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Boundary concentrating solutions for a Hénon-like equation

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This paper is concerned with the existence and qualitative property of solutions for a Hénon-like equation

$$-\Delta u = \|x| - 2|^{\tau} u^{2^*-1-\varepsilon}, \quad u > 0,\ x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega,$$

where $\Omega = \{x \in \mathbb{R}^N : 1 < |x| < 3\}$ with $N \geqslant 4$, $2^* = 2N/(N-2)$, $\tau > 0$ and $\varepsilon > 0$ is a small parameter. For any given $k \in \mathbb{Z}^+$, we construct positive solutions concentrating simultaneously at $2k$ different points for ε sufficiently small, among which k points are near the interior boundary $\{x \in \mathbb{R}^N : |x| = 1\}$ and the other k points are near the outward boundary $\{x \in \mathbb{R}^N : |x| = 3\}$. Moreover, the $2k$ points tend to the boundary of Ω as ε goes to 0.

1. Introduction

We study the Dirichlet problem

$$\left.\begin{aligned} -\Delta u = \psi_\tau(x) u^{2^*-1-\varepsilon}, \quad u > 0,\ x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega = \{x \in \mathbb{R}^N : 1 < |x| < 3\}$ with $N \geqslant 4$, $2^* = 2N/(N-2)$, $\varepsilon > 0$ and $\psi_\tau(x) = ||x| - 2|^{\tau}$, $\tau > 0$. This problem can be regarded as a natural extension to the annular domain of the Hénon equation, which was proposed by Hénon [15] when he studied rotating stellar structures. The Hénon equation

$$\left.\begin{aligned} -\Delta u = |x|^{\tau} u^{2^*-1-\varepsilon}, \quad u > 0,\ x \in B_1(0), \\ u = 0, \quad x \in \partial B_1(0), \end{aligned}\right\} \tag{1.2}$$

where $B_1(0) \subset \mathbb{R}^N$ is a unit ball centred at the origin, has been extensively investigated. The first existence result is due to Ni [17], who obtained at least one radial solution for $\tau > 0$ and $2^* - 1 - \varepsilon \in (1, 2^* - 1 + 2\tau/(N-2))$ via the mountain pass

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theorem in the space of radial functions. The presence of numerical solutions in [11] then shows that for fixed $\tau > 0$ the ground state solution of (1.2) is not radial if ε is sufficiently small and that, for fixed $\varepsilon \in (0, 4/(N-2))$, if τ is large enough, the ground state solution of (1.2) is not radial either. This was proved by Smets *et al.* in [25]. The main idea was to compare the energy of ground state solutions and that of radially symmetric solutions. Later, by minimization under suitable symmetry constraints, Serra [24] proved the existence of at least one non-radial solution for every τ large enough, when $\varepsilon = 0$. Meanwhile, for τ sufficiently small, $\varepsilon = 0$ and a general domain Ω that is not necessarily the unit ball, Hirano [16] obtained at least one positive solution to (1.2) by a constrained variational argument. Byeon and Wang [2, 3] considered the asymptotic behaviour for the solution of (1.2) with τ large. More results about the asymptotic behaviour of ground states of (1.2) can be found in [4, 7, 9] and references therein.

In addition, the Neumann problem for the Hénon equation was investigated in [13], presenting the interesting conclusion for τ large that the ground state is radial, which is contrary to the Dirichlet case. In [10], Carrião *et al.* proved results on existence and multiplicity of non-radial solutions of quasilinear equations of the Hénon type. Calanchi and Ruf considered the case of the system in [5], and proved that the ground state solution of the Hénon-type system is not radially symmetric. For more results we refer the reader to the references therein.

The concentration behaviour of solutions (see [12, 23]) has been studied extensively. It is well known [14] that (1.2) has only radial solutions if $\tau = 0$. Thus, the only possible concentrating point is $x = 0$ if $\tau = 0$. It is necessary that $\tau > 0$ to ensure the existence of multiple concentrating solutions. Cao and Peng proved in [7] that, for ε sufficiently close to 0^+, the ground state solutions of (1.2) possess a unique maximum point whose distance from $\partial B_1(0)$ tends to 0 as $\varepsilon \to 0^+$. Peng [18] and Pistoia and Serra [20] improved the result in [7] for (1.2) for $\tau > 0$ fixed with ε going to 0, and obtained multi-bump solutions that are invariant under the action of the suitable finite subgroup of $\boldsymbol{O}(N)$ and concentrate at the boundary points of $B_1(0)$ as $\varepsilon \to 0$.

Since the weight function $\psi_\tau(x)$ on Ω reproduces a similar qualitative behaviour of $|x|^\tau$ on the unit ball $B_1(0)$ of $\mathbb{R}^N$, Calanchi *et al.* [6] considered (1.1) on the annulus. They proved the existence of two solutions for τ large, and of two additional solutions when ε is close to 0. They also proved the appearance of a symmetry-breaking phenomenon, showing that the least-energy solutions concentrate near the boundary $\partial\Omega$, and hence cannot be radial functions. However, they could not determine near which component of the boundary $\partial\Omega$ the solutions concentrate when $\varepsilon \to 0$. In our paper, we improve the result in [6] for the Hénon-like equation (1.1), and establish the existence of solutions concentrating simultaneously at both the components of the boundary of Ω. Furthermore, we give the rate of concentrating points approaching $\partial\Omega$ as $\varepsilon \to 0$.

To state the main result, we consider the functions

$$U_{y,\lambda}(x) = \frac{[N(N-2)]^{(N-2)/4}\lambda^{(N-2)/2}}{(1+\lambda^2|x-y|^2)^{(N-2)/2}},$$

where $y \in \mathbb{R}^N$, $\lambda > 0$. Then, $U_{y,\lambda}(x)$ satisfies the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$.

We denote by $PU_{y,\lambda}$ the projection onto $H_0^1(\Omega)$ of the function $U_{y,\lambda}$, namely,

$$\left.\begin{aligned} -\Delta PU_{y,\lambda} &= -\Delta U_{y,\lambda} \quad \text{in } \Omega, \\ PU_{y,\lambda} &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

Let $\boldsymbol{O}(N)$ be the set of all orthogonal transformations in $\mathbb{R}^N$ and let G be a finite subgroup of $\boldsymbol{O}(N)$ generated by g, that is, $G = \{g, g^2, \dots, g^k = \mathrm{Id}\}$ for some integer $k \geqslant 1$. The main result in this paper is stated as follows.

Theorem 1.1. *Let $\Omega = \{x \in \mathbb{R}^N : 1 < |x| < 3\}$ with $N \geqslant 4$, $\varepsilon > 0$ and $\tau > 0$. Suppose that $G = \{g, g^2, \dots, g^k = \mathrm{Id}\} \subset \boldsymbol{O}(N)$, $k \geqslant 1$, and G has no fixed points on the boundary of Ω. There then exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, problem (1.1) has a positive solution u_ε of the form*

$$u_\varepsilon(x) = \sum_{i=1}^{k} PU_{g^i z_\varepsilon, \lambda_\varepsilon}(x) + \sum_{i=1}^{k} PU_{g^i(\alpha_\varepsilon z_\varepsilon/|z_\varepsilon|), \tilde{\lambda}_\varepsilon}(x) + v_\varepsilon(x),$$

with $\lambda_\varepsilon > 0$, $\tilde{\lambda}_\varepsilon > 0$, $2 > \alpha_\varepsilon > 1$, $z_\varepsilon \in \Omega$ satisfying

$$\lambda_\varepsilon = O(\varepsilon^{-(N-1)/(N-2)}), \qquad \tilde{\lambda}_\varepsilon = O(\varepsilon^{-(N-1)/(N-2)}),$$

$$v_\varepsilon(g^i x) = v_\varepsilon(x) \quad (i = 1, \dots, k-1),$$

$$(\alpha_\varepsilon - 1) + \operatorname{dist}(z_\varepsilon, \{x \in \mathbb{R}^N : |x| = 3\}) = O(\varepsilon),$$

$$\|v_\varepsilon\|_{H_0^1(\Omega)} = \begin{cases} O(\varepsilon), & N = 4, 5, \\ O(\varepsilon |\ln \varepsilon|^{2/3}), & N = 6, \\ O(\varepsilon^{(N+2)/2(N-2)}), & N > 6. \end{cases}$$

This result provides much finer information on the asymptotic profile of solutions as $\varepsilon \to 0$. The difficulty with the proof is that the local maximum points of the solution tend to both components of $\partial\Omega$ when $\varepsilon \to 0$. Due to the fact that the boundary $\partial\Omega$ has two components and $\varphi_{y,\lambda} := U_{y,\lambda} - PU_{y,\lambda}$ and its first derivatives tend to ∞ as y approaches $\partial\Omega$, we need more precise analysis estimates. The idea of the proof is mainly inspired by that of [18, 22]. We reduce our problem to a finite-dimensional problem by Lyapunov–Schmidt reduction, and then we use Lusternik–Schnirelman theory (see [21, 23]) to solve it.

This paper is organized as follows. In § 2, we give the notation and a crucial preliminary result preparing for Lyapunov–Schmidt reduction. In § 3, we then obtain some important estimates and prove theorem 1.1 by Lusternik–Schnirelman theory. Finally, we collect the detailed estimates in appendix A.

In this paper, we use the following notation.

- C denotes the generic positive constant.
- $O(t)$, $o(t)$ denote $|O(t)| \leqslant C|t|$, $|o(t)|/t \to 0$ as $t \to 0$, respectively.
- For $u \in L^r(\Omega)$, $v \in H_0^1(\Omega)$, define $|u|_r^r = \int_\Omega |u|^r$, $\|v\|^2 = \int_\Omega |\nabla v|^2$.
- B_r stands for the ball $B_r(y)$ centred at y with radius r.
- For simplicity, denote $\mathrm{d}(y^j, \partial\Omega) = \operatorname{dist}(y^j, \partial\Omega)$.

2. Notation and preliminary results

Let $I_\varepsilon(u)$ be the associated functional to (1.1) defined by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^* - \varepsilon}\int_\Omega \psi_\tau |u|^{2^*-\varepsilon} \quad \forall u \in H_0^1(\Omega). \tag{2.1}$$

It is not difficult to verify that $I_\varepsilon \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$\langle I'_\varepsilon(u), v\rangle = \int_\Omega \nabla u \nabla v - \int_\Omega \psi_\tau |u|^{2^*-2-\varepsilon} uv \quad \forall u, v \in H_0^1(\Omega).$$

From critical point theory, the critical point of I_ε is the solution of (1.1). In the following we aim to find the critical point of the functional I_ε.

For $y = (y^1, \dots, y^{2k}) \in \mathbb{R}^N \times \cdots \times \mathbb{R}^N$, $\hat\lambda = (\lambda_1, \dots, \lambda_{2k}) \in \mathbb{R}_+^{2k}$, $\mu > 0$ small, define

$$E^{2k}_{y,\hat\lambda} = \biggl\{ v \in H_0^1(\Omega) \colon \biggl(\frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, v\biggr) = \biggl(\frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, v\biggr) = 0 \quad \text{for } j = 1, \dots, 2k,\ l = 1, \dots, N \biggr\}, \tag{2.2}$$

where $(u, v) = \int_\Omega \nabla u \nabla v$,

$$D^{2k}_\mu = \{(y, \hat\lambda) \in \Omega^{2k} \times \mathbb{R}^{2k} \colon |y^i - y^j| \geqslant c_0 > 0,\ i \neq j,\ 0 < \mathrm{d}(y^j, \partial\Omega) < \mu, \quad \lambda_j\, \mathrm{d}(y^j, \partial\Omega) > \mu^{-1},\ \lambda_j < \mathrm{e}^{\mu/\varepsilon} \text{ for } i, j = 1, \dots, 2k\}, \tag{2.3}$$

where c_0 is a constant. Set

$$M^{2k}_\mu = \{(y, \hat\lambda, v) \colon (y, \hat\lambda) \in D^{2k}_\mu,\ v \in E^{2k}_{y,\hat\lambda},\ \|v\| < \mu\}. \tag{2.4}$$

We build solutions for (1.1) that look like a sum of concentrated solutions for (1.3) centred at several points. More precisely, we seek a solution u_ε of (1.1) having the form $u_\varepsilon(x) = \sum_{j=1}^{2k} PU_{y^j,\lambda_j} + v$ with $(y, \hat\lambda, v) \in M^{2k}_\mu$ for some suitable $\mu > 0$. Define

$$J_\varepsilon(y, \hat\lambda, v) = I_\varepsilon\biggl(\sum_{j=1}^{2k} PU_{y^j,\lambda_j} + v\biggr). \tag{2.5}$$

It is well known (see [1, 22]) that if $\mu > 0$ is small enough,

$$u = \sum_{j=1}^{2k} PU_{y^j,\lambda_j} + v$$

is a positive critical point of $I_\varepsilon(u)$ in $H_0^1(\Omega)$ if and only if $(y, \hat\lambda, v)$ is a critical point of $J_\varepsilon(y, \hat\lambda, v)$ in M^{2k}_μ. On the other hand, by the Lagrange multiplier rule, $(y, \hat\lambda, v) \in M^{2k}_\mu$ is a critical point of $J_\varepsilon(y, \hat\lambda, v)$ if and only if there exist $Y_j \in \mathbb{R}$, $Z_{jl} \in \mathbb{R}$ $(l = 1, \dots, N,\ j = 1, \dots, 2k)$ such that

$$\frac{\partial J_\varepsilon(y, \hat\lambda, v)}{\partial \lambda_j} = Y_j\biggl(\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial \lambda_j^2}, v\biggr) + \sum_{l=1}^N Z_{jl}\biggl(\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial \lambda_j \partial y^j_l}, v\biggr), \quad j = 1, \dots, 2k, \tag{2.6}$$

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$$\frac{\partial J_\varepsilon(y,\hat\lambda,v)}{\partial y_i^j} = Y_j\biggl(\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial\lambda_j\partial y_i^j}, v\biggr) + \sum_{l=1}^{N} Z_{jl}\biggl(\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial y_l^j\partial y_i^j}, v\biggr),$$
$$i = 1,\dots,N,\ j = 1,\dots,2k, \tag{2.7}$$

$$\frac{\partial J_\varepsilon(y,\hat\lambda,v)}{\partial v} = \sum_{j=1}^{2k} Y_j \frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j} + \sum_{j=1}^{2k}\sum_{l=1}^{N} Z_{jl}\frac{\partial PU_{y^j,\lambda_j}}{\partial y_l^j}. \tag{2.8}$$

To prove theorem 1.1, we first give the following proposition to reduce the problem of finding the critical point of J_ε in M_μ^{2k} to a finite-dimensional problem. Throughout this paper, define

$$\mathrm{d}_j := \mathrm{d}(y^j, \partial\Omega) \quad\text{and}\quad \varepsilon_{ij} := \frac{1}{(\lambda_i\lambda_j|y^i - y^j|^2)^{(N-2)/2}} \quad \text{for } i \neq j.$$

Proposition 2.1. *There exist $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that, for $\varepsilon \in (0,\varepsilon_0]$, $\mu \in (0,\mu_0]$, there exists a unique C^1-map*

$$(y,\hat\lambda) \in D_\mu^{2k} \hookrightarrow v_{y,\hat\lambda} \in E_{y,\hat\lambda}^{2k}$$

such that $(y,\hat\lambda, v_{y,\hat\lambda})$ satisfies (2.8) for some Y_j, Z_{jl} $(l = 1,\dots,N,\ j = 1,\dots,2k)$. Furthermore, we have

$$\|v_{y,\hat\lambda}\| = O\biggl(\sum_{j=1,j\neq i}^{2k} \varepsilon_{ij}^{(1/2+\theta)}\biggr) + O\biggl(\sum_{j=1}^{2k}(\mathrm{d}_j + \varepsilon\ln\lambda_j)\biggr) + \sum_{j=1}^{2k} V(\lambda_j, \mathrm{d}_j), \tag{2.9}$$

where

$$V(\lambda,d) = \begin{cases} O(\lambda^{2-N}d^{2-N}), & N = 4,5,\\ O((\lambda d)^{-4}\ln^{2/3}(\lambda d)), & N = 6,\\ O((\lambda d)^{-(N+2)/2}), & N > 6.\end{cases}$$

Proof. Expanding $J_\varepsilon(y,\hat\lambda,v)$, we obtain

$$J_\varepsilon(y,\hat\lambda,v) = J_\varepsilon(y,\hat\lambda,0) + f_\varepsilon(v) + \tfrac{1}{2}\langle A_\varepsilon(v), v\rangle + R_\varepsilon(v), \tag{2.10}$$

where

$$f_\varepsilon(v) = \sum_{j=1}^{2k}\int_\Omega U_{y^j,\lambda_j}^{2^*-1} v - \int_\Omega \psi_\tau\biggl(\sum_{j=1}^{2k} PU_{y^j,\lambda_j}\biggr)^{2^*-1-\varepsilon} v, \tag{2.11}$$

$$\langle A_\varepsilon(v), v\rangle = \int_\Omega |\nabla v|^2 - (2^* - 1 - \varepsilon)\int_\Omega \psi_\tau\biggl(\sum_{j=1}^{2k} PU_{y^j,\lambda_j}\biggr)^{2^*-2-\varepsilon} v^2 \tag{2.12}$$

and

$$D^{(i)}R_\varepsilon(v) = O(\|v\|^{2+\vartheta-i}), \quad i = 0,1,2. \tag{2.13}$$

Here, $\vartheta > 0$ is a constant.

It follows from the fact that f_ε is a continuous form over $E^{2k}_{y,\hat\lambda}$. Then there exists a unique $\hat f_\varepsilon \in E^{2k}_{y,\hat\lambda}$ satisfying

$$f_\varepsilon(v) = (\hat f_\varepsilon, v) \quad \forall v \in E^{2k}_{y,\hat\lambda}. \tag{2.14}$$

Similarly, we obtain that A_ε is a continuous linear operator from $E^{2k}_{y,\hat\lambda}$ to $E^{2k}_{y,\hat\lambda}$. By lemma 2.2 below, we see that, for μ and ε sufficiently small, A_ε is invertible and $\|A_\varepsilon^{-1}\| \leqslant \rho^{-1}$. Using this notation, we have that

$$\frac{\partial J_\varepsilon(y,\hat\lambda,v)}{\partial v}\bigg|_{E^{2k}_{y,\hat\lambda}} = \hat f_\varepsilon + A_\varepsilon v + DR_\varepsilon(v).$$

There is an equivalence between the existence of some constants Y_j, Z_{jl} $(l = 1,\dots,N,\ j=1,\dots,2k)$ such that (2.8) is satisfied and

$$\hat f_\varepsilon + A_\varepsilon v + DR_\varepsilon(v) = 0. \tag{2.15}$$

As in [22], by the implicit function theorem we get $\varepsilon_0 > 0$, $\mu_0 > 0$ and a C^1-map $v_{y,\hat\lambda}\colon (y,\hat\lambda) \in D^{2k}_\mu \hookrightarrow E^{2k}_{y,\hat\lambda}$ for $\varepsilon \in (0,\varepsilon_0]$, $\mu \in (0,\mu_0]$ satisfying (2.15) and

$$\|v_{y,\hat\lambda}\| \leqslant C\|\hat f_\varepsilon\|. \tag{2.16}$$

We now estimate $\|\hat f_\varepsilon\|$. It follows from lemma A.1 in Appendix A and [1] that

$$\begin{aligned}
&\sum_{j=1}^{2k}\int_\Omega U^{2^*-1}_{y^j,\lambda_j} v - \int_\Omega \psi_\tau\biggl(\sum_{j=1}^{2k} PU_{y^j,\lambda_j}\biggr)^{2^*-1-\varepsilon} v \\
&\quad = \sum_{j=1}^{2k}\int_\Omega U^{2^*-1}_{y^j,\lambda_j} v - \int_\Omega \psi_\tau \sum_{j=1}^{2k} PU^{2^*-1-\varepsilon}_{y^j,\lambda_j} v \\
&\qquad + \begin{cases} O\biggl(\displaystyle\int_\Omega \sum_{i=1,\, i\neq j}^{2k} PU^{2^*-2-\varepsilon}_{y^i,\lambda_i} PU_{y^j,\lambda_j} v\biggr), & 2^*-1-\varepsilon > 2, \\ O\biggl(\displaystyle\int_\Omega \sum_{i<j}^{2k} PU^{(2^*-1-\varepsilon)/2}_{y^i,\lambda_i} PU^{(2^*-1-\varepsilon)/2}_{y^j,\lambda_j} v\biggr), & 2^*-1-\varepsilon \leqslant 2 \end{cases} \\
&\quad = O\biggl(\sum_{j=1}^{2k}(\mathrm{d}_j + \varepsilon\ln\lambda_j)\biggr)\|v\| + O\biggl(\sum_{j=1,\, j\neq i}^{2k} \varepsilon_{ij}^{(1/2+\theta)}\biggr)\|v\| + \sum_{j=1}^{2k} V(\lambda_j,\mathrm{d}_j)\|v\|,
\end{aligned}$$

where θ is a small positive constant.

Consequently, we obtain that, for $\mu > 0$ and $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
\|v_{y,\hat\lambda}\| &\leqslant C\|\hat f_\varepsilon\| \\
&= O\biggl(\sum_{j=1,\, j\neq i}^{2k} \varepsilon_{ij}^{(1/2+\theta)}\biggr) + O\biggl(\sum_{j=1}^{2k}(\mathrm{d}_j + \varepsilon\ln\lambda_j)\biggr) + \sum_{j=1}^{2k} V(\lambda_j,\mathrm{d}_j),
\end{aligned}$$

which leads to (2.9). □

LEMMA 2.2. *Let $(y,\hat\lambda)\in D^{2k}_\mu$. For $\mu>0$ and $\varepsilon>0$ sufficiently small, there exists a $\rho>0$ such that*

$$\|A_\varepsilon v\|\geqslant\rho\|v\|\quad\forall v\in E^{2k}_{y,\hat\lambda}.$$

Proof. We argue by contradiction. Suppose that there exist $\varepsilon_n\to 0$, $\mu_n\to 0$, $(y_n,\hat\lambda_n):=(y^{1,n},\dots,y^{2k,n},\lambda_{1,n},\dots,\lambda_{2k,n})\in D^{2k}_{\mu_n}$ and $v_n\in E^{2k}_{y_n,\hat\lambda_n}$ such that

$$\|A_\varepsilon v_n\|=o_n(1)\|v_n\|, \tag{2.17}$$

where $o_n(1)\to 0$ as $n\to\infty$. Without loss of generality we assume that $\|v_n\|=1$.

For $j=1,\dots,2k$, let

$$\tilde v_{j,n}(x)=\lambda_{j,n}^{(2-N)/2}v_n(\lambda_{j,n}^{-1}x+y^{j,n}).$$

Set $v_n(x)=0$ if $x\in\mathbb{R}^N\setminus\Omega$. Then, $\tilde v_{j,n}(x)$ is bounded in $D^{1,2}(\mathbb{R}^N)$, and hence we may assume that there exists $v_j\in D^{1,2}(\mathbb{R}^N)$ such that, as $n\to\infty$,

$$\tilde v_{j,n}(x)\to v_j\quad\text{weakly in } D^{1,2}(\mathbb{R}^N).$$

Now, arguing as in [19], and by the estimates in [1], we can verify that $v_j\equiv 0$, and hence

$$\begin{aligned}
&\int_\Omega\psi_\tau\biggl(\sum_{j=1}^{2k}PU_{y^j_n,\lambda_{j,n}}\biggr)^{2^*-2-\varepsilon_n}v_n^2\\
&\quad\leqslant C\sum_{j=1}^{2k}\int_\Omega PU^{2^*-2}_{y^j_n,\lambda_{j,n}}v_n^2+O\biggl(\int_\Omega\sum_{i\neq j}PU^{(2^*-2-\varepsilon_n)/2}_{y^i_n,\lambda_{j,n}}PU^{(2^*-2-\varepsilon_n)/2}_{y^j_n,\lambda_{j,n}}v_n^2\biggr)\\
&\quad\leqslant C\sum_{j=1}^{2k}\int_{\mathbb{R}^N}U^{2^*-2}_{y^j_n,\lambda_{j,n}}v_n^2+o_n(1)\|v_n\|^2\\
&\quad\leqslant C\sum_{j=1}^{2k}\int_{\mathbb{R}^N\setminus B_{R/\lambda_{j,n}}(y^j_n)}U^{2^*-2}_{y^j_n,\lambda_{j,n}}v_n^2+o_n(1)=o_R(1)+o_n(1),
\end{aligned}$$

where $o_R(1)\to 0$ as $R\to\infty$. Combining this with (2.17), we conclude that $\|v_n\|\to 0$ as $n\to\infty$, which contradicts $\|v_n\|=1$. Therefore, the desired result is proved. □

For the rest of this paper, we take $\lambda_1=\lambda_2=\cdots=\lambda_k=\lambda\in\mathbb{R}_+$ and $\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{2k}=\tilde\lambda\in\mathbb{R}_+$. Let $G=\{g,g^2,\dots,g^k=\mathrm{Id}\}\subset\boldsymbol{O}(N)$. We use d to denote $d_1=\cdots=d_k$ and $\tilde d$ to denote $d_{k+1}=\cdots=d_{2k}$. We define

$$\tilde D^{2k}_\mu=\biggl\{(y,\hat\lambda)=\biggl(z,gz,\dots,g^{k-1}z,\alpha\frac{z}{|z|},\dots,g^{k-1}\biggl(\alpha\frac{z}{|z|}\biggr),\lambda,\dots,\lambda,\tilde\lambda,\dots,\tilde\lambda\biggr):\ (z,\alpha,\lambda,\tilde\lambda)\in\tilde B_\mu\biggr\},$$

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where

$$\tilde{B}_\mu = \bigg\{(z,\alpha,\lambda,\tilde{\lambda}) \in \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \colon 0 < d = \mathrm{d}(z, \{x \in \mathbb{R}^N \colon |x| = 3\}) < \mu, \\ \tilde{\lambda}, \lambda < \mathrm{e}^{\mu/\varepsilon},\ 0 < \tilde{d} = \alpha - 1 < \mu,\ \lambda d > \frac{1}{\mu},\ \tilde{\lambda}\tilde{d} > \frac{1}{\mu}\bigg\}.$$

Next, we show that $v_{y,\hat{\lambda}}$ is invariant under the act of orthogonal transformations of G. This conclusion will be used in proving theorem 1.1.

Lemma 2.3. *If $(y,\hat{\lambda}) \in \tilde{D}^{2k}_\mu$, $v_{y,\hat{\lambda}}$ is obtained in proposition 2.1, then $v_{y,\hat{\lambda}}(g^j x) = v_{y,\hat{\lambda}}(x)$ for $j = 1, \dots, k-1$.*

Proof. Set $\hat{v}(x) = v_{y,\hat{\lambda}}(g^j x)$. Then

$$\begin{aligned} J_\varepsilon(y,\hat{\lambda},\hat{v}) &= I_\varepsilon\bigg(\sum_{i=1}^{2k} PU_{y^i,\lambda_i} + \hat{v}\bigg) = I_\varepsilon\bigg(\sum_{i=1}^{2k} PU_{y^i,\lambda_i}(g^{-j}x) + v_{y,\hat{\lambda}}(x)\bigg) \\ &= I_\varepsilon\bigg(\sum_{i=1}^{2k} PU_{g^j y^i,\lambda_i}(x) + v_{y,\hat{\lambda}}(x)\bigg) = I_\varepsilon\bigg(\sum_{i=1}^{2k} PU_{y^i,\lambda_i}(x) + v_{y,\hat{\lambda}}(x)\bigg) \\ &= J_\varepsilon(y,\hat{\lambda},v_{y,\hat{\lambda}}). \end{aligned}$$

It follows from the uniqueness of the C^1-map obtained in proposition 2.1 that $\hat{v} = v_{y,\hat{\lambda}}$. This verifies the conclusion. □

3. Proof of the main result

In this section, we prove that for the $Y_j, Z_{jl} \in \mathbb{R}$ obtained in proposition 2.1 satisfying (2.8) there exists $(y_\varepsilon, \hat{\lambda}_\varepsilon) \in \tilde{D}^{2k}_\mu$ such that (2.6) and (2.7) are satisfied by $(y_\varepsilon, \hat{\lambda}_\varepsilon, v_{y_\varepsilon,\hat{\lambda}_\varepsilon})$. First, we give some estimates.

Lemma 3.1. *Let $(y,\hat{\lambda}) \in \tilde{D}^{2k}_\mu$ and $v_{y,\hat{\lambda}} \in E^{2k}_{y,\hat{\lambda}}$. For $\mu > 0$ and $\varepsilon > 0$ sufficiently small, for $j = 1, \dots, 2k$, we obtain*

$$\begin{aligned} \frac{\partial J_\varepsilon(y,\hat{\lambda},v)}{\partial \lambda_j} &= \frac{A\varepsilon}{\lambda_j} - \frac{F}{\lambda_j^{N-1}} H(y^j,y^j) \\ &\quad + O\bigg(\frac{\varepsilon \mathrm{d}_j + \varepsilon^2 \ln\lambda_j}{\lambda_j} + \frac{\varepsilon\ln\lambda_j + \mathrm{d}_j}{\lambda_j^{N-1}\mathrm{d}_j^{N-2}} + \frac{1}{\lambda_j^3} + \sum_{i=1,\, i\neq j}^{2k} \frac{1}{\lambda_i^{(N+2)/2}\lambda_j^{N/2}} \\ &\qquad + \frac{1}{\lambda_j^N \mathrm{d}_j^{N-1}} + \sum_{i=1,\, i\neq j}^{2k} \frac{1}{(\lambda_i\lambda_j)^{((N-2)/2)(\theta+1/2)}\lambda_j}\bigg) \\ &\quad + O\bigg(\frac{1}{\lambda_j^2} + \frac{\varepsilon\ln\lambda_j}{\lambda_j} + \sum_{i=1,\, i\neq j}^{2k} \frac{1}{(\lambda_i\lambda_j)^{((N-2)/2)(\theta+1/2)}\lambda_j}\bigg)\|v\|, \end{aligned} \tag{3.1}$$

where θ is a small positive constant, A, F are positive constants depending on N, and $H(y,x)$ denotes the regular part of Green's function $G(y,x)$, which is defined in appendix A.

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Proof. A direct computation shows that

$$\begin{aligned}\frac{\partial J_\varepsilon(y,\hat\lambda,v)}{\partial\lambda_j} &= \int_\Omega \nabla\Big(\sum_{i=1}^{2k} PU_{y^i,\lambda_i}+v\Big)\nabla\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\\ &\quad-\int_\Omega \psi_\tau\Big|\sum_{i=1}^{2k} PU_{y^i,\lambda_i}+v\Big|^{2^*-2-\varepsilon}\Big(\sum_{i=1}^{2k} PU_{y^i,\lambda_i}+v\Big)\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\\ &=: I_1-I_2.\end{aligned}$$

By the orthogonality condition (2.2), lemma A.2 and the estimates in [1], we obtain

$$\begin{aligned}I_1 &= \int_\Omega \nabla PU_{y^j,\lambda_j}\nabla\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}+\sum_{i=1,\,i\neq j}^{2k}\int_\Omega \nabla PU_{y^i,\lambda_i}\nabla\frac{\partial PU_{y^i,\lambda_i}}{\partial\lambda_j}+\int_\Omega\nabla\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\nabla v\\ &= \frac{(N-2)G}{2\lambda_j^{N-1}}H(y^j,y^j)+O\Big(\frac{1}{\lambda_j^{N+1}\mathrm{d}_j^N}\Big)\\ &\quad+\sum_{i=1,\,i\neq j}^{2k}\int_\Omega U_{y^i,\lambda_i}^{2^*-1}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}+O\Big(\frac{1}{\lambda_j^2}\Big)\|v\|\\ &= \frac{(N-2)G}{2\lambda_j^{N-1}}H(y^j,y^j)+O\Big(\frac{1}{\lambda_j^{N+1}\mathrm{d}_j^N}\Big)+O\Big(\sum_{i=1,\,i\neq j}^{2k}\frac{G(y^i,y^j)}{\lambda_i^{(N-2)/2}\lambda_j^{N/2}}\Big)\\ &\quad+O\Big(\sum_{i=1,\,i\neq j}^{2k}\frac{1}{\lambda_i^{(N+2)/2}\lambda_j^{N/2}}\Big)+O\Big(\frac{1}{\lambda_j^2}\Big)\|v\|,\\ I_2 &= \int_\Omega\psi_\tau\Big(\sum_{i=1}^{2k}PU_{y^i,\lambda_i}\Big)^{2^*-1-\varepsilon}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\\ &\quad+(2^*-1-\varepsilon)\int_\Omega\psi_\tau\Big(\sum_{i=1}^{2k}PU_{y^i,\lambda_i}\Big)^{2^*-2-\varepsilon}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}v\\ &\quad+O\Big(\int_\Omega\Big(\sum_{i=1}^{2k}PU_{y^i,\lambda_i}\Big)^{2^*-3-\varepsilon}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}v^2\\ &\qquad\qquad+\int_\Omega\Big|\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\Big||v|^{2^*-1-\varepsilon}\text{ if }2^*>3\Big)\\ &=: I_{21}+I_{22}+I_{23}+I_{24}.\end{aligned}$$

Using lemma A.3 and the estimates in [1], we have

$$\begin{aligned}I_{21} &= \int_\Omega\psi_\tau\sum_{i=1}^{2k}(PU_{y^i,\lambda_i})^{2^*-1-\varepsilon}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\\ &\quad+\begin{cases}O\Big(\displaystyle\int_\Omega\sum_{i=1,\,i\neq l}^{2k}PU_{y^i,\lambda_i}^{2^*-2}PU_{y^l,\lambda_l}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\Big), & 2^*-1-\varepsilon>2,\\ O\Big(\displaystyle\int_\Omega\sum_{i<l}^{2k}PU_{y^i,\lambda_i}^{(2^*-1)/2}PU_{y^l,\lambda_l}^{(2^*-1)/2}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\Big), & 2^*-1-\varepsilon\leqslant 2\end{cases}\end{aligned}$$

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$$= -\frac{\varepsilon A}{\lambda_j} + \frac{(N-2)G+2F}{2\lambda_j^{N-1}} H(y^j, y^j) + O\bigg(\sum_{i=1,\, i\neq j}^{2k} \frac{G(y^i, y^j)}{\lambda_i^{(N-2)/2}\lambda_j^{N/2}} \bigg)$$
$$+ O\bigg(\frac{\varepsilon \ln \lambda_j + \mathrm{d}_j}{\lambda_j^{N-1}\mathrm{d}_j^{N-2}} + \frac{\varepsilon^2 \ln\lambda_j + \varepsilon \mathrm{d}_j}{\lambda_j} + \frac{1}{\lambda_j^N \mathrm{d}_j^{N-1}}$$
$$+ \frac{1}{\lambda_j^3} + \sum_{i=1,\, i\neq j}^{2k} \frac{1}{(\lambda_i\lambda_j)^{((N-2)/2)(\theta+1/2)}\lambda_j} \bigg).$$

It follows from lemma A.4 that

$$I_{22} = O\bigg(\int_\Omega \bigg(\sum_{i=1}^{2k} PU_{y^i,\lambda_i} \bigg)^{2^*-2-\varepsilon} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j} v \bigg)$$
$$= O\bigg(\int_\Omega \sum_{i=1}^{2k} PU_{y^i,\lambda_i}^{2^*-3-\varepsilon} \inf(PU_{y^l,\lambda_l}, PU_{y^i,\lambda_i}) \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j} v \bigg)$$
$$+ O\bigg(\int_\Omega \sum_{i=1}^{2k} PU_{y^i,\lambda_i}^{2^*-2-\varepsilon} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j} v \bigg)$$
$$= O\bigg(\frac{\varepsilon \ln \lambda_j}{\lambda_j} + \frac{1}{\lambda_j(\lambda_j \mathrm{d}_j)^{(N-2)/2}} + \sum_{i=1,\, i\neq j}^{2k} \frac{1}{(\lambda_i\lambda_j)^{((N-2)/2)(\theta+1/2)}\lambda_j} \bigg) \|v\|.$$

Similarly, from lemmas A.5 and A.6, we obtain

$$I_{23} = O(\lambda_j^{-1}\|v\|^2), \qquad I_{24} = O(\lambda_j^{-1}\|v\|^{2^*-1-\varepsilon}).$$

Adding the above estimates, the claim follows. □

Arguing as in the proof of lemma 3.1, from lemmas A.7–A.10 in appendix A and the estimates from [1], we obtain the following lemma.

LEMMA 3.2. *Let $(y, \hat{\lambda}) \in \tilde{D}_\mu^{2k}$ and let $v_{y,\hat{\lambda}} \in E_{y,\hat{\lambda}}^{2k}$. For $\mu > 0$ and $\varepsilon > 0$ sufficiently small, we have, for $j = 1, \dots, 2k$, $i = 1, \dots, N$, that*

$$\frac{\partial J_\varepsilon(y, \hat{\lambda}, v)}{\partial y_i^j} = \frac{G}{\lambda_j^{N-2}} \frac{\partial H(y^j, y^j)}{\partial y_i^j} - D_i \psi_\tau(y^j) B$$
$$+ O\bigg(\varepsilon \ln \lambda_j + \frac{\varepsilon \ln \lambda_j + \mathrm{d}_j}{\lambda_j^{N-2}\mathrm{d}_j^{N-1}} + \frac{1}{\lambda_j^2} + \frac{1}{\lambda_j^{N-1}\mathrm{d}_j^N}$$
$$+ \sum_{l=1,\, l\neq j}^{2k} \frac{1}{(\lambda_l\lambda_j)^{(N-2)/2}\lambda_j} + \sum_{l=1,\, l\neq j}^{2k} \frac{\lambda_j}{(\lambda_i\lambda_j)^{((N-2)/2)(1/2+\theta)}} \bigg)$$
$$+ O\bigg(\frac{1}{\mathrm{d}_j(\lambda_j \mathrm{d}_j)^{(N-2)/2}} + 1 + \varepsilon\lambda_j \ln \lambda_j$$
$$+ \sum_{l=1,\, l\neq j}^{2k} \frac{\lambda_j}{(\lambda_i\lambda_j)^{((N-2)/2)(1/2+\theta)}} \bigg) \|v\|, \tag{3.2}$$

where D_i stands for the derivative with respect to y_i^j, θ is a small positive constant, G, B are positive constants depending on N, $H(y,x)$ is the same as in lemma 3.1.

Let

$$K_\varepsilon(y^1,\dots,y^{2k},\lambda_1,\dots,\lambda_{2k}) = J_\varepsilon(y,\hat\lambda,v_{y,\hat\lambda}),$$
$$L_\varepsilon(z,\alpha,\lambda,\tilde\lambda) = K_\varepsilon\left(z,gz,\dots,g^{k-1}z,\alpha\frac{z}{|z|},\dots,g^{k-1}\left(\alpha\frac{z}{|z|}\right),\lambda,\dots,\lambda,\tilde\lambda,\dots,\tilde\lambda\right)$$

for $(z,\alpha,\lambda,\tilde\lambda)\in\tilde B_\mu$.

Define

$$\begin{aligned}M_\varepsilon = \{(z,\alpha,\lambda,\tilde\lambda)\in\tilde B_\mu\colon \gamma_1\varepsilon\leqslant d=\operatorname{dist}(z,\{x\in\mathbb{R}^N\colon |x|=3\})\leqslant\gamma_2\varepsilon,\\ \lambda\in[\gamma_3\varepsilon^{-(N-1)/(N-2)},\gamma_4\varepsilon^{-(N-1)/(N-2)}],\ \gamma_5\varepsilon\leqslant\tilde d=\alpha-1\leqslant\gamma_6\varepsilon,\\ \tilde\lambda\in[\gamma_7\varepsilon^{-(N-1)/(N-2)},\gamma_8\varepsilon^{-(N-1)/(N-2)}]\},\end{aligned}$$

where $\gamma_i\in(0,+\infty)$, $i=1,\dots,8$, will be chosen later. Next, we have the following lemma.

LEMMA 3.3. *For γ_i, $i=1,\dots,8$, suitably chosen, $L_\varepsilon(z,\alpha,\lambda,\tilde\lambda)$ has at least one critical point in M_ε.*

Proof. First, we estimate $\partial K_\varepsilon/\partial\lambda_j$ and $\partial K_\varepsilon/\partial y_i^j$.

$$\frac{\partial K_\varepsilon}{\partial\lambda_j} = \frac{\partial J_\varepsilon}{\partial\lambda_j} + \left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v}{\partial\lambda_j}\right\rangle,\quad j=1,\dots,2k, \tag{3.3}$$

$$\frac{\partial K_\varepsilon}{\partial y_i^j} = \frac{\partial J_\varepsilon}{\partial y_i^j} + \left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v}{\partial y_i^j}\right\rangle,\quad i=1,\dots,N,\ j=1\dots,2k. \tag{3.4}$$

Since $\partial J_\varepsilon/\partial\lambda_j$ and $\partial J_\varepsilon/\partial y_i^j$ have already been estimated in lemma 3.1 and lemma 3.2, respectively, to estimate $\partial K_\varepsilon/\partial\lambda_j$ and $\partial K_\varepsilon/\partial y_i^j$, we only have to estimate the products

$$\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v}{\partial\lambda_j}\right\rangle\quad\text{and}\quad\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v}{\partial y_i^j}\right\rangle.$$

Writing

$$\frac{\partial v}{\partial\lambda_j} = \omega + \sum_{i=1}^{2k} b_i\frac{\partial PU_{y^i,\lambda_i}}{\partial\lambda_i} + \sum_{i=1}^{2k}\sum_{l=1}^{N} c_l^i\frac{\partial PU_{y^i,\lambda_i}}{\partial y_l^i}, \tag{3.5}$$

where $b_i, c_l^i\in\mathbb{R}$ $(i=1,\dots,2k,\ l=1,\dots,N)$ and $\omega\in E^{2k}_{y,\lambda}$.

We find

$$\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v}{\partial\lambda_j}\right\rangle = \sum_{i=1}^{2k} b_i\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial PU_{y^i,\lambda_i}}{\partial\lambda_i}\right\rangle + \sum_{i=1}^{2k}\sum_{l=1}^{N} c_l^i\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial PU_{y^i,\lambda_i}}{\partial y_l^i}\right\rangle. \tag{3.6}$$

Moreover,

$$\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial PU_{y^i,\lambda_i}}{\partial\lambda_i}\right\rangle = \frac{\partial J_\varepsilon}{\partial\lambda_i},\qquad\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial PU_{y^i,\lambda_i}}{\partial y_l^i}\right\rangle = \frac{\partial J_\varepsilon}{\partial y_l^i}. \tag{3.7}$$

Furthermore, if we take the scalar product in $H_0^1(\Omega)$ of (3.5) with $\partial PU_{y^i,\lambda_i}/\partial\lambda_i$ and $\partial PU_{y^i,\lambda_i}/\partial y_l^i$, respectively ($i = 1,\dots,2k$, $l = 1,\dots,N$), we get a quasi-diagonal linear system with b_j and c_l^j unknown. Solving this system (for details, we refer the reader to [22]), we obtain

$$b_j = O(\|v\|), \quad c_l^j = O\biggl(\frac{\|v\|}{\lambda_j^2}\biggr), \quad j = 1,\dots,2k,\ l = 1,\dots,N. \tag{3.8}$$

Taking account of (3.6) and (3.8), we have that

$$\begin{aligned}\biggl\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial v}{\partial \lambda_j}\biggr\rangle &= \sum_{i=1}^{2k} O(\|v\|)\frac{\partial J_\varepsilon}{\partial \lambda_i} + \sum_{i=1}^{2k}\sum_{l=1}^{N} O\biggl(\frac{\|v\|}{\lambda_i^2}\biggr)\frac{\partial J_\varepsilon}{\partial y_l^i}\\ &= O\biggl(\sum_{i=1}^{2k}\frac{\varepsilon}{\lambda_i} + \sum_{i=1}^{2k}\frac{1}{\lambda_i^{N-1}d_i^{N-2}} + \sum_{i=1}^{2k}\frac{1}{\lambda_i^2}\biggr)\|v\|.\end{aligned}$$

Substituting the above estimate and (3.1) into (3.3), we deduce that

$$\begin{aligned}\frac{\partial K_\varepsilon}{\partial \lambda_j} &= \frac{A\varepsilon}{\lambda_j} - \frac{F}{\lambda_j^{N-1}}H(y^j,y^j)\\ &\quad + O\biggl(\sum_{i=1,\,j\neq i}^{2k}\frac{1}{\lambda_j(\lambda_i\lambda_j)^{(1/2+\theta)((N-2)/2)}}\\ &\qquad\qquad + \frac{\varepsilon \mathrm{d}_j + \varepsilon^2\ln\lambda_j}{\lambda_j} + \frac{1}{\lambda_j^N \mathrm{d}_j^{N-1}} + \frac{1}{\lambda_j^3} + \frac{\varepsilon\ln\lambda_j + \mathrm{d}_j}{\lambda_j^{N-1}\mathrm{d}_j^{N-2}}\biggr)\\ &\quad + O\biggl(\frac{1}{\lambda_j^2} + \sum_{i=1}^{2k}\frac{1}{\lambda_i^{N-1}d_i^{N-2}} + \frac{\varepsilon\ln\lambda_j}{\lambda_j} + \sum_{i=1,\,i\neq j}^{2k}\frac{1}{\lambda_j(\lambda_i\lambda_j)^{(1/2+\theta)((N-2)/2)}}\biggr)\|v\|\\ &=: \frac{A\varepsilon}{\lambda_j} - \frac{F}{\lambda_j^{N-1}}H(y^j,y^j) + V_{\hat\lambda}(\varepsilon, d, \tilde d, \hat\lambda).\end{aligned} \tag{3.9}$$

In a similar way, we also have that

$$\begin{aligned}\frac{\partial K_\varepsilon}{\partial y_i^j} &= \frac{G}{\lambda_j^{N-2}}\frac{\partial H(y^j,y^j)}{\partial y_i^j} - D_i\psi_\tau(y^j)B\\ &\quad + O\biggl(\varepsilon\ln\lambda_j + \frac{\varepsilon\ln\lambda_j + \mathrm{d}_j}{\lambda_j^{N-2}\mathrm{d}_j^{N-1}} + \frac{1}{\lambda_j^{N-1}\mathrm{d}_j^N} + \frac{1}{\lambda_j^2}\\ &\qquad\qquad + \sum_{l=1,\,l\neq j}^{2k}\frac{1}{(\lambda_l\lambda_j)^{(N-2)/2}\lambda_j} + \sum_{l=1,\,l\neq j}^{2k}\frac{\lambda_j}{(\lambda_i\lambda_j)^{((N-2)/2)(1/2+\theta)}}\biggr)\\ &\quad + O\biggl(\frac{1}{\mathrm{d}_j(\lambda_j\mathrm{d}_j)^{(N-2)/2}} + 1 + \varepsilon\lambda_j\ln\lambda_j + \sum_{l=1,\,l\neq j}^{2k}\frac{\lambda_j}{(\lambda_i\lambda_j)^{((N-2)/2)(1/2+\theta)}}\biggr)\|v\|\\ &=: \frac{G}{\lambda_j^{N-2}}\frac{\partial H(y^j,y^j)}{\partial y_i^j} - D_i\psi_\tau(y^j)B + V_{y^j}(\varepsilon, d, \tilde d, \hat\lambda).\end{aligned} \tag{3.10}$$

Set $\Omega_d = \{z \in \Omega\colon \operatorname{dist}(z, \{z \in \Omega\colon |z| = 3\}) > d\}$, and denote by $n(z)$ the unit outward normal at $z \in \partial\Omega_d$. Applying the fact in [21] that, for $d = \operatorname{dist}(z, \{z \in \Omega\colon |z| = 3\})$ sufficiently small,

$$H(z, z) = \frac{1}{2^{N-2}d^{N-2}} + o\biggl(\frac{1}{d^{N-2}}\biggr), \tag{3.11}$$

$$\frac{\partial H(z, z)}{\partial z_i} = \frac{N-2}{2^{N-2}d^{N-1}} n_i(z) + o\biggl(\frac{1}{d^{N-1}}\biggr), \quad i = 1, \dots, N. \tag{3.12}$$

Using the definition of $L_\varepsilon(z, \alpha, \lambda, \tilde{\lambda})$, for $\varepsilon > 0$ sufficiently small and

$$(y^1, y^2, \dots, y^k, y^{k+1}, \dots, y^{2k}) = \biggl(z, gz, \dots, g^{k-1}z, \alpha\frac{z}{|z|}, \dots, g^{k-1}\biggl(\alpha\frac{z}{|z|}\biggr)\biggr),$$

we conclude that

$$\begin{aligned}
\frac{\partial L_\varepsilon}{\partial z_i} &= \frac{\partial K_\varepsilon}{\partial y_i^1} + \sum_{j=2}^{k}\sum_{l=1}^{N} \frac{\partial K_\varepsilon}{\partial y_l^j}\frac{\partial y_l^j}{\partial z_i} + \sum_{j=k+1}^{2k}\sum_{l=1}^{N} \frac{\partial K_\varepsilon}{\partial y_l^j}\frac{\partial y_l^j}{\partial z_i} \\
&= \frac{G}{\lambda^{N-2}}\frac{\partial H(z, z)}{\partial z_i} - D_i\psi_\tau(z)B \\
&\quad + \sum_{j=2}^{2k}\sum_{l=1}^{N}\biggl(\frac{G}{\lambda_j^{N-2}}\frac{\partial H(y^j, y^j)}{\partial y_l^j} - D_l\psi_\tau(y^j)B\biggr)\frac{\partial y_l^j}{\partial z_i} \\
&\quad + O\biggl(\sum_{j=1}^{2k} V_{y^j}(\varepsilon, d, \tilde{d}, \hat{\lambda})\biggr) \\
&= \biggl(\frac{G}{\lambda^{N-2}}\frac{N-2}{2^{N-2}d^{N-1}} - \tau B\biggr)n_i(z) \\
&\quad + \sum_{j=2}^{k}\sum_{l=1}^{N}\biggl(\frac{G}{\lambda_j^{N-2}}\frac{N-2}{2^{N-2}\mathrm{d}_j^{N-1}} - \tau B\biggr)n_l(y^j)\frac{\partial y_l^j}{\partial z_i} \\
&\quad + O\biggl(\sum_{j=1}^{k} V_{y^j}(\varepsilon, d, \tilde{d}, \hat{\lambda}) + d\biggr) + o\biggl(\frac{1}{\lambda^{N-2}d^{N-1}}\biggr) \\
&= k\biggl(\frac{G}{\lambda^{N-2}}\frac{N-2}{2^{N-2}d^{N-1}} - \tau B\biggr)n_i(z) \\
&\quad + O\biggl(\sum_{j=1}^{k} V_{y^j}(\varepsilon, d, \tilde{d}, \hat{\lambda}) + d\biggr) + o\biggl(\frac{1}{\lambda^{N-2}d^{N-1}}\biggr),
\end{aligned}$$

where the third equality is the consequence of the properties of orthogonal transformation and

$$\begin{aligned}
\sum_{l=1}^{N} n_l\biggl(\alpha\frac{z}{|z|}\biggr)\frac{\partial(\alpha z_l/|z|)}{\partial z_i} &= -\alpha\sum_{l=1}^{N} n_l(z)\biggl(\frac{\delta_{il}}{|z|} - \frac{z_i z_l}{|z|^3}\biggr) = -\alpha\biggl(\frac{n_i(z)}{|z|} - \sum_{l=1}^{N} n_l(z)\frac{z_i z_l}{|z|^3}\biggr) \\
&= -\alpha\frac{n_i(z)}{|z|} + \alpha\sum_{l=1}^{N} n_l^2(z)\frac{z_i}{|z|^2} = -\alpha\frac{n_i(z)}{|z|} + \alpha\frac{z_i}{|z|^2} = 0,
\end{aligned}$$

where $\delta_{il} = 1$ if $i = l$ and $\delta_{il} = 0$ if $i \neq l$.

On the other hand,

$$\left.\begin{aligned}\frac{\partial L_\varepsilon}{\partial\lambda}&=\sum_{i=1}^{k}\frac{\partial K_\varepsilon}{\partial\lambda_i}\bigg|_{(z,gz,\dots,g^{k-1}z,\alpha z/|z|,\dots,g^{k-1}(\alpha z/|z|),\lambda,\dots,\lambda,\tilde\lambda,\dots,\tilde\lambda)},\\ \frac{\partial L_\varepsilon}{\partial\tilde\lambda}&=\sum_{i=k+1}^{2k}\frac{\partial K_\varepsilon}{\partial\lambda_i}\bigg|_{(z,gz,\dots,g^{k-1}z,\alpha z/|z|,\dots,g^{k-1}(\alpha z/|z|),\lambda,\dots,\lambda\tilde\lambda,\dots,\tilde\lambda)}.\end{aligned}\right\} \tag{3.13}$$

We are now able to estimate the derivatives of $L_\varepsilon(z,\alpha,\lambda,\tilde\lambda)$ on the boundary ∂M_ε. Using the fact that $d=O(\varepsilon)$, $\tilde d=(\alpha-1)=O(\varepsilon)$, $\lambda=O(\varepsilon^{-(N-1)/(N-2)})$, $\tilde\lambda=O(\varepsilon^{-(N-1)/(N-2)})$, for each $(z,\alpha,\lambda,\tilde\lambda)\in M_\varepsilon$, we have

$$V_{\hat\lambda}(\varepsilon,d,\tilde d,\hat\lambda)=\bar V_{\hat\lambda}(\varepsilon)=O(\varepsilon^{2(N-1)/(N-2)}) \tag{3.14}$$

and

$$V_{y^j}(\varepsilon,d,\tilde d,\hat\lambda)=\bar V_{y^j}(\varepsilon)=O(\varepsilon^{1/(N-2)}). \tag{3.15}$$

Therefore, we obtain

$$\frac{\partial L_\varepsilon}{\partial\lambda}=\sum_{j=1}^{k}\frac{\partial K_\varepsilon}{\partial\lambda_j}=\frac{kA\varepsilon}{\lambda}-\frac{kF}{2^{N-2}}\frac{1}{\lambda^{N-1}d^{N-2}}+\bar V_\lambda(\varepsilon)+o\bigg(\frac{1}{\lambda^{N-1}d^{N-2}}\bigg), \tag{3.16}$$

$$\frac{\partial L_\varepsilon}{\partial n}=k\bigg(\frac{G}{\lambda^{N-2}}\frac{N-2}{2^{N-2}d^{N-1}}-\tau B\bigg)+O\bigg(\sum_{j=1}^{k}V_{y^j}(\varepsilon,\lambda,\mathrm{d}_j)+d\bigg)+o\bigg(\frac{1}{\lambda^{N-2}d^{N-1}}\bigg) \tag{3.17}$$

and

$$\begin{aligned}\frac{\partial L_\varepsilon}{\partial\alpha}&=-k\bigg(\frac{(N-2)G}{2^{N-2}}\frac{1}{(\tilde\lambda)^{N-2}(\alpha-1)^{N-1}}-\tau B\bigg)\\&\quad+O\bigg(\sum_{j=k+1}^{2k}\bar V_{y^j}(\varepsilon)+\tilde d\bigg)+o\bigg(\frac{1}{\tilde\lambda^{N-2}\tilde d^{N-1}}\bigg),\end{aligned} \tag{3.18}$$

$$\frac{\partial L_\varepsilon}{\partial\tilde\lambda}=\sum_{j=k+1}^{2k}\frac{\partial K_\varepsilon}{\partial\lambda_j}=\frac{kA\varepsilon}{\tilde\lambda}-\frac{kF}{2^{N-2}}\frac{1}{\tilde\lambda^{N-1}(\alpha-1)^{N-2}}+\bar V_{\tilde\lambda}(\varepsilon)+o\bigg(\frac{1}{\tilde\lambda^{N-1}\tilde d^{N-2}}\bigg). \tag{3.19}$$

Choosing

$$\begin{aligned}\gamma_1=\gamma_5&=\bigg(\frac12\bigg)^{\sigma}\frac{(N-2)GA}{\tau BF},\\ \gamma_2=\gamma_6&=2^{\sigma}\frac{(N-2)GA}{\tau BF},\\ \gamma_3=\gamma_7&=\frac12\frac{\tau B}{2(N-2)G}\bigg(\frac{F}{A}\bigg)^{(N-1)/(N-2)},\\ \gamma_4=\gamma_8&=\frac32\frac{\tau B}{2(N-2)G}\bigg(\frac{F}{A}\bigg)^{(N-1)/(N-2)},\end{aligned}$$

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where $(N-2)/(N-1) < \sigma < 1$. Then, for $\varepsilon > 0$ sufficiently small, we have, for all $(z, \alpha, \lambda, \tilde{\lambda}) \in M_\varepsilon$, that

$$\frac{\partial L_\varepsilon}{\partial \lambda}(z, \alpha, \gamma_3 \varepsilon^{-(N-1)/(N-2)}, \tilde{\lambda}) < 0 < \frac{\partial L_\varepsilon}{\partial \lambda}(z, \alpha, \gamma_4 \varepsilon^{-(N-1)/(N-2)}, \tilde{\lambda}),$$
$$\frac{\partial L_\varepsilon}{\partial \tilde{\lambda}}(z, \alpha, \lambda, \gamma_7 \varepsilon^{-(N-1)/(N-2)}) < 0 < \frac{\partial L_\varepsilon}{\partial \tilde{\lambda}}(z, \alpha, \lambda, \gamma_8 \varepsilon^{-(N-1)/(N-2)})$$

and

$$\frac{\partial L_\varepsilon}{\partial \alpha}(z, 1+\gamma_6\varepsilon, \lambda, \tilde{\lambda}) > 0 > \frac{\partial L_\varepsilon}{\partial \alpha}(z, 1+\gamma_5\varepsilon, \lambda, \tilde{\lambda}).$$

Combining (3.15) with (3.17) we obtain, for $\varepsilon > 0$ sufficiently small,

$$\frac{\partial L_\varepsilon}{\partial n}(z, \alpha, \lambda, \tilde{\lambda}) < 0 \quad \forall z \in \partial \Omega_{\gamma_2 \varepsilon},$$
$$\frac{\partial L_\varepsilon}{\partial n}(z, \alpha, \lambda, \tilde{\lambda}) > 0 \quad \forall z \in \partial \Omega_{\gamma_1 \varepsilon}.$$

From the Lusternik–Schnirelman theory (see [21, 23]) we deduce that $L_\varepsilon(z, \alpha, \lambda, \tilde{\lambda})$ has at least one critical point in M_ε. □

Proof of theorem 1.1. First, we prove that if $(z_\varepsilon, \alpha_\varepsilon, \lambda_\varepsilon, \tilde{\lambda}_\varepsilon) \in M_\varepsilon$ is a critical point of $L_\varepsilon(z, \alpha, \lambda, \tilde{\lambda})$, then

$$\left.\frac{\partial K_\varepsilon(y^1, \dots, y^{2k}, \lambda_1, \dots, \lambda_{2k})}{\partial y_i^j}\right|_{\substack{(z_\varepsilon, g z_\varepsilon, \dots, g^{k-1} z_\varepsilon, \alpha_\varepsilon z_\varepsilon/|z_\varepsilon|, \dots, \\ g^{k-1}(\alpha_\varepsilon z_\varepsilon/|z_\varepsilon|), \lambda_\varepsilon, \dots, \lambda_\varepsilon, \tilde{\lambda}_\varepsilon, \dots, \tilde{\lambda}_\varepsilon)}} = 0, \tag{3.20}$$

$$\left.\frac{\partial K_\varepsilon(y^1, \dots, y^{2k}, \lambda_1, \dots, \lambda_{2k})}{\partial \lambda_j}\right|_{\substack{(z_\varepsilon, g z_\varepsilon, \dots, g^{k-1} z_\varepsilon, \alpha_\varepsilon z_\varepsilon/|z_\varepsilon|, \dots, \\ g^{k-1}(\alpha_\varepsilon z_\varepsilon/|z_\varepsilon|), \lambda_\varepsilon, \dots, \lambda_\varepsilon, \tilde{\lambda}_\varepsilon, \dots, \tilde{\lambda}_\varepsilon)}} = 0 \tag{3.21}$$

for any $j = 1, \dots, 2k$, $i = 1, \dots, N$. For simplicity, we define $v_\varepsilon = v_{y_\varepsilon, \hat{\lambda}_\varepsilon}$ and

$$(y_\varepsilon, \hat{\lambda}_\varepsilon) = \left(z_\varepsilon, g z_\varepsilon, \dots, g^{k-1} z_\varepsilon, \alpha_\varepsilon \frac{z_\varepsilon}{|z_\varepsilon|}, \dots, g^{k-1}\left(\alpha_\varepsilon \frac{z_\varepsilon}{|z_\varepsilon|}\right), \lambda_\varepsilon, \dots, \lambda_\varepsilon, \tilde{\lambda}_\varepsilon, \dots, \tilde{\lambda}_\varepsilon\right).$$

By lemma 2.3, we have that, for $j = 1, \dots, k$, $(y, \hat{\lambda}) \in \tilde{D}_\mu^{2k}$, $K_\varepsilon(y, \hat{\lambda})$ satisfies

$$K_\varepsilon(g^j y, \hat{\lambda}) = K_\varepsilon(y, \hat{\lambda}).$$

Using the result in [8], we find that (3.20) holds if $(z_\varepsilon, \alpha_\varepsilon, \lambda_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a critical point of $L_\varepsilon(z, \alpha, \lambda, \tilde{\lambda})$.

For the proof of (3.21), according to (3.16) and (3.19), it suffices to prove that,

$$\forall (y, \hat{\lambda}) = \left(z, gz, \dots, g^{k-1} z, \frac{\alpha z}{|z|}, \dots, g^{k-1}\left(\frac{\alpha z}{|z|}\right), \lambda, \dots, \lambda, \tilde{\lambda}, \dots, \tilde{\lambda}\right) \in \tilde{D}_\mu^{2k},$$
$$\frac{\partial K_\varepsilon(y, \hat{\lambda})}{\partial \lambda_i} = \frac{\partial K_\varepsilon(y, \hat{\lambda})}{\partial \lambda_j} \tag{3.22}$$

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for $i,j=1,\dots,k$, or $i,j=k+1,\dots,2k$. We surely have that (3.22) holds, especially for

$$(y,\hat\lambda)=\left(z_\varepsilon, gz_\varepsilon,\dots,g^{k-1}z_\varepsilon,\alpha_\varepsilon\frac{z_\varepsilon}{|z_\varepsilon|},\dots,g^{k-1}\left(\alpha_\varepsilon\frac{z_\varepsilon}{|z_\varepsilon|}\right),\lambda_\varepsilon,\dots,\lambda_\varepsilon,\tilde\lambda_\varepsilon,\dots,\tilde\lambda_\varepsilon\right).$$

Recalling (2.8), we obtain

$$\begin{aligned}\frac{\partial K_\varepsilon(y,\hat\lambda)}{\partial\lambda_i}&=\frac{\partial J_\varepsilon}{\partial\lambda_i}+\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v_{y,\hat\lambda}}{\partial\lambda_i}\right\rangle\\&=\frac{\partial J_\varepsilon}{\partial\lambda_i}-\sum_{j=1}^{2k}Y_j\left\langle\frac{\partial}{\partial\lambda_i}\left(\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j}\right),v_{y,\hat\lambda}\right\rangle\\&\qquad-\sum_{j=1}^{2k}\sum_{l=1}^{N}Z_{jl}\left\langle\frac{\partial}{\partial\lambda_i}\left(\frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}\right),v_{y,\hat\lambda}\right\rangle\\&=\frac{\partial J_\varepsilon}{\partial\lambda_i}-Y_i\left\langle\frac{\partial^2 PU_{y^i,\lambda_i}}{\partial\lambda_i^2},v_{y,\hat\lambda}\right\rangle-\sum_{l=1}^{N}Z_{il}\left\langle\frac{\partial^2 PU_{y^i,\lambda_i}}{\partial\lambda_i\partial y^i_l},v_{y,\hat\lambda}\right\rangle.\end{aligned}$$

Since $(y,\hat\lambda)\in\tilde D^{2k}_\mu$, $v_{y,\hat\lambda}(g^jx)=v_{y,\hat\lambda}(x)$ $(j=1,\dots,k-1)$, we have

$$\frac{\partial J_\varepsilon}{\partial\lambda_i}=\frac{\partial J_\varepsilon}{\partial\lambda_j},$$
$$\left\langle\frac{\partial^2 PU_{y^i,\lambda_i}}{\partial\lambda_i^2},v_{y,\hat\lambda}\right\rangle=\left\langle\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial\lambda_j^2},v_{y,\hat\lambda}\right\rangle$$

and

$$\left\langle\frac{\partial^2 PU_{y^i,\lambda_i}}{\partial\lambda_i\partial y^i_l},v_{y,\hat\lambda}\right\rangle=\left\langle\frac{\partial^2 PU_{y^j,\lambda_j}}{\partial\lambda_j\partial y^j_l},v_{y,\hat\lambda}\right\rangle,$$

$i,j=1,\dots,k$ or $i,j=k+1,\dots,2k$. To obtain (3.22), we need only show that

$$Y_i=Y_j,\quad Z_{il}=Z_{jl},\quad l=1,\dots,N. \tag{3.23}$$

Here, $i,j=1,\dots,k$ or $i,j=k+1,\dots,2k$.

Let Y, $\tilde Y$ and Z_l, $\tilde Z_l$ be determined by the following systems:

$$\begin{aligned}&\left\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial PU_{y^1,\lambda_1}}{\partial\lambda_1}\right\rangle\\&\quad=Y\left\langle\sum_{j=1}^{k}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j},\frac{\partial PU_{y^1,\lambda_1}}{\partial\lambda_1}\right\rangle+\tilde Y\left\langle\sum_{j=k+1}^{2k}\frac{\partial PU_{y^j,\lambda_j}}{\partial\lambda_j},\frac{\partial PU_{y^1,\lambda_1}}{\partial\lambda_1}\right\rangle\\&\qquad+\sum_{l=1}^{N}Z_l\left\langle\sum_{j=1}^{k}\frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l},\frac{\partial PU_{y^1,\lambda_1}}{\partial\lambda_1}\right\rangle\\&\qquad+\sum_{l=1}^{N}\tilde Z_l\left\langle\sum_{j=k+1}^{2k}\frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l},\frac{\partial PU_{y^1,\lambda_1}}{\partial\lambda_1}\right\rangle,\end{aligned}\tag{3.24}$$

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$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle$$
$$= Y\left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle + \tilde{Y}\left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle$$
$$+ \sum_{l=1}^{N} Z_l \left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle$$
$$+ \sum_{l=1}^{N} \tilde{Z}_l \left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle, \quad m = 1, \dots, N, \tag{3.25}$$

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle$$
$$= Y\left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle + \tilde{Y}\left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle$$
$$+ \sum_{l=1}^{N} Z_l \left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle$$
$$+ \sum_{l=1}^{N} \tilde{Z}_l \left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle, \tag{3.26}$$

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle$$
$$= Y\left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle + \tilde{Y}\left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle$$
$$+ \sum_{l=1}^{N} Z_l \left\langle \sum_{j=1}^{k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle$$
$$+ \sum_{l=1}^{N} \tilde{Z}_l \left\langle \sum_{j=k+1}^{2k} \frac{\partial PU_{y^j,\lambda_j}}{\partial y^j_l}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle, \quad m = 1, \dots, N. \tag{3.27}$$

It is not difficult to check that Y, $\tilde{Y}$ and Z_l, $\tilde{Z}_l$ are uniquely determined by (3.24)–(3.27). By the fact that $(y, \hat{\lambda}) \in \tilde{D}^{2k}_\mu$, $y^j = g^{j-1}z$, $y^{k+j} = g^{j-1}(\alpha z/|z|)$ $(j = 1, \dots, k)$, $\lambda_1 = \cdots = \lambda_k = \lambda$ and $\lambda_{k+1} = \cdots = \lambda_{2k} = \tilde{\lambda}$, for such Y, $\tilde{Y}$ and Z_l, $\tilde{Z}_l$, we also have, for $h = 2, \dots, k$, that

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^h,\lambda_h}}{\partial \lambda_h} \right\rangle = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^1,\lambda_1}}{\partial \lambda_1} \right\rangle,$$
$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^h,\lambda_h}}{\partial y^h_m} \right\rangle = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^1,\lambda_1}}{\partial y^1_m} \right\rangle.$$

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Similarly, for $h = k+2, \dots, 2k$,

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^h,\lambda_h}}{\partial \lambda_h} \right\rangle = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial \lambda_{k+1}} \right\rangle,$$
$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^h,\lambda_h}}{\partial y^h_m} \right\rangle = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial PU_{y^{k+1},\lambda_{k+1}}}{\partial y^{k+1}_m} \right\rangle.$$

On the other hand, observe that all the constants Y_j, Z_{jl}, $j = 1, \dots, 2k$, $l = 1, \dots, N$, are uniquely determined by the systems obtained by taking the inner product of (2.8) with $\partial PU_{y^j,\lambda_j}/\partial\lambda_j$, $\partial PU_{y^j,\lambda_j}/\partial y^j_i$, respectively. Therefore, we have $Y_j = Y$, $Z_{jl} = Z_l$, $Y_{k+j} = \tilde{Y}$, $Z_{k+j,l} = \tilde{Z}_l$, $j = 1, \dots, k$, $l = 1, \dots, N$.

Next, by lemma 3.3, (3.20) and (3.21), we deduce that

$$\left(\frac{\partial J_\varepsilon(y,\hat\lambda,v_{y,\hat\lambda})}{\partial \lambda_j} + \left\langle \frac{\partial J_\varepsilon(y,\hat\lambda,v_{y,\hat\lambda})}{\partial v}, \frac{\partial v_{y,\hat\lambda}}{\partial \lambda_j} \right\rangle \right)\bigg|_{(y,\hat\lambda,v_{y,\hat\lambda})=(y_\varepsilon,\hat{\lambda_\varepsilon},v_\varepsilon)} = 0,$$
$$\left(\frac{\partial J_\varepsilon(y,\hat\lambda,v_{y,\hat\lambda})}{\partial y^j_l} + \left\langle \frac{\partial J_\varepsilon(y,\hat\lambda,v_{y,\hat\lambda})}{\partial v}, \frac{\partial v_{y,\hat\lambda}}{\partial y^j_l} \right\rangle \right)\bigg|_{(y,\hat\lambda,v_{y,\hat\lambda})=(y_\varepsilon,\hat{\lambda_\varepsilon},v_\varepsilon)} = 0$$

for $j = 1, \dots, 2k$, $l = 1, \dots, N$.

It follows from (2.8) that $(y_\varepsilon, \hat\lambda_\varepsilon, v_\varepsilon)$ satisfies (2.6) and (2.7). Therefore, $(y_\varepsilon, \hat\lambda_\varepsilon, v_\varepsilon)$ is a critical point of J_ε, and hence

$$u_\varepsilon(x) = \sum_{i=1}^k PU_{g^i z_\varepsilon,\lambda_\varepsilon}(x) + \sum_{i=1}^k PU_{g^i(\alpha_\varepsilon z_\varepsilon/|z_\varepsilon|),\tilde\lambda_\varepsilon}(x) + v_\varepsilon(x)$$

is a critical point of I_ε in $H^1_0(\Omega)$. Moreover, $u_\varepsilon > 0$ in Ω. In fact, multiplying the equation by $u^-_\varepsilon = \max\{-u_\varepsilon, 0\}$ and integrating on Ω and using the Sobolev inequality, we have either $u^-_\varepsilon \equiv 0$ or $\|u^-_\varepsilon\| \geqslant c_0 > 0$. However, $\|v_\varepsilon\| = o(1)$ and $|u^-_\varepsilon|_{2^*} \leqslant |v_\varepsilon|_{2^*}$ imply that $u^-_\varepsilon \equiv 0$; therefore, $u_\varepsilon > 0$ follows from the maximum principle for the weak solution.

Finally, the estimates for α_ε, $\operatorname{dist}(z_\varepsilon, \{x \in \mathbb{R}^N : |x| = 3\})$, λ_ε and $\|v_\varepsilon\|$ follow from the proof of lemma 3.3 and proposition 2.1.

As a result, the proof is complete. □

Appendix A.

From [22] we see that

$$PU_{y,\lambda} = U_{y,\lambda} - \varphi_{y,\lambda}, \varphi_{y,\lambda} = \frac{1}{\lambda^{(N-2)/2}} H(y,\cdot) + f_{y,\lambda},$$

where $H(y,\cdot) = 1/|y-\cdot|^{N-2} - G(y,\cdot)$ in Ω and, for any $y \in \Omega$, G satisfies

$$-\Delta G(y,\cdot) = \rho_N \delta_y \quad \text{in } \Omega,$$
$$G(y,\cdot) = 0 \quad \text{on } \partial\Omega,$$

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where δ_y denotes the Dirac mass at y and $\rho_N = (N-2)\,\mathrm{meas}(S^{N-1})$. It also follows from [22] that

$$f_{y,\lambda} = O\biggl(\frac{1}{\lambda^{(N+2)/2}d^N}\biggr),$$
$$\frac{\partial f_{y,\lambda}}{\partial y_i} = O\biggl(\frac{1}{\lambda^{(N+2)/2}d^{N+1}}\biggr), \qquad \frac{\partial f_{y,\lambda}}{\partial \lambda} = O\biggl(\frac{1}{\lambda^{(N+4)/2}d^N}\biggr),$$
$$|\varphi_{y,\lambda}|_{2^*} = O\biggl(\frac{1}{\lambda^{(N-2)/2}d^{(N-2)/2}}\biggr),$$
$$\biggl|\frac{\partial \varphi_{y,\lambda}}{\partial \lambda}\biggr|_{2^*} = O\biggl(\frac{1}{\lambda^{N/2}d^{(N-2)/2}}\biggr), \qquad \biggl|\frac{\partial \varphi_{y,\lambda}}{\partial y_i}\biggr|_{2^*} = O\biggl(\frac{1}{\lambda^{(N-2)/2}d^{N/2}}\biggr),$$
$$H(y,x) = H(y,y) + \sum_{j=1}^{N} \frac{\partial H(y,y)}{\partial y_j}(x_j - y_j) + O\biggl(\frac{|x-y|^2}{\lambda^2 d^N} + \frac{1}{\lambda^2 d^N}\biggr),$$
$$H(y,y) = O\biggl(\frac{1}{d^{N-2}}\biggr), \qquad \frac{\partial H(y,y)}{\partial y_i} = O\biggl(\frac{1}{d^{N-1}}\biggr)$$

for $d := \mathrm{dist}(y, \partial\Omega)$ small enough.

Lemma A.1. *Let $(y,\lambda) \in D^1_\mu$. For $\mu > 0$ and $\varepsilon > 0$ sufficiently small, we have that*

$$\int_\Omega U^{2^*-1}_{y,\lambda} v - \int_\Omega \psi_\tau P U^{2^*-1-\varepsilon}_{y,\lambda} v = O(d + \varepsilon \ln \lambda)\|v\| + V(\lambda,d)\|v\|,$$

where

$$V(\lambda,d) = \begin{cases} O(\lambda^{2-N}d^{2-N}), & N = 4,5, \\ O((\lambda d)^{-4}\ln^{2/3}(\lambda d)), & N = 6, \\ O((\lambda d)^{-(N+2)/2}), & N > 6. \end{cases}$$

Proof. Let $k_\varepsilon = \lambda^{-(N-2)\varepsilon/2}$. Then $k_\varepsilon = 1 - ((N-2)/2)\varepsilon \ln \lambda + O(\varepsilon^2 \ln^2 \lambda)$. Direct computation yields that

$$\begin{aligned} \int_\Omega (\psi_\tau P U^{2^*-1-\varepsilon}_{y,\lambda} - U^{2^*-1}_{y,\lambda})v &= \int_\Omega \psi_\tau (P U^{2^*-1-\varepsilon}_{y,\lambda} - U^{2^*-1-\varepsilon}_{y,\lambda})v \\ &\quad + \int_\Omega (\psi_\tau - 1) U^{2^*-1-\varepsilon}_{y,\lambda} v + \int_\Omega (k_\varepsilon - 1) U^{2^*-1}_{y,\lambda} v \\ &= \int_{B_d} \psi_\tau (P U^{2^*-1-\varepsilon}_{y,\lambda} - U^{2^*-1-\varepsilon}_{y,\lambda})v \\ &\quad + O\biggl(\int_{\Omega\setminus B_d} U^{2^*-1-\varepsilon}_{y,\lambda} v\biggr) + O(d + \varepsilon \ln \lambda)\|v\| \end{aligned}$$

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$$= O\bigg(|\varphi_{y,\lambda}|_\infty \int_{B_d} U_{y,\lambda}^{2^*-2}|v|\bigg) + O\bigg(\frac{1}{\lambda^{(N+2)/2}d^{(N+2)/2}}\bigg)\|v\| + O(d+\varepsilon\ln\lambda)\|v\|$$

$$= O\bigg(|\varphi_{y,\lambda}|_\infty \bigg(\int_{B_d} U_{y,\lambda}^{2^*(2^*-2)/(2^*-1)}\bigg)^{(2^*-1)/2^*}|v|_{2^*}\bigg) + O\bigg(\frac{1}{\lambda^{(N+2)/2}d^{(N+2)/2}}\bigg)\|v\| + O(d+\varepsilon\ln\lambda)\|v\|$$

$$= V(\lambda,d)\|v\| + O(d+\varepsilon\ln\lambda)\|v\|.$$

□

LEMMA A.2. *Let $(y,\lambda)\in D_\mu^1$. For $\mu>0$ sufficiently small, we have that*

$$\int_\Omega \nabla PU_{y,\lambda}\nabla\frac{\partial PU_{y,\lambda}}{\partial\lambda} = \frac{(N-2)G}{2\lambda^{N-1}}H(y,y) + O\bigg(\frac{1}{\lambda^{N+1}d^N}\bigg),$$

where

$$G = \int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^{(N+2)/2}}$$

depends on N.

Proof. The proof can be found in [22]. □

LEMMA A.3. *Let $(y,\lambda)\in D_\mu^1$. For $\mu>0$ and $\varepsilon>0$ sufficiently small, we have that*

$$\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial\lambda} = -\frac{\varepsilon A}{\lambda} + \frac{(N-2)G+2F}{2\lambda^{N-1}}H(y,y) + O\bigg(\frac{\varepsilon^2\ln\lambda}{\lambda} + \frac{\varepsilon d}{\lambda} + \frac{\varepsilon\ln\lambda}{\lambda^{N-1}d^{N-2}} + \frac{1}{\lambda^N d^{N-1}} + \frac{1+\varepsilon\ln\lambda}{\lambda^3} + \frac{d}{\lambda^{N-1}d^{N-2}}\bigg),$$

where F is strictly positive,

$$A = \int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^N}$$

depends on N.

Proof. The proof follows from

$$\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial\lambda} = \int_{B_d}\psi_\tau(U_{y,\lambda}^{2^*-1-\varepsilon} - (2^*-1-\varepsilon)U_{y,\lambda}^{2^*-2-\varepsilon}\varphi_{y,\lambda})\frac{\partial PU_{y,\lambda}}{\partial\lambda} + O\bigg((1+\varepsilon\ln\lambda)\int_{B_d}U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\bigg|\frac{\partial PU_{y,\lambda}}{\partial\lambda}\bigg|\bigg) + O\bigg(\frac{1}{\lambda^{N+1}d^N}\bigg).$$

With elementary computations, we have that

$$\begin{aligned}
&\int_{B_d} \psi_\tau U_{y,\lambda}^{2^*-1-\varepsilon} \frac{\partial U_{y,\lambda}}{\partial \lambda} \\
&\quad = \psi_\tau(y) \int_{B_d} U_{y,\lambda}^{2^*-1-\varepsilon} \frac{\partial U_{y,\lambda}}{\partial \lambda} \\
&\qquad + k_\varepsilon \int_{B_d} (\psi_\tau(x) - \psi_\tau(y)) U_{y,\lambda}^{2^*-1} (1 + \tfrac{1}{2}(N-2)\varepsilon \ln(1 + \lambda^2 |x-y|^2)) \frac{\partial U_{y,\lambda}}{\partial \lambda} \\
&\quad = \psi_\tau(y) \int_{\mathbb{R}^N} U_{y,\lambda}^{2^*-1-\varepsilon} \frac{\partial U_{y,\lambda}}{\partial \lambda} + k_\varepsilon \int_{B_d} (\psi_\tau(x) - \psi_\tau(y)) U_{y,\lambda}^{2^*-1} \frac{\partial U_{y,\lambda}}{\partial \lambda} \\
&\qquad + O\biggl(\frac{1}{\lambda^{N+1} d^N}\biggr) + O(\varepsilon \ln \lambda + \varepsilon^2 \ln^2 \lambda) \biggl| \int_{B_d} U_{y,\lambda}^{2^*-1} \frac{\partial U_{y,\lambda}}{\partial \lambda} \biggr| \\
&\quad = -\frac{k_\varepsilon A \psi_\tau(y) \varepsilon}{\lambda} + O\biggl(\frac{1 + \varepsilon \ln \lambda}{\lambda^3}\biggr) + O\biggl(\frac{1}{\lambda^{N+1} d^N} + \frac{1 + \varepsilon \ln \lambda}{\lambda^{N+1} d^N}\biggr).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{B_d} \psi_\tau U_{y,\lambda}^{2^*-1-\varepsilon} \frac{\partial \varphi_{y,\lambda}}{\partial \lambda} \\
&\quad = \psi_\tau(y) \int_{B_d} U_{y,\lambda}^{2^*-1-\varepsilon} \frac{\partial \varphi_{y,\lambda}}{\partial \lambda} \\
&\qquad + O\biggl(\int_{B_d} (\psi_\tau(x) - \psi_\tau(y)) U_{y,\lambda}^{2^*-1} (1 + \tfrac{1}{2}(N-2)\varepsilon \ln(1 + \lambda^2 |x-y|^2)) \frac{\partial \varphi_{y,\lambda}}{\partial \lambda} \biggr) \\
&\quad = -\frac{(N-2)G}{2\lambda^{N-1}} H(y,y) \psi_\tau(y) k_\varepsilon + O\biggl(\int_{B_d} |x-y|^2 U_{y,\lambda}^{2^*-1} \frac{\partial \varphi_{y,\lambda}}{\partial \lambda} \biggr) \\
&\qquad + O\biggl(\frac{1}{\lambda^{N+1} d^N} + \frac{\varepsilon \ln \lambda}{\lambda^{N-1} d^{N-2}} \biggr) \\
&\quad = -\frac{(N-2)G}{2\lambda^{N-1}} H(y,y) \psi_\tau(y) + O\biggl(\frac{1}{\lambda^{N+1} d^N} + \frac{\varepsilon \ln \lambda}{\lambda^{N-1} d^{N-2}} + \frac{d}{\lambda^{N-1} d^{N-2}} \biggr)
\end{aligned}$$

and

$$(2^* - 1 - \varepsilon) \int_{B_d} \psi_\tau U_{y,\lambda}^{2^*-2-\varepsilon} \frac{\partial PU_{y,\lambda}}{\partial \lambda} \varphi_{y,\lambda} = -\frac{F}{\lambda^{N-1}} H(y,y) \psi_\tau(y) k_\varepsilon + O\biggl(\frac{1}{\lambda^{N+1} d^N}\biggr).$$

Here, $F > 0$ is a constant.

By the estimates in [22], we get

$$\begin{aligned}
&\int_{B_d} U_{y,\lambda}^{2^*-3} \varphi_{y,\lambda}^2 \biggl| \frac{\partial PU_{y,\lambda}}{\partial \lambda} \biggr| \\
&\qquad = \int_{B_d} U_{y,\lambda}^{2^*-3} \varphi_{y,\lambda}^2 \biggl| \frac{\partial U_{y,\lambda}}{\partial \lambda} \biggr| - \int_{B_d} U_{y,\lambda}^{2^*-3} \varphi_{y,\lambda}^2 \biggl| \frac{\partial \varphi_{y,\lambda}}{\partial \lambda} \biggr|
\end{aligned}$$

$$= O\biggl(\frac{1}{\lambda}|\varphi_{y,\lambda}|_\infty|\varphi_{y,\lambda}|_{2^*}\biggl(\int_{B_d} U_{y,\lambda}^{2^*(2^*-2)/(2^*-1)}\biggr)^{(2^*-1)/2^*}\biggr) + O\biggl(\int_{B_d} U_{y,\lambda}^{2^*-2}\varphi_{y,\lambda}\biggl|\frac{\partial\varphi_{y,\lambda}}{\partial\lambda}\biggr|\biggr)$$

$$= \begin{cases} O\biggl(\dfrac{d}{(\lambda d)^4}\biggr), & N=4,\\ O\biggl(\dfrac{1}{\lambda(\lambda d)^{9/2}}\biggr), & N=5,\\ O\biggl(\dfrac{\ln^{2/3}(\lambda d)}{\lambda(\lambda d)^6}\biggr), & N=6,\\ O\biggl(\dfrac{1}{\lambda(\lambda d)^N}\biggr), & N>6. \end{cases}$$

Combining the above estimates completes the proof. □

LEMMA A.4. *Let* $(y,\lambda)\in D_\mu^1$ *and* $v_{y,\hat\lambda}\in E_{y,\hat\lambda}^{2k}$. *For* $\mu>0$ *and* $\varepsilon>0$ *sufficiently small, we have that*

$$\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-2-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial\lambda}v = O\biggl(\frac{\varepsilon\ln\lambda}{\lambda}+\frac{1}{\lambda^2}+\frac{1}{\lambda(\lambda d)^{(N-2)/2}}\biggr)\|v\|.$$

Proof. The proof follows from

$$\begin{aligned}\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-2-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial\lambda}v &= O\biggl(\frac{1}{\lambda}\int_\Omega \psi_\tau U_{y,\lambda}^{2^*-1-\varepsilon}v\biggr) + O\biggl(\int_\Omega U_{y,\lambda}^{2^*-2}\frac{\partial\varphi_{y,\lambda}}{\partial\lambda}v\biggr)\\ &= O\biggl(\frac{\varepsilon\ln\lambda}{\lambda}+\frac{1}{\lambda^2}+\frac{1}{\lambda(\lambda d)^{(N-2)/2}}\biggr)\|v\|.\end{aligned}$$ □

LEMMA A.5. *Let* $(y,\lambda)\in D_\mu^1$ *and* $v_{y,\hat\lambda}\in E_{y,\hat\lambda}^{2k}$. *For* $\mu>0$ *and* $\varepsilon>0$ *sufficiently small, we have that*

$$\int_\Omega PU_{y,\lambda}^{2^*-3}\frac{\partial PU_{y,\lambda}}{\partial\lambda}v^2 = O(\lambda^{-1}\|v\|^2).$$

Proof. The proof follows from

$$\begin{aligned}\int_\Omega PU_{y,\lambda}^{2^*-3}\frac{\partial PU_{y,\lambda}}{\partial\lambda}v^2 &= O\biggl(\int_\Omega U_{y,\lambda}^{2^*(2^*-3)/(2^*-2)}\biggl|\frac{\partial U_{y,\lambda}}{\partial\lambda}\biggr|^{2^*/(2^*-2)}\biggr)^{(2^*-2)/2^*}\|v\|^2\\ &= O\biggl(\frac{\|v\|^2}{\lambda}\biggr).\end{aligned}$$ □

LEMMA A.6. *Let* $(y,\lambda)\in D_\mu^1$ *and* $v_{y,\hat\lambda}\in E_{y,\hat\lambda}^{2k}$. *For* $\mu>0$ *and* $\varepsilon>0$ *sufficiently small, we have that*

$$\int_\Omega \biggl|\frac{\partial PU_{y,\lambda}}{\partial\lambda}\biggr||v|^{2^*-1-\varepsilon} = O\biggl(\frac{\|v\|^{2^*-1-\varepsilon}}{\lambda}\biggr).$$

Proof. The proof follows from

$$
\begin{aligned}
\int_\Omega |v|^{2^*-1-\varepsilon}\left|\frac{\partial PU_{y,\lambda}}{\partial\lambda}\right|
&= O\biggl(\left|\frac{\partial U_{y,\lambda}}{\partial\lambda}\right|_{2^*}+\left|\frac{\partial\varphi_{y,\lambda}}{\partial\lambda}\right|_{2^*}\biggr)\|v\|^{2^*-1-\varepsilon}\\
&= O\biggl(\frac{1}{\lambda}\biggl(\int_0^{\lambda R}\frac{r^{N-1}(1-r^2)^{2N/(N-2)}}{(1+r^2)^{N^2/(N-2)}}\biggr)^{(N-2)/2N}+\left|\frac{\partial\varphi_{y,\lambda}}{\partial\lambda}\right|_{2^*}\biggr)\|v\|^{2^*-1-\varepsilon}\\
&= O\biggl(\frac{\|v\|^{2^*-1-\varepsilon}}{\lambda}\biggr).
\end{aligned}
$$

□

LEMMA A.7. *Let $(y,\lambda)\in D^1_\mu$. For $\mu>0$ sufficiently small, we have that*

$$
\int_\Omega \nabla PU_{y,\lambda}\nabla\frac{\partial PU_{y,\lambda}}{\partial y_i} = -\frac{G}{\lambda^{N-2}}\frac{\partial H(y,y)}{\partial y_i}+O\biggl(\frac{1}{\lambda^N d^{N+1}}\biggr),
$$

where G is the same as in lemma A.2.

LEMMA A.8. *Let $(y,\lambda)\in D^1_\mu$. For $\mu>0$ and $\varepsilon>0$ sufficiently small, we have that*

$$
\begin{aligned}
\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i} &= -\frac{2G}{\lambda^{N-2}}\frac{\partial H(y,y)}{\partial y_i}+D_i\psi_\tau(y)B\\
&\quad + O\biggl(\frac{\varepsilon\ln\lambda+d}{\lambda^{N-2}d^{N-1}}+\frac{1}{\lambda^{N-1}d^N}+\varepsilon\ln\lambda+\frac{1}{\lambda^2}\biggr),
\end{aligned}
$$

where B is strictly positive.

Proof. The proof follows from

$$
\begin{aligned}
\int_\Omega \psi_\tau PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i}
&= \psi_\tau(y)\int_{B_d} PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i}\\
&\quad + \int_{B_d}(\psi_\tau(x)-\psi_\tau(y))PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i}+O\biggl(\frac{1}{\lambda^N d^{N+1}}\biggr).
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
\psi_\tau(y)\int_{B_d} PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i}
&= \psi_\tau(y)\int_{B_d} U_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i} - (2^*-1-\varepsilon)\psi_\tau(y)\int_{B_d} U_{y,\lambda}^{2^*-2-\varepsilon}\varphi_{y,\lambda}\frac{\partial PU_{y,\lambda}}{\partial y_i}\\
&\quad + O\biggl(\int_{B_d} U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\frac{\partial PU_{y,\lambda}}{\partial y_i}\biggr)
\end{aligned}
$$

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$$
\begin{aligned}
&= -k_\varepsilon\psi_\tau(y)\int_{B_d} U_{y,\lambda}^{2^*-1}\frac{\partial\varphi_{y,\lambda}}{\partial y_i} - (2^*-1-\varepsilon)k_\varepsilon\psi_\tau(y)\int_{B_d} U_{y,\lambda}^{2^*-2}\varphi_{y,\lambda}\frac{\partial PU_{y,\lambda}}{\partial y_i}\\
&\quad + \frac{\psi_\tau(y)}{2^*-\varepsilon}\frac{\partial}{\partial y_i}\int_{\mathbb{R}^N} U_{y,\lambda}^{2^*-\varepsilon} + O\biggl(\int_{B_d} U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\frac{\partial PU_{y,\lambda}}{\partial y_i}\biggr)\\
&\quad + O\biggl(\frac{1}{\lambda^N d^{N+1}} + \frac{\varepsilon\ln\lambda + d}{\lambda^{N-2}d^{N-1}}\biggr)\\
&= -\frac{2G\psi_\tau(y)}{\lambda^{N-2}}\frac{\partial H(y,y)}{\partial y_i} + O\biggl(\frac{\varepsilon\ln\lambda + d}{\lambda^{N-2}d^{N-1}} + \frac{1}{\lambda^{N-1}d^N}\biggr).
\end{aligned}
$$

Here, we have used the estimates (see [22])

$$
(2^*-1)\int_{B_d} U_{y,\lambda}^{2^*-2}\varphi_{y,\lambda}\frac{\partial PU_{y,\lambda}}{\partial y} = \frac{G}{\lambda^{N-2}}\frac{\partial H(y,y)}{\partial y_i} + O\biggl(\frac{1}{\lambda^{N-1}d^N}\biggr)
$$

and

$$
\int_{B_d} U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\frac{\partial PU_{y,\lambda}}{\partial y_i} = \begin{cases}
O\biggl(\dfrac{1}{\lambda^3 d^4}\biggr), & N=4,\\[2ex]
O\biggl(\dfrac{\ln(\lambda d)}{\lambda^5 d^6}\biggr), & N=5,\\[2ex]
O\biggl(\dfrac{(\ln^{2/3}(\lambda d))}{\lambda^6 d^7}\biggr), & N=6,\\[2ex]
O\biggl(\dfrac{1}{\lambda^N d^{N+1}}\biggr), & N>6.
\end{cases}
$$

The direct computation shows that

$$
\begin{aligned}
&\int_{B_d}(\psi_\tau(x)-\psi_\tau(y))PU_{y,\lambda}^{2^*-1-\varepsilon}\frac{\partial PU_{y,\lambda}}{\partial y_i}\\
&\quad = k_\varepsilon\int_{B_d}(\psi_\tau(x)-\psi_\tau(y))U_{y,\lambda}^{2^*-1}\frac{\partial U_{y,\lambda}}{\partial y_i} - k_\varepsilon\int_{B_d}(\psi_\tau(x)-\psi_\tau(y))U_{y,\lambda}^{2^*-1}\frac{\partial\varphi_{y,\lambda}}{\partial y_i}\\
&\qquad - (2^*-1-\varepsilon)k_\varepsilon\int_{B_d}(\psi_\tau(x)-\psi_\tau(y))U_{y,\lambda}^{2^*-2}\varphi_{y,\lambda}\frac{\partial PU_{y,\lambda}}{\partial y_i}\\
&\qquad + O(\varepsilon\ln\lambda)\biggl|\int_{B_d}|x-y|U_{y,\lambda}^{2^*-1}\frac{\partial U_{y,\lambda}}{\partial y_i}\biggr| + O(\varepsilon\ln\lambda)\biggl|\int_{B_d}|x-y|U_{y,\lambda}^{2^*-1}\frac{\partial\varphi_{y,\lambda}}{\partial y_i}\biggr|\\
&\qquad + O\biggl(\int_{B_d}U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\biggl|\frac{\partial PU_{y,\lambda}}{\partial y_i}\biggr|\biggr)\\
&\quad = k_\varepsilon\int_{B_d}D\psi_\tau(y)\cdot(x-y)U_{y,\lambda}^{2^*-1}\frac{\partial U_{y,\lambda}}{\partial y_i} + O\biggl(\biggl|\int_{B_d}(x-y)^3U_{y,\lambda}^{2^*-1}\frac{\partial U_{y,\lambda}}{\partial y_i}\biggr|\biggr)\\
&\qquad + O\biggl(\int_{B_d}U_{y,\lambda}^{2^*-3}\varphi_{y,\lambda}^2\biggl|\frac{\partial PU_{y,\lambda}}{\partial y_i}\biggr|\biggr) + O\biggl(\varepsilon\ln\lambda + \frac{1}{(\lambda d)^{N-1}}\biggr)\\
&\quad = D_i\psi_\tau(y)B + O\biggl(\varepsilon\ln\lambda + \frac{1}{\lambda^2} + \frac{1}{\lambda^{N-1}d^N}\biggr),
\end{aligned}
$$

where B is a strictly positive constant.

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Collecting all the previous estimates, we get the desired result. □

LEMMA A.9. *Let $(y,\lambda) \in D^1_\mu$ and $v_{y,\hat\lambda} \in E^{2k}_{y,\hat\lambda}$. For $\mu > 0$ and $\varepsilon > 0$ sufficiently small, we have that*

$$\int_\Omega \psi_\tau PU^{2^*-2-\varepsilon}_{y,\lambda} \frac{\partial PU_{y,\lambda}}{\partial y_i} v = O\biggl(\varepsilon\lambda\ln\lambda + 1 + \frac{1}{\lambda^{N/2}d^{(N-2)/2}} + \frac{1}{d^{N/2}\lambda^{(N-2)/2}}\biggr)\|v\| + \begin{cases} O\biggl(\dfrac{1}{\lambda^{N-3}d^{N-2}}\biggr)\|v\|, & N<8, \\ O\biggl(\dfrac{(\ln(\lambda d))^{5/8}}{\lambda^5 d^6}\biggr)\|v\|, & N=8, \\ O\biggl(\dfrac{1}{\lambda^{(N+2)/2}d^{(N+4)/2}}\biggr)\|v\|, & N>8. \end{cases}$$

Proof. By direct calculation, we have

$$\begin{aligned}
&\int_\Omega \psi_\tau U^{2^*-2-\varepsilon}_{y,\lambda} \frac{\partial U_{y,\lambda}}{\partial y_i} v \\
&\quad = k_\varepsilon \int_\Omega (\psi_\tau(x) - \psi_\tau(y)) U^{2^*-2}_{y,\lambda} \frac{\partial U_{y,\lambda}}{\partial y_i} v + O\biggl(\varepsilon\ln\lambda \int_\Omega U^{2^*-2}_{y,\lambda}\biggl|\frac{\partial U_{y,\lambda}}{\partial y_i}\biggr||v|\biggr) \\
&\quad = O(\varepsilon\lambda\ln\lambda)\|v\| + O(\|v\|), \\
&\int_\Omega \psi_\tau U^{2^*-2-\varepsilon}_{y,\lambda} \frac{\partial \varphi_{y,\lambda}}{\partial y_i} v \\
&\quad = k_\varepsilon \int_\Omega \psi_\tau(x) U^{2^*-2}_{y,\lambda} \frac{\partial \varphi_{y,\lambda}}{\partial y_i} v + O\biggl(\varepsilon\ln\lambda \int_\Omega U^{2^*-2}_{y,\lambda} \frac{\partial \varphi_{y,\lambda}}{\partial y_i} v\biggr) \\
&\quad = O\biggl(\biggl|\frac{\partial \varphi_{y,\lambda}}{\partial y_i}\biggr|_{2^*}\biggl(\int_\Omega U^{2^*(2^*-2)/(2^*-2)}_{y,\lambda}\biggr)^{(2^*-2)/2^*}\biggr)(1 + O(\varepsilon\ln\lambda))\|v\| \\
&\quad = O\biggl(\frac{\|v\|}{\lambda^{N/2}d^{(N-2)/2}}\biggr), \\
&\int_\Omega \psi_\tau U^{2^*-3-\varepsilon}_{y,\lambda} \varphi_{y,\lambda} \frac{\partial \varphi_{y,\lambda}}{\partial y_i} v \\
&\quad = O\biggl(\int_\Omega U^{2^*-2}_{y,\lambda} \frac{\partial \varphi_{y,\lambda}}{\partial y_i} v\biggr) = O\biggl(\frac{\|v\|}{\lambda^{(N-2)/2}d^{N/2}}\biggr), \\
&\int_\Omega \psi_\tau U^{2^*-3-\varepsilon}_{y,\lambda} \varphi_{y,\lambda} \frac{\partial U_{y,\lambda}}{\partial y_i} v \\
&\quad = O\biggl(|\varphi_{y,\lambda}|_\infty \biggl(\int_{B_d} U^{2^*(2^*-3)/(2^*-2)}_{y,\lambda} \biggl|\frac{\partial \varphi_{y,\lambda}}{\partial y_i}\biggr|^{2^*/(2^*-1)}\biggr)^{(2^*-1)/2^*}\biggr)\|v\| \\
&\qquad + O\biggl(\frac{\|v\|}{\lambda^{(N+2)/2}d^{(N+4)/2}}\biggr)
\end{aligned}$$

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$$= O\left(\frac{\|v\|}{\lambda^{(N+2)/2}d^{(N+4)/2}}\right) + \begin{cases} O\left(\dfrac{1}{\lambda^{N-3}d^{N-2}}\right)\|v\|, & N < 8, \\ O\left(\dfrac{(\ln^{5/8}(\lambda d))}{\lambda^5 d^6}\right)\|v\|, & N = 8, \\ O\left(\dfrac{1}{\lambda^{(N+2)/2}d^{(N+4)/2}}\right)\|v\|, & N > 8. \end{cases}$$

Thus, summing up the above estimates, the claim is proved. □

LEMMA A.10. *Let* $(y,\lambda) \in D^1_\mu$ *and* $v_{y,\hat{\lambda}} \in E^{2k}_{y,\hat{\lambda}}$. *If* $2^* > 3$, *for* $\mu > 0$ *and* $\varepsilon > 0$ *sufficiently small, we have that*

$$\int_\Omega \psi_\tau PU^{2^*-3}_{y,\lambda}\left|\frac{\partial PU_{y,\lambda}}{\partial y_i}\right||v|^2 + \int_\Omega \psi_\tau\left|\frac{\partial PU_{y,\lambda}}{\partial y_i}\right||v|^{2^*-1-\varepsilon} = O(\lambda\|v\|^2).$$

Proof. The computation is similar to lemma A.9; see also [22]. □

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