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FRACTIONAL FOCK–SOBOLEV SPACES

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Abstract. Let $s \in \mathbb{R}$ and $0 < p \leqslant \infty$. The fractional Fock–Sobolev spaces $F^{s,p}_{\mathscr{R}}$ are introduced through the fractional radial derivatives $\mathscr{R}^{s/2}$. We describe explicitly the reproducing kernels for the fractional Fock–Sobolev spaces $F^{s,2}_{\mathscr{R}}$ and then get the pointwise size estimate of the reproducing kernels. By using the estimate, we prove that the fractional Fock–Sobolev spaces $F^{s,p}_{\mathscr{R}}$ are identified with the weighted Fock spaces F^p_s that do not involve derivatives. So, the study on the Fock–Sobolev spaces is reduced to that on the weighted Fock spaces.

§1. Introduction

Let $\mathbb{C}^n$ be the complex n-space and dV be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j, \quad |z| = (z \cdot \overline{z})^{1/2}.$$

For any $0 < p \leqslant \infty$ we let L^p_G denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $f(z)e^{-(1/2)|z|^2}$ is in $L^p(\mathbb{C}^n, dV)$. When $0 < p < \infty$, it is clear that

$$L^p_G = L^p(\mathbb{C}^n, e^{-(p/2)|z|^2}\, dV(z)).$$

We define

$$\|f\|_p = \left[\left(\frac{p}{2\pi} \right)^n \int_{\mathbb{C}^n} |f(z)e^{-(1/2)|z|^2}|^p \, dV(z) \right]^{1/p}.$$

For $p = \infty$ the norm in L^∞_G is defined by

$$\|f\|_\infty = \operatorname{esssup}\{|f(z)|e^{-(1/2)|z|^2} : z \in \mathbb{C}^n\}.$$

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Let F^p denote the space of entire functions in L^p_G. Then F^2 is a closed subspace of the Hilbert space L^2_G (see [15]) with inner product

$$\langle f, g\rangle = \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}\,dV(z).$$

To give a motivation for our study of Fock–Sobolev spaces, recall that the annihilation operator A_j and the creation operator A_j^* from the quantum theory are defined by the commutation relation $[A_j, A_k^*] = \delta_{jk}I$, where I is the identity operator. A natural representation of these operators is achieved on the Fock space F^2, namely,

$$A_j f(z) = \frac{\partial}{\partial z_j} f(z), \qquad A_j^* f(z) = z_j f(z), \quad 1 \leqslant j \leqslant n,\, f \in F^2.$$

Both A_j and A_j^*, as defined above, are densely defined linear operators on F^2 (unbounded though) and satisfy the commutation relation $[A_j, A_k^*] = \delta_{jk}I$. Therefore, it is important to study the operator of multiplication by z_j and the operator of differentiation on the Fock space F^2.

We define the radial derivative $\mathscr{R}$ by

$$\mathscr{R} := \sum_{j=1}^{n}(A_j A_j^* + A_j^* A_j)$$

and the Fock–Sobolev space $F^{s,p}_{\mathscr{R}}$ of fractional order s for which $\mathscr{R}^{s/2}f$ is given by an F^p function. Then $F^{s,2}_{\mathscr{R}}$ is a Hilbert space with inner product

$$\langle f, g\rangle_{F^{s,2}_{\mathscr{R}}} = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \mathscr{R}^{s/2}f(z)\overline{\mathscr{R}^{s/2}g(z)}e^{-|z|^2}\,dV(z)$$

for $f, g \in F^{s,2}_{\mathscr{R}}$. Each point evaluation is a bounded linear functional on $F^{s,2}_{\mathscr{R}}$. So, to each $z \in \mathbb{C}^n$ there corresponds the reproducing kernel K^s_z such that

$$f(z) = \langle f, K^s_z\rangle_{F^{s,2}_{\mathscr{R}}}$$

for $f \in F^{s,2}_{\mathscr{R}}$. Let $K_s(z, w) := K^s_w(z)$. Defining $\Lambda(z, w)$ by

$$\Lambda(z, w) = e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2},$$

we have the following estimates of the reproducing kernel $K_s(z, w)$ for $F^{s,2}_{\mathscr{R}}$.

THEOREM 1.1. *Let $s \in \mathbb{R}$. Then*

$$K_s(z, w) = \mathscr{R}_z^{-s}(e^{z\cdot\overline{w}})$$

and there are positive constants $C = C(s) > 0$ such that

$$|K_s(z, w)| \leqslant C \times \begin{cases} (1 + |z||w|)^{-s}\Lambda(z, w) & \text{if } s > 0, \\ (1 + |z \cdot \overline{w}|)^{-s}\Lambda(z, w) & \text{if } s \leqslant 0, \end{cases}$$

for $z, w \in \mathbb{C}^n$.

It will turn out that polynomially growing/decaying weights quite naturally come into play in the study of our fractional Fock–Sobolev spaces. So, we first introduce such weighted Fock spaces. Given s real we introduce the following norm on F_s^p when $0 < p < \infty$:

$$\|f\|_{F_s^p}^p = \omega_{n,s,p} \int_{\mathbb{C}^n} |(1 + |z|)^s f(z) e^{-(1/2)|z|^2}|^p \, dV(w),$$

where $\omega_{n,s,p}$ is a normalizing constant so that the constant function 1 has norm 1 in F_s^p. When $p = \infty$, we define

$$\|f\|_{F_s^\infty} = \omega_s \sup_{z\in\mathbb{C}^n} [(1 + |z|)^s |f(z)| e^{-(1/2)|z|^2}],$$

where ω_s is a normalizing constant so that the constant function 1 has norm 1 in F_s^∞. Let $L_{G,s}^p$ denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $(1 + |z|)^s f(z)$ is in L_G^p. Then F_s^p is a closed subspace of $L_{G,s}^p$.

It follows that the fractional Fock–Sobolev spaces are realized as the weighted Fock spaces that do not involve derivatives as following Theorem 1.2. So, the study on the Fock–Sobolev spaces is reduced to that on the weighted Fock spaces. It is very convenient to study function theoretic and operator theoretic properties on the weighted Fock spaces instead of the Fock–Sobolev spaces (see [3, 7, 9–11, 13]).

THEOREM 1.2. *Suppose $0 < p \leqslant \infty$ and s is a real number. Then $F_{\mathscr{R}}^{s,p} = F_s^p$ with equivalent norms.*

Constants. In this paper we use the same letter C to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be often specified in parenthesis. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \leqslant CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

§2. Fractional radial derivatives

We note that

$$\mathscr{R} = \sum_{j=1}^{n}(A_j A_j^* + A_j^* A_j) = 2\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n.$$

It is easy to see that $\mathscr{R}$ is unbounded, positive, self-adjoint, and invertible on F^2. In fact, $\mathscr{R}^{-1}$ is a compact operator.

EXAMPLE 2.1. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{z_1^k}{(k+1)\sqrt{k!}}.$$

Then $f \in F^2$, but $\mathscr{R}f \notin F^2$.

For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z)$$

be the orthonormal decomposition of f, where $e_\alpha(z) = z^\alpha/\|z^\alpha\|_2$. Associated with the operator $\mathscr{R}$ is a semigroup $\{B_t\}_{t\geqslant 0}$ defined by the expansion

$$B_t f(z) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)t} c_\alpha e_\alpha(z).$$

We can check that $u(z,t) := B_t f(z)$ is the solution of the heat-type equation:

$$\begin{cases} (\partial_t + \mathscr{R})u = 0 & \text{on } \mathbb{C}^n \times (0,\infty), \\ u(\cdot, 0) = f & \text{on } \mathbb{C}^n. \end{cases}$$

It is easy to see that

$$\|B_t f\|_2^2 \leqslant e^{-2nt}\|f\|_2^2.$$

Thus B_t is contractive. Moreover, we can see that $-\mathscr{R}$ is the infinitesimal generator of $\{B_t\}_{t\geqslant 0}$. That is,

$$B_t = e^{-t\mathscr{R}}.$$

See [5] for more properties concerning the heat semigroup as well as the spectral property of the operator $\mathscr{R}$.

Since $\mathscr{R}$ has discrete spectrum $\{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$, by using the spectral theorem, we define the fractional radial derivative $\mathscr{R}^s$ for $s \in \mathbb{R}$ as following:

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DEFINITION 2.2. Let $s \in \mathbb{R}$. For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z)$$

be the orthonormal decomposition of f. By the spectral theorem, $\mathscr{R}^s$ is given by

$$\mathscr{R}^s f(z) = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s c_\alpha e_\alpha(z), \quad f \in \mathcal{D}om(\mathscr{R}^s).$$

DEFINITION 2.3. Let s be a real number. The Fock–Sobolev space $F_{\mathscr{R}}^{s,p}$ of fractional order s is the space of all entire functions for which $\mathscr{R}^{s/2} f$ is given by an F^p function. The Fock–Sobolev norm of f of fractional order s is defined accordingly,

$$\|f\|_{F_{\mathscr{R}}^{s,p}} = \|\mathscr{R}^{s/2} f\|_p.$$

By using the semigroup, we have the integral representations for the fractional radial derivatives as following. See [2] for analogues in the context of other type of Sobolev spaces.

PROPOSITION 2.4. *Let $f \in F^2$ and $z \in \mathbb{C}^n$. Then the following identities hold:*

(i) *For $0 < s < 1$ we have*

$$\mathscr{R}^s f(z) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{[e^{-t\mathscr{R}} f(z) - f(z)]}{t^s} \frac{dt}{t},$$

where $\Gamma(-s)$ is the gamma function to negative numbers defined by

$$\Gamma(-s) = \frac{\Gamma(-s+n)}{(-s)(-s+1)\cdots(-s+n-1)}$$

choosing n such that $-s+n$ is positive.

(ii) *For $s > 0$ we have*

$$\mathscr{R}^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-t\mathscr{R}} f(z) \frac{dt}{t}.$$

Proof. We prove (i); the proof for (ii) is simpler.

In [4, Proposition 2.2], we calculated the size of Taylor coefficients as following:

$$\left| \frac{\partial^\alpha f(0)}{\alpha!} \right| \lesssim e^{|\alpha|/2} \left(\prod_{j=1}^n \alpha_j^{-\alpha_j/2} \right) \|f\|_{F^2} \tag{2.1}$$

for a given multi-index α where $\alpha_j^{-\alpha_j/2}$ is understood to be 1 when $\alpha_j = 0$.

For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z),$$

be the orthonormal decomposition of f, where $c_\alpha = \partial^\alpha f(0)/\sqrt{\alpha!}$ and $e_\alpha(z) = z^\alpha/\sqrt{\alpha!}$. Note that

$$\int_0^\infty (e^{-t} - 1) \frac{dt}{t^{1+s}} = \Gamma(-s). \tag{2.2}$$

By (2.1) and (2.2), it follows that

$$\begin{aligned}
&\sum_\alpha |c_\alpha||e_\alpha(z)| \int_0^\infty |e^{-(2|\alpha|+n)t} - 1| \frac{dt}{t^{1+s}} \\
&\quad = \sum_\alpha |c_\alpha||e_\alpha(z)| \, (2|\alpha| + n)^s |\Gamma(-s)| \\
&\quad \lesssim \sum_\alpha e^{|\alpha|/2} \left(\prod_{j=1}^n \alpha_j^{-\alpha_j/2} \right) (2|\alpha| + n)^s |z^\alpha|.
\end{aligned}$$

We note that the power series on the right side of the inequality above is convergent for every $z \in \mathbb{C}^n$. By the dominated convergence theorem, we have

$$\begin{aligned}
\Gamma(-s)\mathscr{R}^s f(z) &= \sum_\alpha c_\alpha e_\alpha(z) \, (2|\alpha| + n)^s \Gamma(-s) \\
&= \sum_\alpha c_\alpha e_\alpha(z) \int_0^\infty (e^{-(2|\alpha|+n)t} - 1) \frac{dt}{t^{1+s}} \\
&= \int_0^\infty \sum_\alpha c_\alpha e_\alpha(z) (e^{-(2|\alpha|+n)t} - 1) \frac{dt}{t^{1+s}} \\
&= \int_0^\infty \frac{[e^{-t\mathscr{R}} f(z) - f(z)]}{t^s} \frac{dt}{t}.
\end{aligned}$$

∎

REMARK 2.5. We refer to [4] for another fractional derivatives. In [4], the following derivative $\mathcal{D}^s f$ is given by

$$\mathcal{D}^s f(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\Gamma(n + s + |\alpha|)}{\Gamma(n + |\alpha|)} c_\alpha e_\alpha(z), \qquad f \in \mathcal{D}om(\mathcal{D}^s).$$

We remark that our definition of $\mathscr{R}^s f$ is slightly different from $\mathcal{D}^s f$, but they are asymptotically the same in the sense that $\Gamma(n + s + |\alpha|)/\Gamma(n + |\alpha|) \approx (2|\alpha| + n)^s$ as $|\alpha| \to \infty$ by Stirling's formula.

§3. Estimates of the reproducing kernel for $F^{s,2}_{\mathscr{R}}$

In what follows we use the conventional multi-index notation. Thus for an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},$$

where ∂_j denotes partial differentiation with respect to the jth component. If $z = (z_1, \ldots, z_n)$, then $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

First we get pointwise size estimates for the fractional radial derivatives of the Fock kernel as following.

THEOREM 3.1. *Given s real, there are positive constants $C = C(s) > 0$ such that*

$$|\mathscr{R}_z^s(e^{z\cdot\overline{w}})| \leqslant C \times \begin{cases} (1 + |z\cdot\overline{w}|)^s \Lambda(z, w) & \text{if } s \geqslant 0, \\ (1 + |z||w|)^s \Lambda(z, w) & \text{if } s < 0, \end{cases}$$

for $z, w \in \mathbb{C}^n$.

Proof. Since

$$\begin{aligned} |e^{z\cdot\overline{w}}| &= e^{\mathrm{Re}(z\cdot\overline{w})} = e^{(1/2)|z|^2 + (1/2)|w|^2 - (1/2)|z-w|^2} \\ &\leqslant e^{(1/2)|z|^2 + (1/2)|w|^2 - (1/8)|z-w|^2} = \Lambda(z, w), \end{aligned}$$

the cases $s = 0, 1$ are trivial.

Let $0 < s < 1$. By (i) of Proposition 2.4, we have

$$\mathscr{R}^s(e^{z\cdot\overline{w}}) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathscr{R}} e^{z\cdot\overline{w}} - e^{z\cdot\overline{w}}) \frac{dt}{t^{1+s}}.$$

Now

$$\begin{aligned} e^{-t\mathscr{R}}(e^{z\cdot\overline{w}}) &= \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\alpha!} e^{-(2|\alpha|+n)t} \\ &= \exp(e^{-2t} z\cdot\overline{w}) e^{-nt}. \end{aligned}$$

Thus

$$\mathscr{R}^s(e^{z\cdot\overline{w}}) = \frac{1}{\Gamma(-s)} \int_0^\infty [\exp(e^{-2t} z\cdot\overline{w}) e^{-nt} - e^{z\cdot\overline{w}}] \frac{dt}{t^{1+s}}. \tag{3.1}$$

We write the integral on the right-hand side of (3.1) as the sum of two pieces I_1 and I_2 defined by

$$I_1 = \int_0^1 [\exp(e^{-2t} z\cdot\overline{w}) e^{-nt} - e^{z\cdot\overline{w}}] \frac{dt}{t^{1+s}}$$

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and

$$I_2 = \int_1^\infty [\exp(e^{-2t} z \cdot \overline{w}) e^{-nt} - e^{z \cdot \overline{w}}] \frac{dt}{t^{1+s}}.$$

Given $z, w \in \mathbb{C}^n$, put $x = \mathrm{Re}(z \cdot \overline{w})$ for short. Then

$$\begin{aligned} |I_2| &\lesssim \int_1^\infty [\exp(e^{-2t} x) e^{-nt} + e^x] \frac{dt}{t^{1+s}} \\ &\lesssim \begin{cases} 1 & \text{if } x \leqslant 1, \\ e^x & \text{if } x > 1. \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} |I_1| &\lesssim e^x \int_0^1 |\exp(-(1 - e^{-2t}) z \cdot \overline{w} - nt) - 1| \frac{dt}{t^{1+s}} \\ &= e^x \int_0^1 |E_1(-(1 - e^{-2t}) z \cdot \overline{w} - nt)| \frac{dt}{t^{1+s}}. \end{aligned}$$

Here $E_s(x)$ is the truncated exponential function (see Definition A.1 in Appendix A). Note that (see (A1))

$$\frac{E_1(\lambda)}{\lambda} = \int_0^1 e^{\rho\lambda}\, d\rho \quad \text{for } \lambda \in \mathbb{C}.$$

Then

$$\frac{|E_1(-(1 - e^{-2t}) z \cdot \overline{w} - nt)|}{|(1 - e^{-2t}) z \cdot \overline{w} + nt|} \leqslant \int_0^1 e^{-\rho\{(1-e^{-2t})x + nt\}}\, d\rho.$$

Hence we have

$$\begin{aligned} &\int_0^1 |E_1(-(1 - e^{-2t}) z \cdot \overline{w} - nt)| \frac{dt}{t^{1+s}} \\ &\quad \leqslant \int_0^1 |(1 - e^{-2t}) z \cdot \overline{w} + nt| \int_0^1 e^{-\rho\{(1-e^{-2t})x + nt\}}\, d\rho \frac{dt}{t^{1+s}} \\ &\quad = \int_0^1 \left| \frac{1 - e^{-2t}}{t} z \cdot \overline{w} + n \right| \int_0^1 e^{-\rho\{(1-e^{-2t})x + nt\}}\, d\rho \frac{dt}{t^s}. \end{aligned}$$

Since

$$|1 - e^{-2t}| \approx t \quad \text{for } 0 < t < 1,$$

there exist $c > 0$ such that

$$\int_0^1 |E_1(-(1 - e^{-2t}) z \cdot \overline{w} - nt)| \frac{dt}{t^{1+s}} \lesssim |z \cdot \overline{w}| \int_0^1 \int_0^1 e^{-c\rho t x}\, d\rho \frac{dt}{t^s}.$$

If $x \leqslant 1$, then

$$\int_0^1 \int_0^1 e^{-c\rho t x}\, d\rho\, \frac{dt}{t^s} \lesssim 1.$$

So, in case $x \leqslant 1$, we have

$$\begin{aligned}
|\mathscr{R}^s(e^{z\cdot\overline{w}})| &\lesssim 1 + |z\cdot\overline{w}| \\
&\lesssim (1+|z\cdot\overline{w}|)^s e^{(1/2)|z||w|} \frac{(1+|z\cdot\overline{w}|)^{1-s}}{e^{(1/2)|z||w|}} \\
&\lesssim (1+|z\cdot\overline{w}|)^s e^{(1/2)|z||w|} \\
&\lesssim (1+|z\cdot\overline{w}|)^s e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2}. \qquad (3.2)
\end{aligned}$$

Here we used the following inequality

$$\begin{aligned}
e^{(1/2)|z||w|} &= e^{(3/4)|z||w|-(1/4)|z||w|} \leqslant e^{(3/8)|z|^2+(3/8)|w|^2+(1/4)\mathrm{Re}(z\cdot\overline{w})} \\
&= e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2}.
\end{aligned}$$

If $x > 1$, by Fubini's theorem, it follows that

$$\begin{aligned}
\int_0^1 \int_0^1 e^{-c\rho t x}\, d\rho\, \frac{dt}{t^s} &= \int_0^1 \int_0^1 e^{-c\rho t x} t^{-s}\, dt\, d\rho \\
&\lesssim x^{s-1}\Gamma(1-s) \int_0^1 \rho^{s-1}\, d\rho \\
&\lesssim x^{s-1}.
\end{aligned}$$

Hence, in case $x > 1$, we have

$$|\mathscr{R}^s(e^{z\cdot\overline{w}})| \lesssim e^x |z\cdot\overline{w}| x^{s-1}.$$

For the case $x = \mathrm{Re}(z\cdot\overline{w}) > 1$, we write $\mathrm{Re}(z\cdot\overline{w}) = |z||w|\cos\theta$, where θ is the angle between z and w identified as real vectors in $\mathbb{R}^{2n}$, and $\delta = \cos^{-1}(\frac{1}{4})$. If $|\theta| \leqslant \delta$, then

$$x = \mathrm{Re}(z\cdot\overline{w}) \approx |z\cdot\overline{w}| \approx |z||w|.$$

Hence we have

$$e^x |z\cdot\overline{w}| x^{s-1} \lesssim e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2} |z\cdot\overline{w}|^s. \qquad (3.3)$$

If $\delta < \theta < \pi/2$, then

$$x = \mathrm{Re}(z\cdot\overline{w}) = |z||w|\cos\theta < \tfrac{1}{4}|z||w|.$$

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Hence

$$
\begin{aligned}
e^x|z\cdot\overline{w}|x^{s-1} &\leqslant e^{(1/2)|z||w|}|z\cdot\overline{w}|^s\frac{|z\cdot\overline{w}|^{1-s}}{x^{1-s}e^{(1/4)|z\cdot\overline{w}|}}\\
&\lesssim e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2}|z\cdot\overline{w}|^s. \qquad (3.4)
\end{aligned}
$$

This, together with (3.2), yields the asserted estimate for $0<s<1$.

Now, assume $s>1$. Let m be the greatest nonnegative integer less than s. Then

$$
\begin{aligned}
\mathscr{R}^s(e^{z\cdot\overline{w}}) &= \mathscr{R}^{s-m}\mathscr{R}^m(e^{z\cdot\overline{w}})\\
&= \frac{1}{\Gamma(m-s)}\int_0^\infty\left[\mathscr{R}^m\exp\left(e^{-2t}z\cdot\overline{w}\right)e^{-nt}-\mathscr{R}^m e^{z\cdot\overline{w}}\right]\frac{dt}{t^{1+s-m}}.
\end{aligned}
$$

Note that

$$
\mathscr{R}^m(e^{z\cdot\overline{w}})=\sum_{j=0}^m \ell_j(z\cdot\overline{w})^j e^{z\cdot\overline{w}}
$$

and

$$
\mathscr{R}^m\exp(e^{-2t}z\cdot\overline{w})=\sum_{j=0}^m \ell_j(e^{-2t}z\cdot\overline{w})^j\exp(e^{-2t}z\cdot\overline{w}),
$$

for some nonnegative integers ℓ_j. Thus

$$
\begin{aligned}
&\mathscr{R}^s(e^{z\cdot\overline{w}})\\
&=\frac{1}{\Gamma(m-s)}\sum_{j=0}^m \ell_j(z\cdot\overline{w})^j\int_0^\infty\left[\exp(e^{-2t}z\cdot\overline{w})e^{-(2j+n)t}-e^{z\cdot\overline{w}}\right]\frac{dt}{t^{1+s-m}}.
\end{aligned}
$$

We write the integral on the right-hand side of the above equation as the sum of two pieces J_1 and J_2 defined by

$$
J_1=\int_0^1\left[\exp(e^{-2t}z\cdot\overline{w})e^{-(2j+n)t}-e^{z\cdot\overline{w}}\right]\frac{dt}{t^{1+s-m}}
$$

and

$$
J_2=\int_1^\infty\left[\exp(e^{-2t}z\cdot\overline{w})e^{-(2j+n)t}-e^{z\cdot\overline{w}}\right]\frac{dt}{t^{1+s-m}}.
$$

Then

$$
|J_2|\lesssim\int_1^\infty\left[\exp(e^{-2t}x)e^{-(2j+n)t}+e^x\right]\frac{dt}{t^{1+s-m}}
$$

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and

$$|J_1| \lesssim e^x \int_0^1 |E_1(-(1-e^{-2t})z\cdot\overline{w} - (2j+n)t)| \,\frac{dt}{t^{1+s-m}}$$
$$\lesssim e^x |z\cdot\overline{w}| \int_0^1 \int_0^1 e^{-c\rho t x}\, d\rho\, \frac{dt}{t^{s-m}}.$$

These yield the asserted estimate for $s>1$.

Now for $s>0$, by (ii) of Proposition 2.4, we have

$$\mathscr{R}^{-s}(e^{z\cdot\overline{w}}) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-t\mathscr{R}}(e^{z\cdot\overline{w}})\,\frac{dt}{t^{1-s}}$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty \exp(e^{-2t}z\cdot\overline{w})e^{-nt}\,\frac{dt}{t^{1-s}}.$$

Hence

$$|\mathscr{R}^{-s}(e^{z\cdot\overline{w}})| \leqslant \frac{1}{\Gamma(s)}\int_0^\infty \exp(e^{-2t}x)e^{-nt}\,\frac{dt}{t^{1-s}}.$$

If $x=\mathrm{Re}(z\cdot\overline{w})\leqslant 1$, then

$$|\mathscr{R}^{-s}(e^{z\cdot\overline{w}})| \lesssim \frac{1}{\Gamma(s)}\int_0^\infty e^{-nt}\,\frac{dt}{t^{1-s}} = \frac{1}{n^s}.$$

Now we assume that $x=\mathrm{Re}(z\cdot\overline{w})>1$. Then

$$|\mathscr{R}^{-s}(e^{z\cdot\overline{w}})| \leqslant \frac{1}{\Gamma(s)}\sum_{k=0}^\infty \int_0^\infty \frac{(e^{-2t}x)^k}{k!}e^{-nt}\,\frac{dt}{t^{1-s}}$$
$$= \sum_{k=0}^\infty \frac{x^k}{k!}\frac{1}{\Gamma(s)}\int_0^\infty e^{-(2k+n)t}\,\frac{dt}{t^{1-s}}$$
$$= \sum_{k=0}^\infty \frac{x^k}{(2k+n)^s k!}.$$

By Stirling's formula, it follows that

$$k!(2k+n)^s \approx \Gamma(k+1+s) \quad \text{for large } k.$$

Hence, by Corollary A.4, we have

$$|\mathscr{R}^{-s}(e^{z\cdot\overline{w}})| \lesssim \sum_{k=0}^\infty \frac{x^{k+s}}{\Gamma(k+1+s)}x^{-s}$$
$$= E_s(x)x^{-s}$$
$$\approx e^x x^{-s}, \quad x>1. \tag{3.5}$$

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For the case $x = \mathrm{Re}(z \cdot \overline{w}) > 1$, we write $\mathrm{Re}(z \cdot \overline{w}) = |z||w| \cos\theta$, where θ is the angle between z and w identified as real vectors in $\mathbb{R}^{2n}$, and $\delta = \cos^{-1}(\frac{1}{4})$. It is easily seen from (3.5) that the required estimate holds when $|\theta| \leqslant \delta$, because $x \approx |z||w|$ for such z and w. So, assume $\delta < \theta < \pi/2$. Note $x < \frac{1}{4}|z||w|$ for such z and w. We thus have by our choice of δ

$$\begin{aligned} \frac{e^x}{x^s} &\leqslant \frac{e^{(1/4)|z||w|}}{x^s} \\ &\leqslant e^{(1/4)|z||w|} \\ &= \frac{e^{(1/2)|z||w|}}{(|z||w|)^s} \frac{(|z||w|)^s}{e^{(1/4)|z||w|}} \\ &\lesssim \frac{e^{(1/2)|z||w|}}{(|z||w|)^s}, \quad x > 1. \end{aligned} \tag{3.6}$$

This, together with (3.5), yields the asserted estimate for $x > 1$. This completes the proof. ∎

It is the well-known formula [1] that

$$K_s(z,w) := K_w^s(z) = \sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(w)}$$

where $\{\phi_\alpha\}$ is any orthonormal basis for $F^{s,2}_{\mathscr{R}}$.

LEMMA 3.2. *Let s be real and α be a multi-index of nonnegative integers. Then*

$$\|z^\alpha\|^2_{F^{s,2}_{\mathscr{R}}} = (2|\alpha| + n)^s \alpha!.$$

Proof. Since $\mathscr{R}^{s/2} z^\alpha = (2|\alpha| + n)^{s/2} z^\alpha$, we have

$$\|z^\alpha\|^2_{F^{s,2}_{\mathscr{R}}} = \|\mathscr{R}^{s/2} z^\alpha\|_2^2 = (2|\alpha| + n)^s \|z^\alpha\|_2^2 = (2|\alpha| + n)^s \alpha!.$$ ∎

THEOREM 3.3. *Let $s \in \mathbb{R}$. Then*

$$K_s(z,w) = \mathscr{R}_z^{-s}(e^{z\cdot\overline{w}})$$

and there are positive constants $C = C(s) > 0$ such that

$$|K_s(z,w)| \leqslant C \times \begin{cases} (1 + |z||w|)^{-s}\Lambda(z,w) & \text{if } s > 0, \\ (1 + |z\cdot\overline{w}|)^{-s}\Lambda(z,w) & \text{if } s \leqslant 0, \end{cases}$$

for $z, w \in \mathbb{C}^n$.

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Proof. By Lemma 3.2, we get

$$\begin{aligned} K_s(z,w) &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|^2_{F^{s,2}_{\mathscr{R}}}} \\ &= \sum_{\alpha} \frac{z^\alpha \overline{w}^\alpha}{(2|\alpha|+n)^s \alpha!} \\ &= \mathscr{R}_z^{-s}(e^{z\cdot\overline{w}}). \end{aligned}$$

Hence the size estimates of $K_s(z,w)$ follow from Theorem 3.1. ∎

§4. Auxiliary integral estimates

It follows that the fractional Fock–Sobolev spaces are realized as the weighted Fock spaces that do not involve derivatives. To prove the results we introduce an auxiliary integral estimate for Λ defined by

$$\Lambda(z,w) = e^{(1/2)|z|^2+(1/2)|w|^2-(1/8)|z-w|^2}.$$

To handle the case $1 \leqslant p < \infty$ and for other purposes later, we introduce an integral operator induced by Λ. Given s real, we consider an integral operator L_s defined by

$$L_s\psi(z) := \int_{\mathbb{C}^n} \psi(w)\left(\frac{1+|z|}{1+|w|}\right)^s \Lambda(z,w)e^{-|w|^2}\, dV(w), \quad z\in\mathbb{C}^n$$

for ψ which makes the above integral well-defined.

LEMMA 4.1. [4] *Given s real, the operator L_s is bounded on L^p_G for any $1 \leqslant p \leqslant \infty$.*

The following Jensen-type inequality is needed to handle the case $0 < p \leqslant 1$.

LEMMA 4.2. [4] *Given $0 < p \leqslant 1$, $a > 0$ and s real, there is a constant $C = C(p,a,s) > 0$ such that*

(4.1)
$$\left\{\int_{\mathbb{C}^n} |(1+|z|)^s f(z) e^{-a|z|^2}|\, dV(z)\right\}^p \leqslant C\int_{\mathbb{C}^n} |(1+|z|)^s f(z)e^{-a|z|^2}|^p\, dV(z)$$

for $f\in H(\mathbb{C}^n)$.

LEMMA 4.3. [4] *Let $0 < p < \infty$ and α be an arbitrary real number. Then there is $C = C(p,\alpha) > 0$ such that*

$$\int_{\mathbb{C}^n} (1+|w|)^\alpha \Lambda(z,w)^p e^{-(p/2)|w|^2}\, dV(w) \leqslant C(1+|z|)^\alpha e^{(p/2)|z|^2}.$$

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§5. Fourier type characterization

Cho and Zhu [6] studied Fock–Sobolev spaces of positive integer order. For any positive integer m and $0 < p \leqslant \infty$ we consider the space $F^{m,p}$ consisting of entire functions f on $\mathbb{C}^n$ such that

$$\sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_p < \infty,$$

where $\| \ \|_p$ is the norm in F^p. See [8, 12] for other similar Sobolev spaces. Cho and Zhu [6] proved a useful Fourier type characterization of the Fock–Sobolev space of integer order as following.

THEOREM 5.1. [6] *Suppose $0 < p \leqslant \infty$, m is a nonnegative integer, and f is an entire function on $\mathbb{C}^n$. Then $f \in F^{m,p}$ if and only if the function $z^\alpha f(z)$ is in F^p for all multi-indices α with $|\alpha| = m$. Moreover, $\|f\|_{F^{m,p}}$ is comparable to the norm of the function $|z|^m f(z)$ in L^p_G.*

The purpose of the current paper is to extend the notion of the Fock–Sobolev spaces to the case of fractional orders allowed to be any real number.

THEOREM 5.2. *Let $s \in \mathbb{R}$ and $0 < p \leqslant \infty$. There is a constant $C = C(s, p) > 0$ such that*

$$\|f\|_{F^{s,p}_{\mathscr{R}}} \leqslant C \|f\|_{F^p_s}.$$

Proof. We now consider the cases $0 < p < 1$ and $1 \leqslant p \leqslant \infty$ separately.

Assume $1 \leqslant p \leqslant \infty$. If the function $(1 + |w|)^s f(w)$ is in L^p_G, then

$$f(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} e^{z \cdot \overline{w}} f(w) e^{-|w|^2} \, dV(w), \quad z \in \mathbb{C}^n.$$

Thus we obtain

$$\mathscr{R}^{s/2} f(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathscr{R}^{s/2}(e^{z \cdot \overline{w}}) f(w) e^{-|w|^2} \, dV(w). \tag{5.1}$$

The convergence of the integrals above follows from pointwise estimates for functions in Fock spaces. Hence it follows that

$$\begin{aligned}
|\mathscr{R}^{s/2} f(z)| &\leqslant \frac{1}{\pi^n} \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}(e^{z \cdot \overline{w}})| |f(w)| e^{-|w|^2} \, dV(w) \\
&\lesssim \int_{\mathbb{C}^n} |f(w)| (1 + |w|)^s \left(\frac{1 + |z|}{1 + |w|} \right)^{s/2} \Lambda(z, w) e^{-|w|^2} \, dV(w) \\
&= L_s((1 + |w|)^s |f|)(z).
\end{aligned}$$

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By Lemma 4.1, we have

$$\|\mathscr{R}^{s/2}f\|_p^p \lesssim \|(1+|z|)^s f\|_{L^p_G}.$$

Now let $0 < p < 1$. Then, by Lemma 4.2 and Theorem 3.1,

$$\begin{aligned}|\mathscr{R}^{s/2}f(z)|^p &\lesssim \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}(e^{z\cdot\overline{w}})|^p |f(w)|^p e^{-p|w|^2}\,dV(w)\\ &\lesssim \int_{\mathbb{C}^n} |f(w)|^p(1+|w|)^{sp/2}(1+|z|)^{sp/2}\Lambda(z,w)^p e^{-p|w|^2}\,dV(w)\end{aligned}$$

or

$$\begin{aligned}&\int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(z)|^p e^{-(p/2)|z|^2}\,dV(z)\\ &\quad\lesssim \int_{\mathbb{C}^n} |f(w)|^p(1+|w|)^{sp/2}e^{-p|w|^2}\,dV(w)\\ &\qquad\times \int_{\mathbb{C}^n}(1+|z|)^{sp/2}\Lambda(z,w)^p e^{-(p/2)|z|^2}\,dV(z).\end{aligned}$$

Now, by Lemma 4.3, it follows that

$$\int_{\mathbb{C}^n}(1+|z|)^{sp/2}\Lambda(z,w)^p e^{-(p/2)|z|^2}\,dV(z) \lesssim (1+|w|)^{sp/2}e^{(p/2)|w|^2}.$$

Hence

$$\begin{aligned}&\int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(z)|^p e^{-(p/2)|z|^2}\,dV(z)\\ &\quad\lesssim \int_{\mathbb{C}^n} |f(w)|^p(1+|w|)^{sp}e^{-(p/2)|w|^2}\,dV(w).\end{aligned}$$

∎

THEOREM 5.3. *Suppose $0 < p \leqslant \infty$ and s is a real number. Then there is a constant $C = C(s,p) > 0$ such that*

$$\|f\|_{F^p_s} \leqslant C\|f\|_{F^{s,p}_{\mathscr{R}}}$$

for all $f \in F^{s,p}_{\mathscr{R}}$.

Proof. Let $1 \leqslant p \leqslant \infty$. From the reproducing formula for $\mathscr{R}^{s/2}f$ we obtain

$$f(z) = \mathscr{R}^{-s/2}\mathscr{R}^{s/2}f(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n}\mathscr{R}^{s/2}_w f(w)\mathscr{R}^{-s/2}_z(e^{z\cdot\overline{w}})e^{-|w|^2}\,dV(w).$$

This together with Theorem 3.1 shows that

$$
\begin{aligned}
(1+|z|)^s|f(z)| &\lesssim (1+|z|)^s \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(w)||\mathscr{R}^{-s/2}(e^{z\cdot\overline{w}})|e^{-|w|^2}\,dV(w)\\
&\lesssim \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(w)|\left(\frac{1+|z|}{1+|w|}\right)^{s/2}\Lambda(z,w)e^{-|w|^2}\,dV(w)\\
&= L_s(|\mathscr{R}^{s/2}f|)(z).
\end{aligned}
$$

By Lemma 4.1, we have

$$
\|(1+|z|)^s f\|_{L^p_G} \lesssim \|\mathscr{R}^{s/2}f\|_p.
$$

When $0<p<1$, it follows from Lemma 4.2 and Theorem 3.1 that

$$
\begin{aligned}
|f(z)|^p &\lesssim \left|\int_{\mathbb{C}^n} \mathscr{R}^{s/2}f(w)\mathscr{R}^{-s/2}(e^{z\cdot\overline{w}})e^{-|w|^2}\,dV(w)\right|^p\\
&\lesssim \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(w)\mathscr{R}^{-s/2}(e^{z\cdot\overline{w}})e^{-|w|^2}|^p\,dV(w)\\
&\lesssim \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(w)|^p\frac{e^{(p/2)|z|^2-(p/2)|w|^2-(p/8)|z-w|^2}}{(1+|z|)^{sp/2}(1+|w|)^{sp/2}}\,dV(w).
\end{aligned}
$$

Fubini's theorem shows that the integral

$$
I=\int_{\mathbb{C}^n} |(1+|z|)^s f(z)e^{-(1/2)|z|^2}|^p\,dV(z)
$$

satisfies the following estimates:

$$
I\lesssim \int_{\mathbb{C}^n} |\mathscr{R}^{s/2}f(w)|^p e^{-(p/2)|w|^2}\,dV(w)\int_{\mathbb{C}^n}\left(\frac{1+|z|}{1+|w|}\right)^{sp/2}e^{-(p/8)|z-w|^2}\,dV(z).
$$

Note that

$$
\begin{aligned}
&\int_{\mathbb{C}^n}\left(\frac{1+|z|}{1+|w|}\right)^{sp/2}e^{-(p/8)|z-w|^2}\,dV(z)\\
&\lesssim \int_{\mathbb{C}^n}(1+|z-w|)^{sp/2}e^{-(p/8)|z-w|^2}\,dV(z)\\
&\lesssim 1.
\end{aligned}
$$

The proof is complete. ∎

Theorem 1.2 follows from Theorems 5.2 and 5.3.

Appendix. Truncated exponential functions

Let m be a positive integer. We consider the left truncated exponential function of integer order m, $E_m(\lambda)$, defined by

$$\begin{aligned} E_m(\lambda) &= e^{\lambda} - 1 - \lambda - \frac{\lambda^2}{2!} - \cdots - \frac{\lambda^{m-1}}{(m-1)!} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^{k+m}}{\Gamma(k+1+m)}, \qquad \lambda \in \mathbb{C}, \end{aligned}$$

where Γ is the classical gamma function.

It is easy to check that

$$\frac{E_m(\lambda)}{\lambda^m} = \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} e^{t\lambda}\, dt, \tag{A1}$$

which immediately yields a useful inequality

$$|E_m(\lambda)| \leqslant \left(\frac{|\lambda|}{\operatorname{Re}\lambda}\right)^m E_m(\operatorname{Re}\lambda), \qquad \lambda \in \mathbb{C}. \tag{A2}$$

Now we consider the truncated exponential function of fractional order.

Definition A.1. Let $s \in \mathbb{R}$. We define the generalized exponential function of fractional order s, $E_s(x)$, by

$$E_s(x) = \sum_{k=0}^{\infty} \frac{x^{k+s}}{\Gamma(k+1+s)}, \qquad x \in \mathbb{R}.$$

We have the following integral representation of $E_s(x)$:

Proposition A.2. *Let $s > 0$. Then*

$$E_s(x) = \frac{e^x}{\Gamma(s)} \int_0^x t^{s-1} e^{-t}\, dt, \qquad x \in \mathbb{R}.$$

Proof. Note that the following well-known property of gamma functions

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt \quad \text{when } a, b > 0.$$

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Thus

$$\sum_{k=0}^{\infty} \frac{x^{k+s}}{\Gamma(k+1+s)} = \sum_{k=0}^{\infty} \frac{x^{k+s}}{\Gamma(s)\Gamma(k+1)} \int_0^1 (1-t)^{s-1} t^k \, dt$$
$$= \frac{x^s}{\Gamma(s)} \int_0^1 (1-t)^{s-1} e^{tx} \, dt$$
$$= \frac{e^x}{\Gamma(s)} \int_0^x t^{s-1} e^{-t} \, dt. \qquad \square$$

PROPOSITION A.3. *Let $s = m + r$ where m is a nonnegative integer and $0 \leqslant r < 1$. Then*

$$E_{-s}(x) = \frac{e^x}{\Gamma(1-r)} \int_0^x t^{-r} e^{-t} \, dt + \sum_{k=0}^{m} \frac{x^{k-s}}{\Gamma(k+1-s)}.$$

Proof. We have

$$\sum_{k=0}^{\infty} \frac{x^{k-s}}{\Gamma(k+1-s)} = \sum_{k=m+1}^{\infty} \frac{x^{k-s}}{\Gamma(k+1-s)} + \sum_{k=0}^{m} \frac{x^{k-s}}{\Gamma(k+1-s)}$$
$$= \sum_{k=0}^{\infty} \frac{x^{k+1-r}}{\Gamma(k+2-r)} + \sum_{k=0}^{m} \frac{x^{k-s}}{\Gamma(k+1-s)}$$
$$= \frac{e^x}{\Gamma(1-r)} \int_0^x t^{-r} e^{-t} \, dt + \sum_{k=0}^{m} \frac{x^{k-s}}{\Gamma(k+1-s)}. \qquad \square$$

By Propositions A.2 and A.3, we have the following.

COROLLARY A.4. *Let $s \in \mathbb{R}$. Then*

$$\lim_{x \to \infty} \frac{E_s(x)}{e^x} = 1.$$

REFERENCES

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68**(3) (1950), 337–404.

[2] B. Bongioanni and J. L. Torrea, *Sobolev spaces associated to the harmonic oscillator*, Proc. Indian Acad. Sci. Math. Sci. **116** (2006), 337–360.

[3] H. R. Cho, B. R. Choe and H. Koo, *Linear combinations of composition operators on the Fock–Sobolev spaces*, Potential Anal. **41** (2014), 1223–1246.

[4] H. R. Cho, B. R. Choe and H. Koo, *Fock–Sobolev spaces of fractional order*, Potential Anal. **43** (2015), 199–240.

[5] H. R. Cho, H. Choi and H.-W. Lee, *Boundedness of the Segal–Bargmann Transform on Fractional Hermite–Sobolev Spaces*, J. Funct. Spaces **2017** (2017), Article ID 9176914, 6 pages.
[6] H. R. Cho and K. Zhu, *Fock–Sobolev spaces and their Carleson measures*, J. Funct. Anal. **263**(8) (2012), 2483–2506.
[7] B. R. Choe and J. Yang, *Commutants of Toeplitz operators with radial symbols on the Fock–Sobolev space*, J. Math. Anal. Appl. **415**(2) (2014), 779–790.
[8] B. Hall and W. Lewkeeratiyutkul, *Holomorphic Sobolev spaces and the generalised Segal–Bargmann transform*, J. Funct. Anal. **217** (2004), 192–220.
[9] T. Mengestie, *Schatten class weighted composition operators on weighted Fock spaces*, Arch. Math. (Basel) **101**(4) (2013), 349–360.
[10] T. Mengestie, *Volterra type and weighted composition operators on weighted Fock spaces*, Integral Equations Operator Theory **76**(1) (2013), 81–94.
[11] T. Mengestie, *On trace ideal weighted composition operators on weighted Fock spaces*, Arch. Math. (Basel) **105**(5) (2015), 453–459.
[12] R. Radha and S. Thangavelu, *Holomorphic Sobolev spaces, Hermite and special Hermite semigroups and a Paley–Wiener theorem for the windowed Fourier transform*, J. Math. Anal. Appl. **354**(2) (2009), 564–574.
[13] X. F. Wang, G. F. Cao and J. Xia, *Toeplitz operators on Fock–Sobolev spaces with positive measure symbols*, Sci. China Math. **57**(7) (2014), 1443–1462.
[14] K. H. Zhu, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics **139**, Marcel Dekker, Inc., New York, 1990, xii+258 pp.
[15] K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics **263**, Springer, New York, 2012, x+344 pp.

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