ANZIAM J. **63** (2021), 163–177
doi:10.1017/S1446181121000250

FINITE MATURITY AMERICAN-STYLE STOCK LOANS WITH REGIME-SWITCHING VOLATILITY

XIAOPING LU ✉[1] and ENDAH R. M. PUTRI [2]

(Received 1 May, 2020; accepted 7 April, 2021; first published online 19 August, 2021)

Abstract

We study finite maturity American-style stock loans under a two-state regime-switching economy. We present a thorough semi-analytic discussion of the optimal redeeming prices, the values and the fair service fees of the stock loans, under the assumption that the volatility of the underlying is in a state of uncertainty. Numerical experiments are carried out to show the effects of the volatility regimes and other loan parameters.

2020 *Mathematics subject classification*: 91G20.

Keywords and phrases: stock loans, regime-switching, PDE approach, semi-analytic method, volatility.

1. Introduction

A stock loan is a financial derivative which provides an alternative for stock owners to obtain liquidity without selling their stocks. Since the value of a stock is subject to the fluctuations of the market, the value of the stock loan is affected by market conditions such as economic growth trends. Therefore, the assumption of constant volatility under the standard Black–Scholes (BS) framework, which is used in much of the existing literature on stock loan problems, is inadequate at times [7, 24].

There are various extension studies in the literature of stock loans incorporating a nonconstant volatility. For example, the stochastic volatility model by Wong and Wong [20] and their exponential Levy model [21], and the work of Liang et al. [10] extend the perpetual stock loan valuation under the Levy model to include stock loans with various dividend distributions.

In recent years, various stock loan models with regime-switching have also emerged in the literature. The regime-switching models allow key parameters of the stock

[1] School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia;
e-mail: xplu@uow.edu.au
[2] Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia;
e-mail: endahrmp@mathematics.its.ac.id

to respond to the market dynamics based on the market mode (or "regime") which switches among a finite number of states [24], such that these models are able to capture the market dynamics more accurately than others [11]. However, analytical solutions are only available for some special cases, including the stock loans of European style or American style with infinite maturity. Prager and Zhang [17] introduced a regime-switching model for the finite maturity European-style stock loans, with the stock dynamics switching between geometric Brownian motion and mean-reversion. Zhang and Zhou [27] studied American-type perpetual stock loans under a two-state regime-switching economy, with the states representing the economic conditions, good and bad. The value of their stock loan is governed by a system of coupled BS equations and an analytic solution is obtained using the variational inequality method.

In practice, stock loans have finite maturities and flexible times for redemption of the stock, such that a typical stock loan resembles a finite maturity American call option. The finite horizon and American feature make an analytic solution of finite maturity stock loans unattainable. The additional complexity due to the uncertainty of states makes it even harder to price stock loans under a regime-switching economy. Because of the resemblance of the stock loans to American options, we refer to the literature for the valuation of American options under regime-switching volatility. While there exist closed-form solutions for European options with regime-switching, such as those presented in [3, 15, 16, 28], there is no analytical solution for the American counterpart owing to the presence of free boundaries, except for perpetual options as presented in [5]. Naturally, one is "forced" to look for approximations in the form of either analytical or numerical approximations. There are various numerical approaches, including the tree method [26], finite element methods [6, 23], and the penalty method [9]. To the best of our knowledge, the only analytical approximation method for American options with regime-switching in finite maturity is that of Buffington and Elliott [2]. The price of an American option with regime-switching is decomposed into a European option part and an extra part as the result of American-style privilege. However, the implementation of the method in [2] is not straightforward, and it could be possibly time-consuming. From a practical point of view, it is more valuable to have a method that produces results in a timely manner. The semi-analytic approximation developed by Lu and Putri [14] provides a good balance since it produces an approximate solution efficiently for the original problem, which is not computationally tractable and is difficult to solve directly. Therefore, in this work, we utilize the approximation technique of Lu and Putri [14] to develop our solution for stock loans under a two-state regime-switching economy.

The rest of the paper is organized as follows. Section 2 establishes the governing equation system for stock loans with regime-switching volatility. In Section 3, our solution procedure is discussed. Section 4 presents some numerical examples to illustrate the effects of the loan parameters on the optimal exit prices, the stock loan values and the service fees under a regime-switching economy. The conclusion is given in Section 5.

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2. The stock loan formulation

In this study, the stock loan is set in a two-state regime-switching economy, where the stock (underlying) volatility switches between two states, growth and recession. Thus, the stock price dynamics can be described as follows:

$$dS = (r - \delta)S\,dt + \sigma_X S\,dW_t,$$

where W_t is the standard Wiener process, S is the price of the underlying asset, t is the current time, r is the risk-free interest rate and δ is the constant continuous dividend rate. X is a continuous-time Markov chain with a finite state space,

$$X = \begin{cases} 1 & \text{when the economy is in growth} \\ 2 & \text{when the economy is in recession,} \end{cases}$$

so σ_X stands for the asset volatility in state X, and $\sigma_1 > \sigma_2$. The Markov chain X has a generator matrix

$$\begin{pmatrix} -\xi_{12} & \xi_{12} \\ \xi_{21} & -\xi_{21} \end{pmatrix}$$

with the transition rate from state i to state j being $\xi_{ij} > 0$, $i, j = 1, 2$. Further, W_t and X are assumed to be independent.

Let $V_i(S, t; \sigma_i)$ be the value of the stock loan in state i with finite maturity T. We assume that the risk associated with the regime-switching is diversifiable, so it is not priced separately as in [1, 14, 16, 25, 28]. Therefore, by utilizing Itô's lemma [8] and the principle of risk-neutral valuation, we derive the following coupled partial differential equations (PDEs) under the BS framework:

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta)S\frac{\partial V_1}{\partial S} - rV_1 = \xi_{12}(V_1 - V_2),$$

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - \delta)S\frac{\partial V_2}{\partial S} - rV_2 = \xi_{21}(V_2 - V_1).$$

The condition at the boundary $S = 0$ and the final condition are the same for both V_1 and V_2, that is, for $i = 1, 2$,

$$V_i(0, t) = 0 \quad \text{and} \quad V_i(S, T) = \max(S - qe^{\gamma T}, 0),$$

where q is the initial loan amount and γ is the loan interest rate.

Since the stock loan is of American-type, similar to the case in [14], there are two different optimal exit boundaries corresponding to the two volatility states. Let S_{fi} be the optimal exit price for state i for $i = 1, 2$. A stock loan behaves like an American call (see, for example, [12, 22]), such that at the optimal exit boundary $S = S_{fi}(t)$, the value of the stock loan should take its intrinsic value

$$V_i(S_{fi}, t) = S_{fi} - qe^{\gamma t}.$$

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However, the optimal boundary $S_{fi}(t)$, which is not known in advance, needs to be solved as part of the solution. Therefore, an additional condition is needed:

$$\frac{\partial V_i}{\partial S}(S_{fi}, t) = 1.$$

This condition, which is called the smooth pasting condition, indicates that the partial derivative is continuous across the optimal exit boundary, and financially it reflects the optimal value of the loan contract for the holder, who has the right to exit the contract at the optimal boundary [19].

It is known that the higher the volatility of the underlying asset, the higher the option price for both calls and puts. For an American call, the increased option value means that the option can be exercised later, which implies a higher optimal exercise price, such that $S_{f1}^{BS} > S_{f2}^{BS}$ for $\sigma_1 > \sigma_2$, where S_{fi}^{BS} is the optimal exercise price from the standard BS model. Under a regime-switching economy, it is expected that $S_{f2} < S_{f1}$, but owing to the potential of the economy state to switch from state 1 to state 2, or *vice versa*, $S_{f1} < S_{f1}^{BS}$ and $S_{f2} > S_{f2}^{BS}$ so that $S_{f2}^{BS} < S_{f2} < S_{f1} < S_{f1}^{BS}$. This result is reported in [2, 14, 23] for American calls under a regime-switching economy. It is, therefore, reasonable to assume the same is true for stock loan contracts which behave like American calls with time-dependent strike, and it is proved by Zhang and Zhou [27] for perpetual stock loans with regime switching. For finite maturity stock loans, it becomes obvious in dimensionless variables as can be seen in the next section. Thus, the pricing domain is divided into two regions: the common continuation region ($0 \le S \le S_{f2}$) and the transition region ($S_{f2} \le S \le S_{f1}$). We derive the complete BS-type PDE systems as follows.

In the common continuation region, the value of the stock loan could be $V_1(S, t)$ or $V_2(S, t)$ depending on the volatility state. The values of the stock loan, V_1 and V_2, satisfy the following PDE system:

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + (r-\delta)S\dfrac{\partial V_1}{\partial S} - rV_1 = \xi_{12}(V_1 - V_2) \\ V_1(0,t) = 0 \\ V_1(S,T) = \max(S - qe^{\gamma T}, 0) \\ \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma_2^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} + (r-\delta)S\dfrac{\partial V_2}{\partial S} - rV_2 = \xi_{21}(V_2 - V_1) \\ V_2(0,t) = 0 \\ V_2(S,T) = \max(S - qe^{\gamma T}, 0) \\ V_2(S_{f2}(t), t) = S_{f2} - qe^{\gamma t} \\ \dfrac{\partial V_2}{\partial S}(S_{f2}(t), t) = 1. \end{cases} \qquad 0 \le S \le S_{f2} \qquad (2.1)$$

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In the transition region, $V_2(S,t)$ takes the intrinsic value, that is, $S - qe^{\gamma t}$, so the loan value V_1 satisfies the PDE system

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + (r-\delta)S\dfrac{\partial V_1}{\partial S} - rV_1 = \xi_{12}(V_1 - (S - qe^{\gamma t})) \\ V_1(S,T) = \max(S - qe^{\gamma T}, 0), \quad S_{f2} \le S \le S_{f1} \\ V_1(S_{f1}(t), t) = S_{f1}(t) - qe^{\gamma t} \\ \dfrac{\partial V_1}{\partial S}(S_{f1}(t), t) = 1. \end{cases} \tag{2.2}$$

To ensure the continuity and smoothness of V_1 at the intersection of the two regions, $S = S_{f2}$, the following conditions are enforced:

$$\begin{aligned} \lim_{S\to S_{f2}^-} V_1 &= \lim_{S\to S_{f2}^+} V_1, \\ \lim_{S\to S_{f2}^-} \frac{\partial V_1}{\partial S} &= \lim_{S\to S_{f2}^+} \frac{\partial V_1}{\partial S}. \end{aligned} \tag{2.3}$$

Theoretically, the solutions to the PDE systems (2.1) and (2.2) with (2.3) would give rise to the values of the stock loan and its optimal exit prices. However, not only are the PDE systems linear owing to the unknown optimal boundaries, the "strike" is also time dependent, in addition to the coupling of the systems. As it is well known that the analytical solution of the BS PDE only exists for the simplest case, we attempt to find an analytical approximation to the solution instead.

3. Solution procedure

First, we apply the following transformation of variables to the PDE systems (2.1) and (2.2), and condition (2.3):

$$S = Xqe^{\gamma t},\ S_{fi} = X_{fi}qe^{\gamma t},\ V_i(S,t) = U_i(X,\tau)\, qe^{\gamma t}, \quad t = T - \tau,\ i = 1, 2.$$

The resulting systems of equations are:

$$\begin{cases} -\dfrac{\partial U_1}{\partial \tau} + \dfrac{1}{2}\sigma_1^2 X^2 \dfrac{\partial^2 U_1}{\partial X^2} + (r-\gamma-\delta)X\dfrac{\partial U_1}{\partial X} - (r-\gamma)U_1 = \xi_{12}(U_1 - U_2) \\ U_1(0,\tau) = 0 \\ U_1(X,0) = \max(X-1, 0) \\ -\dfrac{\partial U_2}{\partial \tau} + \dfrac{1}{2}\sigma_2^2 X^2 \dfrac{\partial^2 U_2}{\partial X^2} + (r-\gamma-\delta)X\dfrac{\partial U_2}{\partial X} - (r-\gamma)U_2 = \xi_{21}(U_2 - U_1) \\ U_2(0,\tau) = 0 \\ U_2(X,0) = \max(X-1, 0) \\ U_2(X_{f2},\tau) = X_{f2} - 1 \\ \dfrac{\partial U_2}{\partial X}(X_{f2},\tau) = 1, \end{cases} \tag{3.1}$$

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where $\tau \in [0,T]$, $X \in [0, X_{f2}]$, and

$$\begin{cases} -\dfrac{\partial U_1}{\partial \tau} + \dfrac{1}{2}\sigma_1^2 X^2 \dfrac{\partial^2 U_1}{\partial X^2} + (r-\gamma-\delta)X\dfrac{\partial U_1}{\partial X} - (r-\gamma)U_1 = \xi_{12}[U_1 - (X-1)] \\ U_1(X,0) = X-1 \\ U_1(X_{f1},\tau) = X_{f1} - 1 \\ \dfrac{\partial U_1}{\partial X}(X_{f1},\tau) = 1, \end{cases} \tag{3.2}$$

where $\tau \in [0,T]$, $X \in [X_{f2}, X_{f1}]$.

The continuity conditions are as follows:

$$\begin{aligned} \lim_{X\to X_{f2}^-} U_1 &= \lim_{X\to X_{f2}^+} U_1, \\ \lim_{X\to X_{f2}^-} \frac{\partial U_1}{\partial X} &= \lim_{X\to X_{f2}^+} \frac{\partial U_1}{\partial X}. \end{aligned} \tag{3.3}$$

Clearly, if the term $r-\gamma$ in systems (3.1) and (3.2) is replaced by $\bar{r}$, the resulting systems are the same as those in [14] for the corresponding American option problem. It is worth noting that $\bar{r}$ could take a negative value since the loan interest rate γ is usually higher than the risk-free interest rate r for the loan contract to make financial sense. As a result, unlike American call options, stock loans could reach optimality for some parameter range even when no dividend is paid (see the works of Dai and Xu [4], and Lu and Putri [12]). Thus, we follow the techniques developed in [14] to obtain our solution for the stock loan PDE systems. For ease of reading, we recap some of the important steps here.

We apply the Laplace transform to the PDE systems in (3.1) and (3.2) as well as the conditions in (3). The pricing PDE systems are transformed into a set of ordinary differential equations (ODEs) in Laplace space. The solutions of the ODE system as the result of the Laplace transform of the PDE system (3.1) are:

$$\bar{U}_1(X,p) = \begin{cases} A_1X^{k_1} + A_2X^{k_2} + A_3X^{k_3} + A_4X^{k_4} + \phi(X,p) & \text{if } 1 < X \le X_{f2} \\ A_5X^{k_3} + A_6X^{k_4} & \text{if } 0 \le X \le 1, \end{cases} \tag{3.4}$$

$$\bar{U}_2(X,p) = \begin{cases} B_1X^{k_1} + B_2X^{k_2} + B_3X^{k_3} + B_4X^{k_4} + \phi(X,p) & \text{if } 1 < X \le X_{f2} \\ B_5X^{k_3} + B_6X^{k_4} & \text{if } 0 \le X \le 1, \end{cases} \tag{3.5}$$

where

$$\phi(X,p) = \frac{1}{p+\delta}X - \frac{1}{p+\bar{r}},$$

and $k_1 < k_2 < 0 < k_3 < k_4$ are the roots of the quartic indicial equation for the ODE system; A_i and B_i, $i = 1,2,3\ldots 6$, are some constants in Laplace space.

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The solution of the ODE in Laplace space corresponding to the PDE system (3.2) in the transition region is

$$\bar{U}_1(X,p) = m_1 X^{\gamma_1} + m_2 X^{\gamma_2} + \psi_1 X - \psi_2, \quad X_{f2} \le X \le X_{f1}, \tag{3.6}$$

where

$$\psi_1 = \frac{p+\xi_{12}}{p(p+\delta+\xi_{12})}, \quad \psi_2 = \frac{p+\xi_{12}}{p(p+r+\xi_{12})},$$

m_1 and m_2 are integration constants, and γ_1 and γ_2 are the solutions to the indicial equation of the ODE.

Applying all the necessary conditions of the PDE system to solutions (3.4)–(3.6), after some tedious algebraic manipulations we obtain the following coupled equation system for the computation of $\bar{X}_{f1}$ and $\bar{X}_{f2}$, the optimal exit prices in Laplace space:

$$\begin{pmatrix} (p\bar{X}_{f1})^{-\gamma_1} & 0 \\ 0 & (p\bar{X}_{f1})^{-\gamma_2} \end{pmatrix} F_1(p\bar{X}_{f1}) = \begin{pmatrix} (p\bar{X}_{f2})^{-\gamma_1} & 0 \\ 0 & (p\bar{X}_{f2})^{-\gamma_2} \end{pmatrix} F_2(p\bar{X}_{f2}), \tag{3.7}$$

where functions F_1 and F_2 are listed in the Appendix.

Equation (3.7) is the end of the analytical approach; the highly nonlinear equation has to be solved numerically. However, only a simple MATLAB solver, fsolve, is needed for the solution of the nonlinear equation. Once equation (3.7) is solved for $\bar{X}_{f1}$ and $\bar{X}_{f2}$, it is routine to compute solutions $\bar{U}_1(X,p)$ and $\bar{U}_1(X,p)$ from equations (3.4)–(3.6). A numerical inversion scheme (the Stehfest method [18]) is used to obtain the optimal exit prices and the values of the stock loans in the original space. It is then straightforward to calculate the fair service fees.

4. Numerical experiments

In this section, numerical examples are presented to show the effects of the volatility states on the optimal exit prices, stock loan values and service fees. The effects of different loan parameters on basic properties of the stock loans are also explored. Our calculations are carried out using MATLAB R2015a on an Intel(R) Core(TM)2 Quad CPU Q9550 @2.38GHz RAM 8 GB on a Windows 7 Enterprise Service Pack 1 system.

4.1. Perpetual stock loans To validate our solution technique, we present a comparison of the results of our calculations for maturity T tending to infinity with those from the analytic method in [27], for a perpetual American-style stock loan under a regime-switching economy. For the purpose of comparison, the same parameters are used as in [27].

Table 1 shows the dimensionless optimal exit prices for various values of σ_2^2 with a fixed σ_1^2 value. As shown in the table, the relative errors are far less than 0.1%, indicating the accuracy of our method for the calculation of the optimal exit price of perpetual stock loans.

TABLE 1. Dimensionless optimal exit price of a perpetual stock loan.

$\bar{r} = r - \gamma = -0.03$, $\delta = 0$, $\sigma_1^2 = 0.04$, $\xi_{12} = \xi_{21} = 2$, and $q = 1$						
	Analytic [27]		Our results		Relative error (%)	
σ_2^2	X_{f1}	X_{f2}	X_{f1}	X_{f2}	X_{f1}	X_{f2}
0.001	1.589	1.423	1.589	1.423	0	0
0.005	1.664	1.525	1.663	1.524	0.0601	0.0656
0.01	1.770	1.653	1.769	1.652	0.0565	0.0605
0.02	2.034	1.946	2.033	1.944	0.0492	0.0103
0.03	2.422	2.378	2.420	2.372	0.0826	0.0252

TABLE 2. Dimensionless perpetual stock loan value at $t = 0$.

$r = 0.05$, $\gamma = 0.15$, $\delta = 0$, $\sigma_1 = 0.4$, $\sigma_2 = 0.15$, $\xi_{12} = \xi_{21} = 4$, and $q = 5$						
	Analytic [27]		Our results		Relative error (%)	
X	U_1	U_2	U_1	U_2	U_1	U_2
1	0.034	0.033	0.034	0.033	0	0
2	0.153	0.150	0.153	0.150	0	0
3	0.370	0.362	0.370	0.362	0	0
4	0.691	0.676	0.691	0.676	0	0
5	1.122	1.098	1.122	1.097	0	0.09
6	1.668	1.632	1.667	1.631	0.06	0.06
7	2.332	2.281	2.331	2.281	0.04	0

The values of perpetual American stock loans under a regime-switching economy are also calculated and compared with those obtained in [27], as shown in Table 2. Again, there is good agreement between our results and those obtained by the analytical method, with maximum relative error less than 0.1%.

4.2. Finite maturity stock loans Having validated the accuracy of our method in the previous section, we present our calculations of the optimal exit prices, values and fair service fees of a finite maturity stock loan under a regime-switching economy in dimensional variables.

4.2.1. Optimal exit prices. We first present a comparison of the optimal exit prices under a regime-switching economy with those under the standard BS framework (Figures 1 and 2). The parameters used in the computation are, unless otherwise stated, $\bar{r} = r - \gamma = -0.03$, $\delta = 0$, $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.02$, $\xi_{12} = \xi_{21} = 2$, $q = 1$, and $T = 5$. As expected, the curves of the optimal exit prices for the higher volatility are above the

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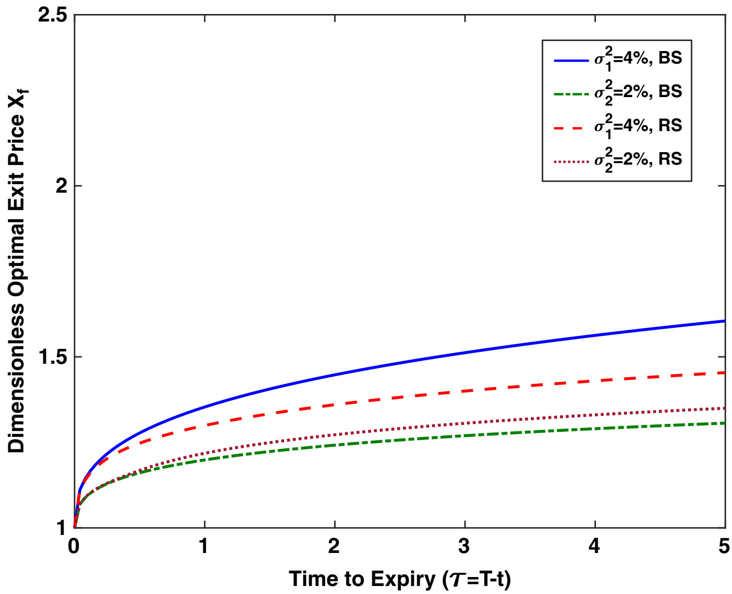

FIGURE 1. Dimensionless optimal exit price.

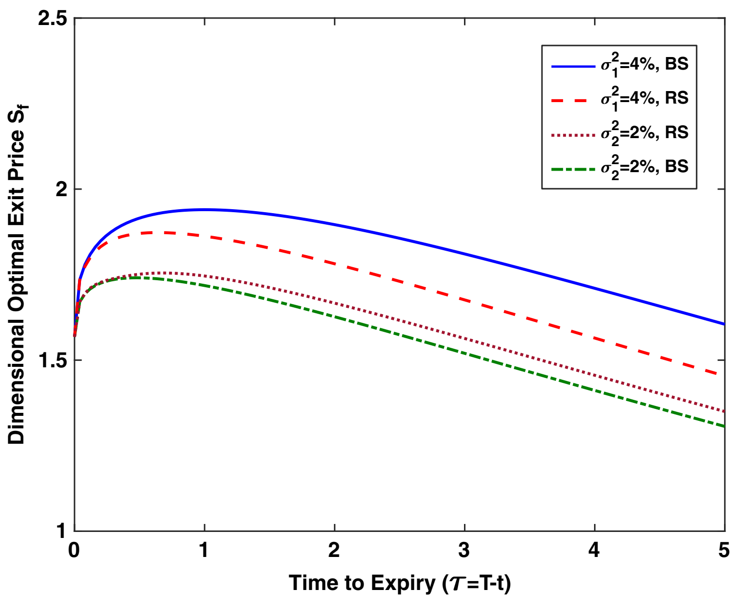

FIGURE 2. Dimensional optimal exit price.

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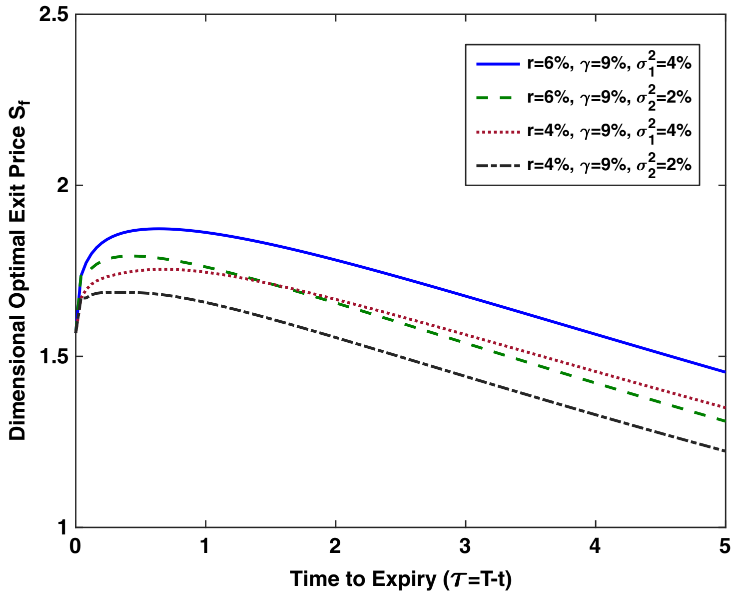

FIGURE 3. Optimal exit price with regime-switching volatility for different risk-free interest rates.

ones for the lower volatility, and the ones with regime-switching volatility are bounded in-between by the ones with constant volatility (the standard BS model).

Under a regime-switching economy, the dimensionless optimal exit prices for stock loans follow the same trend as those for American call options—the dimensionless optimal exit price X_{fi} is finite and monotonically increasing with respect to time to expiry (Figure 1). However, unlike an American call, a stock loan can reach optimality even when no dividend is paid for the underlying stock, as long as $r < \gamma$ for standard stock loans, which is true for financially viable stock loan settings [4, 12]. It is also worth pointing out that the dimensional optimal exit price S_{fi} is not monotonic with respect to time (Figure 2). This is is due to the two competing factors: the time-dependent “strike” ($qe^{\gamma t}$, $t \in [0, T]$) and the “call” nature of the loan near expiry as discussed in [13]. Since the dimensional optimal exit price increases with time, the longer the loan contract, the higher the optimal exit price near maturity. Nevertheless, a stock loan contract should always have a finite maturity, such that its optimal exit price will remain finite.

Figure 3 displays the optimal exit prices for different values of r, with γ fixed at 0.09 and all other parameters the same as before. Clearly, when the risk-free interest rate r is lower, the optimal exit price is also less compared with the one for higher r at the same volatility, consistent with the trend for American calls.

The optimal exit prices for different dividend rates are presented in Figure 4, with other parameters as stated before. Again, as for American call options, higher dividend rates correspond to lower optimal exit prices. The stock price is expected to drop by

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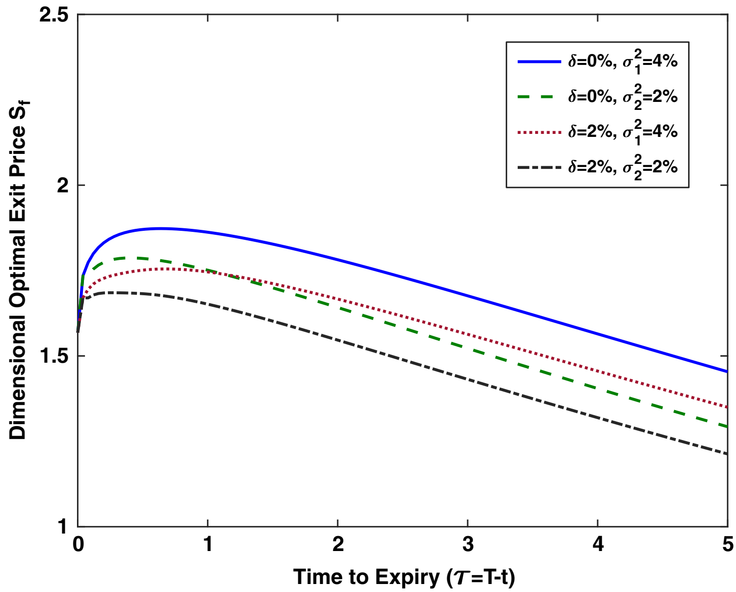

FIGURE 4. Optimal exit price with regime-switching volatility for different dividend rates.

the dividend amount on the ex-dividend date, so a higher amount of dividend paid will lead to a lower optimal exit price and a lower value of the stock loans.

Figure 5 depicts the optimal exit prices corresponding to different loan interest rates γ. It is interesting to note that the curves of the optimal prices under the same economy state intersect. This is a unique feature of the stock loans. A stock loan behaves like an American call, whose optimal exercise price increases if the strike price is higher. On the one hand, the time-dependent "strike" of a stock loan increases exponentially with the loan interest rate γ, so the optimal exit price for the loan with higher γ (higher strike) increases at a faster rate. On the other hand, the initial value of the loan with higher γ is lower, and so is its optimal exit price. As a result, both optimal prices increase at first, with the rate of increase dominated by γ, so the initially lower optimal price surpasses the other optimal price at some time during the lifetime of the loan, $[0, T]$, and eventually both decrease to lower values close to the time of maturity.

4.2.2. Stock loan values. Here we present a comparison of the values of stock loans under a regime-switching economy and under the BS framework at different times for the following loan parameters: $r = 0.06$, $\gamma = 0.09$, $\delta = 0$, $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.02$, $\xi_{12} = \xi_{21} = 2$, $q = 1$ and $T = 5$.

As shown in Tables 3 and 4, the values of the stock loans under a regime-switching economy are bounded by those under the BS model, that is, $V_2^{BS} < V_2^{RS} < V_1^{RS} < V_1^{BS}$. This makes financial sense, because while the economy is in state 1 (growth), there is always a possibility that it will go into state 2 (recession) during the life of the loan

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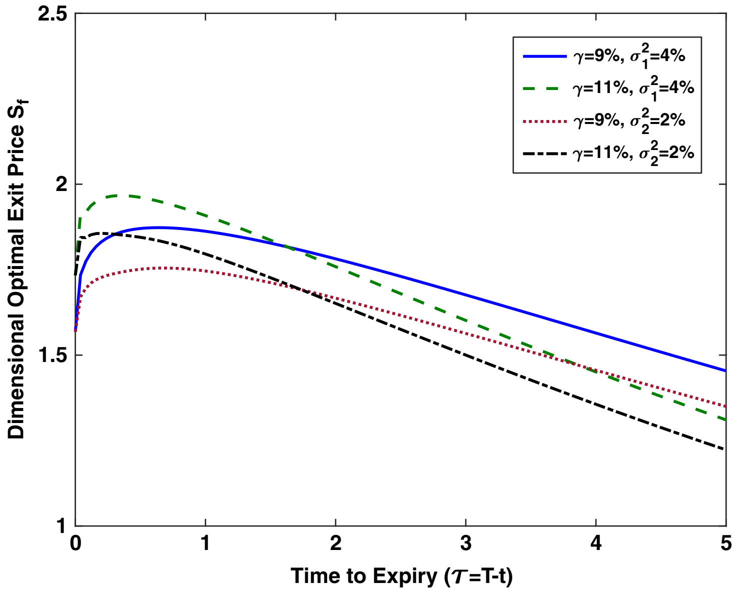

FIGURE 5. Optimal exit price with regime-switching volatility for different loan interest rates.

TABLE 3. Value of stock loan with regime-switching (RS) volatility at $t = 0$.

$S = Xqe^{\gamma t}$	V_1^{BS}	V_1^{RS}	V_2^{RS}	V_2^{BS}	CPU time (s)
$0.4 * 1$	0.869e-3	0.271e-3	0.223e-3	0.016e-3	1.934
$0.6 * 1$	0.012	0.006	0.006	0.002	1.936
$0.8 * 1$	0.051	0.035	0.033	0.019	1.934
$1.0 * 1$	0.129	0.104	0.101	0.080	1.931
$1.2 * 1$	0.248	0.220	0.216	0.205	1.932

TABLE 4. Value of stock loan with regime-switching (RS) volatility at $t = 2$.

$S = Xqe^{\gamma t}$	V_1^{BS}	V_1^{RS}	V_2^{RS}	V_2^{BS}	CPU time (s)
$0.4 * 1.0942$	0.001	0.000	0.000	0.000	1.914
$0.6 * 1.0942$	0.014	0.003	0.002	0.002	1.913
$0.8 * 1.0942$	0.060	0.027	0.025	0.023	1.914
$1.0 * 1.0942$	0.154	0.105	0.101	0.096	1.915
$1.2 * 1.0942$	0.296	0.250	0.245	0.244	1.925

https://doi.org/10.1017/S1446181121000250 Published online by Cambridge University Press

TABLE 5. Fair service fee under regime-switching economy ($T = 5$ and $S_0 = 1$).

LTV	0.8	0.9	1	1.1	1.2
c_1	0.056	0.063	0.104	0.167	0.243
c_2	0.052	0.059	0.101	0.165	0.242

contract, so that $V_1^{RS} < V_1^{BS}$. Similarly, there is always a chance of the economy moving to state 1 from state 2, so that $V_2^{RS} > V_2^{BS}$.

Although the computation is carried out on a relatively slow computer, the time used to calculate a value of stock loan is still less than 2 s. Considering that the computation time includes the calculation of the optimal exit prices, our technique is efficient compared with purely numerical methods such as finite difference methods and simulation methods.

4.2.3. Fair service fees. So far we have seen that under a regime-switching economy, the value of stock loans is affected by the probability of changes in the state of the economy. Naturally, the fair service fee is also dependent on the state of the economy, and it can be computed in a similar way to that for the standard stock loan [12]. The service fee is determined at the beginning of the contract based on the initial value of the stock loan, and it can be calculated as $c_i = V_{i0} - (S_0 - q)$, where c_i is the fair service fee in state i of the economy when the contract is made, S_0 is the initial value of the collateral stock, q is the loan principal, and V_{i0} is the initial value of the stock loan in state i. Since a stock loan has a greater valuein a state of higher volatility, for a loan of the same amount, the fair service fee is higher if the loan is established in the state of growth, and *vice versa*.

Table 5 lists the fair service fees corresponding to various loan-to-value (LTV) ratios, where LTV= q/S_0. A larger loan amount requires a higher service fee as the lender bears more risk. A stock loan contract under a regime-switching economy is marketable for any LTV > $q/S_{f1} = 0.740$ if the economy is initially in state 1, and LTV > $q/S_{f2} = 0.687$ if the economy is in state 2.

5. Conclusion

In this paper, we have formulated a pricing system for a stock loan under a regime-switching economy, and solved the resulting PDEs by using the approach developed by Lu and Putri [14] for the corresponding American option problem. Our results could help the lender to decide on a marketable stock loan and its fair service fee, and help the borrower to decide on an exit strategy, when the state of the economy is subject to change.

The work presented here is for a "standard" stock loan in the sense that the dividend payment is collected by the lender, and paid back to the borrower at maturity or upon exit from the loan. It can also be extended to stock loans with other dividend payment distributions (see the work of Dai and Xu, and Lu and Putri [4, 12]).

Appendix. Functions F_1 and F_2

Here we list the functions F_1 and F_2, and other constants in equation (3.7).

$$F_1(p\bar{X}_{f1}) = a_1 + a_2(p\bar{X}_{f1}),$$

$$a_1 = \frac{1}{\gamma_2 - \gamma_1}\begin{pmatrix}-\gamma_2\\ \gamma_1\end{pmatrix}\left(\frac{1}{p} - \psi_2\right),$$

$$a_2 = \frac{\psi_1}{\gamma_2 - \gamma_1}\begin{pmatrix}\gamma_2 - 1\\ 1 - \gamma_1\end{pmatrix}\left(\frac{1}{p} - \psi_1\right),$$

$$F_2(p\bar{X}_{f2}) = b_1 + b_2(p\bar{X}_{f2}) + b_3\begin{pmatrix}(p\bar{X}_{f2})^{k3} & 0\\ 0 & (p\bar{X}_{f2})^{k4}\end{pmatrix}\begin{pmatrix}A_3\\ A4\end{pmatrix},$$

$$b_1 = -\begin{pmatrix}1 & 1\\ \gamma_1 & \gamma_2\end{pmatrix}^{-1}\left[\begin{pmatrix}1 & 1\\ k_1 & k_2\end{pmatrix}\begin{pmatrix}l_1 & l_2\\ l_1k_1 & l_2k_2\end{pmatrix}^{-1}\begin{pmatrix}\dfrac{r}{p(p+r)}\\ 0\end{pmatrix} + \begin{pmatrix}\dfrac{p}{p(p+r)} - \psi_2\\ 0\end{pmatrix}\right],$$

$$b_2 = \begin{pmatrix}1 & 1\\ \gamma_1 & \gamma_2\end{pmatrix}^{-1}\left[\begin{pmatrix}1 & 1\\ k_1 & k_2\end{pmatrix}\begin{pmatrix}l_1 & l_2\\ l_1k_1 & l_2k_2\end{pmatrix}^{-1}\begin{pmatrix}\dfrac{\delta}{p(p+\delta)}\\ \dfrac{\delta}{p(p+\delta)}\end{pmatrix} + \begin{pmatrix}\dfrac{p}{p(p+\delta)} - \psi_1\\ \dfrac{p}{p(p+\delta)} - \psi_1\end{pmatrix}\right],$$

$$b_3 = -\begin{pmatrix}1 & 1\\ \gamma_1 & \gamma_2\end{pmatrix}^{-1}\left[\begin{pmatrix}1 & 1\\ k_1 & k_2\end{pmatrix}\begin{pmatrix}l_1 & l_2\\ l_1k_1 & l_2k_2\end{pmatrix}^{-1}\begin{pmatrix}l_3 & l_4\\ l_3k_3 & l_4k_4\end{pmatrix} - \begin{pmatrix}1 & 1\\ k_3 & k_4\end{pmatrix}\right].$$

References

[1] P. Boyle and T. Draviam, “Pricing exotic options under regime-switching”, *Insurance Math. Econom.* **40** (2007) 267–282; doi:10.1016/j.insmatheco.2006.05.001.

[2] J. Buffington and R. J. Elliot, “American options with regime-switching”, *Int. J. Theor. Appl. Finance* **5** (2002) 497–514; doi:10.1142/S0219024902001523.

[3] J. Buffington and R. J. Elliott, “Regime-switching and European options”, in: *Stochastic theory and control*, Volume 280 of *Lect. Notes Control Inform. Sci.* (ed. B. Pasik-Duncan), (Springer-Verlag, Berlin, Heidelberg, 2002) 73–82; doi:10.1007/3-540-48022-6_5.

[4] M. Dai and Z. Q. Xu, “Optimal redeeming strategy of stock loans with finite maturity”, *Math. Finance* **21** (2010) 775–793; doi:10.1111/j.1467-9965.2010.00449.x.

[5] X. Guo and Q. Zhang, “Closed-form solutions for perpetual American put options with regime-switching”, *SIAM J. Appl. Math.* **64** (2004) 2034–2049; https://www.jstor.org/stable/4096108.

[6] A. D. Holmes, H. Yang, and S. Zhang, “A front-fixing finite element method for the valuation of American options with regime-switching”, *Int. J. Comput. Math.* **89** (2012) 1094–1111; doi:10.1080/00207160.2012.663911.

[7] Y. Huang, P. A. Forsyth and G. Labahn, “Methods for pricing American options under regime-switching”, *SIAM J. Sci. Comput.* **33** (2011) 2144–2168; doi:10.1137/110820920.

[8] K. Itô, “Stochastic integral”, *Proc. Imp. Acad.* **20** (1944) 519–524; doi:10.3792/pia/1195572786.

[9] A. Q. M. Khaliq and R. Liu, "New numerical scheme for pricing American options with regime-switching", *Int. J. Theor. Appl. Finance* **12** (2009) 319–340; doi:10.1142/S0219024909005245.

[10] Z. Liang, Y. Nie and L. Zhao, "Valuation of infinite maturity stock loans with geometric Lévy model in a risk-neutral framework" (2012) https://www.semanticscholar.org/paper/VALUATION-OF-INFINITE-MATURITY-STOCK-LOANS-WITH-IN-Liang-Nie/7cd27b51fc509822ec02e130bc93fc7cd86f8396.

[11] R. H. Liu, Q. Zhang and G. Yin, "Option pricing in a regime-switching model using the fast Fourier transform", *J. Appl. Math. Stoch. Anal.* **2006** (2006) 018109; doi:10.1155/JAMSA/2006/18109.

[12] X. Lu and E. R. M. Putri, "Semi-analytic valuation of finite maturity stock loans", *Commun. Nonlinear Sci. Numer. Simul.* **27** (2015) 206–215; doi:10.1016/j.cnsns.2015.03.007.

[13] X. Lu and E. R. M. Putri, "Finite maturity margin call stock loans", *Oper. Res. Lett.* **44** (2016) 12–18; doi:10.1016/j.orl.2015.10.007.

[14] X. Lu and E. R. M. Putri, "A semi-analytic valuation of American options under a two-state regime-switching economy", *Phys. A Stat. Mech. Appl.* **538** (2020) 12–18; doi:10.1016/j.physa.2019.122968.

[15] R. S. Mamon and M. R. Rodrigo, "Explicit solutions to European options in a regime-switching economy", *Oper. Res. Lett.* **33** (2005) 581–586; doi:10.1016/j.orl.2004.12.003.

[16] V. Naik, "Option valuation and hedging strategies with jumps in the volatility of asset returns", *J. Finance* **48** (1993) 1969–1984; doi:10.1111/j.1540-6261.1993.tb05137.x.

[17] D. Prager and Q. Zhang, "Stock loan valuation under a regime-switching model with mean-reverting and finite maturity", *J. Syst. Sci. Complex.* **3** (2010) 572–583; doi:10.1007/s11424-010-0146-7.

[18] H. Stehfest, "Algorithm 368: numerical inversion of Laplace transforms [D5]", *Commun. ACM* **13** (1970) 47–49; doi:10.1145/361953.361969.

[19] P. Wilmott, S. Howison and J. Dewynne, *The mathematics of financial derivatives: a student introduction*, 1st edn (Cambridge University Press, Cambridge, 1995); doi:10.1017/CBO9780511812545.

[20] T. W. Wong and H. Y. Wong, "Stochastic volatility asymptotics of stock loans: valuation and optimal stopping", *J. Math. Anal. Appl.* **394** (2012) 337–346; doi:10.1016/j.jmaa.2012.04.067.

[21] T. W. Wong and H. Y. Wong, "Valuation of stock loans using exponential phase-type Levy models", *Appl. Math. Comput.* **222** (2013) 275–289; doi:10.1016/j.amc.2013.07.049.

[22] J. Xia and X. Y. Zhou, "Stock loans", *Math. Fin.* **17** (2007) 307–317; doi:10.1111/j.1467-9965.2006.00305.x.

[23] H. Yang, "A numerical analysis of American options with regime-switching", *J. Sci. Comput.* **44** (2010) 69–91; doi:10.1007/s10915-010-9365-2.

[24] D. Yao, Q. Zhang and X. Y. Zhou, "A regime-switching model for European options", in: *Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems*, Volume 94 of *Int. Ser. Oper. Res. Manag. Sci.* (eds H. Yan, G. Yin and Q. Zhang), (Springer, Boston, MA, 2006) 281–300; doi:10.1007/0-387-33815-2_14.

[25] F. Yi, "American put option with regime-switching volatility (finite time horizon) variational inequality approach", *Math. Methods Appl. Sci.* **31** (2008) 1461–1477; doi:10.1002/mma.984.

[26] F. L. Yuen and H. Yang, "Option pricing with regime-switching by trinomial tree method", *J. Comput. Appl. Math.* **233** (2010) 1821–1833; doi:10.1016/j.cam.2009.09.019.

[27] Q. Zhang and X. Y. Zhou, "Valuation of stock loans with regime-switching", *SIAM J. Control Optim.* **3** (2009) 1229–1250; doi:10.1137/070708998.

[28] S-P. Zhu, A. J. Badran, and X. Lu, "A new exact solution for pricing European options in a two state regime-switching economy", *Comput. Math. Appl.* **64** (2012) 2744–2755; doi:10.1016/j.camwa.2012.08.005.