, we calculate

$$
\begin{aligned}
\mathcal{J}(u_0) &= \hat{M}\left(\int_\Omega \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x, u_0(x))\,\mathrm{d}x \\
&= \hat{M}\left(\int_{\Omega^+} \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,\mathrm{d}x + \int_{\Omega^-} \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,\mathrm{d}x\right) \\
&\quad - \int_{\Omega^+} F(x, u_0(x))\,\mathrm{d}x \\
&\geqslant \hat{M}\left(\int_{\Omega^+} \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,\mathrm{d}x\right) - \int_{\Omega^+} F(x, u_0(x))\,\mathrm{d}x = \mathcal{J}(u_0^+) \geqslant \mathcal{J}(u_0).
\end{aligned}
$$

In fact, these inequalities must be equalities. Since $\hat{M} : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone increasing with $\hat{M}' = M > 0$ in $\mathbb{R}_+$, by Hypothesis (**M1**), the equalities above force $\nabla u_0^-(x) = -\nabla u_0(x) \equiv 0$ a.e. in Ω^-, whence $u_0^-(x) \equiv 0$ a.e. in Ω^-. We have proved $u_0 \geqslant 0$ a.e. in Ω as claimed.

In order to exclude the possibility that $u_0 \equiv 0$ in Ω, we construct a function $u_1 \in W_0^{1,p(x)}(\Omega)$ such that $\mathcal{J}(u_1) < 0 = \mathcal{J}(0)$. To this end, we take an arbitrary nonnegative C^1-function $\phi \in C_c^1(\Omega)$ with compact support in Ω, $\phi \not\equiv 0$ in Ω. In analogy with inequality (3.19), for $0 < t \leqslant 1$, we invoke Hypothesis (**M1**) to estimate

$$
\mathcal{J}(t\phi) = \hat{M}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\phi(x)|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x, t\phi(x))\,\mathrm{d}x \qquad (3.31)
$$

$$
\leqslant \hat{M}\left(\frac{t^{p_-}}{p_-}\int_\Omega |\nabla\phi(x)|^{p(x)}\,\mathrm{d}x\right) - \int_\Omega F(x, t\phi(x))\,\mathrm{d}x.
$$

Recall that $1 \leqslant r \leqslant p_- = \inf_\Omega p(x)$, by our Hypothesis (**p**), inequality (2.2). We take advantage of inequalities (3.25) (for $\hat{M}$) and (3.10) (for F) to estimate the last two terms inequality (3.31) above,

$$
\mathcal{J}(t\phi) \leqslant M(+\infty)\frac{t^r}{r}\int_\Omega |\nabla\phi(x)|^{p(x)}\,\mathrm{d}x - \frac{t^r}{r\varepsilon}\int_\Omega \phi(x)^r\,\mathrm{d}x
$$

$$
= -\frac{t^r}{r}\left(\frac{1}{\varepsilon}\int_\Omega \phi(x)^r\,\mathrm{d}x - M(+\infty)\int_\Omega |\nabla\phi(x)|^{p(x)}\,\mathrm{d}x\right)
$$

for all $t \in [0, t_\varepsilon]$. Choosing $\varepsilon > 0$ small enough, we conclude that $\mathcal{J}(t\phi) < 0$ whenever $0 < t \leqslant t_\varepsilon$. In addition to $u_0 \geqslant 0$ a.e. in Ω, we have proved also $u_0 \not\equiv 0$ in Ω.

Since $u_0 \in W_0^{1,p(x)}(\Omega)$ is a global minimizer for the functional $\mathcal{J} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, it is also a critical point for $\mathcal{J}$ and, hence, a nonnegative weak solution to problem (3.24) provided $u_0 \in L^\infty(\Omega)$. As

$$
M_0 = M\left(\int_\Omega \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,\mathrm{d}x\right)
$$

is a positive constant, $0 < M(0) \leqslant M_0 \leqslant M(+\infty) < \infty$, the Dirichlet problem (3.24) for $u = u_0$ is identical with that in (3.1) with $f(x, u_0)$ replaced

by $M_0^{-1} f(x, u_0)$. The rest of the proof now follows from the conclusion of theorem 3.4. □

REMARK 3.12. The question of uniqueness of general, possibly sign-changing weak solutions to problems of type (3.1) was studied in a number of articles; see, for example, Antontsev, Chipot, and Xie [1], Antontsev and Shmarev [2], Motreanu [16], and references therein. There, the function $f(x, u)$ on the right-hand side of our problem (3.1) is replaced by a somewhat more general reaction function $f(x) - b(x, u)$ for $x \in \Omega$ and $u \in \mathbb{R}$, where $s \mapsto b(x, s) : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and monotone increasing (i.e., nondecreasing) for almost every $x \in \Omega$. Some of the most recent results in [16] require that $s \mapsto b(x, s)$ be even strictly monotone increasing, although two weak comparison results in [16, theorems 4.1 and 4.2] are proved for $s \mapsto b(x, s)$ being nondecreasing only.

All these results are based on the fact that the quasilinear operator $u \mapsto -\Delta_{p(x)}u + b(x, u(x))$ is *monotone* with respect to the $L^2(\Omega)$-induced duality between $W_0^{1,p(x)}(\Omega)$ and its dual space. Of course, a mild additional condition guaranteeing some kind of *strict monotonicity* must be imposed in order to obtain the desired uniqueness result. These hypotheses on the quasilinear operator may be reformulated in terms of convexity properties of the corresponding energy functional; cf. equation (3.7).

According to the original observation in Brézis and Oswald [3] for $p(x) \equiv 2$ (a constant), much stronger convexity properties of this energy functional can be proved for *positive solutions* to problem (3.1): Namely, those first observed in Fleckinger *et al.* [12] for $p(x) \equiv p \in (1, \infty)$ (a constant) and generalized in the present paper for a variable exponent $p(x)$. The mechanism of this approach, based on [3], composes the standard convex energy functional $\mathcal{W}_{p(x),1} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}_+$ defined in equation (1.1) for $r = 1$ with the Nemytskii operator given by the concave function $s \mapsto s^{1/r} : \mathbb{R}_+ \to \mathbb{R}_+$ $(1 < r \leqslant p_- = \inf_\Omega p)$. Somewhat surprisingly, when restricted to the (convex) cone of positive functions in $W_0^{1,p(x)}(\Omega)$, this composition is still convex, even *ray-strictly convex*, as proved in our theorem 2.2. Unlike in Motreanu [16], we are then able to treat problem (3.1) with $s \mapsto b(x, s) : \mathbb{R} \to \mathbb{R}$ strictly monotone decreasing (in the notation of [16]); see Hypothesis (**f2**) and example 3.6.

Acknowledgements

References

1   S. N. Antontsev, M. Chipot and Y. Xie. Uniqueness results for equations of the p(x)-Laplacian type. *Adv. Math. Sci. Appl.* **17** (2007), 287–304.

2   S. N. Antontsev and S. I. Shmarev. Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties. *Nonlinear Anal.* **65** (2006), 728–761.

3   H. Brézis and L. Oswald. Remarks on sublinear elliptic equations. *Nonlinear Anal.* **10** (1986), 55–64.

4   C.-Y. Chen, Y.-C. Kuo and T.-F. Wu. The Nehari manifold of a Kirchhoff type problem involving sign-changing weight functions. *J. Diff. Equ.* **250** (2011), 1876–1908.

5   J. I. Díaz and J. E. Saa. Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires. *Comptes Rendus Acad. Sc. Paris.Série I* **305** (1987), 521–524.

6   L. Diening, P. Harjulehto, P. Hästö and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents.* Lecture Notes in Mathematics, vol. 2017, (Berlin-Heidelberg: Springer-Verlag, 2011).

7   X.-L. Fan. Global C^1, α regularity for variable exponent elliptic equations in divergence form. *J. Diff. Equ.* **235** (2007), 397–417.

8   X.-L. Fan. Existence and uniqueness for the $p(x)$-Laplacian-Dirichlet problems. *Math. Nachr.* **284** (2011), 1435–1445.

9   X.-L. Fan and Q.-H. Zhang. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.* **52** (2003), 1843–1852.

10 X. Fan and D. Zhao. A class of De Giorgi type and H"older continuity. *Nonlinear Anal.* **36** (1999), 295–318.

11 X. Fan, Q. Zhang and D. Zhao. Eigenvalues of $p(x)$-Laplacian Dirichlet problems. *J. Math. Anal. Appl.* **302** (2005), 306–317.

12 J. Fleckinger, J. Hernández, P. Takáč and F. de Thélin. Uniqueness and positivity for solutions of equations with the p-Laplacian. In *Proc. Conf. on Reaction-Diffusion Equations, 1995*, Trieste, Italy (eds. G. Caristi and E. Mitidieri). Lecture Notes in Pure and Applied Mathematics, vol. 194, pp. 141–155 (New York and Basel: Marcel Dekker, 1998).

13 P. Girg and P. Takáč. Bifurcations of positive and negative continua in quasilinear elliptic eigenvalue problems. Ann. Inst. Henri Poincaré. *Anal. Non Linéaire* **9** (2008), 275–327.

14 M. A. Krasnosel'skiĭ and P. P. Zabreĭko. *Geometrical methods of nonlinear analysis* (Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1984).

15 M. Mihăilescu and V. Rădulescu. On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. *Proc. Amer. Math. Soc.* **135** (2007), 2929–2937.

16 V. V. Motreanu. Uniqueness results for a Dirichlet problem with variable exponent. *Comm. Pure. Appl. Anal.* **9** (2010), 1399–1410. Online: doi: 10.3934/cpaa.2010.9.1399.

17 M. del Pino, M. Elgueta and R. F. Manásevich. A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p > 1$. *J. Diff. Equ.* **80** (1989), 1–13.

18 V. Rădulescu and D. Repovš. *Partial differential equations with variable exponents: variational methods and qualitative analysis* (Boca Raton, FL: CRC Press, Taylor and Francis Group, 2015).

19 M. Růžička. *Electrorheological fluids: modeling and mathematical theory.* Lecture Notes in Mathematics, vol. 1748 (Berlin-Heidelberg: Springer-Verlag, 2000).

20 M. Struwe. *Variational methods*, 2nd edn, in *A Series of Modern Surveys in Mathematics*, vol. 34 (Berlin-Heidelberg–New York: Springer-Verlag, 1996).

21 P. Takáč. Nonlinear spectral problems for degenerate elliptic operators. In *Handbook of differential equations: stationary partial differential equations* (eds. M. Chipot and P. Quittner), vol. 1, pp. 385–489 (Amsterdam, The Netherlands: Elsevier Science B.V., 2004).

22 P. Takáč, L. Tello and M. Ulm. Variational problems with a p-homogeneous energy. *Positivity* **6** (2001), 75–94.

23 Q. Zhang. A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. *J. Math. Anal. Appl.* **312** (2005), 24–32.