



On intersections of polynomial semigroup orbits with plane lines

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Abstract. We study intersections of orbits in polynomial semigroup dynamics with lines on the affine plane over a number field, extending previous work of D. Ghioca, T. Tucker, and M. Zieve (2008).

1 Introduction

One of the most studied topics in complex dynamics is the research on orbits of polynomial maps. For a complex number x and polynomials $\mathcal{F} = \{f_1, \dots, f_k\} \subset \mathbb{C}[X]$, one is very interested in understanding the orbit

$$\begin{aligned}\mathcal{O}_{\mathcal{F}}(x) &= \{f_{i_1}(f_{i_2}(\dots(f_{i_n}(x))\dots)) : n \in \mathbb{N}, i_j = 1, \dots, k\}, \\ \mathcal{F}_n &= \{f_{i_1} \circ \dots \circ f_{i_n} : i_j = 1, \dots, k\}.\end{aligned}$$

Considering orbits with $k = 1$, D. Ghioca, T. Tucker, and M. Zieve [8] proved the following:

Let $x_0, y_0 \in \mathbb{C}$ and $f, g \in \mathbb{C}[X]$ with $\deg(f) = \deg(g) > 1$. If $\mathcal{O}_f(x_0) \cap \mathcal{O}_g(y_0)$ is infinite, then f and g have a common iterate.

Such a result provided the first non-monomial cases of the so-called *dynamical Mordell–Lang Conjecture* proposed by Ghioca and Tucker and stated below.

Dynamical Mordell–Lang Conjecture Let f_1, \dots, f_k be polynomials in $\mathbb{C}[X]$, and let V be a subvariety of the affine space \mathbb{A}^k that contains no positive dimensional subvariety that is periodic under the action of (f_1, \dots, f_k) on \mathbb{A}^k . Then $V(\mathbb{C})$ has finite intersection with each orbit of (f_1, \dots, f_k) on \mathbb{A}^k .

For an overview and a more detailed view of the history of the above conjecture, we refer the reader to [2].

The results of [8] were also extended by the same authors to the complex numbers and function fields of characteristic 0, and to cases where the degrees of the polynomials are distinct in [10], in which they also generalised results to cases of a line in a higher dimensional space intersecting a product of multiple orbits defined by

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one map. As a corollary ([10, Corollary 1.5]), they obtained information about the intersection of a higher dimensional line with an orbit defined by a semigroup of polynomial maps that have all but one of its coordinates as the identity. R. Benedetto, D. Ghioca, P. Kurlberg, and T. Tucker [4] studied cases of intersection of orbits of rational functions with curves under some natural conditions, and in [3], the same authors proved that if the conjecture does not hold in the context of endomorphisms of varieties, then the set of iterates landing on a referred subvariety forms a set of density zero. For a discussion with an effective viewpoint and monomial maps, see [13], and for intersection of orbits with finitely generated groups in fields, some analysis is made in [14]. In [15], intersection of orbits and the Mordell–Lang problem is studied on the disk, with non-polynomial mappings.

In this paper, we study the extension of the results of [8] to polynomial semigroup cases with $k \geq 1$ over number fields under some natural conditions. Namely, for sequences $\Phi = (\phi_{i_j})_{j=1}^\infty$ of pairs of univariate polynomials in a finite set \mathcal{F} whose coherent orbit

$$\mathcal{O}_\Phi^c(x, y) = \{(x, y), \phi_{i_1}(x, y), \phi_{i_1}(\phi_{i_2}(x, y)), \phi_{i_1}(\phi_{i_2}(\phi_{i_3}(x, y))), \dots\}$$

intersects the diagonal plane line Δ on infinitely many points.

Among other results, we prove the following theorem.

Theorem 1.1 *Let $x_0, y_0 \in K$, and let $\mathcal{F} = \{\phi_1, \dots, \phi_s\} \subset K[X] \times K[X]$ be a finite set of pairs of polynomials neither of which are conjugate to monomials, with $\phi_i = (f_i, g_i)$ and $\deg f_i = \deg g_i > 1$ for each i . Suppose that $\#(\mathcal{O}_{\mathcal{F}}(x_0, y_0) \cap \Delta) = \infty$. Suppose, moreover, that one of the following conditions is satisfied.*

(i) *There exists a sequence Φ of elements from \mathcal{F} such that*

$$\#\mathcal{O}_\Phi^c(x, y) \cap \Delta = \infty.$$

(ii) *The maps of \mathcal{F} commute with each other.*

Then there exists $\phi = (f, g)$ in the semigroup generated by \mathcal{F} such that $f = g$.

This result implies, for example, that under the conditions of Theorem 1.1, if $\#(\mathcal{O}_{\{f_1, \dots, f_s\}}(x_0) \cap \mathcal{O}_{\{g_1, \dots, g_s\}}(y_0)) = \infty$ with $\#\mathcal{O}_\Phi^c(x, y) \cap \Delta = \infty$, then there exists $\phi = (f, g)$ in the semigroup generated by \mathcal{F} such that $f = g$.

For the strategy of the proof, inspired by [8], we first see that condition (i) on the statement, together with the pigeonhole principle, implies the existence of bivariate diophantine equations with separable variables with infinite solutions, which can be tackled by the definitive work of Bilu and Tichy [5]. Such a description is analysed using results involving decomposition of polynomials and, again, the pigeonhole principle.

In Section 2 we set general notation and definitions used for the paper; in Section 3, we gather necessary results about polynomial equations and decompositions, and our main result is proved in Section 4. Further applications of the result and methods are given in Section 5. Section 6 recalls some known facts about height functions, and further results for polynomials with distinct degrees are exhibited in Section 7.

2 Preliminaries on Sequences and Semigroup Orbits

Throughout the paper, K is assumed to be a fixed number field. We consider $\mathcal{F} = \{\phi_1, \dots, \phi_s\} \subset K[X] \times K[X]$ to be a finite set of pairs of polynomials, with $\phi_i = (f_i, g_i)$. Let $x, y \in K$, and let

$$\mathcal{O}_{\mathcal{F}}(x, y) = \{\phi_{i_n} \circ \dots \circ \phi_{i_1}(x, y) \mid n \in \mathbb{N}, i_j = 1, \dots, s\}$$

denote the *forward orbit of P under \mathcal{F}* .

We set $J = \{1, \dots, s\}$, $W = \prod_{i=1}^{\infty} J$, and let $\Phi_w := (\phi_{w_j})_{j=1}^{\infty}$ be a sequence of polynomials from \mathcal{F} for $w = (w_j)_{j=1}^{\infty} \in W$. In this situation, we let $\Phi_w^{(n)} = \phi_{w_n} \circ \dots \circ \phi_{w_1}$ with $\Phi_w^{(0)} = \text{Id}$, and also

$$\mathcal{F}_n := \{\Phi_w^{(n)} \mid w \in W\}.$$

Precisely, we consider polynomial sequences $\Phi = (\phi_{i_j})_{j=1}^{\infty} \in \prod_{i=1}^{\infty} \mathcal{F}$ and $x, y \in K$, denoting

$$\Phi^{(n)}(x, y) := \phi_{i_n}(\phi_{i_{n-1}}(\dots(\phi_{i_1}(x, y))))$$

The set

$$\begin{aligned} & \{(x, y), \Phi^{(1)}(x, y), \Phi^{(2)}(x, y), \Phi^{(3)}(x, y), \dots\} = \\ & \{(x, y), \phi_{i_1}(x, y), \phi_{i_2}(\phi_{i_1}(x, y)), \phi_{i_3}(\phi_{i_2}(\phi_{i_1}(x, y))), \dots\} \end{aligned}$$

is called the *forward orbit of (x, y) under Φ* , denoted by $\mathcal{O}_{\Phi}(x, y)$.

The point (x, y) is said to be Φ -preperiodic if $\mathcal{O}_{\Phi}(x, y)$ is finite.

For a $x, y \in K$, the \mathcal{F} -orbit of (x, y) is defined as

$$\begin{aligned} \mathcal{O}_{\mathcal{F}}(x, y) &= \left\{ \phi(x, y) \mid \phi \in \bigcup_{n \geq 1} \mathcal{F}_n \right\} = \{ \Phi_w^{(n)}(x, y) \mid n \geq 0, w \in W \} \\ &= \bigcup_{w \in W} \mathcal{O}_{\Phi_w}(x, y). \end{aligned}$$

The point (x, y) is called *preperiodic for \mathcal{F}* if $\mathcal{O}_{\mathcal{F}}(x, y)$ is finite.

We let S be the *shift map* that sends $\Psi = (\psi_i)_{i=1}^{\infty}$ to

$$S(\Psi) = (\psi_{i+1})_{i=1}^{\infty}.$$

We also define the *coherent orbit* of a point (x, y) under a sequence $\Phi = (\phi_{i_j})_{j=1}^{\infty}$ to be the set

$$\mathcal{O}_{\Phi}^c(x, y) = \{(x, y), \phi_{i_1}(x, y), \phi_{i_1}(\phi_{i_2}(x, y)), \phi_{i_1}(\phi_{i_2}(\phi_{i_3}(x, y))), \dots\}.$$

We let Δ denote the diagonal line $\{(x, x) \mid x \in K\}$ in the affine plane $\mathbb{A}^2(K)$.

3 Some Results on Polynomial Composition

The result stated below is a strong fact concerning equations of the form $F(X) = G(Y)$ with infinitely many integral solutions due to Bilu and Tichy [5].

Lemma 3.1 ([8, Corollary 2.2]) *Let K be a number field, S a finite set of nonarchimedean places of K , and $F, G \in K[X]$ with $\deg(F) = \deg(G) > 1$. Suppose $F(X) = G(Y)$ has infinitely many solutions in the ring of S -integers of K . Then $F = E \circ H \circ a$ and $G = E \circ c \circ H \circ b$ for some $E, a, b, c \in \bar{K}[X]$ with a, b and c linear, and $H \in \bar{K}[X]$. Moreover, for fixed K , there are only finitely many possibilities for H .*

The next surprising result shows a certain rigidity on polynomial decomposition.

Lemma 3.2 ([8, Lemma 2.3] (Rigidity)) *Let K be a field of characteristic zero. If $A, B, C, D \in K[X] - K$ satisfy $A \circ B = C \circ D$ and $\deg(B) = \deg(D)$, then there is a linear $l \in K[X]$ such that $A = C \circ l^{-1}$ and $B = l \circ D$.*

Finally, we show, under some conditions, when polynomials from a finite set can be obtained from the same set through composition with linear polynomials.

Lemma 3.3 *Let K be a field of characteristic zero, and suppose $\{F_1, \dots, F_h\} \subset K[X]$ is a finite set of polynomials of degree $d > 1$ with the property that $u \circ F_i \circ v$ is not a monomial whenever $u, v \in K[X]$ are linear for each i . Then the equations $a \circ F_i = F_j \circ b$ have only finitely many solutions in linear polynomials $a, b \in K[X]$ for each $1 \leq i, j \leq h$.*

Proof Suppose $a \circ F_1 = F_2 \circ b$; we denote the coefficients of X^d and X^{d-1} in F_1 by θ_d and θ_{d-1} , and in F_2 by τ_d and τ_{d-1} . We put $\beta_1 = -\theta_{d-1}/d\theta_d$, $\alpha_1 = -F_1(\beta_1)$ and $\beta_2 = -\tau_{d-1}/d\tau_d$, $\alpha_2 = -F_2(\beta_2)$ and see that $\hat{F}_i := \alpha_i + F_i(X + \beta_i)$ ($i = 1, 2$) have no terms of degree $d - 1$ and 0 . Putting $\hat{a} := \alpha_2 + a(x - \alpha_1)$ and $\hat{b} := \beta_2 + b(X + \beta_1)$, we have that $\hat{a} \circ \hat{F}_1 = \hat{F}_2 \circ \hat{b}$, and both have no term of degree $d - 1$. Hence, \hat{b} cannot have a term of degree 0 , neither $\hat{F}_2 \circ \hat{b}$ nor \hat{a} . Hence, we can make $\hat{a} = \delta X$ and $\hat{b} = \gamma X$, which implies that $\delta \hat{F}_1(X) = \hat{F}_2(\gamma X)$. Writing $\hat{F}_1(X) = \sum_i u_i X^i$, $\hat{F}_2 = \sum_i v_i X^i$, we have $\gamma^i = \delta \frac{u_i}{v_i}$ for each non-zero i term. As \hat{F}_1, \hat{F}_2 have at least two terms of distinct degrees, let us say $i > j$, we have $\delta^{i-j} = \frac{u_i v_j}{u_j v_i}$, and there are finitely many possibilities for δ . Since by our construction, $a = -\alpha_2 + \gamma^r \frac{v_r}{u_r}(X + \alpha_1)$ and $b = \beta_2 + \gamma(X - \beta_1)$, there are only finitely many possibilities for a and b . Repeating the same procedure for any pair (F_i, F_j) yields the desired result. ■

4 Proof of Theorem 1.1

Proof We start by letting S be a finite set of nonarchimedean places of K such that the ring of S -integers \mathcal{O}_S contains x_0, y_0 and every coefficient of ϕ_1, \dots, ϕ_s . Then \mathcal{O}_S^2 contains $\phi(x_0, y_0)$ for every $\phi \in \bigcup_{n \geq 1} \mathcal{F}_n$.

By hypothesis, we can suppose first that $\#(\mathcal{O}_{\mathcal{F}}(x_0, y_0) \cap \Delta) = \infty$ with $\#(\mathcal{O}_{\Phi}^c(x_0, y_0) \cap \Delta) = \infty$ for some sequence $\Phi = (F, G) = (\phi_{i_j})_{j=1}^{\infty} = ((f_{i_j}, g_{i_j}))_{j=1}^{\infty}$ of terms belonging to \mathcal{F} , so that $\#\mathcal{O}_F^c(x_0) = \infty$ and $\#\mathcal{O}_G^c(y_0) = \infty$. Let $(n_j)_{j \in \mathbb{N}}$ be such that

$$\phi_{i_1} \circ \dots \circ \phi_{i_{n_j}}(x_0, y_0) \in \Delta$$

for each $j \in \mathbb{N}$.

By the pigeonhole principle, there exists a $t_1 \in \{1, \dots, s\}$ such that for infinitely many j , we have that $\phi_{i_{n_j}} = \phi_{t_1}$, so that $\phi_{i_1} \circ \dots \circ \phi_{i_{n_j}}(x_0, y_0) = \phi_{i_1} \circ \dots \circ \phi_{t_1}(x_0, y_0) \in \Delta$. Again, for the same reason, there must exist a $\phi_{t_2} \in \mathcal{F}$ such that $\phi_{i_{n_j}} = \phi_{t_1}$, $\phi_{i_{n_{j-1}}} = \phi_{t_2}$, and $\phi_{i_1} \circ \dots \circ \phi_{i_{n_j}}(x_0, y_0) = \phi_{i_1} \circ \dots \circ \phi_{t_2} \circ \phi_{t_1}(x_0, y_0) \in \Delta$ for infinitely many j . Obtaining t_n inductively in this way, we can consider the sequence $\Phi^* = (F^*, G^*) = (\phi_k^*)_{k=1}^\infty := (\phi_{t_k})_{k=1}^\infty$, which by its construction satisfies that for every $k \in \mathbb{N}$, the equation $F^{*(k)}(X) = G^{*(k)}(Y)$ has infinitely many solutions in $\mathcal{O}_S \times \mathcal{O}_S$. By Lemma 3.1, for each k , we have $F^{*(k)} = E_k \circ H_k \circ a_k$ and $G^{*(k)} = E_k \circ c_k \circ H_k \circ b_k$ with $E_k \in \overline{K}[X]$, linear $a_k, b_k, c_k \in \overline{K}[X]$, and some $H_k \in \overline{K}[X]$ that comes from a finite set of polynomials. Thus, $H_k = H_l$ for some $k < l$.

If we write $F^{*(l)} = \tilde{F} \circ F^{*(k)}$ and $G^{*(l)} = \tilde{G} \circ G^{*(k)}$ with $(\tilde{F}, \tilde{G}) \in \mathcal{F}_{l-k}$, we have

$$\begin{aligned} \tilde{F} \circ E_k \circ H_k \circ a_k &= F^{*(l)} = E_l \circ H_k \circ a_l, \\ \tilde{G} \circ E_k \circ c_k \circ H_k \circ b_k &= G^{*(l)} = E_l \circ c_l \circ H_k \circ b_l. \end{aligned}$$

By Lemma 3.2, there are linear $u, v \in \overline{K}[X]$ such that

$$H_k \circ a_k = u \circ H_k \circ a_l \quad \text{and} \quad c_k \circ H_k \circ b_k = v \circ c_l \circ H_k \circ b_l.$$

Thus,

$$\tilde{F} \circ E_k \circ u = E_l = \tilde{G} \circ E_k \circ v,$$

and by Lemma 3.2, it follows that $\tilde{F} = \tilde{G} \circ l_1$ for some $l_1 \in \overline{K}[X]$ linear.

Again from the pigeonhole principle, there exists some integer k_0 so that for infinitely many $\ell > k_0$, we have that

$$H_{k_0} = H_\ell.$$

So, for these infinitely many ℓ , we have that there exists some linear polynomial a_ℓ such that

$$\phi_\ell^* \circ \dots \circ \phi_{k_0+1}^* := (F_{k_0, \ell}, G_{k_0, \ell})$$

satisfies

$$F_{k_0, \ell} = G_{k_0, \ell} \circ a_\ell.$$

Inductively, we obtain in this way an infinite $\mathcal{N} \subset \mathbb{N}$ and an infinite sequence $\Psi = (\mathbf{F}, \mathbf{G}) = (\psi_i)_{i=1}^\infty$ of terms in $\bigcup_{n \geq 1} \mathcal{F}_n$ satisfying

$$\mathbf{F}^{(n)} = \mathbf{G}^{(n)} \circ l_n, \text{ with } n \in \mathcal{N}.$$

By Lemma 3.2, this means that $u_n \circ f_{i_j} = g_{i_j} \circ l_n$ for some $1 \leq i_j \leq s$ and u_n linear.

Since $\{l_n \mid n \in \mathcal{N}\}$ is finite by Lemma 3.3, there exist $N > n$ such that $l_N = l_n$. Then, denoting $\mathbf{F}^{(N)} = F_{N-n} \circ \mathbf{F}^{(n)}$ and $\mathbf{G}^{(N)} = G_{N-n} \circ \mathbf{G}^{(n)}$ where $F_{N-n}, G_{N-n} \in \mathcal{F}_m$ for some m , we have

$$\mathbf{F}^{(N)} = \mathbf{G}^{(N)} \circ l_N = G_{N-n} \circ \mathbf{G}^{(n)} \circ l_n = F_{N-n} \circ \mathbf{F}^{(n)} = F_{N-n} \circ \mathbf{G}^{(n)} \circ l_n,$$

and thus

$$F_{N-n} = G_{N-n}$$

as desired.

If, otherwise, we suppose that $\#(\mathcal{O}_{\mathcal{F}}(x_0, y_0) \cap \Delta) = \infty$ and the maps of \mathcal{F} commute, we take t_1 such that $\phi(x_0, y_0) = (\phi_{t_1} \circ \dots)(x_0, y_0) \in \Delta$ for infinitely many ϕ if the semigroup generated by \mathcal{F} , so that, $f_{t_1}(X) = g_{t_1}(Y)$ has infinitely many solutions in $\mathcal{O}_S \times \mathcal{O}_S$. Then we choose t_2 such that $\phi = (\phi_{t_1} \circ \phi_{t_2} \circ \dots)(x_0, y_0) = (\phi_{t_2} \circ \phi_{t_1} \circ \dots)(x_0, y_0) \in \Delta$ for infinitely many $\phi \in \cup_{n \geq 0} \mathcal{F}_n$ (commutativity), so that $(f_{t_2} \circ f_{t_1})(X) = (g_{t_2} \circ g_{t_1})(Y)$ has infinitely many solutions in $\mathcal{O}_S \times \mathcal{O}_S$. In this way, we build a sequence $\Phi^* = (F^*, G^*) = (\phi_{t_n})_{n \in \mathbb{N}}$ such that $F^{*(k)}(X) = G^{*(k)}(Y)$ has infinitely many solutions in $\mathcal{O}_S \times \mathcal{O}_S$ for each $k \in \mathbb{N}$. Then one can proceed as in the first case to achieve the desired conclusion. ■

It turns out that this result is actually true for orbits that intersect an arbitrary line at infinitely many points.

Corollary 4.1 *Under the conditions of Theorem 1.1, with $L : X = l(Y)$ (l linear over K) in place of Δ , there must exist $\phi = (f, g)$ in the semigroup generated by \mathcal{F} such that $f = l \circ g$.*

Proof Suppose $\#(\mathcal{O}_{\mathcal{F}}(x_0, y_0) \cap L) = \infty$. Then defining a new system

$$\mathcal{F}^l := \{(f_1, l \circ g_1 \circ l^{-1}), \dots, (f_s, l \circ g_s \circ l^{-1})\},$$

we have that $\#(\mathcal{O}_{\mathcal{F}^l}(x_0, l(y_0)) \cap \Delta) = \infty$ with the conditions of Theorem 1.1, from which the result follows. ■

5 Further Applications

The two next corollaries are straightforward consequences of Theorem 1.1.

Corollary 5.1 *Let $x_0, y_0 \in K$, and let $\mathcal{F} = \{\phi_1, \dots, \phi_s\} \subset K[X] \times K[X]$ be a set of pairs of polynomials neither of which are conjugate to monomials, with $\phi_i = (f_i, g_i)$ and $\deg f_i = \deg g_i > 1$ for each i . Suppose that $\#(\mathcal{O}_{\{f_1, \dots, f_s\}}(x_0) \cap \mathcal{O}_{\{g_1, \dots, g_s\}}(y_0)) = \infty$, such that for some sequence Φ of terms in $\{f_1, \dots, f_s\} \times \{g_1, \dots, g_s\}$, we have*

$$\mathcal{O}_{\Phi}^c(x_0, y_0) \cap \Delta = \infty.$$

Then there exists $\phi = (f, g)$ in the semigroup generated by \mathcal{F} such that $f = g$.

Proof The result follows from the conditions of Theorem 1.1. ■

Corollary 5.2 *Let $x_0 \in K$, and let $\mathcal{F} = \{f_1, \dots, f_s\} \subset K[X]$ be a set of polynomials neither of which are conjugate to monomials, with $\deg f_1 = \deg f_2 \dots = \deg f_s > 1$. Suppose there are two sequences (trajectories) $\Phi = (f_{u_i})_{i=1}^{\infty}$ and $\Psi = (f_{v_i})_{i=1}^{\infty}$ in the semigroup generated by \mathcal{F} satisfying one of the conditions below:*

- (i) $\mathcal{O}_\Phi^c(x_0) \cap \mathcal{O}_\Psi^c(x_0) = \infty$, or
- (ii) $\#(\mathcal{O}_\Phi(x_0) \cap \mathcal{O}_\Psi(x_0)) = \infty$ and the elements of \mathcal{F} commute.

Then Φ and Ψ have two “words” in common; namely, there exist $m, k \in \mathbb{N}$ such that

$$f_{u_{m+k}} \circ \dots \circ f_{u_m} = f_{v_{m+k}} \circ \dots \circ f_{v_m}.$$

Proof We apply the proof of Theorem 1.1 for \mathcal{F} and $\mathcal{G} = \mathcal{F}$ with (x_0, x_0) . ■

Remark 5.3 In a similar way as in the proof of Corollary 4.1, it can be seen that Corollaries 5.1 and 5.2 can be extended with Δ being replaced by an arbitrary plane line $L : X = l(Y)$ and the set $\{\Phi^{(n_j)}(x_0) = \Psi^{(n_j)}(x_0) \mid n_1 < n_2 < \dots\}$ by $\{\Phi^{(n_j)}(x_0) = l(\Psi^{(n_j)}(x_0)) \mid n_1 < n_2 < \dots\}$, respectively, implying the more general conclusions $f = l \circ g$ and $S^m(\Phi)^{(k)} = l \circ S^m(\Psi)^{(k)}$, respectively.

6 Preliminaries about Height Functions

In order to deal with pairs of polynomials with distinct degrees, we recall known results about certain canonical heights.

Recall that for $x \in \overline{\mathbb{Q}}$, the naive logarithmic height is given by

$$h(x) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log(\max\{1, |x|_v\}),$$

where M_K is the set of places of K , M_K^∞ is the set of archimedean (infinite) places of K , M_K^0 is the set of nonarchimedean (finite) places of K , and for each $v \in M_K$, $|\cdot|_v$ denotes the corresponding absolute value on K whose restriction to \mathbb{Q} gives the usual v -adic absolute value on \mathbb{Q} . Also, we write K_v for the completion of K with respect to $|\cdot|_v$, and we let \mathbb{C}_v denote the completion of an algebraic closure of K_v .

Considering the affine plane over a field L to be $\mathbb{A}^2(L) = L \times L$, there is a way to construct height functions associated with sequences of polynomials. This was done by S. Kawaguchi, by defining and proving the convergence of the sequence $(h(\Phi^{(n)}(P)) / \deg(\Phi^{(n)}))_{n=1}^\infty$, inspired by other classical canonical heights. We point out that in his work, Kawaguchi showed the existence of such heights in a more general context of smooth projective varieties with polarized morphisms.

Lemma 6.1 ([12, Theorem 2.3]) *There is a unique way to attach to each sequence $\Phi = (\phi_i)_{i=1}^\infty$, with $\deg \phi_i \geq 2$ as above, a canonical height function*

$$\hat{h}_\Phi : \mathbb{A}^2(\bar{K}) \longrightarrow \mathbb{R}$$

such that

- (i) $\sup_{P \in \mathbb{A}^2(\bar{K})} |\hat{h}_\Phi(P) - h(P)| \leq O(1)$;
- (ii) $\hat{h}_{S(\Phi)} \circ \phi_1 = (\deg \phi_1) \hat{h}_\Phi$; in particular,

$$\hat{h}_{S^n(\Phi)} \circ \phi_n \circ \dots \circ \phi_1 = (\deg \phi_n) \cdots (\deg \phi_1) \hat{h}_\Phi.$$
- (iii) $\hat{h}_\Phi(P) \geq 0$ for all P ;
- (iv) $\hat{h}_\Phi(P) = 0$ if and only if P is Φ -preperiodic.

We call \hat{h}_Φ a canonical height function (normalized) for Φ .

Considering conditions as above, namely, a number field K , and $\mathcal{H} = \{\phi_1, \dots, \phi_k\}$ now with $\sum_i \deg g_i > k$, the tuple $(\mathbb{A}^2(K), g_1, \dots, g_k)$ becomes a particular case of what we call a dynamical eigensystem of degree $\deg \phi_1 + \dots + \deg \phi_k$. For such, Kawaguchi also proved the following lemma.

Lemma 6.2 ([11, Theorem 1.2.1]) *There exists a unique canonical height function*

$$\hat{h}_{\mathcal{H}} : \mathbb{A}^2(\bar{K}) \rightarrow \mathbb{R}$$

for $(X, \phi_1, \dots, \phi_k)$ characterized by the following two properties:

- (i) $\hat{h}_{\mathcal{H}} = h + O(1)$;
- (ii) $\sum_{j=1}^k \hat{h}_{\mathcal{H}} \circ g_j = (\deg g_1 + \dots + \deg g_k) \hat{h}_{\mathcal{H}}$.

The result below is also well known.

Lemma 6.3 ([8, Lemma 5.4]) *If $l \in K[X]$ is linear, then there exists $c_l > 0$ such that $|h(l(x)) - h(x)| \leq c_l$ for all $x \in \bar{K}$.*

7 Further Results

Finally, we obtain information in some cases of polynomial semigroup orbits with polynomials with distinct degrees, for which the theory of canonical heights is useful.

Proposition 7.1 *Let $x_0, y_0 \in K$, and let $\mathcal{F} = \{\phi_1, \dots, \phi_s\} \subset K[X] \times K[X]$ be a set of pairs of polynomials of degree at least 2, and let $\Phi = (F, G)$ be a sequence of terms in \mathcal{F} such that $\deg(G^{(n)}) = o(\deg(F^{(n)}))$. Then*

$$\#(\mathcal{O}_{\Phi}(x_0, y_0) \cap \Delta) < \infty.$$

In particular, if $\deg f_i > \deg g_i$ for each $i = 1, \dots, s$, then every sequence (trajectory) of \mathcal{F} intersects Δ in only finitely many points.

Proof If x_0 or y_0 is preperiodic for F or G , respectively, then the result is true. Otherwise, Lemma 6.1 states that $\hat{h}_F(x_0) > 0$, so there is a $\delta > 0$ such that every k big enough satisfies

$$h(F^{(k)}(x_0)) > \deg(F^{(k)})\delta.$$

Also, there exists $\varepsilon > 0$ such that

$$h(G^{(k)}(y_0)) < \deg(G^{(k)})\varepsilon,$$

and by the hypothesis, we know that $\deg(F^{(k)})\delta > \deg(G^{(k)})\varepsilon$ for every k large enough. Therefore, $h(F^{(k)}(x_0)) > h(G^{(k)}(y_0))$ and $F^{(k)}(x_0) \neq G^{(k)}(y_0)$ for every k large as wanted. ■

The result above shows, in particular, that if $\#(\mathcal{O}_{\Phi}(x_0, y_0) \cap \Delta) = \infty$, then it cannot be true that $\lim_n \frac{\deg(G^{(k)})}{\deg(F^{(k)})}$ or $\lim_n \frac{\deg(F^{(k)})}{\deg(G^{(k)})}$ is equal to zero, so that the distance

between the canonical heights of x_0 and y_0 associated with F and G , respectively, cannot be increasingly large.

Related to Proposition 7.1 and considering the difference between the degree sum of the coordinates in the sequence, we have that if the n -iterates of a point under the semigroup are all contained in Δ for infinitely many n , then the sum of the degrees of the polynomials in the first coordinate of the generator set is equal to such a sum for the second coordinate polynomials, as in the following proposition.

Proposition 7.2 *Let $x_0, y_0 \in K$, and let $\mathcal{F} = \{\phi_1, \dots, \phi_s\} \subset K[X] \times K[X]$ be a set of pairs of polynomials $\phi_i = (f_i, g_i)$ such that $\sum_i \deg f_i > \sum_i \deg g_i > s$. Suppose that x_0 and y_0 are not preperiodic for $\{f_1, \dots, f_s\}$ and $\{g_1, \dots, g_s\}$, respectively. Then $\{\phi(x_0, y_0) \mid \phi \in \mathcal{F}_n\} \not\subset \Delta$ for all but finitely many numbers n .*

Proof Using Lemma 6.2 and the hypothesis, we proceed similarly as in the previous proof, so that for some positive numbers δ and ε , we have that

$$\begin{aligned} \sum_{f \in \mathcal{F}_k} h(f(x_0)) &= \left(\sum_i \deg f_i \right)^k \delta \\ &> \left(\sum_i \deg g_i \right)^k \varepsilon \\ &> \sum_{g \in \mathcal{F}_k} h(g(y_0)) \end{aligned}$$

for every k large enough, from which the result follows. ■

Remark 7.3 We believe that the results of this paper could be extended to the complex numbers, paying the price of obtaining some function field results to the general context of semigroups of polynomials. For example, [8, Lemma 6.5], which is a specialization limit result of Call and Silverman that is not yet proved for Kawaguchi's canonical height of sequences of maps presented here, and [8, Lemma 6.8] due to Benedetto, which characterizes preperiodic points for polynomials over function fields, and whose proof uses non archimedean analysis. Besides, one would also need to work out versions of [8, Proposition 6.3, Corollary 6.4] for the context of several maps, that do not commute for instance.

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