



# On the fixed points of the Ruelle operator

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*Abstract.* We discuss the relation between the existence of fixed points of the Ruelle operator acting on different Banach spaces, and Sullivan’s conjecture in holomorphic dynamics.

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## 1. Introduction

Let  $\text{Rat}_d(\bar{\mathbb{C}})$  be the set of all rational maps on the Riemann sphere  $\bar{\mathbb{C}}$  of degree  $d$ . When  $R$  is an element of  $\text{Rat}_d(\bar{\mathbb{C}})$ , the postcritical set of  $R$  is given by

$$P_R = \overline{\bigcup_k \bigcup_i R^i(c_k)},$$

where the union is taken over all critical points  $c_k$  of  $R$  and  $i > 0$ . The *Julia set*  $J_R$  is the set of accumulation points of all periodic points of  $R$ , with all isolated points removed. The Fatou set is given by  $F_R = \bar{\mathbb{C}} \setminus J_R$ . The map  $R$  is called *hyperbolic* when  $P_R \cap J_R = \emptyset$ . The Fatou conjecture states that *hyperbolic maps are open and everywhere dense in  $\text{Rat}_d(\bar{\mathbb{C}})$* .

Recall that a rational map  $R$  is *J-stable* if there is an open neighbourhood  $U$  of  $R$  in  $\text{Rat}_d(\bar{\mathbb{C}})$ , such that for  $Q \in U$  there exists a homeomorphism  $h_Q : J_R \rightarrow J_Q$ , quasiconformal in Pesin’s sense (also known as metric quasiconformal), with

$$Q = h_Q \circ R \circ h_Q^{-1}.$$

Due to a result of Mañé, Sad and Sullivan (see [16]), the set of *J-stable* maps forms an open and everywhere dense subset of  $\text{Rat}_d(\bar{\mathbb{C}})$ ; moreover, a *J-stable* map is hyperbolic if and only if there is no invariant Beltrami differential supported on the Julia set. Since hyperbolic maps are *J-stable*, the Fatou conjecture becomes: *every J-stable map is hyperbolic*.

An invariant Beltrami differential  $\mu$  is a  $(-1, 1)$ -differential form locally expressed by  $\mu(z)d\bar{z}/dz$  and whose coefficient  $\mu(z)$  is an  $L_\infty$  function satisfying

$$\mu(z) = \mu(R(z)) \frac{\overline{R'(z)}}{R'(z)}.$$

That is,  $\mu$  is a fixed point of the Beltrami operator, as defined below, acting on the space  $L_\infty(\overline{\mathbb{C}})$  with respect to the planar Lebesgue measure. In this paper, whenever it is clear from the context, we will denote both the differential and its coefficient by the same letter.

Sullivan's conjecture states that *there exists an invariant Beltrami differential supported on the Julia set if and only if  $R$  is a flexible Lattès map*. For the definitions and further properties of Lattès maps, see Milnor's paper [20].

Note that if  $R$  is hyperbolic then  $R^n$  is hyperbolic and hence  $J$ -stable for every  $n > 1$ . We have the following statement (see [3, Theorem 2.1]): if there exists  $n > 1$  such that the iterated map  $R^n$  is  $J$ -stable then  $R$  is hyperbolic. Therefore the Fatou conjecture is true when one considers an iterated rational map. On the other hand, Sullivan's conjecture predicts not only the absence of fixed points for the Beltrami operator but also the lack of periodic points for this operator. In other words, there is no eigenvalue of the Beltrami operator which is a root of unity. Hence Sullivan's conjecture can be interpreted as a spectral problem for a semigroup of Beltrami operators.

According to Sullivan's dictionary between Kleinian groups and holomorphic dynamics, rational maps correspond to finitely generated Kleinian groups. In this setting, we can reformulate Teichmüller theorem (see [9]) as follows.

*Let  $\Gamma$  be a finitely generated Kleinian group. For every  $\Gamma$ -invariant Beltrami differential  $\mu$  inducing a non-trivial quasiconformal deformation (see definition below) on the associated Riemann surface  $S_\Gamma$ , there exists a  $\Gamma$ -invariant holomorphic 2-form  $\phi$  such that  $\int_F \mu\phi \neq 0$  on any fundamental domain  $F$  of  $\Gamma$ .*

In other words, the following separation principle holds: the space of invariant holomorphic 2-forms separates the space of quasiconformally non-trivial invariant Beltrami differentials.

The separation principle is well known in ergodic theory for bounded semigroups of linear endomorphisms of a Banach space. In fact, it is the subject of many ergodic theorems and is one of the oldest principles in this theory.

Due to the observations above, in this article we discuss the following question. *Given a representation of the dynamics of  $R$  into a semigroup of contractions of a suitable Banach space, what consequences arise from the existence of a common non-trivial fixed point of such a representation?* Furthermore, we will discuss what happens when these representations satisfy a separation principle (definitions will be given below).

To keep in line with Sullivan's conjecture we consider representations that arise as versions of complex pull-back or push-forward operators acting on either invariant subspaces  $X \subset L_p(W)$  (not necessarily closed) or spaces which are predual, dual or bidual to  $X$ . Here  $W$  is an  $R$ -invariant set (i.e.  $R(W) \subset W$ ) of positive Lebesgue measure in the Riemann sphere and  $1 \leq p \leq \infty$ .

Throughout our discussion, unless otherwise stated, all  $L_p$  spaces are taken with respect to the planar Lebesgue measure on the dynamical plane. Also we will use one or some combination of the following restrictions on  $R$ .

- (1) The postcritical set  $P_R \neq \overline{\mathbb{C}}$ .
- (2) The postcritical set has Lebesgue measure zero.
- (3) The intersection  $P_R \cap F_R$  is finite and the Fatou set  $F_R$  does not contain parabolic or rotational domains.

In each section we will specify which restrictions apply. But let us note that the class of rational maps satisfying restrictions (1)–(3) is still relevant for the Fatou–Sullivan conjecture. In fact, when  $F_R$  has no rotation or parabolic domains and  $J_R$  is connected, an application of quasiconformal surgery gives that the  $J$ -stability component of  $R$  contains a map  $Q$  which does not admit non-trivial quasiconformal deformations on the Fatou set. Moreover, by the Mañé–Sad–Sullivan theorem  $Q$  is unique up to Möbius conjugacy only in the case where  $J_Q$  does not support an invariant Beltrami differential.

Also, after the celebrated examples of Julia sets of positive measure given by Buff and Cheritat of Cremer polynomials, and by Avila and Lyubich of infinitely renormalizable polynomials, the main conjecture is that the postcritical set either has measure zero or is the whole Riemann sphere. So if  $F_R \neq \emptyset$ , then restriction (2) is natural.

We summarize the main theorems of this paper in the following two theorems. We say that an integrable function  $f$  is *regular* whenever its  $\bar{\partial}$  derivative, in the sense of distributions, is a finite complex-valued measure. Examples of non-regular functions are given by characteristic functions of suitable compact subsets of  $\mathbb{C}$  (see the discussion after Theorem 4.9 and Proposition 4.14). Other notation and definitions will be given in §2.

**THEOREM 1.1.** (Fixed points of Ruelle operator) *Let  $R$  be a rational map. Then the following statements hold true.*

- (1) *No simple function is invariant under Ruelle operator  $R^*$ . Moreover, if  $S_R$  is connected, then a non-zero regular function  $f$  is a fixed point of  $R^*$  if and only if  $R$  is a flexible Lattès map.*
- (2) *(The  $L_p$ -case) Let  $p$  and  $q$  be such that  $1 < p < \infty$  and  $1/p + 1/q = 1$ . The operator  $R_p^* : L_p(\mathbb{C}) \rightarrow L_p(\mathbb{C})$  given by*

$$R_p^*(\phi) = \frac{1}{\sqrt[p]{d}} \sum \phi(\zeta_i) \frac{\zeta_i'}{\zeta_i} |\zeta_i'|^{2/p}$$

*has a non-zero fixed point if and only if  $R$  is a flexible Lattès map.*

- (3) *If  $R$  is mixing with respect to any finite invariant measure absolutely continuous and  $P_R \neq \overline{\mathbb{C}}$ , then  $R^*$  has a non-zero fixed point in  $L_1(J_R)$  if and only if  $R$  is a flexible Lattès map.*

Part (1) follows from Theorem 4.9 and Corollary 4.13. Note that Theorem 4.9 provides the more general case where  $S_R$  is not connected. Part (2) is the content of Theorem 5.1. Part (3) is Theorem 9.2.

For every critical value  $v$  of  $R$  define the operator  $E_v : L_\infty(\mathbb{C}) \rightarrow \ell_\infty$  by the formula

$$E_v(\psi) = \left( \int_{\mathbb{C}} \psi(z) A_n(\gamma_v)(z) |dz|^2 \right)_{n=0}^\infty,$$

where the  $A_n(\gamma_v) = (1/n) \sum_{i=0}^{n-1} R^{*i}(\gamma_v)$  are the Cesàro averages and  $\gamma_v(z) = v(v - 1)/(z(z - 1)(z - v))$ .

**THEOREM 1.2.** (Invariant Beltrami differentials) *Given a rational map  $R$ , the following statements hold true.*

- (1) *If  $P_R$  has zero Lebesgue measure then  $R$  satisfies Sullivan’s conjecture if and only if  $R^*$  is mean ergodic on  $\text{Hol}(R)$  with the topology inherited from  $L_1(J_R)$ .*
- (2) *Assume that  $P_R \neq \overline{\mathbb{C}}$  and that there are no rotational domains. If  $T$  is the Thurston operator for  $R$  then  $T : B_0(S_R) \rightarrow B_0(S_R)$  is mean ergodic. Moreover, let  $\alpha \in B(S_R)$  with  $T(\alpha) = \alpha$  and  $\|\alpha\|_T = 1$ , where  $\|\cdot\|_T$  is the Teichmüller norm. If*

$$\inf_{\phi \in B_0(S_R)} \|\alpha - \phi\|_T < 1$$

*then  $R$  is a flexible Lattès map.*

- (3) *Assume  $P_R \neq \mathbb{C}$ , the set  $P_R \cap F_R$  is finite and  $P_R$  does not admit finite invariant absolutely continuous complex-valued measures. Then  $R$  satisfies Sullivan’s conjecture if and only if the operator  $E_v$  is compact for every critical value  $v$ . Moreover, the operator  $(\text{Id} - T) : B(S_R) \rightarrow B(S_R)$  is compact if and only if  $R$  is postcritically finite.*

Part (1) follows from Theorem 4.6, which is given in a more general situation where  $P_R$  is allowed to have positive Lebesgue measure. Part (2) follows from Theorem 6.8 and Corollary 6.9. Part (3) follows from Theorem 8.4 and Corollary 8.5.

This paper also includes several results not mentioned in the theorems above which are of independent interest; for example, Proposition 4.15 compares the masses over the Julia and Fatou sets for  $\gamma \in \text{Hol}(R)$ . See also Theorems 6.10, 6.11, 7.1 and the discussions in the respective sections.

Most results in this paper are given in terms of ergodic theory and suggest that Sullivan’s conjecture holds true.

## 2. The pull-back and push-forward actions

In this section we give the definition of the Ruelle transfer operator for a rational map  $R$  which is a complex version of the Perron–Frobenius operator. The first instance of this operator probably appeared in Ruelle’s paper [22].

We take the standpoint of the theory of quasiconformal deformations, which deals with a dual interplay between Beltrami differentials and quadratic differentials (i.e. between some  $(-1, 1)$ -forms and  $(2, 0)$ -forms). In fact, there is a natural action on both spaces induced by a rational function  $R$  of degree  $d$ , if we think of  $R$  as a ‘local change of variables’.

Let  $F_{m,n}$  be the space of all  $(m, n)$ -forms  $\alpha(z) = \phi(z) dz^m d\bar{z}^n$ , where  $\phi$  is a complex-valued measurable function on  $\mathbb{C}$ . The pull-back operator acting on  $F_{m,n}$  is given by

$$R_{*(m,n)}(\alpha) = \alpha \circ R = \phi(R(z))R'(z)^m \overline{R'(z)}^n dz^m d\bar{z}^n.$$

The push-forward operator on  $F_{m,n}$  is given by

$$R_{*(m,n)}^*(\alpha) = \sum \alpha(\zeta_i),$$

where the sum is taken over all branches  $\zeta_i$  of  $R^{-1}$ . Therefore we have  $R_{*(m,n)}^* \circ R_{*(m,n)} = \text{deg}(R)\text{Id}$ .

The Beltrami operator is  $B_R = R_{*(-1,1)}$ . The operator  $|B_R| = R_{*(0,0)}$  is called the modulus of  $B_R$  and satisfies  $|B_R|\phi = |B_R(\phi)|$  almost everywhere for every measurable function  $\phi$ .

The Ruelle transfer operator (or the Ruelle operator for short) is  $R^* = R_{*(2,0)}$ , while the operator  $|R^*| = R_{*(1,1)}$  is called the modulus of the Ruelle operator and satisfies  $|R^*(\phi)| \leq |R^*|\phi$  almost everywhere for every measurable function  $\phi$ . The modulus of the Ruelle operator is also known as the Perron–Frobenius operator for the map  $R$  or as the push-forward operator on the space of absolutely continuous measures.

Using coefficients, the Beltrami operator and its modulus are defined by the formulas

$$B_R(\phi) = \phi(R) \frac{\overline{R'}}{R'} \quad \text{and} \quad |B_R|(\phi) = \phi(R),$$

respectively, where  $\phi$  is a measurable function with respect to the Lebesgue measure.

In turn, the Ruelle operator and its modulus are given by

$$R^*(\phi) = \sum \phi(\zeta_i)(\zeta'_i)^2 \quad \text{and} \quad |R^*|(\phi) = \sum \phi(\zeta_i)|\zeta'_i|^2,$$

respectively. Both sums are taken over all branches  $\zeta_i$  of  $R^{-1}$ . We end this section with the following simple facts.

PROPOSITION 2.1. *Let  $A$  be a Lebesgue measurable set,  $\mu \in L_\infty(\mathbb{C})$  and  $\phi \in L_1(\mathbb{C})$ . Then the following statements hold.*

(1)

$$\int_A \mu(z)R^*(\phi(z))|dz|^2 = \int_{R^{-1}(A)} B_R(\mu(z))\phi(z)|dz|^2.$$

(2)

$$\int_A \mu(z)|R^*(\phi(z))|dz|^2 = \int_{R^{-1}(A)} \mu(R(z))\phi(z)|dz|^2.$$

(3) *Define  $R_*(\phi) = R_{*(2,0)}(\phi)/\text{deg}(R) = \phi(R)R^2/\text{deg}(R)$ . Then*

$$\int_A |R_*(\phi(z))||dz|^2 \leq \int_{R(A)} |\phi(z)||dz|^2.$$

(4) *If  $\phi$  is a holomorphic function outside the postcritical set  $P_R$ , then  $R^*\phi$  is also a holomorphic function outside  $P_R$ .*

*Proof.* We begin with part (1). Fix a system of branches of  $R^{-1}$  in the following way. Let  $\tau$  be any differentiable arc containing all critical values of  $R$ . Take  $D = \overline{\mathbb{C}} \setminus \tau$ . Then by the monodromy theorem each of the branches  $\zeta_i$  of  $R^{-1}$  defines a holomorphic function on  $D$ . Set  $D_i = \zeta_i(D)$ . Then  $D_i \cap D_j = \emptyset$  for  $i \neq j$ , so we have

$$\begin{aligned} \int_A \mu(z)R^*(\phi(z))|dz|^2 &= \int_{A \cap D} \mu(z)R^*(\phi(z))|dz|^2 \\ &= \int_{A \cap D} \sum_i \mu(z)\phi(\zeta_i(z))(\zeta'_i(z))^2|dz|^2, \end{aligned}$$

which after a change of variables is equal to

$$\sum_i \int_{\zeta_i(A \cap D)} \mu(R(z)) \frac{\overline{R'}(z)}{R'(z)} \phi(z)|dz|^2 = \int_{R^{-1}(A)} B_R(\mu(z))\phi(z)|dz|^2.$$

Similar computations prove parts (2) and (3).

We now turn to part (4). Again by the monodromy theorem, the Ruelle operator does not depend on the local choice of branches of  $R^{-1}$ . Outside the postcritical set, every branch of  $R^{-1}$  is a local holomorphic function and  $R^{-1}(\overline{\mathbb{C}} \setminus P_R) \subset \overline{\mathbb{C}} \setminus P_R$ , therefore  $R^*(\phi)$  is sum of holomorphic functions, and so is holomorphic.  $\square$

Let  $A$  be an  $R$ -invariant set, that is,  $R(A) \subset A$ . Then  $R^*$  acts on  $L_1(A)$  with  $\|R^*\| \leq 1$ . Indeed, by extending every element in  $L_1(A)$  by zero on  $\mathbb{C} \setminus A$ , the space  $L_1(A)$  can be regarded as a closed subspace of  $L_1(\mathbb{C})$ . By definition, the support  $\text{supp}(R^*f)$  is contained in  $R(\text{supp}(f))$ . If  $A$  is completely  $R$ -invariant (i.e.  $R^{-1}(A) = A$ ), then  $B_R$  acts on  $L_\infty(A)$  with  $\|B_R\| = 1$  and  $B_R$  is dual to  $R^*$ . It is known that if  $f$  is a quasiconformal automorphism of  $\overline{\mathbb{C}}$  then  $R_f = f \circ R \circ f^{-1}$  is rational if and only if its Beltrami coefficient  $\mu_f(z) = \bar{\partial}f/\partial f(z)$  satisfies  $\mu_f(R(z))\overline{(R'(z)/R'(z))} = \mu_f(z)$  almost everywhere. Hence by the measurable Riemann mapping theorem the unit ball of the space of fixed points of  $B_R$  in  $L_\infty(\mathbb{C})$  determines all quasiconformal deformations of the rational map  $R$ .

When  $A$  is backward  $R$ -invariant (i.e.  $R^{-1}(A) \subset A$ ) and  $\phi$  is an integrable function on  $A$ , we have

$$\int_A |R^*(\phi)| \leq \int_A \|R^*\|\phi \leq \int_A |R^*|\phi = \int_{R^{-1}(A)} |\phi| \leq \int_A |\phi|.$$

Here, the integration is with respect to the Lebesgue measure. Therefore the Ruelle operator is a contraction in the  $L_1$  norm.

As another entry of Sullivan’s dictionary the reader may recognize the Ruelle operator as the relative Poincaré theta operator for branched coverings of the sphere onto itself.

### 3. The Thurston operator

Following ideas from Teichmüller theory, we consider the action of the Ruelle operator on spaces of functions that are holomorphic on a given open set.

Let  $K$  be a closed subset of  $\overline{\mathbb{C}}$  and let  $H(K)$  be the subspace of  $L_1(\overline{\mathbb{C}})$  of all functions holomorphic outside of  $K$  with the restricted norm of  $L_1(\overline{\mathbb{C}})$ . If  $K$  is  $R$ -invariant and contains the postcritical set  $P_R$  then the Ruelle operator  $R^*$  is a contractive endomorphism of  $H(K)$ . Define  $S_K = \mathbb{C} \setminus K$ , and let  $A(S_K)$  be the space of all integrable holomorphic functions on  $S_K$  equipped with the  $L_1$  norm, so  $A(S_K)$  is a Banach space. By the discussion on the previous section the Ruelle operator also acts as a contracting endomorphism of  $A(S_K)$ . Every element  $f$  in  $A(S_K)$  extends to an element in  $H(K)$ ; just set  $f(z) = 0$  for all  $z$  in  $K$ . This extension gives a canonical inclusion from  $A(S_K)$  into  $H(K)$ , which is an isomorphism precisely when the Lebesgue measure of  $K$  is zero.

Let  $B(S_K)$  denote the associated Bergman space, that is, the space of all holomorphic functions  $\phi$  on  $S_K$  with  $L_\infty$ -norm

$$\|\phi\| = \sup_{z \in S_K} |\lambda_K^{-2}(z)\phi(z)|,$$

where  $\lambda_K$  denotes the complete hyperbolic metric on  $S_K$ . By the Bers isomorphism theorem (see Theorem 2.1 in Ch. 3, p. 89 of [11]), the space  $B(S_K)$  is linearly isomorphic to  $A^*(S_K)$ , the dual of the Banach space  $A(S_K)$ , by the correspondence that associates

to every  $\phi \in B(S_K)$  the continuous functional  $l_\phi(\psi) = \int_{S_K} \lambda^{-2} \bar{\phi} \psi |dz|^2$  in  $A^*(S_K)$ . Furthermore, there is an equivalent norm on  $B(S_K)$ , called the *Teichmüller norm*, which is the canonical supremum norm of continuous linear functionals on the unit sphere in  $A(S_K)$ .

Let  $T : B(S_K) \rightarrow B(S_K)$  be the dual operator of  $R^*$  up to the identification above. Then  $T$  is a power-bounded operator which is a contraction in the Teichmüller norm. We call the operator  $T$  the *infinitesimal Thurston pull-back operator* (or *Thurston operator* for short). Indeed, as was shown by Douady and Hubbard in [5], when  $R$  is a postcritically finite rational map the operator  $T$  is the derivative of the Thurston pull-back map.

Let  $B_0(S_K)$  be the subspace of all elements in  $B(S_K)$  vanishing at infinity. In other words,  $B_0(S_K)$  is the space of all  $\phi$  in  $B(S_K)$  such that  $|\lambda_K^{-2} \phi(z_j)|$  converges to zero, whenever  $z_j$  is a sequence converging to the boundary  $\partial S_K$ .

Let  $A_*(S_K)$  be the subspace of  $A^*(S_K)$  such that the dual space  $(A_*(S_K))^*$  is isometrically isomorphic to  $A(S_K)$  (see, for instance, Theorem 5 on p. 52 of [9]). The space  $A_*(S_K)$  is constructed as follows. A sequence  $\{\phi_j\}$  in  $A(S_K)$  is *degenerating* if  $\|\phi_j\| = 1$  for all  $j$  and  $\phi_j$  converges pointwise to zero on  $S_K$ . Then  $A_*(S_K)$  is the kernel of the seminorm on  $A^*(S_K)$  given by

$$\beta(l) = \sup_i (\limsup |\phi_i|),$$

where the supremum is taken over all degenerating sequences  $\{\phi_i\}$  in  $A(S_K)$ .

Moreover, the  $*$ -weak topology on  $A(S_K)$  induced by  $A_*(S_K)$  coincides with the topology of pointwise convergence of bounded sequences.

The Bers isomorphism theorem, together with [2, Theorem 1], provides a correspondence between the topologies and the Banach structure of the spaces we are dealing with as follows:

$$\begin{aligned} B(S_K) &\simeq B_0(S_K)^{**} \xleftarrow{f^{**}} A^*(S_K) \xrightarrow{g^{**}} \ell_\infty = c_0^{**}, \\ B_0(S_K)^* &\xleftarrow{f^*} A(S_K) \xleftarrow{g^*} \ell_1, \\ B_0(S_K) &\xleftarrow{f} A_*(S_K) \xrightarrow{g} c_0. \end{aligned}$$

In the notation above,  $f$  is the restriction map of the Bers isomorphism to  $A_*(S_K)$  and, by Lemma 1 and Corollary 1 of pp. 259–260 in [9], it is a surjective map onto  $B_0(S_K)$ . There is an isomorphism  $h$  to its image from  $B_0(S_K)$  into  $c_0$  given by [2, Theorem 1]. The map  $g$  is just the composition  $h \circ f$ . Here  $\ell_\infty$ ,  $\ell_1$  and  $c_0$  denote the spaces of complex-valued sequences that are bounded, absolutely summable and converging to zero, respectively. In this work, we often identify the space  $A(S_K)$  with  $B_0^*(S_K)$  using Bers' isomorphism, so the action  $R^* : B_0^*(S_K) \rightarrow B_0^*(S_K)$  is well defined.

Now we gather together some facts about the geometry and dynamics of operators on  $B(S_K)$  and  $B_0(S_K)$ . First, we need some definitions. A Banach space  $B$  is a *Grothendieck space* if every  $*$ -weak convergent sequence of continuous functionals  $\{l_i\}$  also converges in the weak topology on the Banach space  $B^*$  dual to  $B$ .

Every complemented closed subspace of a Grothendieck space is again a Grothendieck space. Clearly, every reflexive space is a Grothendieck space.

A Banach space  $B$  has the *Dunford–Pettis property* if every weakly compact operator from  $B$  into any Banach space maps weakly compact sets into norm compact sets. As in the case of Grothendieck spaces, the Dunford–Pettis property is also inherited on complemented closed subspaces. A typical example of a Grothendieck space with the Dunford–Pettis property is  $L_\infty(X, \mu)$ , where  $(X, \mu)$  is a positive measure space (see [13]).

A series  $\sum x_n$  in a Banach space  $X$  is called *weakly unconditionally Cauchy (wuC)* if for every  $l \in X^*$  the series  $\sum |l(x_n)|$  is bounded. A Banach space  $X$  is said to have *Pełczyński's property (V)* if every subset  $K \subset X^*$  is relatively weakly compact whenever  $K$  satisfies

$$\limsup_n \sup_{l \in K} |l(x_n)| = 0$$

for every wuC series  $\sum x_n$  in  $X$ .

By results of functional analysis (see, for example, Corollary 3.7, p. 132 in [10]) any closed subspace  $Y \subset c_0$  has Pełczyński's property (V).

Hence we have the following:

- $B_0(S_K)$  has Pełczyński's property (V);
- $B(S_K)$  is a Grothendieck space with the Dunford–Pettis property.

Indeed, by results of Bonnet and Wolf in [2], the space  $B_0(S_K)$  is isomorphic to a closed subspace of  $c_0$ .

By the Bers isomorphism theorem, we have  $L_\infty(S_K) \cong N \oplus B(S_K)$ . So  $B(S_K)$  is a complemented subspace of a Grothendieck space with the Dunford–Pettis property. The space  $N = A(S_K)^\perp \subset L_\infty(S_K)$  is the annihilator of  $A(S_K)$ . In other words,  $N$  consists of the trivial Beltrami differentials.

Again, by a combination of classical results in functional analysis we have the following fact.

- If  $E : B_0(S_K) \rightarrow B_0(S_K)$  is a linear operator then either  $E$  is compact or there exists an infinite-dimensional subspace  $Y$ , which is isomorphic to  $c_0$  such that  $E|_Y$  is an isomorphism onto its image.

Indeed, as noted above  $B_0(S_K)$  has Pełczyński's property (V), then by Lemma 3.3.A and Theorem 3.3.B on p. 128 of [10], either  $E$  is weakly compact or there exists an infinite-dimensional subspace  $Y$ , which is isomorphic to  $c_0$  such that  $E|_Y$  is an isomorphism onto its image. However, if  $E$  is weakly compact then  $E^* : B_0^*(S_K) \rightarrow B_0^*(S_K)$  is also weakly compact. Let us show that every bounded weakly convergent sequence  $\{\psi_n\} \subset A(S_K)$  contains a norm-convergent subsequence. In fact, every bounded sequence in  $A(S_K)$  forms a normal family. Let  $\{\psi_{n_k}\}$  be a subsequence of  $\{\psi_n\}$  converging to its pointwise limit  $\psi$ . Hence,  $\psi_{n_k}$  is weakly convergent and, by the Fatou lemma,  $\psi \in A(S_K)$ . Then, by the uniform integrable convergence in measure theorem (see [23, Theorem 1.5.13]),  $\psi_{n_k}$  converges to  $\psi$  in norm. Therefore  $E^*$ , and hence also  $E$ , are compact operators.

We say that an invariant Beltrami differential  $\mu \in L_\infty(\bar{\mathbb{C}})$  defines a *non-trivial quasiconformal deformation* if and only if  $l_\mu(\psi) = \int_{S_K} \psi(z)\mu(z)|dz|^2$  is a non-zero functional on  $A(S_K)$ ; note that  $l_\mu$  is  $R^*$ -invariant, that is,  $l_\mu(R^*(\psi)) = l_\mu(\psi)$  for all  $\psi \in A(S_K)$ . In other words,  $l_\mu$  induces a non-zero fixed point for  $T$  on  $B(S_K)$ .

Finally, in order to use results given by the second author in [17] without cumbersome recalculations, throughout this paper we will assume that  $\{0, 1, \infty\}$  are fixed points of  $R$ .



This is always the case after passing to a suitable iteration of  $R$  and conjugating with a Möbius map. However, most of our results do not need this normalization. We fix  $K_R = P_R \cup \{0, 1\}$  and set  $S_R = \mathbb{C} \setminus K_R$  and always assume that  $S_R$  is non-empty.

4. Mean ergodicity in holomorphic dynamics

Given an operator  $S$  on a Banach space  $X$ , the  $n$ -Cesàro averages of  $S$  are the operators  $A_n(S)$  defined for  $x \in X$  by

$$A_n(S)(x) = \frac{1}{n} \sum_{i=0}^{n-1} S^i(x).$$

An operator  $S$  on a Banach space  $X$  is called *mean ergodic* if  $S$  is power-bounded, that is, it satisfies  $\|S^n\| \leq M$  for some number  $M$  independent of  $n$ , and the Cesàro averages  $A_n(S)(x)$  converge in norm for every  $x \in X$ .

The topology of convergence in norm is also called the strong topology on  $X$ . If  $A_n(S)$  converges uniformly on the closed unit ball on  $X$  then the operator  $S$  is called *uniformly ergodic*. The following facts can be found, for example, in Krengel’s book [12].

- (1) (*Separation principle*). The operator  $S$  is mean ergodic if and only if  $S$  satisfies the principle of separation of points: if  $x^*$  is a fixed point of  $S^*$  then there exists  $y \in X$  a fixed point of  $S$  such that  $\langle y, x^* \rangle \neq 0$ .
- (2) For a power-bounded operator  $S$ , the equality  $\lim_{n \rightarrow \infty} A_n(S)(x) = 0$  holds if and only if  $x \in \overline{(\text{Id} - S)(X)}$ .
- (3) (*Mean ergodicity lemma*). For a power-bounded operator  $S$ , consider the convex hull  $\text{Conv}(S, x)$  of the orbit of a point  $x$  under  $S$ . Then  $y$  is a weak accumulation point of  $\text{Conv}(S, x)$  if and only if  $y$  is a fixed point of  $S$ . In this situation  $A_n(S)(x)$  converges to  $y$  in norm.  
If  $X$  is a dual space, then  $y$  is a  $*$ -weak accumulation point of  $\text{Con}(S, x)$  if and only if  $y$  is a fixed point of  $S$ . Here a dual space is a space isometrically isomorphic to  $B^*$ , for some Banach space  $B$  where the notion of  $*$ -weak topology is well defined.
- (4) (*Uniform ergodicity lemma*). A power-bounded operator  $S$  acting on a Banach space  $B$  is uniformly ergodic if and only if the subspace  $(\text{Id} - S)B$  is closed or, equivalently, if and only if the point 1 either belongs to the resolvent set or is an isolated eigenvalue of  $S$ . If  $\dim(\text{Fix}(S))$  is finite then the intersection of the spectrum of  $S$  with the unit circle consists of finitely many isolated eigenvalues.

Let  $\text{Hol}(R)$  be the space of all integrable (with respect to the planar Lebesgue measure) rational functions having poles in the forward orbit of the set  $V(R) \cup \{0, 1\}$ , where  $V(R)$  is the set of all critical values of  $R$ . Equivalently,  $\text{Hol}(R)$  consists of all rational functions with simple poles in the forward orbit of  $V(R) \cup \{0, 1\}$  and a zero at infinity of multiplicity at least three. Note that  $\text{Hol}(R)$  is a normed vector space with the norm inherited from  $L_1(\overline{\mathbb{C}})$ . The space  $\text{Hol}(R)$  is not complete and, by the Bers approximation theorem (see [9, Theorem 9]), its completion is  $H(K_R)$  and contains a canonical inclusion of  $A(S_R)$ . Moreover, the completion of  $\text{Hol}(R)$  is equal to  $A(S_R)$  if and only if the Lebesgue measure of  $P_R$  is zero.

We recall some facts from ergodic theory which will be used in this work. A positive measure set  $M \subset \overline{\mathbb{C}}$  is called *wandering* when the sets  $\{R^{-k}(M)\}_k$  are pairwise almost

disjoint Lebesgue measurable sets, that is,  $R^{-i}(M) \cap R^{-j}(M)$  has Lebesgue measure zero whenever  $i \neq j$ . Let  $D(R)$  be the union of all wandering sets. The set  $D(R)$  is called the *dissipative set* and the complement  $C(R) = \mathbb{C} \setminus D(R)$  is the *conservative set*. Similarly, a positive measure set  $M \subset \overline{\mathbb{C}}$  is called *weakly wandering* when there is a sequence  $0 = k_0 < k_1 < k_2 < \dots$  such that the sets  $R^{-k_i}(M)$  are pairwise almost disjoint Lebesgue measurable sets. The *weakly dissipative set*  $W(R)$  is the union of all weakly wandering sets and  $SC(R) = \mathbb{C} \setminus W(R)$  is called the *strongly conservative set*.

The following proposition is a consequence of Theorems 4.6 and 4.11, pp. 141 and 144 in [12], respectively.

**PROPOSITION 4.1.** *Let  $R$  be a rational map. Then the Lebesgue measure of the symmetric difference  $R(SC(R)) \Delta SC(R)$  is zero. Furthermore, if  $SC(R)$  has positive measure there exists a non-negative integrable function  $P$  which is positive on  $SC(R)$  and such that  $P(z)|dz|^2$  defines an invariant probability measure supported on  $SC(R)$ . Moreover, if for a given non-negative measurable function  $\phi$  we have that  $\phi(z)|dz|^2$  defines an invariant probability measure, then  $\text{supp}(\phi)$  is contained in  $SC(R)$ .*

*Proof.* Since the weakly dissipative set is backward  $R$ -invariant, we have  $R(SC(R)) = SC(R)$  almost everywhere. Hence the restriction  $R : SC(R) \rightarrow SC(R)$  is a null-preserving transformation. If  $SC(R)$  has positive measure, then, by Theorem 4.11, p. 144 in [12], there exists a finite  $R$ -invariant measure  $\nu$  on  $SC(R)$  which is equivalent to the Lebesgue measure. Let  $P$  be the Radon–Nikodym derivative of  $\nu$  with respect to the Lebesgue measure, so  $P(z) > 0$  almost everywhere on  $SC(R)$ . Also every finite invariant measure absolutely continuous with respect to Lebesgue is zero on the weakly dissipative set. By Theorem 4.6, p. 141 in [12], the decomposition  $\mathbb{C} = SC(R) \cup W(R)$  is unique up to measure. If  $\phi$  is a non-negative function so that  $\phi|dz|^2$  defines a finite invariant measure then  $\text{supp}(\phi)$  is invariant and, again by [12, Theorem 4.11], we have  $\text{supp}(\phi) \subset SC(R)$ .  $\square$

Due to the classification of Fatou components and Proposition 4.1, it follows that both the conservative set  $C(R)$  and the strongly conservative set  $SC(R)$  intersect the Fatou set  $F_R$  precisely at the union of all rotation domain cycles. Indeed, if a periodic Fatou component  $D$  is not a rotation domain then  $D$  consists of a union of wandering sets.

Now assume  $D$  is a rotation domain. Let  $\tau_\theta$  be an irrational rotation on the unit disk  $\Delta$ . Then for any annulus  $T_r = \{r \leq |z| \leq 1\}$ , the function given by  $P(z) = 1/|z|^2 \text{mod}(T_r)$  for  $z \in T_r$  and zero for  $z$  in  $\Delta \setminus T_r$  defines a  $\tau_\theta$ -invariant probability measure. By using a parametrization of  $D$  and Proposition 4.1, we have that rotation domain cycles are conservative and strongly conservative. Moreover, by Theorem 1.6, p. 117 in [12], we have the following fact.

**LEMMA 4.2.** (Almost everywhere convergence on the dissipative set) *For every  $f \geq 0$  in  $L_1(\overline{\mathbb{C}})$ , the series*

$$\sum_{n=0}^{\infty} |R^{*n}(f)|$$

*converges and is finite almost everywhere on the dissipative set.*

*Proof.* It is enough to show that the series

$$\sum_{n=0}^{\infty} |R^*|^n(f)$$

converges on every wandering set  $W$  of finite positive measure. In fact, we show that the series defines an integrable function. Since  $W$  is wandering, we have, by Proposition 2.1,

$$\int_W \sum_{n=0}^{\infty} |R^*|^n(f) \leq \sum_{n=0}^{\infty} \int_{R^{-n}(W)} f = \int_{\bigcup R^{-n}(W)} f < \|f\|_1. \quad \square$$

Using Proposition 4.1, we reformulate results of Lyubich and McMullen to obtain the following lemma.

LEMMA 4.3. *Let  $R$  be a rational map.*

- *Either  $C(R) \cap J_R \subset P_R$  or  $C(R) = \bar{C}$ .*
- *Either  $SC(R) \cap J_R \subset P_R$  or  $SC(R) = \bar{C}$ .*
- *In the case where  $C(R) = \bar{C}$  but  $P_R \neq \bar{C}$ , there exists a fixed point of the Beltrami operator supported on the Julia set if and only if  $R$  is a flexible Lattès map.*

*Proof.* We follow ideas and arguments of Lyubich and McMullen (see [15] and [18, §3.3]). First, by the Poincaré recurrence theorem for conservative actions (see [1]), we have

$$\liminf_{n \rightarrow \infty} d(x, R^n(x)) = 0$$

for almost every  $x \in C(R)$ , where  $d$  is the spherical metric. Assume that the set  $B = (C(R) \cap J_R) \setminus P_R$  has positive Lebesgue measure. Then  $R(B) \subset B$ , otherwise if  $B$  is not invariant then the set  $X = B \cap R^{-1}(P_R)$  has positive Lebesgue measure. Then  $R^n(X) \cap X$  has zero Lebesgue measure for  $n \neq 1$  since  $R^n(X) \subset P_R$  for all  $n \geq 1$ . This implies that the sets  $\{R^{-k}(X)\}$  are pairwise almost disjoint, thus  $X$  is wandering, which contradicts that  $B \subset C(R)$ . Then again by the Poincaré recurrent theorem we have

$$\limsup_{n \rightarrow \infty} d(R^n(x), P_R) \geq d(x, P_R) > 0$$

for almost every point in  $B$ . Now using Koebe distortion arguments as in [14, §1.19] or [18, §3.3], we obtain that the closure of any invariant positive Lebesgue measure subset  $E$  in  $B$  contains a disk and thus  $\bar{E} = \bar{B} = \bar{C}$ .

In particular, this holds for  $E = (SC(R) \cap J_R) \setminus P_R \subset B$  whenever  $E$  has positive measure since, by Proposition 4.1, the set  $E$  is invariant.

For the third part, by the hypothesis and Poincaré’s theorem for almost every point  $x \in C(R) \setminus P_R$  we have that  $\limsup d(R^n(x), P_R) > 0$ . Hence, if  $J_R$  supports a non-zero invariant Beltrami differential then Theorem 3.17 in McMullen’s book [18] finishes the proof. □

We also use the following proposition.

PROPOSITION 4.4. *Let  $R$  be a rational map and let  $f \in L_1(\bar{C})$  be a fixed point of the Ruelle operator  $R^*$ . Then there exists a fixed point of the Beltrami operator  $\mu \in L_\infty(\bar{C})$*

such that  $\int_{\mathbb{C}} f(z)\mu(z)|dz|^2 \neq 0$ . In fact  $|R^*||f| = |f|$  and  $|f|$  defines an absolutely continuous finite invariant measure that satisfies  $|f|/f = \mu$  almost everywhere on the support of  $f$ .

*Proof.* This summarizes the results given in Lemma 11 and Corollary 12 in [17].  $\square$

Note that Lemma 4.3 and Proposition 4.4 give necessary conditions for Sullivan's and Fatou's conjectures.

The following lemma is a consequence of part 3 of Lemma 5 and part 1 of Theorem 3 in [17]. For  $v$  in  $\mathbb{C}$ , define the function  $\gamma_v$  by

$$\gamma_v(z) = \frac{v(v-1)}{z(z-1)(z-v)}.$$

Throughout this paper, the Cesàro averages  $A_n(\gamma_v)$  of functions of the form  $\gamma_v$  will be always taken with respect to the Ruelle operator  $R^*$ .

LEMMA 4.5. *Let  $R$  be a rational map.*

- (1) *If the Fatou set  $F_R$  contains a periodic attracting domain  $V$  then there exist an invariant Beltrami differential  $\mu$  supported on the grand orbit of  $V$  and a critical value  $v_0$  such that  $\int_{\mathbb{C}} \mu(z)\gamma_{v_0}(z)|dz|^2 \neq 0$ .*
- (2) *If  $\mu$  is an invariant Beltrami differential in  $L_{\infty}(J_R)$  then  $\mu \neq 0$  if and only if there exists a critical value  $v_0$  such that  $\int_{\mathbb{C}} \mu(z)\gamma_{v_0}(z)|dz|^2 \neq 0$ .*

The following theorem gives a connection between Sullivan's conjecture and mean ergodicity on a suitable subspace of  $L_1(\mathbb{C})$ .

THEOREM 4.6. *Let  $R$  be a rational map such that  $SC(R) \cap P_R$  has Lebesgue measure zero. Then  $R$  satisfies Sullivan's conjecture if and only if  $R^*$  is mean ergodic on  $\text{Hol}(R)$  with the topology inherited from  $L_1(J_R)$ .*

*Proof.* Assume that  $R$  satisfies Sullivan's conjecture. Then either there is no invariant Beltrami differential supported on the Julia set or  $R$  is a flexible Lattès map. If there is no invariant Beltrami differential supported on  $J_R$  then the Beltrami operator on  $L_{\infty}(J_R)$  does not have fixed points. Hence  $(\text{Id} - R^*)(L_1(J_R))$  is dense in  $L_1(J_R)$ . Otherwise, by the Hahn–Banach theorem, there would exist a non-zero continuous functional  $\mathcal{L}$  on  $L_1(J_R)$  such that  $(\text{Id} - R^*)(L_1(J_R)) \subset \ker(\mathcal{L})$ . By the Riesz representation theorem there exists  $\alpha \in L_1(J_R)$  which is fixed by the Beltrami operator and representing the functional  $\mathcal{L}$ , which is a contradiction. Then  $A_n(R^*)(f)$  converges to zero for all  $f$  in  $L_1(J_R)$ . In particular, this happens when  $f \in \text{Hol}(R)$ . Thus  $R^*$  is mean ergodic.

If  $R$  is a flexible Lattès map then, since  $R$  is postcritically finite, the space  $A(S_R)$  is finite-dimensional. So  $A(S_R)$  coincides with the subspace  $\text{Hol}(R)$  and, by the mean ergodicity lemma, the Ruelle operator  $R^*$  is mean ergodic. Note that for this part of the proof we do not need that  $SC(R) \cap P_R$  has Lebesgue measure zero.

Conversely, assume that  $R^*$  is mean ergodic in  $\text{Hol}(R)$ . To finish the theorem it is sufficient to show that if the Julia set  $J_R$  supports an invariant Beltrami differential then  $R$  is a flexible Lattès map. Indeed, by Lemma 4.5, every fixed point of the Beltrami operator supported on the Julia set defines a continuous invariant functional  $\mathcal{L}$  on  $L_1(J_R)$  which

is non-zero on  $\text{Hol}(R)$ . Let  $\phi$  in  $\text{Hol}(R)$  be such that  $\mathcal{L}(\phi) \neq 0$ . By mean ergodicity  $A_n(R^*)(\phi)$  converges to some non-zero element  $f$  in  $L_1(J_R)$ . By Proposition 4.4,  $|f|$  defines an absolutely continuous finite invariant measure, hence  $\text{supp}(f) \subset SC(R)$ . Since  $SC(R) \cap P_R$  has measure zero, it follows that by Lemma 4.3 the map  $R$  is a flexible Lattès map. □

Let us note that Theorem 4.6 holds even when  $S_R = \emptyset$ . Moreover, when  $P_R \neq J_R$ , we can consider the space  $A(S_R)$  instead of  $\text{Hol}(R)$  and get the same conclusion as in the previous theorem. As mentioned in the introduction, conjecturally, for a map  $R$  with non-empty Fatou set, the condition on the postcritical set of the previous theorem is always fulfilled. On the other hand, the mean ergodicity of Ruelle operator is a rather simple consequence of geometric conditions of  $S_R$ . For example, the Cesàro averages  $A_n(\gamma_V)$  are weakly convergent on measurable subsets  $Y$  of  $S_R$  of finite hyperbolic area (see [4] and discussion therein). Hence, the existence of fixed points of the Ruelle operator is the main obstruction to extending Theorem 4.6 in full generality.

In general, as the following corollary shows, the Ruelle operator for rational maps is not mean ergodic on  $A(S_R)$  or  $L_1(\mathbb{C})$ .

**COROLLARY 4.7.** *Let  $R$  be a rational map.*

- (1) *If  $F_R$  contains an attracting periodic component then  $R^* : A(S_R) \rightarrow A(S_R)$  is not mean ergodic.*
- (2) *If  $F_R$  contains a periodic non-rotational component then  $R^* : L_1(\mathbb{C}) \rightarrow L_1(\mathbb{C})$  is not mean ergodic.*

*Proof.* Part (1) is proved by contradiction. Let  $V$  be an attracting domain of  $F_R$  and assume that  $R^*$  is mean ergodic on  $A(S_R)$ . Then by part (1) of Lemma 4.5, there exists a Beltrami differential  $\mu$ , supported on the grand orbit of  $V$ , such that the functional  $l_\mu(\phi) = \int_{\mathbb{C}} \mu(z)\phi(z)|dz|^2 \neq 0$  on  $A(S_R)$  and  $l_\mu(R^*\phi) = l_\mu(\phi)$ . By the mean ergodicity lemma, there exists  $\psi \in A(S_R)$  with  $R^*(\psi) = \psi$  such that  $l_\mu(\psi) \neq 0$ . This implies that the restriction  $f$  of  $\psi$  on the grand orbit of  $V$  is non-zero and it is a fixed point of  $R^*$ . By Proposition 4.4, the function  $|f|$  defines a finite invariant measure. However, the grand orbit of  $V$  belongs to the dissipative set. Hence, using arguments similar to the previous theorem, we can show that  $f$  is zero almost everywhere on the grand orbit of  $V$ . This is a contradiction.

We now turn to part (2). Under this hypothesis, it is enough to show that the grand orbit of any non-rotational periodic domain supports a non-zero invariant Beltrami differential. We conclude this part using the classification of Fatou components and arguments similar to those of part (1). In part (1) we already considered the attracting case, so by the classification of Fatou components, we have to consider the cases where there is a Fatou component  $V$  which is either parabolic or superattracting. In the parabolic case, let  $\Phi$  be a linearization function defined on the grand orbit of  $V$ , that is, a function satisfying  $\Phi(R(z)) = \Phi(z) + 1$ . Then a short computation shows that the function  $\mu = \overline{\Phi}'/\Phi'$  gives a non-zero invariant Beltrami differential. In the superattracting case, let  $\Phi$  now be the Böttcher coordinate around a neighbourhood  $U$  of the superattracting cycle. Let  $\nu(z) = \overline{\Phi}'/\overline{\Phi}\Phi'$ . Then for  $z \in U$  we have  $\nu(R(z))(\overline{R}'(z)/R'(z)) = \nu(z)$ . Using the dynamics of

$R$ , we can extend to a non-zero invariant Beltrami differential defined on the grand orbit of  $U$ . □

*Remark.* By the corollary above, in order to deal with mean ergodicity of the Ruelle operator on  $A(S_R)$ , we will often assume that  $R$  does not accept invariant Beltrami differentials defining non-trivial quasiconformal deformations supported on the Fatou set. According to [19, Theorems 6.2 and 6.8], this is the same as saying that there are no Herman rings and if  $c$  is a critical point in  $F_R$ , then either

- (1)  $\mathcal{O}_+(c) = \bigcup_n R^n(c)$  is finite, or
- (2)  $\mathcal{O}_+(c)$  accumulates to a parabolic point  $p$  and there is no other critical point  $\tilde{c} \in F_R$  with forward orbit  $\mathcal{O}_+(\tilde{c})$  accumulating to  $p$  and  $\mathcal{O}_+(\tilde{c}) \cap \mathcal{O}_+(c) = \emptyset$ .

In other words, any non-zero invariant functional on the space  $A(S_R)$  is induced by an  $R^*$ -invariant Beltrami differential supported on the Julia set. Indeed we have the following equivalent statement.

**PROPOSITION 4.8.** *Any non-zero invariant functional on the space  $A(S_R)$  is induced by an  $R^*$ -invariant Beltrami differential supported on the Julia set if and only if the Cesàro averages  $A_n(f)$  converge to zero with respect to the  $L_1$  norm over the Fatou set for every  $f \in A(S_R)$ .*

*Proof.* Let  $A(F_R)$  be the space of holomorphic Lebesgue integrable functions on  $F_R$ . Then  $R^*$  acts on  $A(F_R)$  with  $\|R^*\| \leq 1$ .

We prove the proposition by contradiction. If there exists  $f \in A(S_R)$  such that  $\int_{F_R} |A_n(f)|$  does not converge to zero, then by the separation principle on the space  $A(F_R)$  there exists a non-zero  $R^*$ -invariant continuous functional  $l$  with  $l(f|_{F_R}) \neq 0$ .

On the other hand, by the Hahn–Banach extension and Riesz representation theorems there exists a function  $v \in L_\infty(F_R)$  representing  $l$  on  $A(F_R)$ , that is,  $l(g) = \int_{F_R} gv|dz|^2$  for every  $g \in A(F_R)$ . We claim that we can choose  $v$  to be an invariant Beltrami differential supported on  $F_R$ . Indeed, if  $v$  is not invariant then take any  $*$ -weak accumulation point of the sequence  $v_n = (1/n) \sum_{i=0}^{n-1} B^i(v)$  where  $B$  is the Beltrami operator. Then by the mean ergodicity lemma  $v_\infty \in L_\infty(F_R)$  is an invariant Beltrami differential supported on the Fatou set. Moreover,  $v_\infty$  defines the same functional on  $A(F_R)$  as  $v$ .

Indeed, consider a subsequence  $n_i$  such that  $v_{n_i} \rightarrow v_\infty$  in the  $*$ -weak topology on  $L_\infty(F_R)$  so we have

$$\begin{aligned} \int_{F_R} g(z)v_\infty(z)|dz|^2 &= \lim_{i \rightarrow \infty} \int_{F_R} v_{n_i}(z)g(z)|dz|^2 \\ &= \lim_{i \rightarrow \infty} \int_{F_R} v(z)A_{n_i}(g(z))|dz|^2 \\ &= \lim_{i \rightarrow \infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} l(R^{*j}(g)) = l(g). \end{aligned}$$

Hence  $L(g) = l(g|_{F_R})$  for  $g \in A(S_R)$  gives a non-zero continuous  $R^*$ -invariant functional on  $A(S_R)$  induced by an invariant Beltrami differential supported on  $F_R$  as claimed.

To get a contradiction we need to show that there is no Beltrami differential  $\mu$  supported on  $J_R$  such that

$$L(g) = \int_{\mathbb{C}} \mu(z)g(z)|dz|^2 = \int_{\mathbb{C}} \nu(z)g(z)|dz|^2$$

for any  $g \in A(S_R)$ .

Assume that there is such a  $\mu$ . Let  $\phi(a) = \int_{\mathbb{C}} \gamma_a(z)(\nu(z) - \mu(z))|dz|^2$  be the potential function for the invariant differential  $\nu - \mu$ . Then  $\phi$  is a continuous function on  $\mathbb{C}$ . Since  $\gamma_a \in A(S_R)$  for  $a \in P_R$ , we have  $\phi(a) = 0$ . Now we follow the proof of Theorem 3 in [17]. By invariance we have  $\phi(R(a)) = R'(a)\phi(a)$ , which implies that  $\phi(a) = 0$  for every repelling periodic point  $a$ , and hence on  $J_R$ . Therefore  $\bar{\partial}\phi = \nu - \mu = 0$  almost everywhere in  $J_R$ , which is a contradiction.  $\square$

Later on, we will discuss the relation between the topologies on  $\text{Hol}(R)$  induced by  $L_1$  norms over the Fatou and the Julia set, respectively.

Next we give some conditions under which the Ruelle operator does not have a fixed point. We call an integrable function  $f$  *regular* if the derivative  $\bar{\partial}f$ , taken in the sense of distributions, is a finite complex-valued measure. Examples of non-regular functions are given by characteristic functions of suitable compact sets. See the remark after the proof of the following theorem.

**THEOREM 4.9.** *Let  $R$  be a rational map. Assume that the postcritical set  $P_R$  is such that either*

- *the diameter with respect to the spherical metric of all components  $D$  of  $S_R$  is uniformly bounded away from zero, or*
- *$J_R \cap P_R$  is contained in the union of the boundaries of the components of  $S_R$ .*

*Then  $R^*$  has a regular non-zero fixed point if and only if  $R$  is a flexible Lattès map.*

*Proof.* Assume that  $R$  is a flexible Lattès map. Then  $R$  has an invariant Beltrami differential  $\mu$  unique up to multiplication by scalars (see Milnor [20]). This differential  $\mu$  defines a non-zero functional  $l_\mu$  on  $A(S_R)$  given by the pairing

$$l_\mu(\phi) = \int_{\mathbb{C}} \phi(z)\mu(z)|dz|^2.$$

Since  $A(S_R)$  is finite-dimensional,  $R^*$  is mean ergodic on  $A(S_R)$ , and by the separation principle there exists a non-zero fixed point  $f \in A(S_R)$  of the Ruelle operator, which is unique up to multiplication by scalars. Since  $f$  is an integrable holomorphic function outside finitely many points of  $\mathbb{C}$ , the function  $f$  is rational with simple poles only. Hence the distributional derivative  $\bar{\partial}f$  is a finite combination of Dirac measures supported on the poles of  $f$ .

The Beltrami differential  $\mu$  is a unique fixed point of the Beltrami operator. Since the Beltrami operator  $B : L_1(\mathbb{C}) \rightarrow L_1(\mathbb{C})$  is dual to  $R^*$ , it follows by the separation principle that the operator  $R^*$  is mean ergodic on  $L_1(\mathbb{C})$ . Hence  $f \in L_1(\mathbb{C})$  is a unique fixed point of the Ruelle operator up to scalar multiplication.

Now let  $f$  be a non-zero regular fixed point of the Ruelle operator. Then by Proposition 4.4 the function  $|f|$  is the density of a finite invariant measure and there is an invariant Beltrami differential  $\mu$  with  $\mu = |f|/f$  on  $\text{supp}(f)$ . Hence  $\text{supp}(f) \subset SC(R)$

by Proposition 4.1. Then by Lemma 4.3 either  $R$  is a flexible Lattès map or the support  $\text{supp}(f)$  is contained in the postcritical set.

Without loss of generality we may assume that  $P_R$  is a proper subset of  $\mathbb{C}$ . Let  $\nu$  be such that  $d\nu = \bar{\partial}f$  and set

$$F(z) = \int_{\mathbb{C}} \frac{d\nu(t)}{t - z}.$$

Since the support of  $\nu$  belongs to the support of  $f$ , the map  $F(z)$  is holomorphic outside  $P_R$ . Note that  $F(z) = f(z)$  holds for Lebesgue almost every point. Indeed, by Weyl’s lemma there exists an entire function  $h(z)$  such that  $h(z) = F(z) - f(z)$  almost everywhere. Since  $f(z)$  has compact support,  $F(z)$  converges to zero as  $z$  converges to  $\infty$ , thus  $h(z) = 0$ . In particular,  $F(z)$  is identically zero outside the support of  $\nu$ .

We claim that the first condition of Theorem 4.9 implies that  $\nu$  is identically zero on  $\mathbb{C}$ . Recall that the generalized Mergelyan theorem (see, for example, Theorem 10.4 in Gamelin’s paper [8]) states: *if the diameters of all components of the complement of a compact set  $K$  on the plane  $\mathbb{C}$  are bounded away from zero then any continuous function which is holomorphic on the interior of  $K$  is a uniform limit of rational functions with poles outside of  $K$* . Since  $P_R$  satisfies the conditions of the generalized Mergelyan theorem and has empty interior, any continuous function on  $P_R$  is a uniform limit of rational functions with poles outside  $P_R$ . Given a finite set of complex numbers  $b_i$  and points  $a_i$  in  $\mathbb{C} \setminus P_R$ , define  $r(z) = \sum (b_i/(z - a_i))$  and we get

$$\int r d\nu = \sum b_i F(a_i) = 0.$$

Hence  $\nu$  represents a zero functional on the space of continuous functions on  $P_R$ . By the Riesz representation theorem the measure  $\nu$  is null as claimed.

Thus  $f(z) = 0$  almost everywhere, which is a contradiction.

We now assume that  $P_R$  satisfies the second condition and follow closely the arguments of part 3 of Proposition 14 in [17]. Let  $\{Y_i\}$  be the family of all components of  $\mathbb{C} \setminus P_R$ . Then we claim that  $f(z) = 0$  almost everywhere on  $\bigcup_i \partial Y_i \subset P_R$  whenever  $F(z) = f(z)$  almost everywhere.

Otherwise, there exist a component  $Y_{i_0}$  and  $E \subset \partial Y_{i_0}$  with  $m(E) > 0$  and  $\int_E f \neq 0$ . Since  $P_R$  is compact we can assume that  $\infty$  belongs to  $Y_{i_0}$  by conjugating by a Möbius map. Then the function  $F_E(z) = \int_E (|dt|^2/(t - z))$  is a continuous function on the plane which is holomorphic outside  $\partial Y_{i_0}$ . Again, by the generalized Mergelyan theorem  $F_E$  is a uniform limit of rational functions with poles in  $Y_{i_0}$ . Hence, using similar arguments to those above, we obtain  $\int F_E(z) \bar{\partial}f(z) = 0$ . Applying Fubini’s theorem, we compute

$$\begin{aligned} 0 &= \int F_E(z) \bar{\partial}f(z) = \int \bar{\partial}f(z) \int_E \frac{|dt|^2}{t - z} \\ &= \int_E |dt|^2 \int \frac{\bar{\partial}f(z)}{t - z}. \\ &= - \int_E F(t) |dt|^2 = - \int_E f(t) |dt|^2. \end{aligned}$$

This is a contradiction, so we have the claim.



Now if  $P_R \subset \bigcup \partial Y_i$  then by the claim  $f(z) = 0$  almost everywhere in  $\mathbb{C}$ . This contradiction finishes the proof.  $\square$

Let us note that if  $P_R \cap J_R \subset \partial V$ , where  $V$  is a component of the Fatou set  $F_R$  then the second condition of the theorem is always satisfied. Indeed, in this situation only finitely many components of  $F_R$  contain  $P_R \cap J_R$  on its boundary and hence  $P_R \cap J_R$  belongs to the boundary of finitely many components of  $S_R$ . So if  $R$  has a completely invariant Fatou component then  $R$  satisfies the second condition of the theorem. In the other hand, we do not know an example of a rational map  $R$  such that  $S_R$  has infinitely many components. On the discussion above we saw that the convergence of Cesàro averages on subspaces of  $L_1(\mathbb{C})$  is closely related to the existence of non-trivial invariant Beltrami differentials under some conditions.

*Remark.* The arguments of the proof of Theorem 4.9 also provide explicit examples of compact sets whose characteristic functions are not regular. Indeed, let  $K$  be a compact subset satisfying the generalized Mergelyan theorem (e.g. a positive Lebesgue measure Cantor set). If the characteristic function  $\chi_K$  is regular, by Weyl’s lemma we have  $\chi_K(x) = \int (\bar{\partial} \chi_K / (z - x))$  almost everywhere, which contradicts the generalized Mergelyan theorem.

**COROLLARY 4.10.** *Let  $R$  be a rational map satisfying the conditions of Theorem 4.9. Suppose there exists a critical value  $v \in V(R)$  such that the total variation of  $\bar{\partial} A_n(\gamma_v)$  is uniformly bounded. Then  $R$  is not structurally stable.*

*Proof.* Since the total variation of the sequence  $\bar{\partial} A_n(\gamma_v)$  is uniformly bounded, it is  $*$ -weakly precompact when acting on continuous functions.

Let  $m_0$  be a complex-valued measure which is a  $*$ -weak accumulation point of this sequence. Since  $\text{supp}(\bar{\partial} A_n(\gamma_v)) \subset P_R$ , we have  $\text{supp}(m_0) \subset P_R$ . Considering  $\bar{\partial} A_n(\gamma_v)$  as measures, a straightforward computation shows that

$$A_n(\gamma_n)(z_0) = - \int_{\mathbb{C}} \frac{\bar{\partial} A_n(\gamma_v)(t)}{t - z_0} = - \int_{P_R} \frac{\bar{\partial} A_n(\gamma_v)(t)}{t - z_0}$$

for every  $z_0$  outside  $P_R$ .

If  $m_0 \neq 0$  then, as in Theorem 4.9, by the generalized Mergelyan theorem the integral  $-\int_{\mathbb{C}} (dm_0(t)/(t - z))$  is non-zero and is an accumulation point of  $A_n(\gamma_v)$  in the topology of pointwise convergence on  $S_R$ . Therefore, this integral is a regular non-zero fixed point. By Theorem 4.9, the map  $R$  is a Lattès map which is not structurally stable.

The remaining the case is when  $\bar{\partial} A_n(\gamma_v)$  converges to zero in the  $*$ -weak topology. Let  $\mu \in L_\infty(\mathbb{C})$  and consider its potential function

$$F_\mu(z) = - \frac{z(z - 1)}{\pi} \int \frac{\mu(\zeta) |d\zeta|^2}{\zeta(\zeta - 1)(\zeta - z)}.$$

Recall that  $F_\mu$  is continuous on  $\mathbb{C}$  and satisfies  $\bar{\partial} F_\mu(z) = \mu(z)$  in the sense of distributions. We claim that if  $\mu$  is a fixed point of the Beltrami operator then we have  $\int_{\mathbb{C}} \mu(z) \gamma_v(z) |dz|^2 = 0$ .

Indeed, since  $\int F_\mu(z)\bar{\partial}A_n(\gamma_v)(z)$  converges to zero and  $\mu$  is invariant, we have that

$$\begin{aligned} \int_{\mathbb{C}} F_\mu(z)\bar{\partial}A_n(\gamma_v)(z) &= - \int_{\mathbb{C}} \bar{\partial}F_\mu(z)A_n(\gamma_v)(z)|dz|^2 \\ &= - \int_{\mathbb{C}} \mu(z)A_n(\gamma_v)(z)|dz|^2, \end{aligned}$$

which by duality and the invariance of  $\mu$  implies

$$\int_{\mathbb{C}} \mu(z)A_n(\gamma_v)(z)|dz|^2 = \int_{\mathbb{C}} \mu(z)\gamma_v(z)|dz|^2 = 0$$

as claimed.

If  $R$  is structurally stable and  $\mu$  is an invariant differential then by part (3) of Lemma 5 in [17], the potential  $F_\mu$  satisfies for  $a \in \mathbb{C}$  the equation

$$F_\mu(R(a)) - R'(a)F_\mu(a) = -R'(a) \sum_{c_i} \frac{1}{R''(c_i)} F_\mu(R(c_i))\gamma_a(c_i), \tag{*}$$

where the sum is taken over all critical points  $c_i$  with  $i = 1, \dots, 2\text{deg}(R) - 2$ . Moreover, by [17, Theorem 3] there exists a  $(2\text{deg}(R) - 2)$ -dimensional space  $X$  of invariant Beltrami differentials, so that the correspondence

$$\beta : \mu \mapsto F_\mu(R(a)) - R'(a)F_\mu(a)$$

is a linear isomorphism of  $X$  onto its image. But by the claim,

$$F_\mu(v) = \int_{\mathbb{C}} \mu(z)\gamma_v(z)|dz|^2 = 0$$

for every invariant Beltrami differential  $\mu$ , in particular for  $\mu \in X$ . Since  $v$  is a critical value, we get, using the right part of equation (\*), that the space  $\beta(X)$  has dimension at most  $2\text{deg}(R) - 3$ , which is a contradiction. □

As an immediate corollary we have the following result.

**COROLLARY 4.11.** *Assume the Julia set  $J_R$  has positive Lebesgue measure. Then  $J_R$  does not support a non-zero invariant Beltrami differential if and only if, for any critical value  $v$ , the sequence  $\bar{\partial}A_n(\gamma_v)$  converges to zero on every continuous function  $\phi$  on  $J_R$  with distributional derivative  $\phi_{\bar{z}} \in L_\infty(J_R)$ .*

*Proof.* The proof is by contradiction. First, note that for every continuous function  $\phi$  with distributional derivative  $\phi_{\bar{z}}$  we have

$$\int \phi \bar{\partial}A_n(\gamma_v) = - \int \phi_{\bar{z}} A_n(\gamma_v(z))|dz|^2.$$

By duality,

$$- \int \phi_{\bar{z}} A_n(\gamma_v(z))|dz|^2 = -\frac{1}{n} \sum_{j=1}^n \int B^j(\phi_{\bar{z}})\gamma_v(z)|dz|^2,$$

where  $B$  is the Beltrami operator. If there exists  $\phi$  such that the sequence  $\int \phi \bar{\partial} A_n(\gamma_v)$  does not converge to zero, then any accumulation point of  $-(1/n) \sum_{j=1}^n B^j(\phi_{\bar{z}})$  is a non-zero invariant Beltrami differential. Conversely, if there is a non-zero invariant Beltrami differential  $\mu$ , then its potential  $F_\mu$  is continuous and  $(F_\mu)_{\bar{z}} = \mu$ , and

$$-F_\mu(v) = \int \mu(z) A_n(\gamma_v(z)) |dz|^2 = \int F_\mu \bar{\partial} A_n(\gamma_v)$$

converges to zero. So  $-F_\mu(v) = 0$  for all critical value  $v$ . This contradicts part (2) of Lemma 4.5. □

In the following statements we show that there are no fixed points of the Ruelle operator among the examples of non-regular functions mentioned above.

**PROPOSITION 4.12.** *Assume that a function  $g = f + c\chi_A$ , where  $c$  is a constant and  $\chi_A$  is the characteristic function of a measurable set  $A$  such that  $A \setminus \text{supp}(f)$  has positive measure. If  $g$  is a fixed point of the Ruelle operator then  $R^*(f) = f$  and  $c = 0$ .*

*Proof.* If  $B = A \setminus \text{supp}(f)$ , then by Propositions 4.1 and 4.4 we have that  $D = B \cap SC(R)$  has positive measure and  $\mu(z) = |g(z)|/g(z) = \mu(R(z))(\overline{R'(z)})/R'(z)$  almost everywhere on  $\text{supp}(g)$ . Then there exists  $k$  such that the measure of  $D \cap R^k(D)$  is positive, and hence the set  $C = D \cap (R^k)^{-1}(D)$  has positive measure. Moreover, we have  $\mu = |c|/c$  on  $C$ . If  $c \neq 0$  then by invariance of  $\mu$ , we have that  $(R^k)'$  is real-valued on  $C$  and thus  $C \subset ((R^k)')^{-1}(\mathbb{R})$ , which contradicts  $m(C) > 0$ . □

As an immediate corollary we have that a simple function cannot be a fixed point of the Ruelle operator.

**COROLLARY 4.13.** *If  $f = \sum c_i \chi_{A_i}$  where the  $A_i$  are distinct measurable sets, then  $f$  is a fixed point of the Ruelle operator if and only if  $c_i = 0$  for all  $i$ .*

Let us show that for any infinite closed set  $K$  in  $\overline{\mathbb{C}}$ , the space  $H(K)$  always contains a non-regular function, even when  $K$  has zero Lebesgue measure. On the other hand,  $H(K)$  contains characteristic functions if and only if  $K$  has positive measure. Nevertheless, by Lemma 4.3, Proposition 4.4 and Theorem 4.6, if the postcritical set has measure zero then any fixed point of the Ruelle operator is necessarily a regular function. In general, we conjecture that any fixed point of the Ruelle operator is necessarily a regular function. In the last section we will discuss the existence of fixed points when the postcritical set has positive measure.

**PROPOSITION 4.14.** *Let  $K$  be an infinite closed subset of  $\overline{\mathbb{C}}$ . Then  $H(K)$  contains non-regular functions.*

*Proof.* It is enough to show the proposition when  $K$  is a bounded infinite closed subset of  $\mathbb{C}$ . Otherwise we can compose with a Möbius map. For  $\mu \in L_\infty(\mathbb{C})$  the assignment  $\mu \mapsto F_\mu|_K$  defines a continuous compact operator  $S : L_\infty(\mathbb{C}) \rightarrow C(K)$  (see Theorem 7 of Ch. 3, p. 56 of [9]). Then the dual operator  $S^* : M(K) \rightarrow L_\infty^*(\mathbb{C})$  defined on the space

$M(K)$  of all finite complex-valued measures with total variation as norm is also compact. By direct computation we have that if  $\nu \in M(K)$  then

$$S^*(\nu)(z) = \int \gamma_a(z) d\nu(a).$$

Hence by Fubini’s theorem the image of  $S^*$  belongs to  $H(K)$ . If  $U \in M(K)$  is the closed unit ball, then  $S^*(U)$  is closed in  $H(K)$ .

If every element in  $H(K)$  is regular so  $H(K) = \bigcup_{n \geq 0} (S^*(nU))$  then by Baire’s category theorem there exists  $n_0$  such that  $S^*(n_0U)$  has non-empty interior. But, by the compactness of  $S^*$  any ball inside  $S^*(n_0U)$  is finite-dimensional, which implies that  $H(K)$  has finite dimension and hence  $K$  is finite. This is a contradiction. □

We endow the space  $\text{Hol}(R)$  with two non-complete norms. The first is given by  $\|f\|_1 = \int_{F_R} |f|$  and the second by  $\|f\|_2 = \int_{J_R} |f|$  for  $f \in \text{Hol}(R)$ . Let us call  $\text{Hol}_1$  and  $\text{Hol}_2$  the respective normed spaces. If the Fatou set is empty then  $\text{Hol}_1 = 0$ . Similarly,  $\text{Hol}_2 = 0$  when the Julia set has measure zero. The operator  $R^*$  is a contraction on each space.

Next we show that any compatibility between these two topologies on  $\text{Hol}(R)$  gives rise to a sort of rigidity of the dynamics of  $R$ .

PROPOSITION 4.15. *Fix a rational map  $R$  with  $F_R \neq \emptyset$ .*

- (1) *If the identity map  $\text{Id} : \text{Hol}_1 \rightarrow \text{Hol}_2$  is continuous, then there are no fixed points of the Beltrami operator on the Julia set  $J_R$ .*
- (2) *If the Fatou set  $F_R$  admits an invariant Beltrami differential defining a non-zero functional on  $\text{Hol}_1$ , then  $\text{Id} : \text{Hol}_2 \rightarrow \text{Hol}_1$  is continuous if and only if  $J_R$  has Lebesgue measure zero.*

*Proof.* For part (1), suppose that there is a non-zero fixed point  $\mu$  of the Beltrami operator on the Julia set. Then  $\mu$  defines a non-trivial invariant continuous linear functional  $l_\mu$  on  $\text{Hol}_2$ . Since  $\text{Id}$  is a continuous operator, the functional  $l(\alpha) = l_\mu(\text{Id}(\alpha))$  is a continuous invariant linear functional on  $\text{Hol}_1$  and hence extends to the completion  $\overline{\text{Hol}_1} \subset L_1(F_R)$  of  $\text{Hol}_1$  with respect to its norm. By the Hahn–Banach and Riesz representation theorems there exists a function  $\nu \in L_\infty(F_R)$  such that

$$l(\phi) = \int_{F_R} \phi(z)\nu(z)|dz|^2$$

for all  $\phi \in \overline{\text{Hol}_1}$ .

As in the proof of Proposition 4.8, we can assume that  $\nu$  is a fixed point of the Beltrami operator supported on the Fatou set.

In particular, for  $\gamma_a \in \text{Hol}(R)$  with  $a \in P_R$ , the continuous functions  $F_{\nu_\infty}(a) = \int_{\mathbb{C}} \gamma_a(z)\nu_\infty(z)|dz|^2$  and  $F_\mu(a) = \int_{\mathbb{C}} \gamma_a(z)\mu(z)|dz|^2$  coincide on the orbit of all critical values, and hence on  $P_R$ . However, this contradicts [17, Theorem 3], using an analogous argument to the proof of Proposition 4.8.

For part (2), notice that if the Julia set has measure zero, then  $\text{Hol}_2$  consists only of the zero function. So assume that the Lebesgue measure of the Julia set is not zero. Let  $\mu$  a fixed point of Beltrami operator with

$$l_\mu(\phi) = \int_{F_R} \phi(z)\mu(z)|dz|^2 \neq 0.$$

Applying the same arguments as in part (1), we get a continuous linear functional on  $\text{Hol}_2$  defined by a fixed point of the Beltrami operator now supported on the Julia set. This again, by an analogous argument of Proposition 4.8 gives a contradiction, hence the Lebesgue measure of  $J_R$  is zero.  $\square$

Now define  $X = (\text{Id} - R^*)(\text{Hol}(R))$  and let  $X_1$  and  $X_2$  be the closures of  $X$  in the spaces of  $\text{Hol}_1$  and  $\text{Hol}_2$ , respectively. The proof of Theorem 4.6 shows that if there is no invariant Beltrami differential supported on a Julia set of positive measure, then we get  $X_2 = \text{Hol}_2$  which coincides with  $\text{Hol}(R)$  as a set and hence also  $X_1 \subset X_2$ . We will prove the converse in Proposition 4.17 below. First we need a technical result.

LEMMA 4.16. *Let  $l$  be a linear functional on  $\text{Hol}(R)$ . If  $X_1 \subset \ker(l)$  then  $l$  is continuous on  $\text{Hol}_1$ .*

*Proof.* Let  $W$  be the finite-dimensional space of all linear combinations of the functions  $\gamma_v$ , with  $v$  a critical value of  $R$ . We will show first that  $\text{Hol}_1$  equals the sum  $X_1 + W$ ; note that  $X_1 \cap W$  may be non-zero. Indeed, by definition the space  $\text{Hol}_1$  is the linear span of  $\gamma_a(z)$  where  $a$  is an element in the union of the forward orbits of all critical values. For every critical value  $v$  of  $R$ , by Lemma 5 of [17], we have

$$R^*(\gamma_v) = \frac{1}{R'(v)}\gamma_{R(v)} + w$$

for some  $w$  in  $W$ . Since  $R^*(\gamma_v) - \gamma_v \in X_1$  and  $-w + \gamma_v \in W$ , the element  $\gamma_{R(v)}$  is a sum of elements in  $X_1$  and  $W$ . But  $X_1$  is invariant under  $R^*$ , so by an induction argument we conclude that  $\gamma_{v_n} \in X_1 + W$  for every  $v_n = R^n(v)$ . Therefore the space  $X_1$  has finite codimension on  $\text{Hol}_1$ .

If  $X_1 \subset \ker(l)$  then  $l$  projects to a linear functional  $\mathcal{L}$  defined on the finite-dimensional space  $\text{Hol}_1/X_1$ . As  $X_1$  is closed this implies that both  $\mathcal{L}$  and  $l$  are continuous.  $\square$

PROPOSITION 4.17. *Suppose that the Julia set  $J_R \neq \overline{\mathbb{C}}$  and has positive Lebesgue measure. Then the only invariant Beltrami differential supported on  $J_R$  is zero if and only if  $X_1 \subset X_2$ .*

*Proof.* If there is no invariant Beltrami differential supported on the Julia set, then, as in Theorem 4.6,  $X_2 = \text{Hol}_2$ . Since the Lebesgue measure of  $J_R$  is positive,  $\text{Hol}_2$  coincides as a set with  $\text{Hol}(R)$ . In other words, every element  $\omega \in \text{Hol}(R)$  can be approximated by elements of the form  $\alpha_i - R^*(\alpha_i)$  with  $\alpha_i \in \text{Hol}(R)$  in  $L_1(J_R)$  and hence  $X_2$  contains  $X_1$ .

In the other direction, assume  $X_1 \subset X_2$ . As in Lemma 4.16, we have  $\text{Hol}(R) = X_1 + F$  for some finite-dimensional vector space  $F \subset W$ . For an invariant Beltrami differential  $\mu$  supported on the Julia set, the assignment  $l_\mu(\phi) = \int_{J_R} \phi(z)\mu(z)|dz|^2$  defines an invariant linear continuous functional on  $\text{Hol}_2$  for which we have  $X_2 \subset \ker(l_\mu)$ . Lemma 4.16 shows that  $l_\mu$  is continuous on  $\text{Hol}_1$ , so by the Riesz representation theorem there is a function  $v \in L_\infty(F_R)$  so that  $l_\mu(\phi) = \int_{F_R} \phi(z)v(z)|dz|^2$ . However, since  $\mu$  is invariant, using analogous arguments to the proof of Proposition 4.15, we again get a contradiction to [17, Theorem 3].  $\square$

5. Action on  $L_p$  spaces

If we want to extend the theory to  $L_p(K)$  spaces (with  $K$  completely invariant), we need to modify the operators somehow in an *ad hoc* manner. In the formulas below,  $d$  denotes the degree of  $R$ . Let  $p, q$  be such that  $1/p + 1/q = 1$ .

The action of  $R$  by *pull-back* on  $L_p$  is given by

$$R_{*p}(\phi) = \frac{1}{\sqrt[p]{d}} (\phi \circ R) |R'|^{2/p} \frac{R'}{R'(z)},$$

when  $\phi \in L_p$ .

Similarly, the *push-forward* action of  $R$  on  $L_q$  (for  $1 < q < \infty$ ) is defined by

$$R_q^*(\phi) = \frac{1}{\sqrt[q]{d}} \sum \phi(\zeta_i) \frac{\zeta_i'}{\zeta_i} |\zeta_i'|^{2/q},$$

where the sum is taken over all branches  $\zeta_i$  of  $R^{-1}$ , that is, they satisfy  $R(\zeta_i(z)) = z$  for almost all  $z$ . The constants are suitably chosen so that  $R_p^*$  and  $R_{*q}$  are mutually dual. Indeed, if  $\phi \in L_p$  then for any  $g \in L_q$ , we have

$$\int R_{*p}(\phi(z)) \overline{g(z)} |dz|^2 = \frac{1}{\sqrt[p]{d}} \int \phi(R(z)) |R'(z)|^{2/p} \frac{\overline{R'(z)}}{R'(z)} \overline{g(z)} |dz|^2$$

and, after changing variables with  $R(z) = t$ , is equal to

$$\frac{1}{\sqrt[p]{d}} \int \phi(t) \left[ \sum_{i=0}^d \left( \overline{g} |R'|^{2/p} \frac{\overline{R'}}{R'} \right) (\zeta_i(t)) |\zeta_i'(t)|^2 \right] |dt|^2$$

for all branches  $\zeta_i$  of  $R^{-1}$ . By direct calculation, for every  $i$  we have

$$\left( |R'|^{p/2} \frac{\overline{R'}}{R'} \right) (\zeta_i) |\zeta_i'|^2 = |\zeta_i'|^{2/q} \frac{\zeta_i'}{\zeta_i}.$$

Hence the previous expression is equal to

$$\int \phi(t) \overline{[R_q^*(g)](t)} |dt|^2.$$

Moreover, we have  $R_p^* \circ R_{*p} = \text{Id}$  on  $L_p$ .

We now have two continuous families of contractions depending on  $p > 0$ , which are mutually dual for  $p \geq 1$  and include the Ruelle and Beltrami operators, for  $p = 1$  and  $p = \infty$ , respectively.

The next theorem is the  $L_p$  version of the action of Ruelle operators (compare with the previous section). Unfortunately, for  $1 < p < \infty$ , the Ruelle operator on  $L_p$  does not detect whether there is an invariant Beltrami differential without an associated fixed point for the Ruelle operator.

**THEOREM 5.1.** *Let  $K$  be a completely invariant set of positive measure. Given  $p$  with  $1 < p < \infty$ , the operator  $R_p^*$  has a fixed point in  $L_p(K)$  if and only if  $R$  is a flexible Lattès map.*

*Proof.* If  $R_p^*$  has a fixed point then its dual  $R_{*q}$  has a fixed point on  $L_q(K)$ . Let  $\psi$  be a fixed point of  $R_{*q}$ . Then  $|\psi|/\psi$  is an invariant Beltrami differential. On the other hand, the function  $f = |\psi|^q$  is integrable and satisfies

$$f(z) = \frac{f(R(z))|R'(z)|^2}{\deg(R)}.$$

In this case  $\text{supp}(f)$  is completely invariant. By Proposition 4.1 and the discussion afterwards, either  $R$  is a Lattès map or  $\text{supp}(f) \subset C(R) \cap P_R \subset J_R$ . But the latter is not possible. Indeed, let  $E$  be the operator  $E : C(J_R) \rightarrow C(J_R)$  on the space of complex-valued continuous functions  $\phi$  on  $J_R$  given by

$$E_R(\phi)(z) = \frac{1}{\deg(R)} \sum_i \phi(\zeta_i(z)),$$

where the sum is taken over all branches  $\zeta_i$  of  $R^{-1}$ .

Let us recall Lyubich’s theorem [15, Theorem 5] which states that every continuous functional invariant with respect to  $E$  is induced by a multiple of the maximal entropy measure.

Let  $\nu(z)$  be the measure such that  $d\nu(z) = f(z) |dz|^2$ . Now

$$\int_{J_R} E_R(\phi) d\nu = \frac{1}{\deg(R)} \int_{J_R} \left[ \sum_i \phi(\zeta_i(z)) \right] f(z) |dz|^2.$$

By the arguments and computations of Proposition 2.1, the latter is equal to

$$\frac{1}{\deg(R)} \int_{J_R} \phi(z) f(R(z)) |R'(z)|^2 |dz|^2 = \int_{J_R} \phi d\nu.$$

Then  $\nu$  is a multiple of the maximal entropy measure. By Zdunik’s theorem (see [24])  $R$  is a postcritically finite rational map. Hence  $f(z) = 0$  almost everywhere since  $\text{supp}(f) \subset C(R) \cap P_R$ , which is a contradiction. Thus  $R$  is a flexible Lattès map.

Conversely, if the map  $R$  is a flexible Lattès map, then there exists an integrable function  $f_0$  such that  $f_0 = f_0(R)(R')^2/\deg R$  and  $\mu(z) = \overline{f_0(z)}/|f_0(z)|$  is a fixed point of the Beltrami operator. Therefore  $\psi = |f_0|^{1/p}\mu$  is a fixed point for  $R_{*p}$  on  $L_p(K)$  and induces a fixed point of the Ruelle operator in  $L_p(K)$ . Since the dual operator satisfies  $R_q^* \circ R_{*q} = \text{Id}$ , the converse follows immediately.  $\square$

Note that the theorem above shows that if  $R_{p_0}^*$  has a non-zero fixed point in  $L_{p_0}(K)$  for some  $p_0$ , then  $R_p^*$  has a fixed point in  $L_p(K)$  for all  $p$  with  $1 < p < \infty$ .

### 6. Fixed points of bidual actions and uniform ergodicity

In this section, we start with the following lemma which is a summary of results due to Lotz in [13].

LEMMA 6.1. *Let  $B$  be a Grothendieck space with the Dunford–Pettis property and let  $S : B \rightarrow B$  be a power-bounded operator.*

- (1) *If  $S$  is mean ergodic, then  $S$  is uniformly ergodic.*
- (2) *If the space  $\text{Fix}(S^*)$  of fixed points of the dual of  $S$  is separable then  $S$  is uniformly ergodic and the Cesàro averages uniformly converge to a compact projection.*

Part (1) is the content of [13, Theorem 5]. Part (2) is the content of [13, Theorem 7].

For the rational map  $R$ , let  $T : B(S_R) \rightarrow B(S_R)$  be the Thurston operator as defined in §3. We have the following theorem.

**THEOREM 6.2.** *Let  $R$  be a rational map such that  $P_R$  is not the whole sphere, the conservative set  $C(R)$  does not contain any Fatou component and 1 belongs to the spectrum  $\sigma(R^*)$  on  $A(S_R)$ . Then the following four conditions are equivalent.*

- (1) *The space  $\text{Fix}(T^*)$  is separable.*
- (2) *The operator  $T$  is mean ergodic.*
- (3) *The Ruelle operator  $R^*$  is uniformly ergodic.*
- (4) *The map  $R$  is a flexible Lattès map.*

*Proof.* By the discussion on §3, the space  $B(S_R)$  is a Grothendieck space with the Dunford–Pettis property. Then Lemma 6.1 applied to the operator  $T$  gives the implications from (1) to (2) and from (2) to (3).

To show that (3) implies (4), note that, since  $R^*$  is a contraction,  $R^*$  is a power-bounded operator. By the uniform ergodicity lemma and the assumption, the value 1 is an isolated eigenvalue of  $\sigma(R^*)$ . Then the Ruelle operator  $R^*$  has a non-zero fixed point  $\phi$  in  $A(S_R)$ . Hence by Proposition 4.4, the modulus  $|\phi|$  defines an invariant finite measure such that the support of  $|\phi|$  contains a component of  $S_R$ . Then by Proposition 4.1 and Lemma 4.3 the map  $R$  is a flexible Lattès map.

Now we show that (4) implies (1). For a flexible Lattès map, the space  $A(S_R)$  is finite-dimensional, and hence  $A^*(S_R)$  and its dual are finite-dimensional too, which implies (1).  $\square$

The condition that 1 belongs to the spectrum is necessary for the discussion around Sullivan’s conjecture. Otherwise, the Beltrami operator does not have fixed points. However, in this situation there are open questions:

- (1) Does there exist a rational map  $R$  with infinite postcritical set such that the norm of  $R^*$  on  $A(S_R)$  is strictly smaller than 1?
- (2) Or more generally, is it true that if 1 does not belong to the spectrum of  $R^*$  on  $A(S_R)$  then  $R$  is postcritically finite?

By the uniform ergodicity lemma, the conditions of the questions above imply that  $R^*$  is uniformly ergodic on  $A(S_R)$ .

The following corollary gives a partial answer to these questions for the class of  $J$ -stable rational maps.

**COROLLARY 6.3.** *If  $R$  is  $J$ -stable such that  $P_R \neq J_R$  then the following statements are equivalent.*

- (1) *The operator  $R^*$  is uniformly ergodic on  $A(S_R)$ .*
- (2) *The map  $R$  is hyperbolic and postcritically finite.*

*Proof.* Condition (2) implies (1) since, in this case,  $A(S_R)$  is finite-dimensional.

For the converse, first note that 1 does not belong to  $\sigma(R^*)$ . Indeed, if 1 belongs to  $\sigma(R^*)$  then, by Theorem 6.2,  $R$  is a flexible Lattès map, which contradicts  $J$ -stability. Hence every invariant Beltrami differential supported on  $S_R$  defines a trivial



quasiconformal deformation. By Lemma 4.5, any non-zero invariant Beltrami differential supported on the Julia set defines a non-trivial quasiconformal deformation. Hence by Theorem E in [16], the map  $R$  is hyperbolic. Moreover, by Theorem D in [16], we have that  $R$  is postcritically finite.  $\square$

If the Julia set  $J_R$  has positive measure and does not support non-zero invariant Beltrami differentials, then the action of the Ruelle operator on  $L_1(J_R)$  is mean ergodic because  $\lim (1/n) \sum_{i=0}^{n-1} B^i(\mu) = 0$  for all  $\mu \in L_\infty(J_R)$  by the mean ergodicity lemma. Thus, by duality the Cesàro averages of the Ruelle operator converge to zero in the weak topology and hence in the strong topology by the mean ergodicity lemma. In contrast with this fact, we show that the action of  $R^*$  on  $L_1(J_R)$  is uniformly ergodic only when  $J_R$  has Lebesgue measure zero. In slightly more generality, we prove the following theorem.

**THEOREM 6.4.** *Let  $R$  be a rational map. If  $K$  is a completely invariant set of positive Lebesgue measure, then the Ruelle operator is not uniformly ergodic on  $L_1(K)$ .*

*Proof.* By contradiction. If the measure of  $K$  is positive and  $R^*$  is uniformly ergodic in  $L_1(K)$ , we claim that  $R^*$  is an automorphism of  $L_1(K)$ . It is enough to show that  $R^*$  is injective since  $R^*$  is surjective by the relation  $R^* \circ R_* = \text{Id}$ .

For every  $\lambda$  with  $|\lambda| < 1$  and  $\phi$  non-zero element in  $L_1(K)$ , the element

$$\phi_\lambda = \sum_{n=0}^{\infty} \lambda^n R_*^n(\phi) \neq 0$$

and satisfies  $\lambda R_*(\phi_\lambda) = \phi_\lambda - \phi$ . If  $\phi$  is a non-zero element in  $\text{Ker}(R^*)$  then  $R^*(\phi_\lambda) = \lambda\phi_\lambda$ . It follows that 1 is not an isolated eigenvalue in the spectrum, which contradicts the uniform ergodicity of  $R^*$ . Therefore  $R^*$  is an isomorphism as we claimed. This, in turn, implies

$$R_* \circ R^* = \text{Id}$$

on  $L_1(K)$ . Now let  $x_1$  and  $x_2$  be different fixed points of  $R$ , and fix a point  $b$  different from  $x_1$  and  $x_2$ . Then the restriction of  $\omega_b(z) = ((x_1 - b)(x_2 - b))/((z - x_1)(z - x_2)(z - b))$  to  $K$  is integrable. Since  $\omega_b$  is rational, and the equation  $R_* \circ R^*(\omega_b) = \omega_b$  holds almost everywhere on a set of positive measure, then it holds on the whole Riemann sphere. If we take  $b$  such that neither  $b$ ,  $R(b)$  nor  $R^{-1}(b)$  are critical values of  $R$ , then  $R^*(\omega_b)$  has a non-trivial pole on  $R(b)$ , which implies that  $R_* \circ R^*(\omega_b)$  has poles in  $R^{-1}(R(b))$  which are different from the poles of  $\omega_b$ . This contradiction finishes the proof.  $\square$

We have the following corollary which complements Corollary 6.3 in the case where  $P_R = \overline{\mathbb{C}}$ . In this case, since  $S_R = \emptyset$ , we consider the uniform ergodicity over  $\text{Hol}(R)$  equipped with the  $L_1$  topology on  $\mathbb{C}$ .

**COROLLARY 6.5.** *Let  $R$  be a rational map with  $P_R = \overline{\mathbb{C}}$ . Then  $R^*$  is not uniformly ergodic on  $\text{Hol}(R)$  equipped with the topology of  $L_1(\mathbb{C})$ .*

*Proof.* Since  $P_R = \overline{\mathbb{C}}$ , we have that  $\text{Hol}(R)$  is everywhere dense in  $L_1(\mathbb{C})$ . If  $R^*$  is uniformly ergodic in  $\text{Hol}(R)$ , it is also uniformly ergodic on  $L_1(\mathbb{C})$ , which contradicts Theorem 6.4.  $\square$

Let us observe that the Thurston operator  $T$  leaves  $B_0(S_R)$  invariant. Since the Ruelle operator is continuous in the  $*$ -weak topology on  $B_0(S_R)$ , it follows that  $R^*$  is dual to the restriction of  $T$  to  $B_0(S_R)$ , so we have  $(T|_{B_0(S_R)})^{**} = T$ .

By the facts discussed in §3, either every endomorphism  $Q$  on  $B_0(S_R)$  is compact or there exists an infinite-dimensional subspace  $E$  of  $B_0(S_R)$  such that the restriction  $Q|_E : E \rightarrow B_0(S_R)$  is an isomorphism onto its image. Let us consider  $Q = \text{Id} - T$  and investigate two extreme situations: first, when  $Q : B_0(S_R) \rightarrow B_0(S_R)$  is an isomorphism onto its image, that is, the space  $E$  is the whole of  $B_0(S_R)$ ; and second, when  $Q$  is a compact operator. Our goal is to show that these two extreme situations imply the uniform ergodicity of the operator  $T$ , which is equivalent to the uniform ergodicity of  $R^*$ .

**THEOREM 6.6.** *If  $Q = \text{Id} - T$  is either an isomorphism onto its image or a compact operator, then  $T$  is uniformly ergodic.*

*Proof.* If  $Q : B_0(S_R) \rightarrow B_0(S_R)$  is an isomorphism onto its image, then the subspace  $(\text{Id} - T)B_0(S_R)$  is closed, hence  $T$  is uniformly ergodic by the uniform ergodicity lemma.

Now assume  $Q$  is compact. As a consequence of the spectral decomposition theorem, for any  $\epsilon > 0$  there is a splitting  $B_0(S_R) = F_\epsilon \oplus X_\epsilon$  in which both subspaces,  $F_\epsilon$  and  $X_\epsilon$ , are invariant under  $Q$ , with  $\dim(X_\epsilon) < \infty$  and such that the norm of  $Q$  restricted to  $F_\epsilon$  is less than  $\epsilon$ . As  $F_\epsilon$  and  $X_\epsilon$  are  $Q$ -invariant these are also  $T$ -invariant. Consider the restriction  $Q|_{F_\epsilon}$ . Since the norm on  $F_\epsilon$  is small, it follows that  $(\text{Id} - Q)|_{F_\epsilon}$  is invertible. The von Neumann series yields  $(\text{Id} - Q)^{-1} = \sum_{n=0}^\infty Q^n$ , at least on  $F_\epsilon$ . This means that  $(\text{Id} - Q)^{-1}|_{F_\epsilon}$  is a compact operator as it is the norm limit of compact operators. We have  $\text{Id} - Q = T$ , and by above  $T$  is invertible on  $F_\epsilon$  and its inverse  $T^{-1}|_{F_\epsilon}$  is a compact operator. Hence  $T|_{F_\epsilon}$  is a compact isomorphism, which implies that  $F_\epsilon$  has finite dimension. Therefore  $B_0(S_R)$  has finite dimension too since  $X_\epsilon$  is finite-dimensional and  $T$  is a contraction over a finite-dimensional space, thus is uniformly ergodic by the uniform ergodicity lemma. □

As an immediate corollary we have the following result.

**COROLLARY 6.7.** *Under the conditions of the last theorem, the operator  $Q$  is compact if and only if  $R$  is postcritically finite.*

*Proof.* If  $R$  is postcritically finite, then  $B_0(S_R)$  has finite dimension. Since  $Q$  is continuous,  $Q$  is compact. Conversely, if  $Q$  is compact, as in the proof of the previous theorem, the linear space  $A(S_R)$ , which is dual to  $B_0(S_R)$ , has finite dimension. This is only possible when  $P_R$  is finite. □

The following is the main theorem of this section.

**THEOREM 6.8.** *Assume that the conservative set  $C(R)$  does not intersect the Fatou set  $F_R$ . Then the Thurston operator  $T$  is mean ergodic on  $B_0(S_R)$ . Moreover,  $T$  has a non-zero fixed point in  $B_0(S_R)$  if and only if  $R$  is a flexible Lattès map.*

*Proof.* Assume that there is no non-zero fixed point of  $R^* : B_0^*(S_R) \rightarrow B_0^*(S_R)$ . Then by arguments similar to those of the proof of Theorem 4.6 we have that  $(I - T)(B_0(S_R))$

is dense in  $B_0(S_R)$ . This implies that for  $\phi \in B_0(S_R)$  the averages  $(1/n) \sum_{i=0}^{n-1} T^i(\phi)$  converge to zero. Then  $T$  is mean ergodic in  $B_0(S_R)$ .

If  $R^*$  has a fixed point, by Proposition 4.4, Lemma 4.3 and the arguments presented in the proof of Theorem 6.2 it follows that  $R$  is a flexible Lattès map and  $T$  is mean ergodic on  $B(S_R)$  and hence on  $B_0(S_R)$ .

Finally, by the separation principle of §4, a mean ergodic operator  $T$  has a non-trivial fixed point if and only if its dual  $R^*$  has a non-trivial fixed point. By the arguments above this is the case precisely when  $R$  is a flexible Lattès map. □

The following corollary shows that if  $T$  has a fixed point in  $B(S_R)$  close enough to  $B_0(S_R)$  in Teichmüller distance, with respect to the Teichmüller norm, then  $R$  is a Lattès map.

**COROLLARY 6.9.** *Let  $R$  be as in Theorem 6.8. Let  $\alpha \in B(S_R)$  with  $T(\alpha) = \alpha$  and  $\|\alpha\|_T = 1$ , where  $\|\cdot\|_T$  is the Teichmüller norm. If*

$$\text{dist}(\alpha, B_0(S_R)) = \inf_{\phi \in B_0(S_R)} \|\alpha - \phi\|_T < 1,$$

*then  $R$  is a flexible Lattès map.*

*Proof.* Given the hypothesis, we can find  $\phi_0 \in B_0(S_R)$  subject to  $\|\alpha - \phi_0\|_T < 1$ . By Theorem 6.8,  $T$  is mean ergodic. If  $T$  has a fixed point in  $B_0(S_R)$  then  $R$  is a Lattès map. Otherwise the Cesàro averages  $A_n(\phi)$  converge to zero for every  $\phi$  in  $B_0(S_R)$ . In particular, we have

$$1 = \|\alpha\|_T = \lim \|A_n(\alpha - \phi_0)\|_T \leq \|\alpha - \phi_0\|_T < 1,$$

which is a contradiction. Thus  $T$  has a fixed point in  $B_0(S_R)$  and  $R$  is a Lattès map. □

We say that  $R$  is *dissipative* if the dissipative set of  $R$  is the whole Riemann sphere; in other words, the conservative set has Lebesgue measure zero.

**THEOREM 6.10.** *Let  $R$  be a dissipative map. Then, for any  $\alpha$  in  $B_0(S_R)$ , the orbit under the Thurston operator  $T^n(\alpha)$  converges weakly to zero.*

*Proof.* Since the conservative set of  $R$  has Lebesgue measure zero, by Lemma 4.2, we have that  $\sum_n R^{*n}(\phi)(z)$  absolutely converges almost everywhere in  $\mathbb{C}$  for every  $\phi$  in  $A(S_R)$ .

In particular, it follows that  $R^{*n}\phi$  converges to zero almost everywhere on  $\mathbb{C}$ . But  $R^{*n}\phi$  is a bounded sequence in  $A(S_R)$ . Therefore  $R^{*n}\phi$  defines a normal family of holomorphic functions on  $S_R$ . Hence  $R^{*n}\phi$  converges pointwise to zero. By duality, the  $T$  orbit of any element in  $B_0(S_R)$  weakly converges to zero. □

An operator satisfying the conclusion of the previous theorem is called *weakly asymptotic*.

**THEOREM 6.11.** *Suppose that the measure of the postcritical set  $P_R$  is zero and that the image of the closed unit ball in  $B_0(S_R)$  under  $\text{Id} - T$  is closed. Then  $R$  satisfies Sullivan’s conjecture.*

*Proof.* We use [7, Theorems 2.3 and 3.3], which state that, for a separable Banach space  $X$  and power-bounded operator  $T$ , if the image of the closed unit ball under  $(\text{Id} - T)$  is closed then either  $T$  is uniformly ergodic or the space  $\overline{(\text{Id} - T)(X)}$  contains an isomorphic copy of an infinite-dimensional dual Banach space.

Now let us show that  $T$  is uniformly ergodic on  $B_0(S_R)$ . By contradiction, if  $T$  is not uniformly ergodic then the space  $\overline{(\text{Id} - T)(B_0(S_R))}$  contains an infinite-dimensional dual Banach space  $Y$ . By [2, Theorem 1],  $B_0(S_R)$  is isomorphic to a closed subspace of  $c_0$ . On the other hand, any closed infinite-dimensional subspace of  $c_0$  contains a closed subspace isomorphic to  $c_0$ . Hence  $Y$  contains an isomorphic copy of  $c_0$ . Let  $h : c_0 \rightarrow Y$  be this isomorphism onto the image. Since  $Y$  is a dual space, the closed unit ball in  $Y$  is  $*$ -weak compact, so the isomorphism  $h$  can be extended to a continuous linear operator  $O : \ell_\infty \rightarrow Y$ . Then  $O^* : Y^* \rightarrow (\ell_\infty)^*$  is continuous with respect to the  $*$ -weak topology. As  $Y$  is separable any bounded set of linear functionals contains a subsequence converging in the  $*$ -weak topology. Since  $\ell_\infty$  is a Grothendieck space,  $O^*$  is a weakly compact operator. Thus  $O$  and hence  $h$  are weakly compact. But a separable Banach space with weakly compact unit ball is reflexive; this contradicts the fact that  $c_0$  is not reflexive. Therefore  $T$  is uniformly ergodic on  $B_0(S_R)$ .

If  $T$  has a fixed point in  $B(S_R)$ , then  $\text{Id} - T$  is not invertible in  $B(S_R)$ . Then  $(\text{Id} - T) : B_0(S_R) \rightarrow B_0(S_R)$  is not an isomorphism. Thus by the uniform ergodicity lemma the value 1 is an isolated eigenvalue for  $T$  in  $B_0(S_R)$ . Then the value 1 is an isolated eigenvalue for  $R^*$  in  $A(S_R)$ . Hence  $R^*$  is mean ergodic on  $A(S_R)$ . Therefore by hypothesis  $R^*$  is mean ergodic in  $\text{Hol}(R)$  with topology inherited by  $L_1(J_R)$ . Then Theorem 4.6 finishes the proof.  $\square$

Let us add some comments about the closeness of the image of the closed unit ball under  $(\text{Id} - T)$ . Since  $B(S_R)$  is a dual space, the image of its closed unit ball under  $(\text{Id} - T)$  is always closed. When the image of the closed unit ball of  $B_0(S_R)$  under  $(\text{Id} - T)$  is not closed, consider the space

$$X = \text{cl}((\text{Id} - T)^{-1}(B_0(S_R))) \subset B(S_R).$$

Then again on  $X$ , the image of the closed unit ball under  $(\text{Id} - T)$  is closed. But  $X$  is invariant under  $T$  and  $B_0(S_R) \oplus \text{Fix}(T, B(S_R)) \subset X$ . Then we have the following result.

**PROPOSITION 6.12.** *Let  $X$  be as in the discussion above. Assume that  $X$  satisfies one of the following conditions:*

- (1)  $X$  is separable;
- (2)  $X$  is a Grothendieck space;
- (3) the operator  $T : X \rightarrow X$  is mean ergodic.

*Then  $T : B(S_R) \rightarrow B(S_R)$  is uniformly ergodic.*

*Proof.* If  $X$  is separable, since the image of the closed unit ball under  $(\text{Id} - T)$  is closed and  $\overline{(\text{Id} - T)(X)} = B_0(S_R)$ , we have that, as in the proof of Theorem 6.11, either  $T$  is uniformly ergodic or  $B_0(S_R)$  contains an isomorphic copy of infinite-dimensional dual Banach space. But as the proof of Theorem 6.11 shows,  $B_0(S_R)$  cannot contain such a copy. Hence  $T$  is uniformly ergodic.

If  $X$  is a Grothendieck space, then  $(\text{Id} - T) : X \rightarrow B_0(S_R)$  is weakly compact. It follows that  $(\text{Id} - T) : B_0(S_R) \rightarrow B_0(S_R)$  is also weakly compact. Recall that either every endomorphism  $Q$  of  $B_0(S_R)$  is compact or there exists an infinite-dimensional subspace  $Y$ , isomorphic to  $c_0$ , such that the restriction of  $Q$  on  $Y$  is an isomorphism onto its image. We conclude that  $(\text{Id} - T)$  is a compact endomorphism of  $B_0(S_R)$ , and by duality  $(\text{Id} - T)$  is also compact on  $B(S_R)$ . Thus  $T$  is uniformly ergodic by Theorem 6.6.

Finally, if  $T : X \rightarrow X$  is mean ergodic then [7, Proposition 3.8] states precisely that the image of the closed unit ball in  $B_0(S_R)$  under  $(\text{Id} - T)$  is closed and the arguments of Theorem 6.11 apply again. □

In the next section we analyse further properties of  $X$  in a more general setting.

### 7. Hamilton–Krushkal sequences

In this section, to avoid cumbersome calculations and definitions, we consider rational maps  $R$  satisfying two conditions.

- First, there is no non-trivial quasiconformal deformation supported on the Fatou set. That is, by definition, every invariant Beltrami differential supported on the Fatou set defines a zero functional on  $A(S_R)$ . Other characterizations are given by Proposition 4.8.

As mentioned in the introduction, this condition does not impose any restriction on the study of Sullivan’s conjecture if the Julia set is connected.

- Second, the postcritical set  $P_R$  does not support an absolutely continuous invariant measure with respect to the Lebesgue measure.

In other words, the measure of the intersection of  $P_R$  with the strongly conservative set  $SC(R)$  is zero. In the last section we will discuss and give partial results for the case where  $P_R \cap SC(R)$  has positive measure.

Let us consider the elements  $\gamma_v(z)$  with  $v$  a critical value. Let  $D = \{A_n(\gamma_{v_i})\}$  be the set of Cesàro averages for all  $\gamma_{v_i}$  with  $v_i$  in the critical value set  $V(R)$ . A sequence  $\{\phi_k\}$  in  $A(S_R)$  is called *degenerating non-normalized*, if there are constants  $C$  and  $\epsilon > 0$  with  $\epsilon < \|\phi_k\| < C$  such that  $\phi_k$  converges to zero pointwise.

By Bers’ isomorphism theorem, we identify  $A^*(S_R)$  with the space  $B(S_R)$  and consider the seminorm on  $B(S_R)$  given by

$$K(l) = \sup_k \limsup (|l(\phi_k(z))|),$$

where the supremum is taken over all sequences  $\{\phi_k\}$  in  $D$ .

Note that a sequence in  $D$  is either degenerating or precompact in norm. Indeed, if the sequence  $\{A_n(\gamma_v)\}$  is not degenerating and does not converge in norm, then there exists a subsequence which converges pointwise and locally uniformly to a non-zero limit which is fixed by the Ruelle operator. By Lemma 4.3 and Proposition 4.4, the map  $R$  is a Lattès map. By Theorem 6.2, the Ruelle operator  $R^*$  is uniformly ergodic. Then  $A_n(\gamma_v)$  converges in norm for every critical value  $v$ , which is a contradiction.

The *Hamilton–Krushkal space*  $HK(R)$  is the zero set of  $K$ . Since we have  $K(l) \leq \|l\|$  on  $B(S_R)$ , the space  $HK(R)$  is a closed subspace of  $B(S_R)$ .

A subspace  $Y$  of a Banach space  $X$  is called *coseparable* whenever  $X/Y$  is separable.

THEOREM 7.1. *A rational map  $R$  satisfies Sullivan’s conjecture if and only if  $HK(R)$  is coseparable in  $A^*(S_R)$ .*

*Proof.* If there is no invariant Beltrami differential supported on the Julia set then, by Theorem 4.6, the action of  $R^*$  on  $L_1(J_R)$  is mean ergodic and the Cesàro averages converge to zero in  $L_1(J_R)$ . Thus by Proposition 4.8 we have that  $R^*$  is mean ergodic on  $A(S_R)$ . So we get  $HK(R) = A^*(S_R)$  and that the quotient is separable. If  $R$  is Lattès, then  $A^*(S_R)$  is finite-dimensional and so is  $HK(R)$ .

Conversely, assume that  $HK(R)$  is coseparable. Then there exist a countable set  $\{\alpha_i\}$  of elements in  $A^*(S_R)$  such that  $S = \{\alpha_i\} + HK(R)$  is dense in  $A^*(S_R)$ . By induction, and a diagonal argument, we can pick a sequence  $\{n_k\}$  such that for every  $i$  and every critical value  $v$  the sequence  $\alpha_i(A_{n_k}(\gamma_v))$  converges. Since  $R^*$  is a contraction and  $S$  is an everywhere dense subset of  $A^*(S_R)$ , the Cesàro averages  $A_{n_k}(\gamma_v)$  converge weakly for every  $v$ . By the mean ergodicity lemma, the sequence  $A_{n_k}(\gamma_v)$  converges in norm for all  $v$ . Let  $\mu$  be a non-zero invariant Beltrami differential. By Lemma 4.5 there exists a critical value  $v_0$  such that  $\int_{\mathbb{C}} \mu(z)\gamma_{v_0}(z) \neq 0$ . Thus the limit  $f_0 = \lim A_{n_k}(\gamma_{v_0})$  is a non-zero fixed point of the Ruelle operator in  $A(S_R)$ . Since the measure of  $P_R \cap SC(R)$  is zero and  $SC(R)$  contains  $\text{supp}(f)$ , we have that  $SC(R) = \overline{\mathbb{C}}$  by part (2) of Lemma 4.3. But  $\mu(z)$  not zero almost everywhere, so by part (3) of Lemma 4.3 the map  $R$  is a flexible Lattès map. □

From the arguments in the previous proof, we have the following corollary.

COROLLARY 7.2. *If  $HK(R)$  is coseparable then the Ruelle operator  $R^*$  is mean ergodic on  $A(S_R)$  equipped with the norm inherited from  $L_1(J_R)$ .*

Also, we have the following result.

COROLLARY 7.3. *The space  $HK(R)$  is coseparable if and only if  $\text{codim}(HK(R)) = 0$ .*

### 8. Amenability and compactness

In this section our goal is to give compactness conditions for suitable operators under which a map  $R$  satisfies the Sullivan conjecture. We will keep the technical assumptions given at the beginning of the previous section.

For every critical value  $v$  of  $R$  define the operator  $E_v : B(S_R) \rightarrow \ell_\infty$  by the formula

$$E_v(\psi) = \left( \int_{\mathbb{C}} \lambda^{-2}(z)\overline{\psi}(z)A_n(\gamma_v)(z)|dz|^2 \right)_{n=0}^\infty.$$

In particular, an element  $\phi \in A^*(S_R)$  belongs to  $HK(R)$  whenever  $E_v(\phi) \in c_0$  for every critical value  $v$ . On the image of  $E_v$ , the Thurston operator  $T$  acts as

$$\hat{T}_v(E_v(\psi)) = E_v(T(\psi)).$$

This formula defines  $\hat{T}_v$  as a linear endomorphism, not necessarily continuous, of the image of  $E_v$ .

A mean  $m$  on  $\ell_\infty$  is a positive linear functional which satisfies three conditions.

- $m(1, 1, 1, \dots) = 1$ .
- If  $\sigma$  is the shift  $\sigma(a_1, a_2, \dots) = (a_2, a_3, \dots)$ , then  $m(x) = m(\sigma(x))$  for any  $x \in \ell_\infty$ .
- $\liminf |a_i| \leq |m(a_1, a_2, \dots)| \leq \limsup |a_i|$ .

A mean is also known as a *Banach limit* on  $\ell_\infty$ .

A linear operator  $O : X \rightarrow X$  (not necessarily continuous) defined on a linear subspace  $X$  (not necessarily closed) of  $\ell_\infty$  has an invariant mean if there is a mean on  $\ell_\infty$  with non-zero restriction to  $X$  that satisfies  $m(O(\alpha)) = m(\alpha)$  for  $\alpha \in X$ . We denote by  $M(O)$  the set of all invariant means for  $O$ .

LEMMA 8.1. *The set  $M(\hat{T}_v)$  is empty if and only if  $E_v(A^*(S_R))$  consists exclusively of sequences that converge to zero.*

*Proof.* By definition means are invariant under the shift and bounded by the supremum of the elements of the sequence; this implies that  $c_0 \subset \ker(m)$  for every mean  $m$ . Hence, if  $E_v(A^*(S_R)) \subset c_0$ , we get  $M(\hat{T}_v) = \emptyset$ .

Conversely, if there is an element  $h \in A^*(S_R)$  with  $E_v(h) \in E_v(A^*(S_R)) \setminus c_0$  then there is a subsequence  $\{n_j\}$  such that  $h(A_{n_j}(\gamma_v))$  converges to a non-zero number  $a$ . By duality this implies that  $A_{n_j}^*(h)$  allows a subsequence which converges  $*$ -weakly to a non-zero element  $l_0 \in A^*(S_R)$  that satisfies  $T(l_0) = l_0$ . Then  $E_v(l_0) = (l_0(\gamma_v), l_0(\gamma_v), \dots)$  and, since  $E_v(A^*(S_R))$  is a subspace of  $\ell_\infty$ , we conclude that  $E_v(A^*(S_R))$  contains the constant sequence 1. This implies that  $E_v(A^*(S_R))$  intersects the space of convergent sequences in a non-empty set. On convergent sequences, the functional  $l : \{c_i\} \mapsto \lim c_i$  is continuous. By the Banach limit theorem there exists an extension  $\mathcal{L}$  to all of  $\ell_\infty$ , which is a mean.

Next we show that  $\mathcal{L}$  is  $\hat{T}_v$ -invariant on  $E_v(A^*(S_R))$ . In fact, we have

$$|T(h)(A_n(\gamma_v)) - h(A_{n+1}(\gamma_v))| \leq \frac{4\|h\|\|\gamma_v\|}{n},$$

for  $h \in A^*(S_R)$ . So the difference

$$|T(h)(A_n(\gamma_v)) - h(A_{n+1}(\gamma_v))|$$

converges uniformly to zero as  $n$  tends to  $\infty$  on any ball of  $A^*(S_R)$ , and hence  $(\sigma - \hat{T}_v)E_v(A^*(S_R)) \subset c_0$ . Thus  $(\sigma - \hat{T}_v)E_v(A^*(S_R))$  belongs to  $\ker(\mathcal{L})$ . The invariance of  $\mathcal{L}$  with respect to  $\sigma$  implies the invariance of  $\mathcal{L}$  with respect to  $\hat{T}_v$ . Therefore  $M(\hat{T}_v)$  is non-empty. □

Next we show the finiteness of  $M(\hat{T}_v)$  in very special cases.

THEOREM 8.2. *The set  $M(\hat{T}_v)$  is finite if and only if  $M(\hat{T}_v)$  contains at most one element or, equivalently, if and only if  $E_v(A^*(S_R))$  consists exclusively of convergent sequences.*

*Proof.* The first equivalence is clear after one notices that  $M(\hat{T}_v)$  is convex, so we just worry about the second.

If  $M(\hat{T}_v)$  is empty, by Lemma 8.1 we are done. Otherwise, again by the arguments in Lemma 8.1, if  $E_v(A^*(S_R)) \subset c$  then the only invariant mean is given by the restriction of the limit functional,  $\{a_n\} \mapsto \lim a_n$ . Conversely, we assume that there is only one invariant

mean  $m$  and set  $X = \overline{E_v(A^*(S_R))}$ . Again, by the same arguments given in Lemma 8.1, the space  $E_v(A^*(S_R))$  contains the element  $(1, 1, 1, \dots)$ .

Now we claim that if  $L$  is a mean then  $L(x) = m(x)$  for all  $x \in X$ . Indeed, by Lemma 8.1, we have that  $(\sigma - \hat{T}_v)E_v(A^*(S_R)) \subset c_0$ . It follows that  $L(\hat{T}_v(x)) = L(x)$  for all  $x \in E_v(A^*(S_R))$ . Since  $(1, 1, 1, \dots) \in E_v(A^*(S_R))$ , we have  $L|_{E_v(A^*(S_R))} \neq 0$ . By uniqueness  $L(x) = m(x)$  on  $E_v(A^*(S_R))$ , so by the continuity of  $L$  and  $m$  we get our claim. In particular,  $E_v(A^*(S_R)) \cap \ker(m) \subset \bigcap_L \ker(L)$  where the intersection is taken over all means  $L$ .

The space  $X$  admits the decomposition  $X = \mathbb{C} \cdot (1, 1, 1 \dots) \oplus (\ker(m) \cap X)$ , and there is a fixed point of the Beltrami operator  $\mu$  such that  $E_v^{-1}(X) = \mathbb{C}\mu \oplus E_v^{-1}(\ker(m))$ . By the Banach limit theorem (see [12, Theorem 4.1]) and the claim above, for every  $h \in E_v^{-1}(\ker(m))$  we get

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} h(A_j(\gamma_v)) = 0.$$

But we have

$$\frac{1}{k} \sum_{j=0}^{k-1} h(A_j(\gamma_v)) = h\left(\frac{1}{k} \sum_{j=0}^{k-1} A_j(\gamma_v)\right),$$

and thus the sequence  $f_k = (1/k) \sum_{j=0}^{k-1} A_j(\gamma_v)$  is a weakly convergent sequence of integrable holomorphic functions. Let  $\phi_0$  be a weak limit of  $f_k$ . Then  $\phi_0 \in A(S_R)$  and  $\{f_k\}$  is a bounded sequence which, in particular, converges to  $\phi_0$  in  $*$ -weak topology, hence  $\phi_0$  is a pointwise limit of  $\{f_k\}$ . Since  $\mu$  is invariant and  $\int \mu \gamma_v \neq 0$ , we have that  $\int \mu \phi_0 \neq 0$ . Now recall that the sequence  $\{A_n(\gamma_v)\}$  is either degenerating or converges by norm. If  $\{A_n(\gamma_v)\}$  is degenerating then  $f_k$  is also degenerating. This implies that  $\phi_0 = 0$ , which contradicts the existence of  $\mu$ . Thus  $\{A_n(\gamma_v)\}$  converges in norm. Then  $E_v(h)$  is a convergent sequence for every  $h \in A^*(S_R)$ . □

We have the following result.

**THEOREM 8.3.** *The map  $R$  satisfies Sullivan’s conjecture if and only if  $M(\hat{T}_v)$  is finite for every critical value  $v$ .*

*Proof.* If  $M(\hat{T}_v)$  is finite then, by Theorem 8.2, it consists of at most one element  $m_v$ . The space  $E_v(A^*(S_R))$  consists only of convergent sequences for every critical value  $v$ . By definition  $HK(R)$  contains all elements  $h$  in  $A^*(S_R)$  such that  $E_v(h)$  is a sequence converging to zero for every critical value  $v$ . In other words,  $\bigcap_v \ker(m_v(E_v(A^*(S_R)))) \subset HK(R)$ . But the space  $\bigcap_v \ker(m_v(E_v(A^*(S_R))))$  has finite codimension in  $A^*(S_R)$ .

This implies that  $HK(R)$  is coseparable in  $A^*(S_R)$ , and by Theorem 7.1 the map  $R$  satisfies Sullivan’s conjecture.

By Corollary 7.3 the space  $HK(R)$  coincides with  $A^*(S_R)$ , and by Lemma 8.1 and Theorem 8.2, the converse follows. □

Now we prove the following theorem.

**THEOREM 8.4.** *Let  $R$  be a rational map without rotational domains. Then for any given critical value  $v$  of  $R$ , the following statements are equivalent.*



- (1) The space  $M(\hat{T}_v)$  is finite.
- (2) The restriction  $E_v : B_0(S_R) \rightarrow \ell_\infty$  is weakly compact.
- (3) The restriction  $E_v : HK(R) \rightarrow \ell_\infty$  is compact.
- (4) The operator  $E_v : B(S_R) \rightarrow \ell_\infty$  is compact.

*Proof.* Clearly, we have that (4) implies (3) and (3) implies (2).

Let us show that (2) implies (4). Recall that  $R$  does not admit non-trivial quasiconformal deformations on the Fatou set, that the measure of  $P_R \cap SC(R)$  is zero, and  $F_R$  does not have rotational domains. Hence by Theorem 6.8, the Thurston operator  $T$  is mean ergodic on  $B_0(S_R)$ , so  $E_v(B_0)$  consists of convergent sequences. Since  $\ell_1$  is isometrically isomorphic to  $c^*$ , it follows that, by duality  $E_v^* : \ell_1 \rightarrow A(S_R) = B_0^*(S_R)$  is given by  $E_v^*({a_n}) = \sum_n a_n A_n(\gamma_v)$ . If  $E_v$  is weakly compact on  $B_0(S_R)$  then  $E_v^*$  is weakly compact on  $\ell_1$ . The image of the canonical basis of  $\ell_1$  is  $\{A_n(\gamma_v)\}$ . By the mean ergodicity lemma, the sequence  $\{A_n(\gamma_v)\}$  is precompact in norm. Then  $E_v^*$  is a compact operator. By duality the operator  $E_v^{**} : B \rightarrow \ell_\infty$  given by  $E_v|_{B_0(S_R)}^{**}(l) = E_v(l) = \{l(A_n(\gamma_v))\}$  is compact.

Now let us show that (1) implies (2). By Theorem 8.2,  $E_v(B(S_R))$  consists of convergent sequences. In other words,  $E_v$  defines a continuous operator from a Grothendieck space into a separable space. Since the unit ball in  $\ell_1$  is sequentially precompact in the  $*$ -weak topology, by definition of a Grothendieck space, we have that  $E_v$  is a weakly compact operator on  $B(S_R)$ . Hence the restriction of  $E_v$  on  $B_0(S_R)$  is also weakly compact.

Finally, let us show that (2) implies (1). By Theorem 8.2 it is enough to show that  $E_v(B(S_R))$  consists of convergent sequences. But  $E_v$  is a compact operator on  $B(S_R)$  and  $E_v(B(S_R)) = E_v^{**}(B_0(S_R) \subset E_v(B_0(S_R)))$ . By mean ergodicity of  $T$  on  $B_0(S_R)$ , the set  $E_v(B_0(S_R))$  consists only of convergent sequences. □

The following corollary is one of the main results of this paper.

**COROLLARY 8.5.** *A rational map  $R$  satisfies Sullivan’s conjecture if and only if either the operator  $\text{Id} - T$  is compact or the operator  $E_v$  is compact for every critical value  $v$ .*

*Proof.* First we prove the implication  $\Rightarrow$ . If  $R$  is either a Lattès map or the Julia set does not support a non-zero invariant Beltrami differential. If  $R$  is Lattès the operator  $\text{Id} - T$  is compact. The rest follows from Theorems 8.3 and 8.4.

Next we prove the implication  $\Leftarrow$ . If  $\text{Id} - T$  is compact then, by Theorem 6.6, the operator  $T$  is uniformly ergodic on  $B(S_R)$ . If  $R$  admits an invariant non-zero Beltrami differential supported on the Julia set then 1 belongs to the spectrum of  $T$ . Then Theorem 6.2 implies that the map  $R$  is Lattès. If  $E_v$  is compact for every critical value  $v$ , then  $R$  satisfies Sullivan’s conjecture by Theorems 8.3 and 8.4. □

Note that if  $\text{Id} - T$  is compact then the operators  $E_v$  are compact but the converse is not true. A map  $R$  where  $E_v \circ (\text{Id} - T)$  is not compact, for some critical value  $v$ , would serve as a counterexample to Sullivan’s conjecture. However, we have the following observation which is one of the main motivations of the present work.

**PROPOSITION 8.6.** *The operator  $E_v \circ (\text{Id} - T) : B(S_R) \rightarrow \ell_\infty$  is compact for every critical value  $v$ .*

*Proof.* A bounded sequence on  $B(S_R)$  contains a subsequence which is  $*$ -weakly convergent. Let  $\mu_i$  be a sequence on  $B(S_R)$  with  $*$ -weak limit  $\mu_0$ . Define  $\omega_i = E_v \circ (\text{Id} - T)(\mu_i)$  and  $\omega_0 = E_v \circ (\text{Id} - T)(\mu_0)$ . We will prove that  $\omega_i$  converges to  $\omega_0$  in norm.

From the definition we have

$$\begin{aligned} \|\omega_i - \omega_0\| &= \sup_n \left| \int_{S_R} A_n(\gamma_v) \cdot (\text{Id} - T)(\mu_i) - A_n(\gamma_v) \cdot (\text{Id} - T)(\mu_0) \right| \\ &= \sup_n \left| \int_{S_R} A_n(\gamma_v) \cdot (\text{Id} - T)(\mu_i - \mu_0) \right|. \end{aligned}$$

However, as  $T$  is dual to the Ruelle operator we get

$$\begin{aligned} \left| \int_{S_R} A_n(\gamma_v) \cdot (\text{Id} - T)(\mu_i - \mu_0) \right| &= \left| \int_{S_R} (\text{Id} - R^*)(A_n(\gamma_v)) \cdot (\mu_i - \mu_0) \right| \\ &\leq \frac{2\|\gamma_v\|}{n} \|\mu_i - \mu_0\|. \end{aligned}$$

Since  $\|\gamma_v\| \|\mu_i - \mu_0\|$  is bounded and  $\mu_i$  converges  $*$ -weakly to  $\mu_0$ , we have that  $\|\omega_i - \omega_0\|$  converges to zero as  $i \rightarrow \infty$ . Hence  $E_v \circ (\text{Id} - T)$  is compact.  $\square$

In general, the compactness of the composition  $E_v \circ (\text{Id} - T)$  does not imply the compactness of any of the factors. But this implication is true, by Theorem 8.4, if and only if Sullivan’s conjecture holds true.

For every  $v$ , the operator  $E_v$  has a canonical extension on  $L_\infty(\overline{\mathbb{C}})$  with the same defining formula. When  $P_R \neq \overline{\mathbb{C}}$  and  $SC(R) \cap P_R$  has Lebesgue measure zero, the extension of  $E_v$  on  $L_\infty(\overline{\mathbb{C}})$  is compact if and only if  $E_v$  is compact on  $B(S_R)$ . The extension of the operators  $E_v$  on  $L_\infty(\overline{\mathbb{C}})$ , which we also denote by  $E_v$ , gives a sort of “marking” for a rational map  $R$ . Furthermore, the operators  $E_v$  induce a topology on the rational maps as follows.

A sequence of rational maps  $R_i$  converges to  $R_0$  in the  $v$ -sense, where  $v$  is a critical value of  $R_0$ , if and only if for a given  $\mu \in L_\infty(\mathbb{C})$  there exists a sequence of critical values  $v_i$  of  $R_i$  such that  $E_{v_i}(\mu) \rightarrow E_v(\mu)$  in  $\ell_\infty$ . When  $R_i$  converges to  $R_0$  in the  $v$ -sense we will say that  $R_0$  is a  $v$ -limit of  $R_i$ .

**PROPOSITION 8.7.** *If  $R_0$  is a  $v$ -limit of  $R_i$  such that  $M(\hat{T}_{v_i})$  is finite for all critical values  $v_i$  of  $R_i$ , then  $M(\hat{T}_v)$  is also finite.*

*Proof.* Let  $c$  be the space of convergent sequences. By Theorem 8.2, it is enough to show that the image  $E_v$  belongs to  $c$ . By the hypothesis, for every critical value  $w$  of  $R_i$  the image of  $E_w$  belongs to  $c$ . As the space  $c$  is closed in  $\ell_\infty$ , the image of  $E_v$  is also a subset of  $c$ .  $\square$

In other words, roughly speaking, any  $v$ -limit of rational maps satisfying Sullivan’s conjecture also satisfies Sullivan’s conjecture. Moreover, in general not every accumulation point in the  $v$ -sense has the same degree as the approximating elements. Further details on this topology will be the subject of a forthcoming work.

9. A mixing condition

In this section we show that the Ruelle operator does not have fixed points when  $R$  satisfies a kind of mixing condition over its strongly conservative set.

We say that  $R$  satisfies the  $M$ -condition if, for an invariant ergodic probability measure  $\nu$  which is absolutely continuous with respect to the Lebesgue measure supported on the Julia set  $J_R$ ,  $R$  satisfies the following two properties.

- (1) If  $A$  and  $B$  are  $\nu$ -measurable subsets of  $SC(R) \cap J_R$ , then  $\lim \nu(B \cap R^{-n}(A)) = \nu(A)\nu(B)$ .
- (2) There exists a  $\nu$ -measurable set  $A_\nu \subset SC(R) \cap J_R$  with  $\nu(A_\nu) > 0$  such that the sequence of functions

$$B^n(\chi_{SC(R)}(x)) = \frac{\overline{(R^n)'(x)}}{(R^n)'(x)}$$

is precompact in the topology of convergence in measure on  $A_\nu$ .

The reader might recognize in the first property the classical mixing condition for invariant probability measures. We will comment on the second property at the end of this section. If there is no invariant absolutely continuous probability measure, then the  $M$ -condition is vacuously satisfied. This is the case when the Lebesgue measure of the strongly conservative set is zero.

PROPOSITION 9.1. Assume  $R$  satisfies the first property of the  $M$ -condition. Let  $\nu$  be an invariant ergodic probability measure absolutely continuous with respect to Lebesgue. Let  $W = \text{supp}(\nu)$  be the support of  $\nu$ . Then for every  $\phi$  in  $L_\infty(W, \nu)$ , the sequence

$$(|R^*|)^{*n}(\phi)$$

converges  $*$ -weakly to a constant.

*Proof.* The proof follows from classical ergodic theory, and for the reader's convenience we include it here. Let  $\phi$  be a non-negative function such that  $d\nu = \phi(z)|dz|^2$ . Since  $\nu$  is an invariant probability measure we have  $|R^*|(\phi) = \phi$ . Now consider the space  $L_1(W, \nu)$  and the operator  $S$  on  $L_1(W, \nu)$  given by  $S(g) = (1/\phi)|R^*|(g\phi)$  with dual  $S^*(\omega) = \omega(R)$  for  $\omega$  in  $L_1^*(W, \nu)$ . Note that  $S$  and  $S^*$  are contractions in both  $L_1(W, \nu)$  and  $L_\infty(W, \nu)$ . By well-known ergodic theorems (see, for instance, Ch. 6 of Dunford and Schwartz [6]), both  $S$  and  $S^*$  are contracting mean ergodic operators on  $L_p(W, \nu)$  for all  $1 \leq p < \infty$ . The first part of the  $M$ -condition implies that, for every  $f$  and  $g$  in  $L_2(W, \nu)$ , we have

$$\lim \int S^n(f) \bar{g} d\nu = \lim \int f \overline{S^{*n}(g)} d\nu = \lim \int f \overline{g(R^n)} d\nu = \int f d\nu \int \bar{g} d\nu.$$

Since  $\nu$  is a probability measure, we get the chain of inclusions

$$L_\infty(W, \nu) \subset L_2(W, \nu) \subset L_1(W, \nu),$$

and  $L_2(W, \nu)$  defines an everywhere dense subspace in  $L_1(W, \nu)$ . Hence the orbits of  $S$  and  $S^*$  converge weakly in  $L_1(W, \nu)$  and  $*$ -weakly in  $L_\infty(W, \nu)$ , respectively. Let  $f_0$  be an element in  $L_2(W, \nu)$ . Then the weak limits of  $S^n(f_0)$  and  $S^{*n}(f_0)$  are fixed points for  $S$  and  $S^*$ , respectively. But  $\nu$  is ergodic, so the spaces of fixed points of  $S$  and  $S^*$  consist only of constants. The conclusion of the proposition follows from the equality  $S^*(\mu) = \mu(R) = |R^*|(\mu)$ . □

THEOREM 9.2. Assume that  $R$  is not a Lattès map, that it satisfies the  $M$ -condition, and  $P_R \neq \mathbb{C}$ . Then there is no non-zero fixed point of the Ruelle operator in  $L_1(J_R)$ .

*Proof.* Without loss of generality we can assume that  $P_R$  is bounded. Now, assume that there exists a non-zero fixed point  $f$  of the Ruelle operator in  $L_1(J_R)$ . Then by Proposition 4.4 there exists a fixed point of the Beltrami operator  $\mu$  with  $\mu(z) = \overline{f(z)}/|f(z)|$  almost everywhere on  $\text{supp}(f)$  and  $|f| = \mu f$  is the density of a finite invariant measure  $\nu$ . After normalization we can assume that  $\nu$  is a probability measure. By Proposition 4.1, we have that  $\text{supp}(f) \subset SC(R)$ . Since  $\mu(z)$  is not zero almost everywhere and  $R$  is not a Lattès map, by the part (3) of Lemma 4.3 the support of  $f$  is a bounded measurable subset of the postcritical set  $P_R$ .

For  $S$  and  $S^*$  as in the proof of Proposition 9.1, consider the operator given by

$$Z(g) = \frac{1}{|f|} R^*(g|f|),$$

which defines an endomorphism of  $L_1(\text{supp}(f), \nu)$ . In this situation we have that

$$Z^*(\alpha)(z) = B(\alpha)(z) = \alpha(R)(z) \frac{\overline{R'(z)}}{R'(z)}$$

defines an endomorphism of  $L_\infty(\text{supp}(f), \nu)$ . We obtain  $Z(g) = \overline{\mu} S(\mu g)$  and  $Z^*(\alpha) = \mu S^*(\overline{\mu} \alpha)$ . By Proposition 9.1 the orbits of  $S$  and  $S^*$  converge weakly to constants; hence the orbits of the operators  $Z$  and  $Z^*$  converge weakly to scalar multiples of  $\overline{\mu}$  and  $\mu$ , respectively. Let  $c_g$  be the constant such that  $Z^{*n_i}(g)$  converges weakly to  $c_g \mu$ .

Let  $z_0$  be a density point of  $\text{supp}(f)$  and a continuity point of  $\mu$ . Since  $\text{supp}(f)$  is a subset of the strongly conservative set  $SC(R)$  and almost every point  $\text{supp}(f)$  is recurrent, we can assume that  $z_0$  is also recurrent. This implies that there exists a sequence  $\{n_i\}$  such that

$$|\mu(R^{n_i}(z_0)) - \mu(z_0)| \rightarrow 0.$$

But  $\mu$  is invariant, so that  $|\overline{((R^{n_i})' / (R^{n_i})')}(z_0) - 1|$  converges to zero. Using together that  $\text{supp}(f)$  is bounded and the  $M$ -condition holds, we can assume that the previous sequence converges pointwise almost everywhere in  $A_\nu \subset \text{supp}(f)$ . In this case we have  $\overline{(R^{n_i})' / R^{n_i}}(z_0) = (Z^*)^{n_i}(\chi_{\text{supp}(f)})(z_0)$ . By the Lebesgue dominated convergence theorem  $\overline{(R^{n_i})' / R^{n_i}}$  converges to its pointwise almost everywhere limit in the  $L_1$  norm on  $A_\nu$ . As norm and weak limits agree whenever they both exist we have  $c_{\chi_{\text{supp}(f)}} = 1/\mu(z_0)$ . But  $c_{\chi_{\text{supp}(f)}}$  does not depend on the point  $z_0$  or on the sequence  $\{n_i\}$ . Therefore  $\mu(z) = 1/c_{\chi_{\text{supp}(f)}}$  for almost every  $z$  in  $A_\nu$ . Since  $A_\nu \subset SC(R)$  there exists a natural  $k_0$  such that  $\nu(A_\nu \cap R^{k_0}(A_\nu)) > 0$ . Hence for a density point  $y$  of  $A_\nu \cap R^{k_0}(A_\nu)$  there exists a density point  $x \in A_\nu$  so that  $y = R^{k_0}(x)$  and by invariance we have

$$c_{\chi_{\text{supp}(f)}}(x) = \mu(x) = \mu(y) \frac{\overline{(R^{k_0})'(x)}}{(R^{k_0})'(x)} = c_{\chi_{\text{supp}(f)}}(y) \frac{\overline{(R^{k_0})'(x)}}{(R^{k_0})'(x)}.$$

Then, again as in the proof of Proposition 4.12, we have  $A_\nu \cap (R^{k_0})^{-1}(A_\nu \cap R^{k_0}(A_\nu)) \subset ((R^{k_0})')^{-1}(\mathbb{R})$ . But  $\nu$  is absolutely continuous with respect to the Lebesgue measure and  $\nu(A_\nu) > 0$ . This contradiction completes the proof. □

As an immediate corollary we have the following result.

**COROLLARY 9.3.** *If  $R$  satisfies the conditions of Theorem 9.2 and there exists a non-zero invariant Beltrami differential  $\mu$ , then  $\text{supp}(\mu) \cap SC(R)$  has Lebesgue measure zero.*

Finally, let us comment on the  $M$ -condition. According to Rees (see [21]), the known examples of rational maps for which the strongly conservative set has positive Lebesgue measure forms a set of positive Lebesgue measure consisting of ergodic maps  $R$  with  $SC(R) = \overline{\mathbb{C}}$ . In other words, there exists a unique invariant absolutely continuous probability measure  $\rho$  on  $\overline{\mathbb{C}}$  so that for any pair  $A, B$  of measurable subsets there exists  $n_0 \in \mathbb{N}$  such that the Lebesgue measure of  $R^{n_0}(A) \cap R^{n_0}(B)$  is positive. In ergodic theory this corresponds to the fact that the operator  $S$  of Proposition 9.1 has strongly convergent orbits in  $L_1(\mathbb{C}, \rho)$ . The first part of our definition of the  $M$ -condition is equivalent to weak convergence of orbits in  $L_1(\mathbb{C}, \rho)$  with respect to  $|R^*|$ , and this is the classical definition of mixing dynamical systems with respect to a not necessarily invariant measure.

Now, concerning the second property of the  $M$ -condition: the precompactness in measure of a family of bounded measurable functions on probability measure spaces is a rather simple consequence of Komlós theorem (see, for example, [1, p. 39]) which states that: *if  $(X, \alpha)$  is a probability measure space and  $f_n$  is a sequence in  $L_1(X, \alpha)$  with  $\sup \|f_n\|_1 < \infty$ , then there exist  $f_0 \in L_1(X, \alpha)$  and a subsequence  $\{g_k := f_{n_k}\} \subset \{f_n\}$  such that for any subsequence  $g_{k_i}$  of  $g_k$  we have*

$$\frac{1}{m} \sum_{i=1}^m g_{k_i} \rightarrow f_0,$$

$\alpha$ -almost everywhere for  $m \rightarrow \infty$ . We call the sequence  $g_k$  a *Komlós subsequence* of  $f_n$ . Then we have the following proposition.

**PROPOSITION 9.4.** *Let  $(X, \alpha)$  be a probability space and  $g_k$  be a Komlós subsequence of a sequence of measurable functions  $f_n$  with  $|f_n| \leq M < \infty$  almost everywhere. Then there exists a subset  $A \subset X$  with  $\alpha(A) > 0$  such that  $g_k$  converges pointwise almost everywhere on  $A$ .*

*Proof.* Let  $f_0$  be a measurable function satisfying the Komlós theorem for  $g_k$ . Then  $|f_0| \leq M$  almost everywhere. Let  $x_0 \in X$  be a point such that the functions  $f_0$  and  $g_k$  are well defined at  $x_0$  and  $(1/N) \sum_{k=1}^N g_k(x_0) \rightarrow f_0(x_0)$  as  $N \rightarrow \infty$ .

Let us show that the bounded sequence of complex numbers  $\{g_k(x_0)\}$  converges to  $f_0(x_0)$ . Since  $\{g_k(x_0)\}$  is bounded, it is sufficient to check that the accumulation set of  $g_k(x_0)$  consists of a single point  $f_0(x_0)$ . Assume that there exists a subsequence  $g_{k_i}(x_0)$  converging to  $b \neq f_0(x_0)$ . Then the Cesàro averages

$$\frac{1}{N} \sum_{i=1}^N g_{k_i}(x_0) \rightarrow b$$

as  $N \rightarrow \infty$ . But this contradicts that  $g_k$  is a Komlós subsequence. □

Since pointwise convergence almost everywhere implies convergence in measure in finite measure spaces, we conclude that the second condition is always fulfilled on  $SC(R)$ .

Let us note that if  $\mu$  is an invariant Beltrami differential then by Birkhoff’s theorem we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \bar{\mu}(R^i(z)) = \bar{\mu}(z) \left[ \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{\overline{R^i(z)}}{R^i(z)} \right) \right] \rightarrow \int \bar{\mu}(z) d\nu(z).$$

Here the convergence is almost everywhere and in the  $L_1$  norm on  $\text{supp}(\nu)$ , where  $\nu$  is an invariant absolutely continuous probability measure. So the Cesàro averages  $(1/n) \sum_{i=0}^{n-1} B^i(\chi_{SC(R)})$  converge almost everywhere to a multiple of  $\mu$  on  $\text{supp}(\mu) \cap \text{supp}(\nu) \subset SC(R)$ . If every Komlós subsequence of any subsequence of  $B^n(\chi_{SC(R)})$  converges to a multiple of  $\mu$  then the whole sequence  $B^n(\chi_{SC(R)})$  converges to  $\mu$  pointwise almost everywhere and hence converges in norm in  $L_1(\nu)$  by the Lebesgue dominated theorem.

Therefore, the second part of the  $M$ -condition is an analogue of weak mixing of the complex Perron–Frobenius operator acting on  $L_1(\nu)$  as the operator  $Z$  defined on the proof of Theorem 9.2.

Finally, the next observation is the main motivation of this section. This proposition follows from classical ergodic theorems and a well-known fact, due to Sullivan, which states that a measurable set  $A \subset J_R$  has zero Lebesgue measure whenever the iterates  $R^n$  are injective on  $A$  and the Lebesgue measure of  $R^n(A) \cap R^m(A)$  is zero for all distinct  $m, n > 0$ .

**PROPOSITION 9.5.** *Assume that a rational map  $R$  is injective on  $P_R \cap J_R$  and the Lebesgue measure of  $P_R \cap SC(R)$  is zero. Then the Lebesgue measure of  $P_R$  is zero if and only if  $R$  is mixing on its conservative part with respect to the Lebesgue measure restricted on  $P_R$ .*

*Proof.* If the Lebesgue measure of  $P_R$  is zero then clearly  $R$  is mixing on  $P_R$  with respect to Lebesgue measure.

Assume that the Lebesgue measure of  $P_R$  is positive. Let us consider the dynamics of  $R$  restricted to  $P_R$  and let  $C(P_R)$  and  $D(P_R)$  be the conservative and dissipative parts of this action, respectively. As  $R$  is injective on  $P_R \cap J_R$ , by Sullivan’s lemma stated above, the Lebesgue measure of  $D(P_R)$  is zero. Since the measure of  $SC(R) \cap P_R$  is zero, then by Proposition 4.1 there are no invariant absolutely continuous measures on  $P_R$ .

Assume that  $R$  is mixing on the conservative part of  $P_R$ . Using Theorem 1.4 on p. 255 of [12] which states that every positive contraction  $E$  on a  $L_1$  space has orbits strongly convergent to zero whenever  $E$  has weakly convergent orbits and  $E$  has no non-zero fixed points, for every  $\phi \in L_1(C(R))$  the orbit of  $|R^*|^n(\phi)$  converges strongly to zero. As  $R(C(R)) = C(R)$ , we have

$$0 = \lim_{n \rightarrow \infty} \int_{C(R)} |R^{*n}| \chi_{C(R)} = \lim_{n \rightarrow \infty} \int_{R^{-n}(C(R))} \chi_{C(R)} = \int \chi_{C(R)}.$$

Thus the Lebesgue measure of  $C(R)$  is zero and hence  $P_R$  has also Lebesgue measure zero. □

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