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CHARACTER GRAPHS WITH NONBIPARTITE HAMILTONIAN COMPLEMENT

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Abstract

For a finite group G, let $\Delta(G)$ denote the character graph built on the set of degrees of the irreducible complex characters of G. In this paper, we obtain a necessary and sufficient condition which guarantees that the complement of the character graph $\Delta(G)$ of a finite group G is a nonbipartite Hamiltonian graph.

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1. Introduction

Let G be a finite group and let $\mathrm{cd}(G)$ be the set of all character degrees of G, that is, $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, where $\mathrm{Irr}(G)$ is the set of all complex irreducible characters of G. The set of prime divisors of character degrees of G is denoted by $\rho(G)$. It is well known that the character degree set $\mathrm{cd}(G)$ may be used to provide information on the structure of the group G. For example, in the late 1990s, Huppert conjectured that the non-Abelian simple groups are essentially determined by the set of their character degrees. He verified the conjecture on a case-by-case basis for many non-Abelian simple groups, including the Suzuki groups, many of the sporadic simple groups and a few of the simple groups of Lie type [4].

A useful way to study the character degree set of a finite group G is to associate a graph to $\mathrm{cd}(G)$. One of these graphs is the character graph $\Delta(G)$ of G [9]. Its vertex set is $\rho(G)$ and two vertices p and q are joined by an edge if the product pq divides some character degree of G. The main questions in this research area concern the relationships between the group structure of G and certain graph-theoretical features of $\Delta(G)$. For instance, Lewis and White [7] proved that $\Delta(G)$ has three connected components if and only if $G = S \times A$, where $S \cong \mathrm{PSL}_2(2^n)$ for some integer $n \geqslant 2$ and A is an Abelian group. Also, in [10], it was shown that if the character graph $\Delta(G)$ of

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a finite group G is k-regular of odd order for some positive integer k, then $\Delta(G)$ is a complete graph. For further results of this type, we refer to the survey by Lewis [6].

An important family of graphs is the class of Hamiltonian graphs. Let Γ be a simple graph with n vertices. Any cycle of Γ of length n is called a Hamilton cycle. We say that Γ is Hamiltonian if it contains a Hamilton cycle. In [2], it was shown that the character graph $\Delta(G)$ of a solvable group G is Hamiltonian if and only if $\Delta(G)$ is a block with at least three vertices. In this paper, we give a necessary and sufficient condition on the structure of a finite group G which guarantees that the complement of $\Delta(G)$ is a nonbipartite Hamiltonian graph. Note that when the complement of $\Delta(G)$ is a Hamiltonian graph of odd order, then it is automatically nonbipartite.

Our main result is the following theorem. For an integer $n \geqslant 1$, we denote the set of prime divisors of n by $\pi(n)$.

Theorem 1.1. *Let G be a finite group. The complement of the character graph $\Delta(G)$ is a nonbipartite Hamiltonian graph if and only if $G \cong \mathrm{SL}_2(2^f) \times A$, where $f \geqslant 2$ is an integer, $||\pi(2^f + 1)| - |\pi(2^f - 1)|| \leqslant 1$ and A is an Abelian group.*

2. Preliminaries

In this paper, all groups are assumed to be finite and all graphs are simple and finite. For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$. If H is a subgroup of G and $\theta \in \mathrm{Irr(H)}$, we denote by $\mathrm{Irr(G} \mid \theta)$ the set of irreducible characters of G lying over θ and define $\mathrm{cd(G} \mid \theta) := \{\chi(1) \mid \chi \in \mathrm{Irr(G} \mid \theta)\}$. We use Clifford's theorem [5, Theorem 6.11] and Gallagher's theorem [5, Corollary 6.17]. If $N \lhd G$ and $\theta \in \mathrm{Irr(N)}$, the inertia subgroup of θ in G is denoted by $I_G(\theta)$.

Lemma 2.1 [11]. *Let N be a normal subgroup of a group G so that $G/N \cong S$, where S is a non-Abelian simple group. Let $\theta \in \mathrm{Irr(N)}$. Then either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(G/N)$ for some $\chi \in \mathrm{Irr(G} \mid \theta)$ or θ is extendible to $\theta_0 \in \mathrm{Irr(G)}$ and $G/N \cong A_5$ or $\mathrm{PSL}_2(8)$.*

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simple graph with the vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. If $E(\Gamma) = \emptyset$, we call Γ an empty graph. When C is a cycle of Γ of odd (even) order, then C is called an odd (even) cycle of Γ. The complement of Γ and the induced subgraph of Γ on $X \subseteq V(\Gamma)$ are denoted by Γ^c and $\Gamma[X]$, respectively. We now state some results on character graphs needed in the next section.

Lemma 2.2 [1]. *Let G be a finite group and let π be a subset of the vertex set of $\Delta(G)$ such that $|\pi|$ is an odd number larger than 1. Then π is the set of vertices of a cycle in $\Delta(G)^c$ if and only if $O^{\pi'}(G) = S \times A$, where A is Abelian, $S \cong \mathrm{SL}_2(u^\alpha)$ or $S \cong \mathrm{PSL}_2(u^\alpha)$ for a prime $u \in \pi$ and a positive integer α, and the primes in $\pi - \{u\}$ are alternately odd divisors of $u^\alpha \pm 1$.*

We will make use of Dickson's list of the subgroups of $\mathrm{PSL}_2(q)$, which can be found as Hauptsatz II.8.27 of [3]. We also use the fact that the Schur multiplier of $\mathrm{PSL}_2(q)$ is trivial unless $q = 4$ or q is odd.

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Lemma 2.3 [12]. *Let $G \cong \mathrm{PSL}_2(q)$, where $q \geqslant 4$ is a power of a prime p.*

(a) *If q is even, then $\Delta(G)$ has three connected components, $\{2\}$, $\pi(q-1)$ and $\pi(q+1)$, and each component is a complete graph.*

(b) *If q is odd and $q > 5$ is odd, then $\Delta(G)$ has two connected components, $\{p\}$ and $\pi((q-1)(q+1))$.*

 (i) *The connected component $\pi((q-1)(q+1))$ is a complete graph if and only if $q-1$ or $q+1$ is a power of 2.*

 (ii) *If neither of $q-1$ or $q+1$ is a power of 2, then $\pi((q-1)(q+1))$ can be partitioned as $\{2\} \cup M \cup P$, where both the sets $M = \pi(q-1) - \{2\}$ and $P = \pi(q+1) - \{2\}$ are nonempty. The subgraph of $\Delta(G)$ corresponding to each of the subsets M, P is complete, all primes are adjacent to 2 and no prime in M is adjacent to any prime in P.*

Lemma 2.4 [8]. *Let p be a prime and $f \geqslant 2$ be an integer such that $q = p^f \geqslant 5$ and $S \cong \mathrm{PSL}_2(q)$. If $q \neq 9$ and $S \leqslant G \leqslant \mathrm{Aut(S)}$, then G has irreducible characters of degrees $(q+1)[G : G \cap \mathrm{PGL}_2(q)]$ and $(q-1)[G : G \cap \mathrm{PGL}_2(q)]$.*

3. Proof of Theorem 1.1

In this section, we wish to prove our main result.

Lemma 3.1. *Let G be a finite group. Then every prime $p \in \rho(G)$ is a vertex of an odd cycle in $\Delta(G)^c$ if and only if $G \cong \mathrm{SL}_2(2^f) \times A$, where $f \geqslant 2$ is an integer and A is an Abelian group.*

Proof. If $G \cong \mathrm{SL}_2(q) \times A$, where $q = 2^f$ for some integer $f \geqslant 2$ and A is Abelian, then, by Lemma 2.3, we have nothing to prove. Conversely, suppose that every prime $p \in \rho(G)$ is a vertex of an odd cycle in $\Delta(G)^c$. We complete the proof with the following steps.

Step 1. If $\pi \subseteq \rho(G)$ is the set of vertices of an odd cycle of $\Delta(G)^c$, then $2 \in \pi$. On the contrary, suppose that $\pi_1 \subseteq \rho(G)$ is the set of vertices of an odd cycle of $\Delta(G)^c$ and $2 \notin \pi_1$. Since $\Delta(G)^c$ is not bipartite, Lemma 2.2 implies that G is nonsolvable and $2 \in \rho(G)$. Thus, by assumption, there exists $\pi_2 \subseteq \rho(G)$ such that π_2 is the set of vertices of an odd cycle in $\Delta(G)^c$ and $2 \in \pi_2$. By Lemma 2.2, for $i = 1$ and 2, $N_i := O^{\pi_i'}(G) = S_i \times A_i$, where A_i is Abelian, $S_i \cong \mathrm{SL}_2(u_i^{\alpha_i})$ or $S_i \cong \mathrm{PSL}_2(u_i^{\alpha_i})$ for a prime $u_i \in \pi_i$ and a positive integer α_i, and the primes in $\pi_i - \{u_i\}$ are alternately odd divisors of $u_i^{\alpha_i} + 1$ and $u_i^{\alpha_i} - 1$. Let $H := S_1 S_2$. Note that $u_2 = 2$. It is easy to see that $H/Z(H) \cong \mathrm{PSL}_2(u_1^{\alpha_1}) \times \mathrm{PSL}_2(u_2^{\alpha_2})$. Thus, using Lemma 2.3, 2 is adjacent to all vertices in π_2, which is impossible. Hence, $2 \in \pi_1$ and we are done.

Step 2. There exists a unique normal subgroup L of G such that $L \cong \mathrm{SL}_2(2^f)$ for some integer $f \geqslant 2$, $\rho(G) = \rho(L)$ and $G/R(G)$ is an almost simple group with socle $S := LR(G)/R(G) \cong L$. Let $p \in \rho(G)$. By assumption, there exists $\pi \subseteq \rho(G)$ so that π is the set of vertices of an odd cycle in $\Delta(G)^c$ and $p \in \pi$. Then $2 \in \pi$ by Step 1.

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By Lemma 2.2, $N := O^{\pi'}(G) = L \times A$, where A is Abelian, $L \cong \mathrm{SL}_2(2^f)$ for a positive integer $f \geqslant 2$, and the primes in $\pi - \{2\}$ are alternately odd divisors of $2^f + 1$ and $2^f - 1$. Since $2 \in \pi$, by the Feit–Thompson theorem, G/N is solvable. Therefore, so is G/L. This proves Step 2.

Step 3. *2 is an isolated vertex of* $\Delta(G)$. Using Step 2, there exists a unique normal subgroup L of G so that $L \cong \mathrm{SL}_2(2^f)$ for some integer $f \geqslant 2$ and $\rho(G) = \pi(L)$. For a contradiction, suppose that 2 is adjacent to some vertex $p \in \rho(G) - \{2\}$ in $\Delta(G)$. As $p \in \pi(2^{2f} - 1)$ and $\Delta(L) \subseteq \Delta(G)$, using Lemma 2.3, we see that p is not a vertex of an odd cycle in $\Delta(G)^c$, which is a contradiction.

Step 4. $G \cong S \times R(G)$*, where* $S \cong \mathrm{SL}_2(2^f)$ *for some integer* $f \geqslant 2$ *and* $R(G)$ *is Abelian.* By Step 2, $G/R(G)$ is an almost simple group with socle $S \cong \mathrm{SL}_2(2^f)$, where $f \geqslant 2$ is an integer and $\rho(G) = \pi(S)$. We claim that $G/R(G) = S$. On the contrary, suppose that $G/R(G) \neq S$. If $f = 2$, then $6 \in \mathrm{cd(G/R(G))} \subseteq \mathrm{cd(G)}$, which is a contradiction with Step 3. Thus, $f \geqslant 3$. By Lemma 2.4, since $\rho(G) = \pi(S)$, there exists $p \in \pi(S)$ such that p is adjacent to all vertices in $\pi(2^{2f} - 1) - \{p\}$. But this is a contradiction as p is a vertex of an odd cycle in $\Delta(G)^c$. Thus, $G/R(G) = S$. Since $\rho(G) = \pi(S)$ and every prime $p \in \rho(G)$ is a vertex of an odd cycle in $\Delta(G)^c$, there exist $p_1 \in \pi(2^f - 1)$ and $p_2 \in \pi(2^f + 1)$ such that p_1 and p_2 are nonadjacent vertices in $\Delta(G)$. Note that by Step 3, 2 is an isolated vertex of $\Delta(G)$. Hence, the induced subgraph of $\Delta(G)^c$ on $\pi := \{2, p_1, p_2\}$ is a triangle. Let H be the last term of the derived series of G. Since G is nonsolvable, H is nontrivial. Let $N := H \cap R(G)$. As $H/N \cong HR(G)/R(G) \lhd G/R(G) \cong S$, we deduce that $H/N \cong S$. We claim that $N = 1$. On the contrary, suppose that $N \neq 1$. Then there exists a nonprincipal linear character $\lambda \in \mathrm{Irr(N)}$. If $f = 2$ or 3, then $S \cong A_5$ or $\mathrm{PSL}_2(8)$ and $\Delta(S)$ is an empty graph. Using Lemma 2.1, for some $\chi \in \mathrm{Irr(H} \mid \lambda)$, we find that $\chi(1)$ is divisible by two distinct primes in $\pi(S)$. This is a contradiction as $\Delta(G)^c[\pi]$ is a triangle. Hence, $f \geqslant 4$. Suppose that $I := I_H(\lambda)$ and $M := I/N$. Since H is perfect and the Schur multiplier of S is trivial, $M \neq S$. Thus, M is contained in a maximal subgroup L of S. From Dickson's list, the index $[S : M]$ is divisible by one of the numbers

$$2^{f-1}(2^f + 1), \quad 2^{f-1}(2^f - 1), \quad 2^f + 1, \quad 2^{f-a}\,\frac{(2^{2f} - 1)}{2^{2a} - 1},$$

where $f/a \geqslant 2$ is prime in the last case. If $2 \mid [S : M]$, then, using Clifford's theorem, 2 is adjacent to some prime $p \in \pi(2^{2f} - 1)$, which is impossible as 2 is an isolated vertex of $\Delta(G)$. Also, if M is an elementary Abelian 2-group, then, by Clifford's theorem, some $m \in \mathrm{cd(H} \mid \lambda)$ is divisible by $2^{2f} - 1$ and this is a contradiction as $\Delta(G)^c[\pi]$ is a triangle. Therefore, by Dickson's list, M is a Frobenius group of order $2^f n$, where $n \mid (2^f - 1)$. Thus, by Clifford's and Gallagher's theorems, there exists $m \in \mathrm{cd(H} \mid \lambda)$ such that m is divisible by either $2(2^f + 1)$ or $2^{2f} - 1$, which is again a contradiction. Hence, $N = 1$ and $G \cong S \times R(G)$. If $p \in \rho(R(G))$, then, as $\rho(G) = \pi(S)$, it follows that p is an isolated vertex in $\Delta(G)^c$, which is again a contradiction. Thus, $R(G)$ is Abelian and the proof is completed. □

Proof of Theorem 1.1. Suppose that $G \cong \mathrm{SL}_2(2^f) \times A$, where $f \geqslant 2$ is an integer. Then $||\pi(2^f + 1)| - |\pi(2^f - 1)|| \leqslant 1$ and A is an Abelian group. Using Lemma 2.3, we are done. Conversely, suppose that $\Delta(G)^c$ is a nonbipartite Hamiltonian graph. Using Lemma 3.1, it is enough to show that every prime $p \in \rho(G)$ is a vertex of an odd cycle in $\Delta(G)^c$. If $|\rho(G)|$ is odd, we have nothing to prove. Hence, we can assume that $|\rho(G)|$ is even. Since $\Delta(G)^c$ is not bipartite, we deduce that there exists $\pi \subseteq \rho(G)$ such that $\Delta(G)^c[\pi]$ is an odd cycle. Since $\Delta(G)^c$ is Hamiltonian, it is easy to see that every prime $p \in \rho(G)$ is a vertex of an odd cycle in $\Delta(G)^c$ and the proof is completed. □

References

[1] Z. Akhlaghi, C. Casolo, S. Dolfi, E. Pacifici and L. Sanus, 'On the character degree graph of finite groups', *Ann. Mat. Pura Appl.*, to appear, 20 pages.
[2] M. Ebrahimi, A. Iranmanesh and M. A. Hosseinzadeh, 'Hamiltonian character graphs', *J. Algebra* **428** (2015), 54–66.
[3] B. Huppert, *Endliche Gruppen I*, Die Grundlehren der mathematischen Wissenschaften, 134 (Springer, Berlin–New York, 1967).
[4] B. Huppert, 'Some simple groups which are determined by the set of their character degrees I', *Illinois J. Math.* **44** (2000), 828–842.
[5] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press, San Diego, CA, 1976).
[6] M. L. Lewis, 'An overview of graphs associated with character degrees and conjugacy class sizes in finite groups', *Rocky Mountain J. Math.* **38**(1) (2008), 175–211.
[7] M. L. Lewis and D. L. White, 'Connectedness of degree graphs of non-solvable groups', *J. Algebra* **266**(1) (2003), 51–76.
[8] M. L. Lewis and D. L. White, 'Non-solvable groups with no prime dividing three character degrees', *J. Algebra* **336** (2011), 158–183.
[9] O. Manz, R. Staszewski and W. Willems, 'On the number of components of a graph related to character degrees', *Proc. Amer. Math. Soc.* **103**(1) (1988), 31–37.
[10] Z. Sayanjali, Z. Akhlaghi and B. Khosravi, 'On the regularity of character degree graphs', *Bull. Aust. Math. Soc.* **100**(3) (2019), 428–433.
[11] H. P. Tong-Viet, 'Groups whose prime graphs have no triangles', *J. Algebra* **378** (2013), 196–206.
[12] D. L. White, 'Degree graphs of simple linear and unitary groups', *Comm. Algebra* **34**(8) (2006), 2907–2921.

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