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RESEARCH ARTICLE

On the second-order excess wealth order and its properties

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Abstract

In the literature, some stochastic orders have been extended to the higher orders in different scenarios. In this paper, inspired by interesting properties of the excess wealth order and its wide range application particularly in comparing the tail variability of risks, we consider the second-order excess wealth order and study its main properties. We obtain two results characterizing the proposed order. We also investigate its relationship with other well-known variability orders and criteria to compare risks. An application of the results in comparing the epoch times of two nonhomogeneous poisson processes is also given.

1. Introduction

Stochastic orders are statistical tools for comparing between random variables in the sense of ageing, variability and other different point of views. Variability orders are the ones that are used to compare the dispersion of random variables. Among them, the excess wealth order also known as right spread order [15,32] is such well-known variability order which is defined through the excess wealth (or the right spread) function. Let X and Y be two nonnegative random variable with their distribution functions F and G and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Besides, F^{-1} and G^{-1} denote their corresponding inverse functions. The excess wealth function associated to X is defined as

$$W_X(p) = E[(X - F^{-1}(p))_+] = \int_{F^{-1}(p)}^{\infty} \bar{F}(x)\,\mathrm{d}x = \int_p^1 (1-u)\,\mathrm{d}F^{-1}(u)$$
$$= \int_p^1 [F^{-1}(u) - F^{-1}(p)]\,\mathrm{d}u, \quad p \in (0,1),$$

where $(Z)_+ = \max\{0, Z\}$, which is well defined when X has finite mean.

A random variable X is said to be smaller than the random variable Y in excess wealth order (denoted by $X \leq_{\mathrm{ew}} Y$) if

$$W_X(p) \leq W_Y(p), \quad \text{for all } p \in (0,1).$$

The reader can find more details and results about this order in Belzunce [7], Muller and Stoyan [27], Li and Shaked [24], Ahmad and Kayid [1], Kochar *et al.* [22], Shaked and Shanthikumar [33] and Belzunce *et al.* [10]. It is also mentioned that the excess wealth order has been considered as a method of comparing risks in actuarial sciences [11,14,34,35] and a tool used in reliability theory [19,21]. It also has an appealing role in the context of extreme risk analysis and auction theory [20] and analysis of packet transmission processes [25]. In addition, Belzunce *et al.* [8] have classified the excess wealth order within a family of dispersion-type variability orders. Ahmad *et al.* [2] have given a result characterizing the increasing mean inactivity time class of life distributions by means of the excess wealth order. Bartoszewicz and Skolimowska [6] have studied the preservation of the excess wealth order under exponential mixtures. Xie and Zhuang [36] have established several stochastic comparisons of simple spacings of generalized order statistics in the excess wealth order. Some recent results on comparison of order statistics and related statistics based on the excess wealth order can be find in a review paper by Balakrishnan and Zhoa [4]. Furthermore, Fernández-Ponce *et al.* [16] proposed and studied a multivariate generalization of the excess wealth function and used it for a multivariate stochastic comparison. Ortega-Jiménez *et al.* [28] have used the excess wealth order in providing sufficient conditions for comparing several distances between pairs of random variables.

In the context of reliability, when the burn-in test is used in order to eliminate early failures of the produced units and is continued until the $100p$ percent of the units fail, the producer uses the excess function to obtain the expected additional lifetime of the units. In actuarial literature, the excess wealth function is the net premium for a stop-loss contract with fixed intention $x = F^{-1}(p)$. Sordo [35] has pointed out that there is growing interest in the use of certain tail conditional characteristics as measures of risk, which are informative about the magnitude and variability of the losses beyond the value-at-risk. In addition, Sordo [35] has suggested that the tail variabilities of risks should be compared by means of the excess wealth order. However, as it is the nature of such partial orderings, none of the random variables X and Y may dose not dominate the other one at the sense of the excess wealth order. Regarding the above suggestion, it is natural to ask whether the excess wealth function can still be used, at least in another way, for comparing the tail variabilities of risks. This leads us to the notion of the higher order of the excess wealth order.

In the context of stochastic orders, the higher orders of the stochastic orders arise in different situations in theoretical and applied statistics (cf. [26,38] for the higher order (inverse) stochastic dominance and Ramos and Sordo [30] for the second-order absolute Lorenz order). In this paper, inspired by the suggestion in Sordo [35] about the usefulness of the excess wealth order in comparing the tail variabilities of risks and investigating another measure for this purpose, and following the definition of the second-order stochastic dominance order, we consider the second-order excess wealth order as the following.

Definition 1.1. *A random variable X is said to be smaller than the random variable Y in the second-order excess wealth order (denoted by $X \leq_{\text{sew}} Y$) if*

$$\int_p^1 W_X(u)\,\mathrm{d}u \leq \int_p^1 W_Y(u)\,\mathrm{d}u, \quad \textit{for all } p \in [0,1].$$

To have an interpretation of this order, one can see that

$$\begin{aligned}\int_p^1 W_X(u)\,\mathrm{d}u &= \int_p^1 E[(X - F^{-1}(u))_+]\,\mathrm{d}u = \int_{F^{-1}(p)}^{\infty} E[(X-x)_+]\,\mathrm{d}F(x) \\ &= E[(X_2 - X_1)_+ I(X_1 > F^{-1}(p))],\end{aligned} \tag{1.1}$$

where X_1 and X_2 are two independent copies of X and $I(\cdot)$ is the indicator function. That is, if X is a risk, $\int_p^1 W_X(u)\,\mathrm{d}u$ measures the distance between two risks beyond the value $F^{-1}(p)$ along the tail of the population of X. Therefore, the second-order excess wealth order can be considered as another measure for comparing the riskiness of two probability distributions. The longer the distance, the more dangerous the risk. In the context of reliability, measure (1.1) gives the positive difference between the lifetime of the two units that have survived the burn-in period. The plots of the density function, $W_X(p)$ and $\int_p^1 W_X(u)\,\mathrm{d}u$, are shown in Figure 1 for two choices of the parameters of Weibull distribution. One can see that the heavier right tail distribution has greater $\int_p^1 W_X(u)\,\mathrm{d}u$. The plot also shows that $\int_p^1 W_X(u)\,\mathrm{d}u$ reveals the difference in right tail of the distributions better than $W_X(p)$, specially for small p's.

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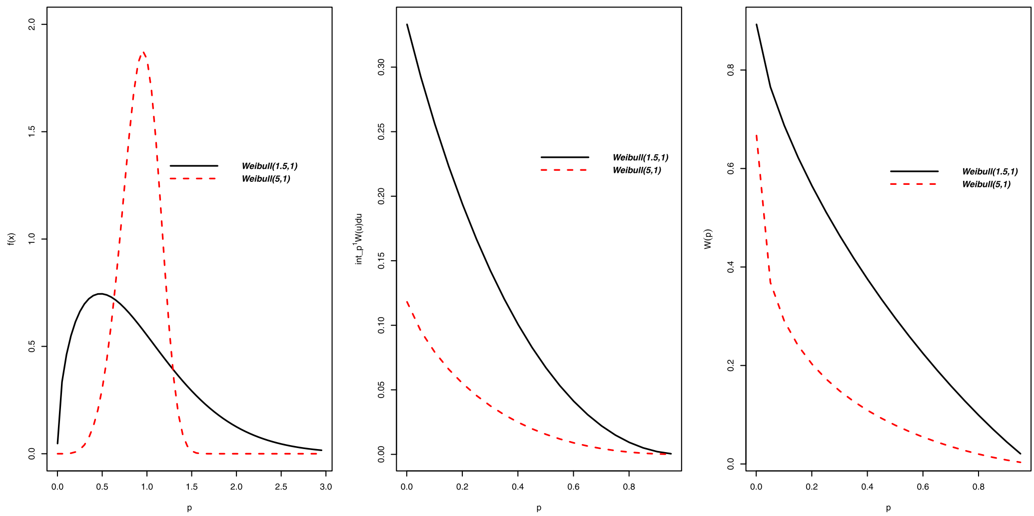

Figure 1. *Plots of the density* $\int_p^1 W(u)\,\mathrm{d}u$ *and* $W(p)$ *for Weibull distribution.*

It is worth to mention that using the degree n stop-loss function

$$\pi_X^n(x) = E[(X-x)_+^n], \quad n = 0, 1, 2 \ldots, \quad \pi_X^0(x) = \bar{F}(x),$$

Hurlimann [18] has considered a generalization of the usual right spread (or excess wealth) order and called a random variable X precedes Y in bi-degree (n, m) right spread order, written $X \leq_{RS}^{(n,m)}$, if $\pi_X^n((\pi_X^m)^{-1}(p)) \leq \pi_Y^n((\pi_Y^m)^{-1}(p))$, for, $1, 2, 3 \ldots$, $m = 0, 1, 2, \ldots, n-1$, and all p in support of $(\pi_X^m)^{-1}$. Regarding Theorem 2.1 in Hurlimann [17], one can see that for $n = 2$ and $m = 0$, the order $X \leq_{RS}^{(2,0)}$ is equivalent that

$$\int_p^1 W_X(u)\,\mathrm{d}F^{-1}(u) \leq \int_p^1 W_Y(u)\,\mathrm{d}G^{-1}(u),$$

which is not the same as the above second-order excess wealth order.

We study the basic properties of this new stochastic order. The rest of the paper is organized as follows. First, we recall some definitions in preliminaries section. In Section 3, we give some main results for the order. Two characterization results are also given in this section. Section 4 is devoted to the relationship between the proposed order and some well-known variability orders and the tail variability measures of risks. An application of the proposed order is also given in this section. Finally, some conclusions are given in Section 5.

2. Preliminaries

Before proceeding to give the main results of the paper, we overview some preliminary concepts of ageing and stochastic orders. (For more details of these concepts, see, e.g., [23,33].)

Let X and Y be two nonnegative random variables with their density functions f and g, distribution functions F and G and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Furthermore, $m_X(t) = \int_t^\infty \bar{F}(x)\,\mathrm{d}x/\bar{F}(t)$ denotes the mean residual life function corresponding to random variable X (m_Y is defined analogously). Throughout this paper, we assume that these functions all exist and increasing (decreasing) means nondecreasing (nonincreasing).

Definition 2.1. *(i) The random variable X is said to be new better (worse) than used in expectation, NBUE(NWUE), if* $m_X(t) \leq (\geq) m_X(0)$*, for all* $t > 0$.

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(ii) *The random variable X is said to be increasing (decreasing) mean residual life (IMRL(DMRL)) if $m_X(t)$ is increasing (decreasing) in t.*
(iii) *The random variable X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$, for all $x > 0$.*
(iv) *The random variable X is said to be smaller than Y in the likelihood ratio ordering (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x.*
(v) *The random variable X is said to be smaller than Y in the DMRL order (denoted by $X \leq_{\mathrm{dmrl}} Y$) if $m_Y(G^{-1}(u))/m_X(F^{-1}(u))$ is increasing in $u \in [0, 1]$.*
(vi) *The random variable X is said to be smaller than Y in the NBUE order (denoted by $X \leq_{\mathrm{nbue}} Y$) if $m_X(F^{-1}(u))/m_Y(G^{-1}(u)) \leq E[X]/E[Y]$, for all $u \in [0, 1]$.*
(vii) *The random variable X is said to be smaller than Y in the increasing convex order (denoted by $X \leq_{\mathrm{icx}} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$, for all increasing convex functions ϕ.*
(viii) *The random variable X is said to be smaller than Y is Lorenz order (denoted by $X \leq_{\mathrm{L}} Y$) if $(1/E(X)) \int_0^u F^{-1}(v)\,\mathrm{d}v \geq (1/E(Y)) \int_0^u G^{-1}(v)\,\mathrm{d}v$, for all $u \in [0, 1]$.*

The above stochastic orders are related as the following.

$$X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{st}} Y \Rightarrow X \leq_{\mathrm{icx}} Y, \quad X \leq_{\mathrm{dmrl}} Y \Rightarrow X \leq_{\mathrm{nbue}} Y \Rightarrow X \leq_{\mathrm{L}} Y$$

3. Some basic properties and characterizations

In this section, we investigate the main properties of the proposed order such as its characterization and its closure properties. First, note that

$$\begin{aligned}
\int_p^1 W_X(u)\,\mathrm{d}u &= \int_p^1 \int_{F^{-1}(u)}^{\infty} \bar{F}(x)\,\mathrm{d}x\,\mathrm{d}u = \int_{F^{-1}(p)}^{\infty} \int_y^{\infty} \bar{F}(x)\,\mathrm{d}x\,\mathrm{d}F(y) \\
&= \int_{F^{-1}(p)}^{\infty} \int_{F^{-1}(p)}^{x} \bar{F}(x)\,\mathrm{d}F(y)\,\mathrm{d}x \\
&= \int_{F^{-1}(p)}^{\infty} \bar{F}(x)[F(x) - p]\,\mathrm{d}x && (3.1) \\
&= \int_p^1 (1-u)(u-p)\,\mathrm{d}F^{-1}(u) && (3.2) \\
&= \int_p^1 (2u - p - 1)F^{-1}(u)\,\mathrm{d}u, && (3.3)
\end{aligned}$$

where the latter equation is obtained by using the integration by parts. Belzunce *et al.* [9] have provided their measure for some statistical parametric models. For comparison purposes, Table 1 gives the $\int_p^1 W_X(u)\,\mathrm{d}u$ for the same statistical models.

It is clear that the excess wealth order implies the second-order excess wealth order. However, the following example shows that the reverse is not necessarily true.

Example 3.1. *Let X and Y be distributed as Weibull and gamma with the density functions $f(x) = 2xe^{-x^2}$ and $g(x) = \frac{1}{\sqrt{2}}x^{-0.5}e^{-x}$, $x > 0$, respectively. Figure 2 shows that $X \leq_{\mathrm{sew}} Y$ but the excess wealth order dose not hold.*

It is easy to see that in similar with the excess wealth order, the order $\leq_{\mathrm{sew}}$ is also location independent. That is, if $X \leq_{\mathrm{sew}} Y$ then $X + c \leq_{\mathrm{sew}} Y$ and $X \leq_{\mathrm{sew}} Y + c$ for all $c \in (-\infty, \infty)$. The following theorem gives a characterization result for the second-order excess wealth order. For a proof, we will need the following lemma.

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Table 1. *The* $\int_p^1 W_X(u)\,\mathrm{d}u$ *for some distributions.*

Distribution name	$F(x)$	$\int_p^1 W_X(u)\,\mathrm{d}u$
Uniform(a, b)	$F(x) = \frac{x-a}{b-a},\ a \le x \le b$	$\frac{1}{6}(b-a)(1-p)^3$
Potential(α, λ)	$F(x) = (\lambda x)^\alpha,\ 0 \le x \le \frac{1}{\lambda}$	$\frac{\alpha}{\lambda(\alpha+1)(2\alpha+1)}[(2\alpha+1)(p^{1/\alpha+1} - p) - p^{1/\alpha+2} + 1]$
Pareto(α, λ)	$F(x) = 1 - (\frac{\lambda}{x})^\alpha,\ x \ge \lambda$	$\frac{\lambda\alpha}{(\alpha-1)(2\alpha-1)}(1-p)^{-1/\alpha+2}$
Exponential(λ)	$F(x) = 1 - e^{-\lambda x}, x \ge 0$	$\frac{1}{2\lambda}(1-p)^2$

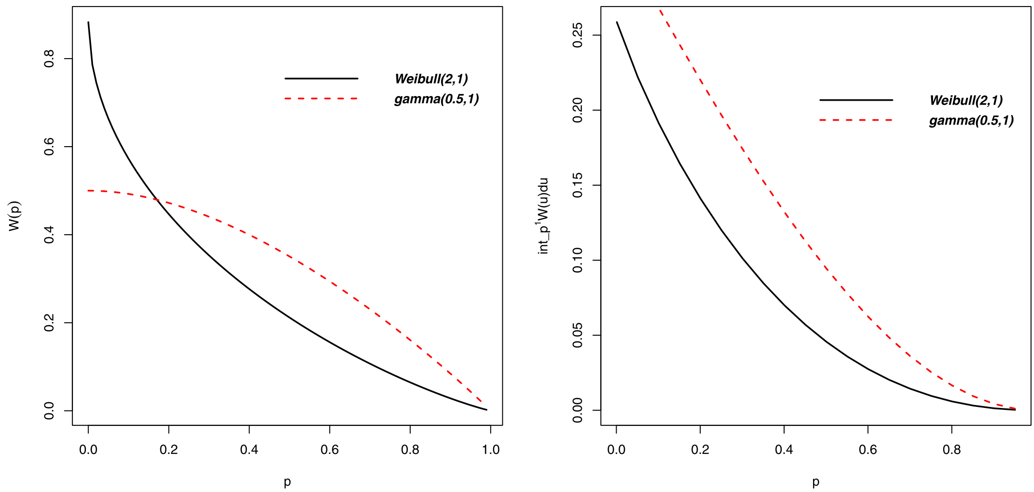

Figure 2. *Plots of* $W(p)$ *and* $\int_p^1 W(u)\,\mathrm{d}u$ *for distributions given in Example 3.1.*

Lemma 3.1 (Barlow and Proschan [5, p. 120**.** *).] Let* $W(x)$ *be a measure, not necessarily positive, for which* $\int_t^\infty \mathrm{d}W(x) \ge 0$ *for all t, and let* $h(x) \ge 0$ *be increasing. Then*

$$\int_t^\infty h(x)\,\mathrm{d}W(x) \ge 0, \quad \textit{for all } t.$$

Theorem 3.1. $X \le_{\text{sew}} Y$ *if and only if,*

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(u) \le \int_0^1 (1-u)\phi(u)\,\mathrm{d}G^{-1}(u), \tag{3.4}$$

for any increasing convex function $\phi : [0, 1] \to \mathrm{R}$*, such that* $\phi(0) = 0$.

Proof. For the only if part, it is known that (cf. [12]), given an increasing and convex function ϕ, then ϕ is continuous and there exists a positive and increasing function h such that

$$\phi(b) - \phi(a) = \int_a^b h(v)\,\mathrm{d}v. \tag{3.5}$$

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We have

$$\begin{aligned}\int_0^1 (1-u)\phi(u)\,\mathrm{d}G^{-1}(u) - \int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(u)\\ &= \int_0^1 (1-u)\phi(u)[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)]\\ &= \int_0^1 \int_0^u (1-u)h(v)\,\mathrm{d}v[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)]\\ &= \int_0^1 \int_v^1 (1-u)[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)]h(v)\,\mathrm{d}v \end{aligned} \tag{3.6}$$

where the second equality comes from (3.5) and the assumption $\phi(0) = 0$. Under the hypothesis,

$$\int_p^1 \int_v^1 (1-u)[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)]\,\mathrm{d}v = \int_p^1 [W_Y(v) - W_X(v)]\,\mathrm{d}v \geq 0.$$

The result now follows from Lemma 3.1.

For the if part, note that, for a fixed $p \in [0,1]$, the function $\phi(u) = (u-p)^+ = \max\{u-p, 0\}$ is increasing and convex such that $\phi(0) = 0$ and

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(u) = \int_p^1 (1-u)(u-p)\,\mathrm{d}F^{-1}(u) = \int_p^1 W_X(u)\,\mathrm{d}u,$$

where, the last equality is just Eq. (3.2). Thus, the result easily follows. □

One may consider

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(u) = \int_0^\infty \bar{F}(x)\phi(F(x))\,\mathrm{d}x,$$

as a generalized version of the cumulative residual entropy (CRE, introduced by Rao *et al.* [31]) which gives the CRE by taking $\phi(u) = -\ln(1-u)$.

By using the integration by parts, one can also see that the inequality (3.4) is equivalent to that

$$\int_0^1 W_X(u)\,\mathrm{d}\phi(u) \leq \int_0^1 W_Y(u)\,\mathrm{d}\phi(u), \quad \text{or} \quad \int_0^1 \phi(u)\,\mathrm{d}W_X(u) \geq \int_0^1 \phi(u)\,\mathrm{d}W_Y(u). \tag{3.7}$$

Belzunce [7] gives a characterization of the excess wealth order in terms of the increasing convex order. In general, neither of the orders $\leq_{\text{sew}}$ and $\leq_{\text{icx}}$ implies the other (let, for example, X be distributed as Weibull with the shape parameter equal to 4 and 0.9 and the scale parameter 1 and Y be distributed as gamma with the shape parameter equal to 2.2 and 0.8 and scale parameter 1). However, the following theorem also gives a characterization of the second-order excess wealth order in terms of the increasing convex order.

Theorem 3.2. *Let X_1, X_2 and Y_1, Y_2 be independent copies of continuous random variables X and Y, respectively. Then, $X \leq_{\text{sew}} Y$ if and only if*

$$(X_2 - X_1)_+ I(X_1 > F^{-1}(p)) \leq_{\text{icx}} (Y_2 - Y_1)_+ I(Y_1 > G^{-1}(p)), \quad \textit{for all } p \in [0,1]. \tag{3.8}$$

Proof. The only if part follows from Eq. (1.1) and the fact that the order $\leq_{\text{icx}}$ implies $E(X) \leq E(Y)$.

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To prove the if part, one can observe that the survival function of $\mathcal{X}_p = (X_2 - X_1)_+ I(X_1 > F^{-1}(p))$ at $x \geq 0$ is given by

$$\bar{K}_{\mathcal{X}_p}(x) = \int_{F^{-1}(p)}^{\infty} \bar{F}(x+z)\,\mathrm{d}F(z) = \int_p^1 \bar{F}(x + F^{-1}(u))\,\mathrm{d}u.$$

The $\bar{K}_{\mathcal{Y}_p}(\cdot)$, the survival function of $\mathcal{Y}_p = (Y_2 - Y_1)_+ I(Y_1 > G^{-1}(p))$ is given analogously. We observe that for a fixed p, $\lim_{t\to\infty} \int_t^{\infty} [\bar{K}_{\mathcal{Y}_p}(x) - \bar{K}_{\mathcal{X}_p}(x)]\,\mathrm{d}x = 0$ and under the hypothesis, $E[\mathcal{X}_p] \leq E[\mathcal{Y}_p]$. This implies that the number of sign changes of $\bar{K}_{\mathcal{X}} - \bar{K}_{\mathcal{Y}}$ is at most one. The result now follows from Theorem 4.A.22 in Shaked and Shanthikumar [33, p. 194]. □

For any nonnegative random variable X and for any $a > 0$, we have

$$\int_p^1 W_{aX}(u)\,\mathrm{d}u = a\int_p^1 W_X(u)\,\mathrm{d}u,$$

from which, for any two nonnegative random variables X and Y we get that if $X \leq_{\text{sew}} Y$, then $aX \leq_{\text{sew}} aY$. The result for a general transform of X is given in the following theorem.

Theorem 3.3. *If $X \leq_{\text{sew}} Y$, then $\varphi(X) \leq_{\text{sew}} \varphi(Y)$, for all increasing convex functions φ.*

Proof. First, note that, the survival and quantile functions of the random variable $\varphi(X)$ are $\bar{F}(\varphi^{-1}(x))$ and $\varphi(F^{-1}(p))$, respectively. Using Eqs. (3.3) and (3.5), we have

$$\begin{aligned}\int_p^1 W_{\varphi(Y)}(u)\,\mathrm{d}u - \int_p^1 W_{\varphi(X)}(u)\,\mathrm{d}u &= \int_p^1 (2u-p-1)[\varphi(G^{-1}(u)) - \varphi(F^{-1}(u))]\,\mathrm{d}u \\ &= \int_p^1 \int_{F^{-1}(u)}^{G^{-1}(u)} (2u-p-1)h(v)\,\mathrm{d}v\,\mathrm{d}u. \qquad (3.9)\end{aligned}$$

Now, we obtain a lower bound for (3.9). First, assume that $G^{-1}(u) > F^{-1}(u)$. Then, we get

$$\int_{F^{-1}(u)}^{G^{-1}(u)} h(v)\,\mathrm{d}v \geq h(F^{-1}(u))[G^{-1}(u) - F^{-1}(u)].$$

On the other hand, if $G^{-1}(u) < F^{-1}(u)$, then

$$\int_{F^{-1}(u)}^{G^{-1}(u)} h(v)\,\mathrm{d}v = -\int_{G^{-1}(u)}^{F^{-1}(u)} h(v)\,\mathrm{d}v \geq h(F^{-1}(u))[G^{-1}(u) - F^{-1}(u)].$$

Therefore, we obtain

$$\int_p^1 \int_{F^{-1}(u)}^{G^{-1}(u)} (2u-p-1)h(v)\,\mathrm{d}v\,\mathrm{d}u \geq \int_p^1 (2u-p-1)h(F^{-1}(u))[G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u.$$

By the hypothesis,

$$\int_p^1 (2u-p-1)[G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u = \int_p^1 [W_Y(u) - W_X(u)]\,\mathrm{d}u \geq 0.$$

Since $h(F^{-1}(u))$ is a positive and increasing function, applying Lemma 3.1 now gives that

$$\int_p^1 (2u-p-1)h(F^{-1}(u))[G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u \geq 0,$$

which completes the proof. □

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As an application in reliability theory, let $X_1, X_2, \ldots, X_n$ be the independent random lifetimes of the components of a parallel system which are copies of X. Consider another parallel systems with $Y_1, Y_2, \ldots, Y_n$ being its components lifetime which are independent and are copies of Y. The following theorem shows that if $X \leq_{\text{sew}} Y$, then the lifetime of the systems are also ordered in the sense of $\leq_{\text{sew}}$.

Theorem 3.4. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent copies of X and Y, respectively. If $X \leq_{\text{sew}} Y$, then*

$$\max\{X_1, X_2, \ldots, X_n\} \leq_{\text{sew}} \max\{Y_1, Y_2, \ldots, Y_n\}.$$

Proof. It is sufficient to consider only the case $n = 2$. Using Eq. (3.2), $X \leq_{\text{sew}} Y$ is equivalent to

$$\int_{\sqrt{p}}^{1} (1-u)(u-\sqrt{p})[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)] \geq 0, \quad \text{for all } p \in [0,1]. \tag{3.10}$$

On the other hand, the quantile functions of $\max\{X_1, X_2\}$ and $\max\{Y_1, Y_2\}$ are $F^{-1}(\sqrt{u})$ and $G^{-1}(\sqrt{u})$, respectively. Hence, to prove $\max\{X_1, X_2\} \leq_{\text{sew}} \max\{Y_1, Y_2\}$, we need to show that

$$\int_{p}^{1} (1-u)(u-p)[\mathrm{d}G^{-1}(\sqrt{u}) - \mathrm{d}F^{-1}(\sqrt{u})] \geq 0, \quad \text{for all } p \in [0,1],$$

or, equivalently, that

$$\int_{\sqrt{p}}^{1} (1+u)(u+\sqrt{p})(1-u)(u-\sqrt{p})[\mathrm{d}G^{-1}(u) - \mathrm{d}F^{-1}(u)] \geq 0, \quad \text{for all } p \in [0,1],$$

which is obtained by using the inequality (3.10) and applying Lemma 3.1. □

Consider also two series systems with $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ being their components lifetime which are independent and are copies of X and Y, respectively. For these series systems, the following result gives the reversed preservation property of the order $\leq_{\text{sew}}$; that is, if $\min\{X_1, X_2, \ldots, X_n\} \leq_{\text{sew}} \min\{Y_1, Y_2, \ldots, Y_n\}$, then the lifetimes of components in two systems are also ordered in the sense of $\leq_{\text{sew}}$.

Theorem 3.5. *For any integer $n > 0$, if*

$$\min\{X_1, X_2, \ldots, X_n\} \leq_{\text{sew}} \min\{Y_1, Y_2, \ldots, Y_n\},$$

then $X \leq_{\text{sew}} Y$.

Proof. The survival functions of $\min\{X_1, X_2, \ldots, X_n\}$ and $\min\{Y_1, Y_2, \ldots, Y_n\}$ are given by $\bar{F}_{(n)}(x) = \bar{F}^n(x)$ and $\bar{G}_{(n)}(x) = \bar{G}^n(x)$, respectively. Notice also that $G_{(n)}^{-1} F_{(n)}(x) = G^{-1}F(x)$. It is not difficult to see from Eq. (3.1) that $X \leq_{\text{sew}} Y$ is equivalent to

$$\int_t^\infty \bar{F}(x)[\bar{F}(t) - \bar{F}(x)]\, \mathrm{d}(G^{-1}F(x) - x) \geq 0, \quad \text{for all } t \geq 0. \tag{3.11}$$

Thus, the assumption is equivalent to, for all $t \geq 0$,

$$\int_t^\infty \bar{F}^n(x)[\bar{F}^n(t) - \bar{F}^n(x)]\, \mathrm{d}(G^{-1}F(x) - x) \geq 0.$$

Using the fact that

$$\bar{F}^n(t) - \bar{F}^n(x) = [\bar{F}(t) - \bar{F}(x)] \sum_{i=1}^{n} \bar{F}^{n-i}(t)\bar{F}^{i-1}(x),$$

we obtain

$$\begin{aligned}
&\int_t^{\infty} \bar{F}(x)[\bar{F}(t) - \bar{F}(x)]\,\mathrm{d}(G^{-1}F(x) - x)\\
&\quad = \int_t^{\infty} [\bar{F}^{n-1}(x)\sum_{i=1}^{n}\bar{F}^{n-i}(t)\bar{F}^{i-1}(x)]^{-1}\bar{F}^{n}(x)[\bar{F}^{n}(t) - \bar{F}^{n}(x)]\,\mathrm{d}(G^{-1}F(x) - x) \geq 0,
\end{aligned}$$

where, the inequality follows from Lemma 3.1 and the fact that

$$\left[\bar{F}^{n-1}(x)\sum_{i=1}^{n}\bar{F}^{n-i}(t)\bar{F}^{i-1}(x)\right]^{-1},$$

is increasing in x. Thus, the proof is complete. □

Some other properties of the second excess wealth order in reliability theory can also be tracked. For the residual random variable $X_t = X - t|X > t$ with survival function $\bar{F}_{X_t}(x) = \bar{F}(x+t)/\bar{F}(t)$ and corresponding inverse function $F_{X_t}^{-1}(p) = F^{-1}(p\bar{F}(t) + F(t)) - t$, we get that

$$W_{X_t}(p) = (1-p)m_X(F^{-1}(p\bar{F}(t) + F(t))) = \frac{W_X(p\bar{F}(t) + F(t))}{\bar{F}(t)}$$

and

$$\begin{aligned}
\int_p^1 W_{X_t}(u)\,\mathrm{d}u &= \int_p^1 (1-u)m_X(F^{-1}(u\bar{F}(t) + F(t)))\,\mathrm{d}u\\
&= \frac{1}{\bar{F}^2(t)}\int_{p\bar{F}(t)+F(t)}^{1} W_X(u)\,\mathrm{d}u. \qquad (3.12)
\end{aligned}$$

Now, we have the following result which is a characterization of the DMRL(IMRL) and NBUE(NWUE) distributions in terms of the second-order excess wealth order.

Theorem 3.6.

(a) $X \in$ DMRL(IMRL) *if and only if* $X_t \geq_{\text{sew}} (\leq_{\text{sew}})X_{t'}$, *for all* $t < t'$.
(b) $X \in$ DMRL(IMRL) *if and only if* $X_t \geq_{\text{sew}} (\leq_{\text{sew}})X$, *for all* $t > 0$.
(c) $X \in$ NBUE(NWUE) *if and only if* $X \leq_{\text{sew}} (\geq_{\text{sew}})Y$, *where* Y *is an exponentially distributed random variable with mean* $E(X)$.

Proof. We give proof of parts (b) and (c). For part (b), regarding Eq. (3.12), one can see that $X_t \geq_{\text{sew}} (\leq_{\text{sew}})X$ if and only if

$$\begin{aligned}
\int_p^1 W_{X_t}(u)\,\mathrm{d}u &= \int_p^1 (1-u)m_X(F^{-1}(u\bar{F}(t) + F(t)))\,\mathrm{d}u\\
&\leq (\geq)\int_p^1 (1-u)m_X(F^{-1}(u))\,\mathrm{d}u = \int_p^1 W_X(u)\,\mathrm{d}u, \quad \text{for all } p \in [0,1],
\end{aligned}$$

which holds if and only (the only if parts holds using the integral comparison theorem) $m_X(F^{-1}(u\bar{F}(t) + F(t))) \leq (\geq)m_X(F^{-1}(u))$ for all $u \in [0,1]$ or equivalently $m_X(F^{-1}(u_2)) \leq (\geq)m_X(F^{-1}(u_1))$ for all $u_1 < u_2$. This implies that $m_X(F^{-1}(u))$ or $m_X(u)$ is decreasing (increasing). That is $X \in$ DMRL(IMRL).

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To prove part (c), we see from Table 1 that if Y is distributed as exponential with mean $E(X)$, then $\int_p^1 W_Y(u)\,\mathrm{d}u = (E(X)(1-p)^2)/2$. On the other hand, if

$$\int_p^1 W_X(u)\,\mathrm{d}u = \int_p^1 (1-u)m_X(F^{-1}(u))\,\mathrm{d}u \le (\ge)\frac{E(X)(1-p)^2}{2}, \quad \text{for all } p \in [0,1],$$

then

$$\int_p^1 (1-u)[m_X(F^{-1}(u)) - E(X)]\,\mathrm{d}u \le (\ge)0,$$

which follows that $m_X(x) \le (\ge)E(X)$ for all $x \ge 0$. This means that $X \in NBUE(NWUE)$ if and only if $X \le_{\text{sew}} (\ge_{\text{sew}})Y$, where Y is an exponentially distributed random variable with mean $E(X)$. □

We end this section by the following result which shows that the second-order excess wealth order is preserved under convergence.

Theorem 3.7. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be two sequences of random variables such that X_n converges in distribution to a random variable X and Y_n to a random variable Y, where X and Y are continuous random variables with interval supports. If $X_n \le_{\text{sew}} Y_n$ for all $n \in N$, where $E[(X_n)_+] \to E[(X)_+]$ and $E[(Y_n)_+] \to E[(Y)_+]$ then $X \le_{\text{sew}} Y$.*

Proof. Under the given conditions, Theorem 2.21 in Belzunce *et al.* [9] provides that $W_{X_n}(u)$ and $W_{Y_n}(u)$ converge pointwise to $W_X(u)$ and $W_Y(u)$, respectively, for all $u \in (0,1)$. This along with Theorem 15.1(iii) in Billingsley [13, p. 201] imply that $\int_p^1 W_{X_n}(u)\,\mathrm{d}u$ and $\int_p^1 W_{Y_n}(u)\,\mathrm{d}u$ converge to $\int_p^1 W_X(u)\,\mathrm{d}u$ and $\int_p^1 W_Y(u)\,\mathrm{d}u$, respectively. The result now follows from the assumption that $X_n \le_{\text{sew}} Y_n$ for all $n \in N$. □

4. Relationship with other stochastic orders

It is well-known that most of the classical variability orders agree with the comparison of variances and Gini's mean differences. This means that if, for example, $X \le_{\text{ew}} Y$, then $\text{Var}(X) \le \text{Var}(Y)$ and $\text{GMD}(X) \le \text{GMD}(Y)$ (cf. [33, p. 166]), where $\text{GMD}(X) = E|X_1 - X_2|$ (and analogously for $\text{GMD}(Y)$) is the Gini mean difference corresponding to X with X_1 and X_2 being two independent copies of X. The Gini mean difference is a dispersion measure that shares many properties with the variance (cf. [37]). The order $\le_{\text{sew}}$ is also in agreement with comparison among the Gini mean differences. To see this, let $\phi(x) = x$ which is increasing and convex with $\phi(0) = 0$. Therefore, it follows from (3.4) that if $X \le_{\text{sew}} Y$ then

$$\text{GMD}(X) = 2\int_0^\infty \bar{F}(x)F(x)\,\mathrm{d}x \le 2\int_0^\infty \bar{G}(x)G(x)\,\mathrm{d}x = \text{GMD}(Y).$$

The following theorem shows that the ordered Gini mean differences along with the order $\le_{\text{dmrl}}$ implies the second-order excess wealth order.

Theorem 4.1. *If $X \le_{\text{dmrl}} Y$ and $\text{GMD}(X) \le \text{GMD}(Y)$, then $X \le_{\text{sew}} Y$.*

Proof. First, we observed that

$$\int_0^1 W_X(u)\,\mathrm{d}u = \frac{1}{2}\text{GMD}(X), \tag{4.1}$$

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In addition, according to the definition (cf. [33, p. 221]), $X \leq_{\rm dmrl} Y$ is equivalent to that $m_X(F^{-1}(u))/m_Y(G^{-1}(u)) = W_Y(u)/W_X(u)$ is increasing in $u \in [0,1]$ which is the same as the condition

$$\frac{\frac{1}{\mathrm{GMD}(Y)} W_Y(u)}{\frac{1}{\mathrm{GMD}(X)} W_X(u)} \quad \text{is incrasing in } u \in [0,1].$$

Note also that $(W_X(u)/\mathrm{GMD}(X))(W_Y(u)/\mathrm{GMD}(Y))$ is a density function corresponding to a random variable, say, $Z_X(Z_Y)$. Then, the above condition is equivalent to that $Z_X \leq_{\rm lr} Z_Y$ which implies that $Z_X \leq_{\rm st} Z_Y$. Now, this along with the hypothesis $\mathrm{GMD}(X) \leq \mathrm{GMD}(Y)$ gives that

$$\frac{\int_p^1 W_X(u)\,\mathrm{d}u}{\mathrm{GMD}(X)} \leq \frac{\int_p^1 W_Y(u)\,\mathrm{d}u}{\mathrm{GMD}(Y)} \leq \frac{\int_p^1 W_Y(u)\,\mathrm{d}u}{\mathrm{GMD}(X)},$$

which implies that $X \leq_{\rm sew} Y$. □

One can also easily show that if $X \leq_{\rm nbue} Y$ and $E(X) \leq E(Y)$, then $X \leq_{\rm sew} Y$.

Remark 4.1. *Asadi and Zohrevand [3] have defined the dynamic cumulative entropy by* $\mathcal{E}(X,t) = -\int_0^\infty \bar{F}_t(x)\log(\bar{F}_t(x))\,\mathrm{d}x$, *where* $\bar{F}_t(x) = \bar{F}(x+t))/\bar{F}(t)$, $x,t \geq 0$ *is the survival function of the residual lifetime* $X_t = X - t|X > t$ *corresponding to random variable* X. *It is clear that* $\mathcal{E}(X,t)$ *is in the form (3.4) with* $\phi(u) = -\log(1-u)$. *This follows that if* $X \leq_{\rm sew} Y$, *then* $\mathcal{E}(X,t) \leq \mathcal{E}(Y,t)$ *for all* $t \geq 0$.

Remark 4.2. *Rajesh and Sunoj [29] have called X is less uncertainty in cumulative Tsallis entropy than Y (denoted by* $X \leq^{\rm LCTU} Y$*) if* $\zeta_\alpha^X(t) \leq \zeta_\alpha^Y(t)$, *for all* $t \geq 0$, *where* $\zeta_\alpha^X(t) = (1/(\alpha-1))\int_0^\infty \bar{F}_t(x)[1-\bar{F}_t^{\alpha-1}(x)]\,\mathrm{d}x$, $\alpha > 0$, $\alpha \neq 1$ *is the cumulative Tsallis entropy corresponding to random variable X (*$\zeta_\alpha^Y(t)$ *is defined analogously). It is readily seen that* $\zeta_\alpha^X(t)$ *is in the form (3.4) with* $\phi(u) = 1-(1-u)^{\alpha-1}$, $1 < \alpha \leq 2$. *Thus, if* $X \leq_{\rm sew} Y$, *then* $X \leq^{\rm LCTU} Y$, *for* $1 < \alpha \leq 2$.

It is not difficult to see that the excess wealth function can be expressed in terms of the function

$$A_X(p) = \int_0^p [F^{-1}(u) - E(X)]\,\mathrm{d}u, \quad 0 \leq p \leq 1,$$

as

$$W_X(p) = -A_X(p) - (1-p)A_X'(p), \tag{4.2}$$

where $A_X'(p) = (\mathrm{d}/\mathrm{d}p)A_X(p)$. $A_X(p)$ is called the absolute Lorenz curve (cf. [30]) and takes the values $A_X(0) = A_X(1) = 0$ which follow that

$$A_X(p) = -\int_p^1 [F^{-1}(u) - E(X)]\,\mathrm{d}u, \quad 0 \leq p \leq 1.$$

Now, we have the following theorem which shows that the dilation order implies the second-order excess wealth order.

Theorem 4.2. *If* $X \leq_{\rm dil} Y$, *then* $X \leq_{\rm sew} Y$.

Proof. Using Eq. (4.2) and the integration by parts, we get that

$$\begin{aligned}\int_p^1 W_X(u)\,\mathrm{d}u &= -\int_p^1 A_X(u)\,\mathrm{d}u - \int_p^1 (1-u)\,\mathrm{d}A_X(u)\\ &= pA_X(p) + \int_p^1 u\,\mathrm{d}A_X(u) - \int_p^1 (1-u)\,\mathrm{d}A_X(u)\\ &= -p\int_p^1 \mathrm{d}A_X(u) + \int_p^1 u\,\mathrm{d}A_X(u) - \int_p^1 (1-u)\,\mathrm{d}A_X(u)\\ &= \int_p^1 [2u-1-p]\,\mathrm{d}A_X(u)\\ &= \int_0^1 [2u-1-p]I(u\geq p)\,\mathrm{d}A_X(u). \end{aligned} \tag{4.3}$$

This along with Theorem 2.2 in Ramos and Sordo [30] now follow the result. □

The following example shows that the order $\leq_{\mathrm{sew}}$ dose not necessarily imply $X \leq_{\mathrm{dil}} Y$.

Example 4.1. *Let X and Y be random variables with distribution functions $F(x) = x/3$ and $G(x) = \sqrt{x/3}$, $0 < x < 3$, respectively. Then, we have $A_X(p) = \frac{3}{2}(p^2 - p)$, $A_Y(p) = p^3 - p$, $0 \leq p \leq 1$. Also,*

$$\int_p^1 W_X(u)\,\mathrm{d}u = 0.5(1 - 3p - 3p^2 - p^3), \qquad \int_p^1 W_Y(u)\,\mathrm{d}u = 0.5 - p + p^3 - 0.5p^4.$$

It is not difficult to verify that $A_X(p) < A_Y(p)$ for $p < 0.5$ and $A_X(p) > A_Y(p)$ for $p > 0.5$ which by Theorem 2.2 in Ramos and Sordo [30] means that the dilation order dose not hold. Though,

$$\int_p^1 W_Y(u)\,\mathrm{d}u - \int_p^1 W_X(u)\,\mathrm{d}u = 0.5(p(1-p^3) + 3p^3 + 3p^2) \geq 0, \quad \textit{for all } p \in [0,1].$$

One can see that Eq. (4.2) is a first-order differential equation in $A_X(p)$ whose solution can be found as

$$A_X(p) = -(1-p)\int_0^p \frac{W_X(u)}{(1-u)^2}\,\mathrm{d}u. \tag{4.4}$$

At the case where the right endpoint of the support of X, $u_X = F^{-1}(1) = \inf\{x : F(x) = 1\}$ is finite, it can also be found that

$$A_X(p) = (1-p)\int_p^1 \frac{W_X(u)}{(1-u)^2}\,\mathrm{d}u + (1-p)[E(X) - u_X]. \tag{4.5}$$

As we already mentioned, the orders $\leq_{\mathrm{sew}}$ and $\leq_{\mathrm{icx}}$ do not imply each other. However, using the above equation, we are able to show that the order $\leq_{\mathrm{sew}}$ implies the $\leq_{\mathrm{icx}}$ for distributions with finite supports.

Theorem 4.3. *Let X and Y be two nonnegative random variables with finite corresponding right endpoint of the support u_X and u_Y, respectively, such that $u_Y \geq u_X$. If $X \geq_{\mathrm{sew}} Y$, then $X \leq_{\mathrm{icx}} Y$.*

Proof. Under the hypothesise, by Eq. (4.5) and applying Lemma 3.1, we get that for all $p \in [0,1]$

$$\begin{aligned}A_X(p) - A_Y(p) &= (1-p)\int_p^1 \frac{[W_X(u) - W_Y(u)]}{(1-u)^2}\,\mathrm{d}u\\ &\quad + (1-p)[E(X) - E(Y) + u_Y - u_X]\\ &\geq (1-p)[E(X) - E(Y)],\end{aligned}$$

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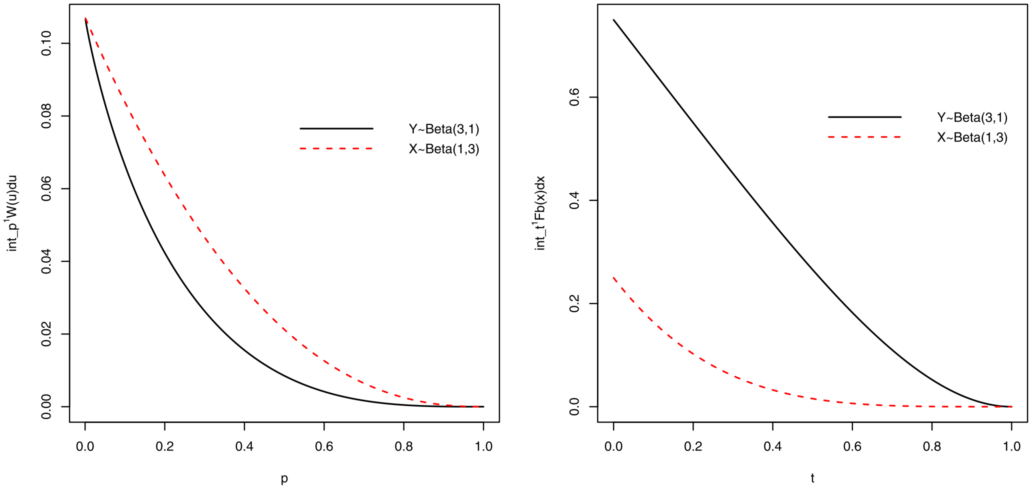

Figure 3. *Plots of* $\int_p^1 W(u)\,\mathrm{d}u$ *and* $\int_t^1 \bar{F}(x)\,\mathrm{d}x$ *for distributions given in Example 4.2.*

or equivalently

$$-\int_p^1 [F^{-1}(u) - E(X)]\,\mathrm{d}u + \int_p^1 [G^{-1}(u) - E(Y)]\,\mathrm{d}u \geq (1-p)[E(X) - E(Y)].$$

That is,

$$\int_p^1 F^{-1}(u)\,\mathrm{d}u \leq \int_p^1 G^{-1}(u)\,\mathrm{d}u, \quad \text{for all } p \in [0,1],$$

which is equivalent to $X \leq_{\text{icx}} Y$ (see Theorem 4.A.3 in [33, p. 183]). □

Example 4.2. *Let* X *and* Y *be distributed as* Beta(1, 3) *and* Beta(3, 1) *with distribution functions* $F(x) = 1 - (1-x)^3$ *and* $G(x) = x^3$, $0 < x < 1$, *respectively. Then, we have*

$$\int_t^1 \bar{F}(x)\,\mathrm{d}x = \frac{(1-t)^4}{4}, \quad \int_p^1 W_X(u)\,\mathrm{d}u = \frac{3}{4}(1+p)(1-p)^{4/3} - 2\int_p^1 u(1-u)^{1/3}\,\mathrm{d}u,$$
$$\int_t^1 \bar{G}(x)\,\mathrm{d}x = \frac{1-t^4}{4}, \quad \int_p^1 W_Y(u)\,\mathrm{d}u = \frac{6}{7}(1-p^{7/3}) - \frac{3}{4}(1+p)(1-p^{4/3}).$$

Figure 3 depicts that $X \geq_{\text{sew}} Y$ *and* $X \leq_{\text{icx}} Y$.

For $p \in [0,1]$, denote recursively $F_1^{-1}(p) = F^{-1}(p)$, and $F_n^{-1}(p) = \int_p^1 F_{n-1}^{-1}(u)\,\mathrm{d}u$, for $n = 2, 3, \ldots$ (and analogously for G_n^{-1}). For any positive integer m and all $p \in [0,1]$, if $F_m^{-1}(p) \leq G_m^{-1}(p)$, then it is denoted $X \leq_m^{-1} Y$ (see [33, p. 213]). The result of the next theorem is similar with that in the above theorem between the orders $\leq_{\text{sew}}$ and $\leq_3^{-1}$.

Theorem 4.4. *Let* X *and* Y *be two nonnegative random variables with finite corresponding right endpoint of the support* u_X *and* u_Y, *respectively, such that* $u_Y \geq u_X$. *If* $X \geq_{\text{sew}} Y$, *then* $X \leq_3^{-1} Y$.

Proof. From Eqs. (4.4) and (4.5), we have

$$\begin{aligned}\int_p^1 A_X(u)\,\mathrm{d}u &= -\int_p^1 (1-u)\int_0^u \frac{W_X(v)}{(1-v)^2}\,\mathrm{d}v\,\mathrm{d}u\\ &= -\frac{(1-p)^2}{2}\int_0^p \frac{W_X(v)}{(1-v)^2}\,\mathrm{d}v - \frac{1}{2}\int_p^1 W_X(v)\,\mathrm{d}v\\ &= \frac{(1-p)}{2}A_X(p) - \frac{1}{2}\int_p^1 W_X(v)\,\mathrm{d}v\\ &= \frac{(1-p)^2}{2}\int_p^1 \frac{W_X(v)}{(1-v)^2}\,\mathrm{d}v + \frac{(1-p)^2}{2}[E(X)-u_X] - \frac{1}{2}\int_p^1 W_X(v)\,\mathrm{d}v,\\ &= \int_p^1 \left[\frac{(1-p)^2}{2(1-v)^2} - \frac{1}{2}\right] W_X(v)\,\mathrm{d}v + \frac{(1-p)^2}{2}[E(X)-u_X]. \end{aligned} \tag{4.6}$$

where the second equality follows from integration by parts. Now, using the hypothesis and applying Lemma 3.1, we see that for all $p \in [0,1]$

$$\begin{aligned}\int_p^1 A_X(u)\,\mathrm{d}u - \int_p^1 A_Y(u)\,\mathrm{d}u &= \int_p^1 \left[\frac{(1-p)^2}{2(1-v)^2} - \frac{1}{2}\right][W_X(v) - W_Y(v)]\,\mathrm{d}v\\ &\quad + \frac{(1-p)^2}{2}[E(X) - E(Y) + u_Y - u_X]\\ &\geq \frac{(1-p)^2}{2}[E(X) - E(Y)],\end{aligned}$$

or equivalently

$$\int_p^1\int_u^1 F^{-1}(v)\,\mathrm{d}v\,\mathrm{d}u \leq \int_p^1\int_u^1 G^{-1}(v)\,\mathrm{d}v\,\mathrm{d}u,$$

which completes the proof. □

As a corollary, the above theorem along with Theorem 4.A.72 in Shaked and Shanthikumar [33, p. 213] follow that if $X \geq_{\text{sew}} Y$ and $u_Y \geq u_X$, then

$$E[\max\{X_1, X_2, \ldots, X_k\}] \leq E[\max\{Y_1, Y_2, \ldots, Y_k\}], \quad k \geq 2,$$

where $X_i's[Y_i's]$ are independent copies of $X[Y]$.

It is known that [33, p. 165] $X \leq_{\text{ew}} Y$ if and only if,

$$\frac{G_2^{-1}(p) - F_2^{-1}(p)}{(1-p)} = \frac{1}{(1-p)}\int_p^1 [G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u \quad \text{is increasing in } p \in (0,1). \tag{4.7}$$

In the next theorem, we give a similar equivalence for the second-order excess wealth order.

Theorem 4.5. *$X \leq_{\text{ew}} Y$ if and only if,*

$$\frac{G_3^{-1}(p) - F_3^{-1}(p)}{(1-p)^2} = \frac{1}{(1-p)^2}\int_p^1\int_u^1 [G^{-1}(v) - F^{-1}(v)]\,\mathrm{d}v\mathrm{d}u \quad \textit{is increasing in } p \in [0,1]. \tag{4.8}$$

Proof. First, we have from Eq. (4.2) that

$$\int_p^1 W_X(u)\,\mathrm{d}u = (1-p)A_X(p) - 2\int_p^1 A_X(u)\,\mathrm{d}u,$$

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which implies that $X \leq_{\rm sew} Y$ if and only if, for all $p \in [0, 1]$

$$2\int_p^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u \geq (1-p)[A_X(p) - A_Y(p)], \tag{4.9}$$

or, equivalently,

$$\frac{1}{(1-p)^2}\int_p^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u \quad \text{is increasing in } p \in [0, 1]. \tag{4.10}$$

On the other hand, one can write

$$\int_p^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u = \int_p^1 \int_u^1 [G^{-1}(v) - F^{-1}(v)]\,\mathrm{d}v\,\mathrm{d}u + \frac{(1-p)^2}{2}[E(X) - E(Y)].$$

Therefore, the condition (4.10) is equivalent to that

$$\frac{1}{(1-p)^2}\int_p^1 \int_u^1 [G^{-1}(v) - F^{-1}(v)]\,\mathrm{d}v\,\mathrm{d}u \quad \text{is increasing in } p \in [0, 1].$$

This completes the proof. □

Under Definition 3.1 in Ramos and Sordo [30], X is said to be smaller than Y in the second-order absolute Lorenz order if $\int_p^1 A_X(u)\,\mathrm{d}u \geq \int_p^1 A_Y(u)\,\mathrm{d}u$, for all $p \in [0, 1]$. The following theorem shows that the second-order absolute Lorenz order implies the second excess wealth order.

Theorem 4.6. *If $X \leq_{\rm sew} Y$, then $\int_p^1 A_X(u)\,\mathrm{d}u \geq \int_p^1 A_Y(u)\,\mathrm{d}u$, for all $p \in [0, 1]$.*

Proof. First, note that the inequality (4.9) gives that

$$\int_0^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u \geq 0.$$

On the other hand, it follows from (4.10) that

$$\int_p^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u \geq (1-p)^2 \int_0^1 [A_X(u) - A_Y(u)]\,\mathrm{d}u.$$

These two latter inequalities now give the result. □

Example 4.3. *For the random variables in Example 4.1, we have*

$$\int_p^1 A_X(u)\,\mathrm{d}u - \int_p^1 A_Y(u)\,\mathrm{d}u = 0.25p^2[1 + 2p(1-p)] \geq 0.$$

We also had $\int_p^1 W_X(u)\,\mathrm{d}u \leq \int_p^1 W_Y(u)\,\mathrm{d}u$.

Using Eq. (3.3), one can write

$$\int_p^1 W_X(u)\,\mathrm{d}u = E(X)\int_p^1 (2u - p - 1)\,\mathrm{d}L_X(u) = E(X)\int_p^1 [1 - 2L_X(u)]\,\mathrm{d}u, \tag{4.11}$$

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where $L_X(u) = (1/E(X)) \int_0^u F^{-1}(v)\,\mathrm{d}v$ is the Lorenz curve corresponding to X. The following result gives the relationship between the Lorenz and second-order excess wealth orders.

Theorem 4.7. *If $X \leq_{\mathrm{L}} Y$ and $E(X) \leq E(Y)$, then $X \leq_{\mathrm{sew}} Y$.*

Proof. Under the hypothesises and by using Eq. (4.11), we obtain

$$\frac{\int_p^1 W_X(u)\,\mathrm{d}u}{E(X)} = \int_p^1 [1 - 2L_X(u)]\,\mathrm{d}u \leq \int_p^1 [1 - 2L_Y(u)]\,\mathrm{d}u = \frac{\int_p^1 W_Y(u)\,\mathrm{d}u}{E(Y)} \leq \frac{\int_p^1 W_Y(u)\,\mathrm{d}u}{E(X)},$$

which completes the proof. □

Belzunce *et al.* [9] have developed a new criterion to compare risks based on the notion of expected proportional shortfall. They have said that X is smaller than Y in the expected proportional shortfall order (in short, the PS order), denoted by $X \leq_{\mathrm{PS}} Y$, if

$$\frac{W_X(p)}{F^{-1}(p)} \leq \frac{W_Y(p)}{G^{-1}(p)}, \quad \text{for all } p \in D_X \cap D_Y,$$

where $D_X = \{p \in (0,1) : F^{-1}(p) > 0\}$ (and analogously for D_Y). The following theorem connects the second-order excess wealth order with the PS order.

Theorem 4.8. *If $X \leq_{\mathrm{PS}} Y$ and $E(X) \leq E(Y)$, then $X \leq_{\mathrm{sew}} Y$.*

Proof. The result easily follows from the assumption, Theorem 4.7 and Theorem 2.19 in Belzunce *et al.* [9]. □

An application of the excess wealth order in comparing the epoch times of two nonhomogeneous poisson processes (NHPP) has been given in Belzunce *et al.* [8]. We end this section by considering the same comparison problem with respect to the second-order excess wealth order.

Let $T_{1,n}$ and $T_{2,n}$, $n \geq 1$ be the epoch points of two NHPPs with intensity functions λ_1 and λ_2, respectively, such that $\int_t^\infty \lambda_i(u)\,\mathrm{d}u = \infty$, $i = 1, 2$, for all $t \geq 0$. Let also X and Y be two nonnegative random variables with hazard rates λ_1 and λ_2 and with distribution functions F and G, respectively. The distribution functions of $T_{1,n}$ and $T_{2,n}$ can be expressed as (see [8])

$$F_n(t) = \Phi_n(F(t)), \quad \text{and} \quad G_n(t) = \Phi_n(G(t)), \quad t \geq 0,$$

respectively, where $\Phi_n(p) = \Gamma_n(-\ln(1-p))$ for $p \in (0,1)$, and Γ_n is the distribution function of a gamma distribution with scale parameter 1 and shape parameter n.

Theorem 4.9. *If $X \leq_{\mathrm{sew}} Y$, then $T_{1,n} \leq_{\mathrm{sew}} T_{1,n}$ for all $n \geq 1$.*

Proof. Regarding Theorem 3.1, the assertion follows if we show that, for any increasing and convex function ϕ

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F_n^{-1}(u) \leq \int_0^1 (1-u)\phi(u)\,\mathrm{d}G_n^{-1}(u),$$

or equivalently,

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(\Phi_n^{-1}(u)) \leq \int_0^1 (1-u)\phi(u)\,\mathrm{d}G_n^{-1}(\Phi_n^{-1}(u)).$$

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Note that

$$\int_0^1 (1-u)\phi(u)\,\mathrm{d}F^{-1}(\Phi_n^{-1}(u)) = \int_0^1 \bar{\Phi}_n(u)\phi(\Phi_n(u))\,\mathrm{d}F^{-1}(u)$$
$$= \int_0^1 (1-u)\phi^*(u)\,\mathrm{d}F^{-1}(u),$$

where $\phi^*(u) = (\bar{\Phi}_n(u)\phi(\Phi_n(u)))/(1-u)$. It is not difficult to see that $\phi^*(u)$ is an increasing and convex function. Now, the assumption along with Theorem 3.1 implies that

$$\int_0^1 (1-u)\phi^*(u)\,\mathrm{d}F^{-1}(u) \leq \int_0^1 (1-u)\phi^*(u)\,\mathrm{d}G^{-1}(u).$$

This completes the proof. □

5. Conclusion

Inspired by interesting properties of the excess wealth order and its application for tail variability comparing in risk analysis, in this paper, we considered a new tail variability measure through extending the excess wealth order to its second-order. At the case where the excess wealth functions of two risks cross each other (Example 3.1), we are not able to compare tail variability of risks based on the excess wealth order. In such cases, one may use the second-order excess wealth order for the comparing purpose ensuring that the excess wealth function has still been involved in comparison and the included tail variability information been used. For proposed order, we obtained the main properties and were able to give two characterization results. Its property and an application in the reliability theory were also given. We were able to establish the preservation property of the order under parallel systems and its reversed preservation property under series systems. The preservation properties for the general systems and other closure and preservation properties can be investigated as future works. The relationship between the proposed second-order excess wealth order and some well-known stochastic orders was studied. We have shown that the dilation order, and under a condition the $\leq_{\text{dmrl}}$, $\leq_{\text{nbue}}$, the Lorenz and the expected proportional shortfall orders imply the second-order excess wealth order. We have also demonstrated that the proposed second-order excess wealth order implies the second Lorenz order. It was also shown that the order follows the increasing convex order for finite support distributions. We ended up the paper by giving an application of the proposed order in comparing the epoch times of two nonhomogeneous poisson processes. Indeed, the areas where the excess wealth and other variability orders have been applied, can be investigated for potential applications of the $\leq_{\text{sew}}$ order.

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