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Modal Quantifiers, Potential Infinity, and Yablo sequences

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Abstract

When properly arithmetized, Yablo's paradox results in a set of formulas which (with local disquotation in the background) turns out to be consistent, but ω-inconsistent. Adding either uniform disquotation or the ω-rule results in inconsistency. Since the paradox involves an infinite sequence of sentences, one might think that it doesn't arise in finitary contexts. We study whether it does. It turns out that the issue depends on how the finitistic approach is formalized. On one of them, proposed by M. Mostowski, all the paradoxical sentences simply fail to hold. This happens at a price: the underlying finitistic arithmetic itself is ω-inconsistent. Finally, when studied in the context of a finitistic approach which preserves the truth of standard arithmetic (developed by one of the authors), the paradox strikes back — it does so with double force, for now the inconsistency can be obtained without the use of uniform disquotation or the ω-rule.

1 Introduction

Yablo (1993) provided a by now famous example of a semantic paradox which, according to the author, does not involve self-reference. Recall the paradox arises when one considers the following sequence of sentences:

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Y_0 For any $k > 0$, Y_k is false.
Y_1 For any $k > 1$, Y_k is false.
Y_2 For any $k > 2$, Y_k is false.
$\vdots$
Y_n For any $k > n$, Y_k is false.
$\vdots$

Take any sentence Y_n from the sequence and ask what would happen if it was true. Suppose it is. Then, things are as it says, and for any $j > n$ Y_j is false. In particular Y_{n+1} is false and also for any $j > n + 1$ Y_j is false.

But the second conjunct is exactly what Y_{n+1} states, so it turns that Y_{n+1} is true after all. The assumption that Y_n is true led therefore to a contradiction. So it is false. This means that not all sentences following Y_n are false, and so one of them, say Y_k, is true. But then, we can again obtain a contradiction by repeating for Y_k the same reasoning that we have just given for Y_n. So, whether Y_n is true, or false, a contradiction follows. Hence the paradox.

The most complete source on Yablo's paradox is currently a seminal book by Roy Cook (2014). It can be treated both as a comprehensive guide to the mathematical foundations behind formalizing the paradox, as well as a philosophical monograph concerning the (alleged) circularity of the paradox and its generalizations. Cook provides an extensive examination of the concept of circularity and analyzes in which characterizations (in different languages) Yablo's paradox can be considered truly non-circular. Further, he shows how Yabloesque techniques might be generalized to generate (apparently) non-circular versions of other paradoxes. We believe that our results provide new insights into a slightly different thread of the debate on the paradox than the ones that are of focus in Cook's book. We do not engage in the discussion of circularity at all, and explore meta-logical consequences of introducing a new and independently motivated way of formalizing the paradox within a modal, finitistic, arithmetical setting.

A fruitful study of the paradox formalized over arithmetic performed e.g. in (Ketland, 2005) and AUTHOR'S PAPER has revealed that the reasoning has the following interesting feature. In order to derive the contradiction one needs to use a strong assumption concerning the notion of truth: namely one has to assume "for all n, Y_n if and only if $\ulcorner Y_n \urcorner$ is true." $\forall n\ (Y_n \equiv Tr(Y_n))$. If we wanted to replace this *uniform disquotation* with an infinity of *local disquotation* instances, contradiction could be obtained only if we used some infinitary inference rule (requiring an infinite number of premises) such as the ω-rule. We'll begin the paper by surveying the relevant results

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in the next section.

So far, the story is rather well-known. What is somewhat less known, is that there is a way of handling the paradox which relies on finitistic assumptions. After all, if the world is finite, there aren't enough things in the world to interpret all sentences from the Yablo sequence, and the last interpreted one is vacuously true without any threat of paradox. Yablo's paradox can be thought of as an infinitary version of the Liar paradox, so perhaps thinking it can be dealt with by tackling the notion of infinity isn't extremely implausible.

Of course, the finitist owes us a story about how they make sense of arithmetic, and how the whole thing should be studied by formal methods. It so happens, that formal tools for this task have already been developed (Mostowski, 2001a), (Mostowski, 2001b), (Mostowski and Zdanowski, 2005). In what follows we'll explain what the finitist story about arithmetic is, and we'll use it to study the Yablo paradox in the finitistic setting. On this approach, it will turn out that things are as we expected: Yablo sentences are all false in potentially infinite domains, despite the fact that the framework is rich enough to incorporate sufficiently strong arithmetic.

There is, however, a glitch. We'll argue that the way quantifiers are handled in this finitistic setting results in a somewhat unusual arithmetical theory. For instance, in a *potentially infinite* domain it turns out that the sentence "there is a greatest number" comes out true without making "n is the greatest number" true for any n. If your goal, as a finitist, is not to revise current mathematics, but to make sense of it in terms of potential infinity, this approach isn't for you.

It turns out that there is another formal approach to potential infinity developed in AUTHOR'S PAPER,which has already been used to obtain standard arithmetic, and to make sense of abstraction principles (in the neologicist sense). In the third part of this paper we study how this framework handles Yablo's paradox. It turns out that the price of making potential infinity digestible to classical mathematicians is that the Yablo paradox strikes back, with even more power than in the standard arithmetical setting.[1]

[1]There are some similarities between the potentialist approach to thinking about quantification (especially higher-order) developed in (Urbaniak, 2008a,b, 2010, 2012, 2014) and (Urbaniak, 2016), and the one present in (Linnebo, 2013; Linnebo and Shapiro, 2017; Hamkins and Linnebo, 2017), and (Linnebo, 2018). While there is some overlap of ideas and interests (Urbaniak and Linnebo worked together in Bristol in 2009), Linnebo's approach is not motivated by nominalist considerations and leads to different technical developments. The discussion of these differences lies beyond the scope of this paper.

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Before presenting our results, we would like to mention it is worth noting that in a recent paper on *A Logical Foundation of Arithmetic* by Joongol Kim (2015) the author developed an approach to formalizing foundations of artihmetic by logical means only, using modal semantics for expressing logical necessity. He gave an account of how arithmetical reasoning can be captured in a purely logical framework. This use of a modal framework is justified by the need to provide a criterion of identity for numbers, formulated with reference to contexts involving sentences indirectly referring to cardinalities, such as a simple *there are* n F*s*. Kim develops an *Adverbial Logic for Arithmetic* (ALA), demonstrates that basic concepts of arithmetic can be explicitly defined in the language of ALA, and that PA can be derived from those definitions by what he claims to be logical means alone. A crucial distinction that is made in Kim's paper — and which is loosely related to the modal semantics of FM-domains we introduce in this paper — is the one between generic individual terms and numerical terms of the language of arithmetic (for the details, see the original paper), and some heavy lifting is done by their semantics in ALA. What is important in our approach, which we apply to a finitistically motivated account of arithmetic (rather than Kim's approach stemming from the debate on neologicism), is that we are not concerned with distinguishing between these types of terms explicitly, but rather defining quantifier clauses so that their semantics includes information concerning the role of terms both as expressions referring to possible sizes of finite arithmetical models, as well as to the positions in such models.

2 Arithmetization of Yablo sentences

Let's start by going over the results pertaining to Yablo sentences obtained in the standard arithmetical setting.

2.1 Existence of Yablo sequences

One might ask how we actually know that Yablo sequences exist in formal theories. This is a legitimate question since we're moving from the paradox as formulated hand-wavily in natural language to its properly defined formalized counterpart. It is possible to construct a Yablo sequence within a given theory, but in order to do so, we need to use a general version of the diagonal lemma (for formulae with two free variables in the language containing the truth predicate):

Diagonal Lemma 1. Let $\mathtt{T}$ be a first-order theory in the language $\mathcal{L}_{Tr}$ (a language of arithmetic extended by a truth predicate), containing Robinson's arithmetic Q. Then, for any $\mathcal{L}_{Tr}$-formula $\varphi(x, y)$ there is a $\mathcal{L}_{Tr}$-formula $\psi(x)$ such that:

$$\mathtt{T} \vdash \ \psi(x) \equiv \varphi(x, \overline{\ulcorner \psi(x) \urcorner}).$$

⊣

The expression $Qx\ P(\overline{\ulcorner \varphi(\dot{x}) \urcorner})$, where $Q \in \{\forall, \exists\}$ is to be read as follows: *for all natural numbers x (there exists a natural number x such that), the result of substituting a numeral denoting x for a variable free in φ has property P.* When the intended meaning is clear from the context, we sometimes omit the dot notation.

Definition 1 (Yablo Formula). $Y(x)$ is a Yablo formula in a theory $\mathtt{T}$ iff it satisfies the *Yablo condition*, i.e.:

$$\mathtt{T} \vdash \forall x(Y(x) \equiv \forall w > x \neg Tr(\overline{\ulcorner Y(\dot{w}) \urcorner})).$$

⊣

This also gives rise to a natural way of defining sentences belonging to a Yablo sequence.

Definition 2 (Yablo Sentence). φ is a Yablo sentence in a theory $\mathtt{T}$ iff it is obtained by substituting a numeral for x in Yablo formula $Y(x)$. ⊣

This enables us to prove that we are not dealing with fictitious entities, but with properly defined formal objects. Next, we prove:

Theorem 1. [Existence of Yablo Formula (Priest, 1997)] Let $\mathtt{T}$ be a theory in a language $\mathcal{L}_{Tr}$ containing Peano Arithmetic $\mathtt{PA}$. Then there exists a Yablo formula in $\mathtt{T}$.

Proof. Let $\varphi(x, y) = \forall w > x\ \neg Tr(sub(y, \ulcorner y \urcorner, name(w)))$. By the Diagonal Lemma, there is a formula $Y(x)$ such that:

$$\mathtt{T} \vdash Y(x) \equiv \forall w > x\ \neg Tr(sub(\overline{\ulcorner Y(x) \urcorner}, \ulcorner y \urcorner, name(w))).$$

Therefore by the fact that *sub* represents the appropriate function, we have:

$$\mathtt{T} \vdash Y(x) \equiv \forall w > x\ \neg Tr(\overline{\ulcorner Y(\dot{w}) \urcorner}).$$

Thus we may conclude that $Y(x)$ is a Yablo formula. Hence, Yablo sentences exist. □

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2.2 Consistency of Yablo sequences

An interesting question one might ask is exactly what principles about Yablo sentences lead to the inconsistency of a formal theory. Despite the fact that on the level of natural language it is not difficult to derive contradiction from the definition of Yablo sequence, the formal counterpart (to be specified below) is only ω-inconsistent and consistent. Most of the results from this section were originally obtained by Ketland (2005); but Theorem 4 is new.

Definition 3 (ω-consistency). Let $\mathtt{T}$ be a first-order theory in the arithmetical language $\mathcal{L}$. $\mathtt{T}$ is ω-consistent if there is no $\varphi(x) \in Frm_{\mathcal{L}}$ such that simultaneously:

$$\forall n \in \omega \ \ \mathtt{T} \vdash \neg\varphi(\overline{n})$$
$$\mathtt{T} \vdash \exists x \varphi(x)$$

⊣

If there is such a formula φ, then we say that T is an ω-inconsistent theory.

Definition 4 ($\mathtt{PA_F}$). Let $\mathcal{L}_F$ be a language of arithmetic extended by one monadic predicate F.

$$\mathtt{PA_F} := \mathtt{PA} \cup \{F(\overline{n}) \equiv \forall x > \overline{n}\, \neg F(x) \ : n \in \omega\}$$ ⊣

Lemma 1. $\mathtt{PA_F}$ is ω-inconsistent.

Proof. Work in $\mathtt{PA_F}$. Fix an $n \in \omega$ and assume $F(\overline{n})$. By the definition of $\mathtt{PA_F}$, we have:

$$\forall x > \overline{n}\, \neg F(x). \tag{$\star$}$$

In particular, ($\star$) entails $\forall x > \overline{n+1}\, \neg F(x)$. This sentence is however equivalent to $F(\overline{n+1})$. But from ($\star$), $\neg F(\overline{n+1})$ follows. Contradiction. So $\neg F(\overline{n})$.

Since the choice of n was arbitrary, this means:

$$\forall n \in \omega \quad \mathtt{PA_F} \vdash \ \neg F(\overline{n}). \tag{1}$$

But by the definition of $\mathtt{PA_F}$ we then have:

$$\forall n \in \omega \quad \mathtt{PA_F} \vdash \exists x > \overline{n}\, F(x).$$

In particular:

$$\mathtt{PA_F} \vdash \exists x\, F(x). \tag{2}$$

(1) with (2) jointly mean that $\mathtt{PA_F}$ is ω-inconsistent, as advertised.

□

Lemma 2. $\mathtt{PA_F}$ is consistent.

Proof. Consider any nonstandard model $\mathcal{M}$ of $\mathtt{PA}$. Pick a nonstandard element $a \in M$ (M is the domain of $\mathcal{M}$) and let $A = \{a\}$. Extend the language of $\mathtt{PA}$ with a monadic predicate F and put $F^{\mathcal{M}} = A$. Since a is nonstandard, we get

$$\forall n \in \omega \ \ (\mathcal{M}, A) \models \neg F(n).$$

But since A is non-empty, $(\mathcal{M}, A) \models \exists x\, F(x)$. Moreover, because there is a non-standard witness for the formula $F(x)$, we have:

$$\forall n \in \omega \ \ (\mathcal{M}, A) \models \ \exists x > n\, F(x).$$

Hence we obtain $\forall n \in \omega \ \ (\mathcal{M}, A) \models \ F(n) \equiv \forall x > n\, \neg F(x)$ (because both sides of the equivalence are false in the model). But because $\mathcal{M}$ is already a model of $\mathtt{PA}$, the last statement lets us conclude that $(\mathcal{M}, A) \models \ \mathtt{PA_F}$, which by soundness means that $\mathtt{PA_F}$ is consistent. □

Definition 5 (Local Arithmetical Disquotation).

$$AD = \{Tr(\overline{\ulcorner \varphi \urcorner}) \equiv \varphi \ : \varphi \in Sent_{\mathcal{L}}\}$$

⊣

Definition 6 (Local Yablo Disquotation).

$$YD = \{Tr(\overline{\ulcorner Y(\overline{n}) \urcorner}) \equiv Y(\overline{n}) \ : Y(\overline{n}) \text{ belongs to the Yablo sequence}\}.$$

⊣

Definition 7 ($\mathtt{PA_D}$). Let $\mathtt{PAT}^-$ be a theory obtained from $\mathtt{PA}$ by extending the language of arithmetic with a truth predicate Tr. Further, let $\mathtt{PAT}$ be obtained from $\mathtt{PAT}^-$ by allowing the induction scheme to apply also to formulae containing Tr).[2] Let $\mathtt{PA_D} = \mathtt{PAT} \cup AD \cup YD$ and let $\mathtt{PA_D^-}$ be $\mathtt{PA_D}$ with the induction axiom scheme restricted to formulae without the truth predicate.

Theorem 2. $\mathtt{PA_D^-}$ is ω-inconsistent.

[2] $\mathtt{PAT}$ is then not really a theory of truth, with Tr being just a new predicate, without any substance to it.

Proof. By Definition 1 (of Yablo formulae) and Theorem 1 we obtain:

$$\forall n \in \omega \;\; \mathtt{PA_D} \vdash Y(\overline{n}) \equiv \forall x > \overline{n}\, \neg Tr(\overline{\ulcorner Y(\dot{x}) \urcorner}).$$

By the fact that $\mathtt{PA_D}$ contains YD we get:

$$\forall n \in \omega\, \mathtt{PA_D} \vdash Tr(\overline{\ulcorner Y(\overline{n}) \urcorner}) \equiv \forall x > \overline{n}\, \neg Tr(\overline{\ulcorner Y(\dot{x}) \urcorner}).$$

Let $F(x) := Tr(\overline{\ulcorner Y(\dot{x}) \urcorner})$. With this definition we get:

$$\forall n \in \omega \;\; \mathtt{PA_D} \vdash F(\overline{n}) \equiv \forall x > \overline{n} \neg F(x).$$

This means $\mathtt{PA_D}$ contains $\mathtt{PA_F}$, and so applying Lemma 1 to this fact ends the proof. □

It immediately follows:

Corollary 1. $\mathtt{PA_D}$ is ω-inconsistent.

However:

Theorem 3. $\mathtt{PA_D^-}$ is consistent.

Proof. Consider any nonstandard model $\mathcal{M}$ of $\mathtt{PA}$. Take

$$Th_{\mathcal{L}}(\mathcal{M}) = \{\ulcorner \varphi \urcorner : \mathcal{M} \models \varphi \text{ and } \varphi \in Sent_{\mathcal{L}}\}.$$

Further, let us denote $t(x) := \ulcorner Y(\dot{x}) \urcorner$ (t assigns to any number n, the code of the n-th Yablo sentence; the value of t depends on the value of x which occurs as a free variable in the formula $Y(x)$). The formula $t(x) = y$ is true of infinitely many standard numbers, and so by overspill there are nonstandard numbers b and c such that $t^{\mathcal{M}}(b) = c$, i.e. $c = \ulcorner Y(b) \urcorner$. Let $S = Th_{\mathcal{L}}(\mathcal{M}) \cup \{c\}$. We extend the language of $\mathtt{PA}$ with the truth predicate Tr and put $Tr^{\mathcal{M}} = S$. We then have

$$\forall n \in \omega \quad (\mathcal{M}, S) \models \exists x > n \;\, Tr(\ulcorner Y(\dot{x}) \urcorner),$$

since we have $(\mathcal{M}, S) \models Tr(c)$ and hence $\forall n \in \omega \;\; (\mathcal{M}, S) \models \; b > n \wedge Tr(\ulcorner Y(b) \urcorner)$. S contains the codes of all true arithmetical sentences, so:

$$(\mathcal{M}, S) \models AD.$$

The codes of standard sentences $Y(\overline{n})$ do not belong to S — such sentences are neither arithmetical (because they contain the truth predicate) nor coded by c (because c codes a non-standard Yablo sentence). Thus we conclude that

$$\forall n \in \omega \;\; (\mathcal{M}, S) \models \neg Tr(\ulcorner Y(\bar{n}) \urcorner).$$

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Additionally, by Definition 1 and Theorem 1 we have:

$$\forall n \in \omega \ \ \mathtt{PAT}^{-} \vdash Y(\overline{n}) \equiv \forall x > n\, \neg Tr(\ulcorner Y(\dot{x}) \urcorner).$$

Since $(\mathcal{M}, S)$ is a model of $\mathtt{PAT}^{-}$, we obtain:

$$\forall n \in \omega\, (\mathcal{M}, S) \models Y(\overline{n}) \equiv \forall x > n\, \neg Tr(\ulcorner Y(\dot{x}) \urcorner).$$

But from this it follows that

$$\forall n \in \omega\, (\mathcal{M}, S) \models \neg Y(\overline{n}). \tag{3}$$

Therefore we may conclude that

$$(\mathcal{M}, S) \models YD$$

which ends the proof. □

Note that in what follows some of the arguments are model-theoretic: in such contexts, we sometimes talk about particular numbers without assuming that they are standard, and so without assuming that there are standard numerals. For this reason we'll follow the shady practice of not using the bar notation for numerals in such contexts.

Theorem 4. $\mathtt{PA}_{\mathtt{D}}$ is consistent.

Proof. We will show that $\mathtt{PA}_{\mathtt{D}}$ is finitely satisfiable. Consider any finite subset Δ of $\mathtt{PA}_{\mathtt{D}}$. There is only a finite number of elements of YD (that is, instances of disquotation schema for Yablo sentences) in Δ. Hence, there is a greatest m such that there is an element of YD for $Y(m)$ in Δ. Without loss of generality, we may assume that for all $n \leq m$ Δ contains elements of YD for sentences $Y(n)$. Now take a model $\mathcal{M} \models \mathtt{PA}$ and expand it by:

$$Tr^{\mathcal{M}} := \{\ulcorner Y(m) \urcorner\} \cup \{\ulcorner \varphi \urcorner :\ \varphi \in Sent_{\mathcal{L}} \cap \Delta \ \text{ and } \ \mathcal{M} \models \varphi\}.$$

By this definition we have:

(1) $(\mathcal{M}, Tr^{\mathcal{M}}) \models Tr(\ulcorner Y(m) \urcorner)$,

(2) $\forall k < m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models \neg Tr(\ulcorner Y(k) \urcorner)$,

(3) $\forall k > m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models \neg Tr(\ulcorner Y(k) \urcorner)$,

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and for all arithmetical $\varphi \in \Delta$ that hold in $\mathcal{M}$:

$$(4) \quad (\mathcal{M}, Tr^{\mathcal{M}}) \models Tr(\ulcorner \varphi \urcorner).$$

From (1) we easily get that: $\forall k < m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models \exists x > k\, Tr(\ulcorner Y(\dot{x}) \urcorner)$, and hence by the definition of Yablo sentences:

$$(5) \quad \forall k < m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models \neg Y(k).$$

By (3) we obtain: $(\mathcal{M}, Tr^{\mathcal{M}}) \models \forall x > m\, \neg Tr(\ulcorner Y(\dot{x}) \urcorner)$, and hence, again, by the definition of Yablo sentences:

$$(6) \quad (\mathcal{M}, Tr^{\mathcal{M}}) \models Y(m).$$

(1) and (6) give us then:

$$(\mathcal{M}, Tr^{\mathcal{M}}) \models Y(m) \equiv Tr(\ulcorner Y(m) \urcorner),$$

whereas by (2) and (5) we have:

$$\forall k < m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models Y(k) \equiv Tr(\ulcorner Y(k) \urcorner).$$

These two statements are obviously equivalent to: $\forall n \leq m \ (\mathcal{M}, Tr^{\mathcal{M}}) \models Y(n) \equiv Tr(\ulcorner Y(n) \urcorner)$, and it follows that:

$$(*) \quad (\mathcal{M}, Tr^{\mathcal{M}}) \models \Delta \cap YD.$$

From (4) and definition of $Tr^{\mathcal{M}}$ we immediately obtain: $(\mathcal{M}, Tr^{\mathcal{M}}) \models \varphi \equiv Tr(\ulcorner \varphi \urcorner)$ for all arithmetical $\varphi \in \Delta$, which means that:

$$(**) \quad (\mathcal{M}, Tr^{\mathcal{M}}) \models \Delta \cap AD.$$

Let $Ind(\psi)$ denote the instance of the induction scheme for the $\mathcal{L}_{Tr}$ formula ψ. Since $\mathcal{M} \models \mathtt{PA}$, clearly for all arithmetical ψ, if $Ind(\psi) \in \Delta$, $\mathcal{M} \models Ind(\psi)$.

For ψ containing Tr such that $Ind(\psi) \in \Delta$, consider that the only atomic ψ that we care about is of the form $Tr(k)$. The instance of induction for $Tr(k)$ will have the following form:

$$Tr(0) \wedge \forall x\, (Tr(x) \rightarrow Tr(x+1)) \rightarrow \forall x\, Tr(x)$$

Clearly, the antecedent is false: 0 doesn't code a true sentence, and it is false that if a number codes a true sentence then so does its successor. It is folklore result that if a

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predicate is inductive, the induction scheme can be extended to all formulae containing it.[3]

So, by the reasoning above we can even make a stronger claim that for any $\psi \in \mathcal{L}_{Tr}$:

$$(***) \quad (\mathcal{M}, Tr^{\mathcal{M}}) \models Ind(\psi).$$

It follows from $(*)$, $(**)$ and $(***)$ that $(\mathcal{M}, Tr^{\mathcal{M}}) \models \Delta$. By compactness, this means that $\mathtt{PA_D}$ is satisfiable and therefore consistent. □

Corollary 2. $\mathtt{PA_D}$ is a conservative extension of $\mathtt{PA}$.

Proof. Let us consider an arithmetical sentence φ such that $\mathtt{PA} \nvdash \varphi$. This entails that $\mathtt{PA} \cup \{\neg\varphi\}$ is consistent. Then there is a nonstandard model $\mathcal{M}$ of $\mathtt{PA}$ such that $\mathcal{M} \models \neg\varphi$. Since in the proof of Theorem 4 it did not actually matter what arithmetical extension of $\mathtt{PA}$ we started with, we can repeat the reasoning with $Th(\mathcal{M}) \supset \mathtt{PA_D} \cup \{\neg\varphi\}$, and so there is a model $\mathcal{M}'$ elementarily equivalent to $\mathcal{M}$ such that $(\mathcal{M}', Tr^{\mathcal{M}'}) \models \mathtt{PA_D}$. Obviously, since by the construction: $\neg\varphi \in Tr^{\mathcal{M}'}$, we have $(\mathcal{M}', Tr^{\mathcal{M}'}) \not\models \varphi$, and so $\mathtt{PA_D} \nvdash \varphi$. □

It is however still possible to derive a contradiction from the Yablo sequence, but in order to achieve this we would have to use a generalized version of the truth principle governing the Yablo sequence — it would be necessary to add to our arithmetical theory principles that would be strong enough to prove a version of the disquotation schema *uniform* for all Yablo sentences.

Definition 8 (Uniform Yablo Disquotation).

$$\forall x (Tr(\overline{\ulcorner Y(\dot{x}) \urcorner}) \equiv Y(x)) \tag{UYD}$$

Theorem 5. Let $\mathtt{S} = \mathtt{PAT} + \mathtt{UYD}$. S is inconsistent.

Proof. We are working in $\mathtt{S}$. By the definition of Yablo formula and Theorem 1, we have

$$\forall x \, (Y(x) \equiv \forall w > x \neg Tr(\overline{\ulcorner Y(\dot{w}) \urcorner})).$$

From (UYD) we then derive:

$$\forall x \, (Y(x) \equiv \forall w > x \, \neg Y(\dot{w})). \tag{4}$$

[3] This folklore result is usually proved by induction on formula complexity.

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From this and unraveling of $Y(w)$ we infer:

$$\forall x\,(Y(x) \equiv \forall w > x \exists z > w\, Tr(\ulcorner Y(\dot{z})\urcorner)).$$

By (UYD) again we get

$$\forall x\,(Y(x) \equiv \forall w > x\, \exists z > w\, Y(z)). \tag{5}$$

By extendability ($\forall x \exists\, y\, x < y$) and transitivity of $<$ (both clearly provable in $\mathtt{S}$) and (5) we get:

$$\forall x\,(Y(x) \equiv \exists w > x\, Y(w)). \tag{6}$$

By (4) and (6) we have:

$$\forall x\,((\forall w > x \neg Y(w)) \equiv (\exists w > x Y(w)))$$

Contradiction. □

Definition 9 (ω-rule). The ω-rule is the following infinitary inference rule defined for arithmetical formulae φ:

$$\frac{\varphi(0),\ \varphi(\overline{1}),\ \varphi(\overline{2}),\ ...,\ \varphi(\overline{n}),\ ...}{\forall x\, \varphi(x)}$$

Definition 10 ($\mathtt{T}^{\omega}$). Let $\mathtt{T}$ be an axiomatizable first-order theory in the language $\mathcal{L}_{Tr}$. If α is an ordinal, a sequence $(\varphi_0, ..., \varphi_\alpha)$ of formulae is a derivation in ω-logic (ω-derivation) from $\mathtt{T}$ if and only if, for each ordinal $\beta \leq \alpha$:

1. φ_β is an axiom of $\mathtt{T}$ or
2. there are $\gamma < \beta$ and $\delta < \beta$, such that $\varphi_\delta = (\varphi_\gamma \rightarrow \varphi_\beta)$, or
3. (3) $\varphi_\beta = \forall x \psi(x)$ for some $\psi(x) \in Frm_{\mathcal{L}_{Tr}}$ with exactly one free variable and there is an injective function $f : \omega \rightarrow \beta$ such that $\forall n \in \omega\ \ \varphi_{f(n)} = \psi(n)$ (this condition means that φ_β has been introduced by means of the ω-rule).

We say that φ is a theorem of $\mathtt{T}$ in ω-logic if there is an ω-derivation of φ from $\mathtt{T}$. We also call such φ an ω-consequence of $\mathtt{T}$ and denote it as follows: $\mathtt{T}^{\omega} \vdash \varphi$. $\mathtt{T}^{\omega}$ is a set of sentences that are theorems of $\mathtt{T}$ in ω-logic. ⊣

Theorem 6. Let $\mathtt{PA}_{\mathtt{D}}^{\omega-} = (\mathtt{PAT}^{-} \cup AD \cup YD)^{\omega}$. $\mathtt{PA}_{\mathtt{D}}^{\omega-}$ is inconsistent.

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Proof. One way to see why this holds is by reference to Theorem 2: adding the ω-rule to an ω-inconsistent system results in an inconsistent system. But we find the slow-motion version of the argument somewhat informative, so here it is.

Since $\mathtt{PA}_{\mathtt{D}}^{\omega -}$ is an extension of $\mathtt{PAT}^{-}$, by Theorem 1 Yablo sentences exist in this theory. We are working in $\mathtt{PA}_{\mathtt{D}}^{\omega -}$. Fix $n \in \omega$ and suppose $Y(\overline{n})$ for *reductio*. By the definition of Yablo sentence, we therefore have $\forall x\,(x > \overline{n} \Rightarrow \neg Tr(\overline{\ulcorner Y(\dot{x})\urcorner}))$. Hence

$$\neg Tr(\overline{\ulcorner Y(\overline{n+1})\urcorner}) \tag{7}$$

as well as $\forall x\,(x > \overline{n+1} \Rightarrow \neg Tr(\overline{\ulcorner Y(\dot{x})\urcorner}))$. But the last sentence is equivalent to $Y(\overline{n+1})$. By the appropriate disquotation instance we therefore get

$$Tr(\overline{\ulcorner Y(\overline{n+1})\urcorner})$$

which contradicts (7). Since the choice of n was arbitrary, we have:

$$\forall n \in \omega \quad \mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash \neg Y(\overline{n}).$$

In particular:

$$(*) \quad \mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash \neg Y(\overline{23}).$$

By the fact that we have a disquotation instance for every Yablo sentence in our theory we also have:

$$\forall n \in \omega \quad \mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash \neg Tr(\overline{\ulcorner Y(\overline{n})\urcorner}).$$

It is worth noting that so far we've only used the resources of $\mathtt{PA}_{\mathtt{D}}$ in this proof. Now we'll need to use the specific means of $\mathtt{PA}_{\mathtt{D}}^{\omega -}$. By applying the ω-rule we therefore obtain:

$$\mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash \forall x\, \neg Tr(\overline{\ulcorner Y(\dot{x})\urcorner}).$$

In particular we have

$$\mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash \forall x\,(x > \overline{23} \rightarrow \neg Tr(\overline{\ulcorner Y(\dot{x})\urcorner})).$$

But this means we have:

$$(**) \quad \mathtt{PA}_{\mathtt{D}}^{\omega -} \vdash Y(\overline{23}).$$

It immediately follows from $(*)$ and $(**)$ that $\mathtt{PA}_{\mathtt{D}}^{\omega -}$ is inconsistent. □

Let's sum up the situation so far. We have a paradox in natural language. Formalizing it, as long as we work in a disquotational theory of truth without any other assumptions, even those theories which prove the existence of Yablo sentences are still

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consistent (albeit ω-inconsistent). To obtain a contradiction, we need either inferential rules that go beyond the standard means of first-order logic (namely: the ω-rule), or a stronger uniform principle of disquotation for Yablo sentences.

Since the paradox somehow involves the concept of infinity (as the essential role of the ω-rule suggests), we'll turn our attention to a formalization of Yablo's paradox in a setting that takes a somewhat different approach to infinity. It is the framework developed by the late Marcin Mostowski and others in (Mostowski, 2001a), (Mostowski, 2001b) and (Mostowski and Zdanowski, 2005), meant as a formalization of the concept of potential infinity. We'll describe how one could go about avoiding the paradox by taking the distinction between potential and actual infinity seriously.

3 Potentially infinite domains and *sl*-semantics

In this section we consider the so-called sl-semantics of FM-domains. This framework was motivated by considerations in computational foundations of mathematics and the search for the semantics under which first-order sentences would be interpreted in potentially infnite domains.

Potentially infinite domains are in this framework understood as sequences of finite models increasing without any finite bound. Once we turn to actually finite domains, we need to overcome a small technical obstacle concerning the arithmetical language: it seems that the reference of some singular terms should come out undefined. One way to go would be to employ the apparatus of partial functions. A simpler method, however, uses relational symbols instead of function symbols. This is the one that we'll employ.

Let $R \subseteq \omega^r$ be an r-ary relation on natural numbers. By $R^{(n)}$ we denote $R \cap \{0, 1, ..., n-1\}^r$, the restriction of this relation to first n natural numbers. For any model $\mathcal{A}$ over some fixed signature $\sigma = (R_1, ..., R_k)$ (in particular, one can take the signature to comprise the relational counterpart of standard arithmetical functions such as addition or multiplication) we define the FM-domain of $\mathcal{A}$ as follows:

$$FM(\mathcal{A}) = \{\mathcal{A}_n : \ n = 1, 2, ...\}$$

where

$$\mathcal{A}_n = (\{0, 1, ..., n-1\}, R_1^{(n)}, ..., R_k^{(n)}).$$

That is, $FM(\mathcal{A})$ is the set of all finite initial segments of natural numbers, with the arithmetical relations appropriately restricted.

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For the purposes of this section, by $\mathbb{N}$ we denote the standard model of arithmetic $(\omega, +, \times, 0, s, <)$, where the arithmetical functions are interpreted as relations, so we have that:

$$FM(\mathbb{N}) = \{\mathbb{N}_n : \ n = 1, 2, ...\},$$

where

$$\mathbb{N}_n = (\{0, 1, ..., n-1\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)}).$$

(Note that this means that numerals are replaced with predicates, so that if in a finite point a numerical would fail to refer, its predicative counterpart refers to the empty set, and if it wouldn't, its predicative counterpart refers to an appropriate singleton. Thus, in the standard arithmetical language each n-ary symbol is replaced by the corresponding $n+1$-ary one; we'll keep referring to it as $\mathcal{L}$ with no threat of misunderstanding.)

Ultimately, we'll be evaluating formulas in $FM(\mathbb{N})$; but in order to do that, we first need to evaluate them in the finite segments that $FM(\mathbb{N})$ contains. Satisfaction at finite models involved is understood according to the standard notion of satisfaction. Satisfaction in $FM(\mathbb{N})$ is understood, intuitively, as satisfaction in all sufficiently large segments from $FM(\mathbb{N})$.

Definition 11 ($sl(FM(\mathbb{N}))$). For any $\varphi \in Sent_{\mathcal{L}}$ we say that φ is sl-true in $FM(\mathbb{N})$ (true in sufficiently large models, hence the abbreviation sl):

$$FM(\mathbb{N}) \models_{sl} \varphi \text{ if and only if } \exists m \, \forall k \ (k \geq m \ \Rightarrow \ \mathbb{N}_k \models \varphi).$$

Let us then denote:

$$sl(FM(\mathbb{N})) = \{\varphi \in Sent_{\mathcal{L}} : \ FM(\mathbb{N}) \models_{sl} \varphi\}.$$

$sl(FM(\mathbb{N}))$ is called the *sl-theory* of $FM(\mathbb{N})$. More generally we could say that for a given class $\mathcal{K}$ of finite models:

$$sl(\mathcal{K}) = \{\varphi \in Sent_{\mathcal{L}_\sigma} : \ \exists n \, \forall \mathcal{M} \in \mathcal{K} \ (card(\mathcal{M}) \geq n \ \Rightarrow \ \mathcal{M} \models \varphi)\}.$$

Now we can use the notion to define a consequence relation by saying:

$$\Gamma \models_{sl} \varphi \text{ iff } \forall \mathcal{K} \ [\Gamma \subseteq sl(\mathcal{K}) \Rightarrow \varphi \in sl(\mathcal{K})]$$

Obviously, for a given vocabulary, for any class $\mathcal{K}$ of finite models and for any set of sentences Δ we say that $\mathcal{K} \models_{sl} \Delta$ if and only if $\mathcal{K} \models_{sl} \varphi$ for any $\varphi \in \Delta$. $\dashv$

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3.1 Yablo Sequences under sl-semantics

We consider the language obtained by adjoining the truth predicate Tr to the arithmetical language $\mathcal{L}$ and we modify $(FM(\mathbb{N}))$ by adding to its every element $\mathbb{N}_k$ an interpretation T_k of the truth predicate Tr.

Definition 12 ($FM(\mathbb{N})^T$). Let $\mathcal{K} = \{(\mathbb{N}_k, T_k) : k \in \omega \text{ and } T_k \subseteq \{0, ..., k-1\}\}$. An $FM(\mathbb{N})^T$-domain is any subset of $\mathcal{K}$ such that for any natural m it contains exactly one model of the cardinality m. ⊣

We obviously have to ensure that Yablo sentences exist in $sl((FM(\mathbb{N})^T))$. For this, we need to state some important facts concerning the sl-theory of FM-domains, leaving them without a proof. Further details may be found in (Mostowski, 2001a), (Mostowski, 2001b), (Mostowski and Zdanowski, 2005), (Mostowski, 2016).

Theorem 7 (Representability of substitution and naming). There exist formulae $\boldsymbol{Sub}(x, y, z)$ and $\boldsymbol{Name}(x)$ such that for any $FM(\mathbb{N})^T$-domain the following hold:

- $FM(\mathbb{N})^T \models_{sl} \boldsymbol{Sub}(\ulcorner\varphi(x)\urcorner, \ulcorner x\urcorner, \ulcorner t\urcorner) = \ulcorner\varphi(t)\urcorner$ for any formula φ and any term t.
- $FM(\mathbb{N})^T \models_{sl} \boldsymbol{Name}(n) = \ulcorner\overline{n}\urcorner$ for any natural number n.

From the existence of the formulae $\mathbf{Sub}(x, y, z)$ and $\mathbf{Name}(x)$ we get:

Theorem 8 (FM Diagonal Lemma). The following hold for any $FM(\mathbb{N})^T$:

- For any $\mathcal{L}_{Tr}$-formula $\varphi(x)$ with exactly one free variable there is an $\mathcal{L}_{Tr}$-sentence ψ such that $FM(\mathbb{N})^T \models_{sl} \psi \equiv \varphi(\ulcorner\psi\urcorner)$.
- For any $\mathcal{L}_{Tr}$-formula $\varphi(x, y)$ with exactly two free variables there is an $\mathcal{L}_{Tr}$-formula $\psi(x)$ such that $FM(\mathbb{N})^T \models_{sl} \psi(x) \equiv \varphi(x, \ulcorner\psi(x)\urcorner)$.

Theorem 9 (FM-Undefinability of Truth). There is no arithmetical formula $\psi(x)$ such that for all arithmetical sentences φ:

$$FM(\mathbb{N}) \models_{sl} \psi(\ulcorner\varphi\urcorner) \equiv \varphi.$$

Corollary 3 (Existence of Yablo sentences in $FM(\mathbb{N})^T$-domains). There exists a formula $Y(x)$ such that for any $FM(\mathbb{N})^T$-domain we have:

$$\forall n \in \omega \;\; FM(\mathbb{N})^T \models_{sl} Y(n) \equiv \forall x\,(x > n \rightarrow \neg Tr(\ulcorner Y(\dot{x})\urcorner)),$$

i.e. Yablo sentences exist in $sl(FM(\mathbb{N})^T)$.

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Proof. Obvious by the fact that the diagonal lemma is sl-true in $FM(\mathbb{N})^T$. □

Theorem 10 (Yablo sentences are false in the limit). For any class $\mathcal{K}$ of finite models, if $\mathcal{K} \models_{sl} AD + YD$, then for all $n \in \omega$ $\mathcal{K} \models_{sl} \neg Y(n)$. In fact, AD isn't essential (it's only added to ensure Tr behaves like a truth predicate): for any $n \in \omega$ we have $YD \models_{sl} \neg Y(n)$.

Proof. Let us fix a class $\mathcal{K}$. For the sake of contradiction, suppose that there is a Yablo sentence that is not false at sufficiently large models, that is which for any size of a model is true at some model in $\mathcal{K}$ of at least that size, that is:

$$\exists n \forall k \exists \mathcal{M} \in \mathcal{K} \ (card(\mathcal{M}) \geq k \ \wedge \ \mathcal{M} \models Y(n)).$$

Let us take take such n. Let us fix k and take $\mathcal{M} \in \mathcal{K}$ (without loss of generality we could assume that $\mathcal{M} = \mathbb{N}^T_k$ for some natural k) with $card(\mathcal{M}) \geq k$ such that $\ulcorner Y(n+1) \urcorner \in |\mathcal{M}|$ and:

(i) $\mathcal{M} \models Y(n)$.

(ii) $\mathcal{M} \models Y(n+1) \equiv \forall x(x > n+1 \rightarrow \neg Tr(\ulcorner Y(\dot{x}) \urcorner))$.

(iii) $\mathcal{M} \models Y(n+1) \equiv Tr(\ulcorner Y(n+1) \urcorner)$.

(iv) $\mathcal{M} \models Y(n) \equiv \forall x \, (x > n \rightarrow \neg Tr(\ulcorner Y(\dot{x}) \urcorner))$ (this last condition means that $\mathcal{M}$ is sufficiently large to satisfy the defining formula of Yablo sentences).

(i) will be satisfied by the assumption of the proof, (ii) and (iv) follow from Corollary 3 and (iii) results from the assumption that $\mathcal{K} \models_{sl} YD$ once we have enough numbers to code all the formulas needed for the claims to hold.

From (i) and the fact that $\mathcal{M}$ models and therefore the defining formula for Yablo sentences we obtain:

$$\mathcal{M} \models \neg Tr(\ulcorner Y(n+1) \urcorner), \tag{8}$$

as well as:

$$\mathcal{M} \models \forall x(x > n+1 \rightarrow \neg Tr(\ulcorner Y(\dot{x}) \urcorner)). \tag{9}$$

Now, from (8), by (iii) we have that:

$$\mathcal{M} \models \neg Y(n+1),$$

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and from (9), by (ii) we have that:

$$\mathcal{M} \models Y(n+1),$$

which gives a contradiction that ends the proof. □

We will now show a construction of a class $\mathcal{K}$ such that $\mathcal{K} \models_{sl} YD + AD$, which means that Theorem 10 holds non-vacuously. It turns out that $FM(\mathbb{N})^Y$ is such a class.

Definition 13 ($FM(\mathbb{N})^Y$ and $sl(FM(\mathbb{N})^Y)$). We fix a formula $Y(x)$ satisfying the condition specified in Corollary 3. A family of models $FM(\mathbb{N})^Y$ is an $FM(\mathbb{N})^T$-domain $\{\mathbb{N}_k^Y : \ k \in \omega\}$ such that $\mathbb{N}_k^Y = (\mathbb{N}_k, T_k)$, where $T_k = TA_k \cup TY_k$, and:

$$TA_k = \{\ulcorner\varphi\urcorner : \ \varphi \in Sent_{\mathcal{L}} \text{ and } \mathbb{N}_k \models \varphi\} \cap |\mathbb{N}_k|$$
$$TY_k = \{\ulcorner Y(m)\urcorner : \ \ulcorner Y(m)\urcorner \in |\mathbb{N}_k| \wedge \ulcorner Y(m+1)\urcorner \notin |\mathbb{N}_k|\}.$$

If $|\mathcal{M}| = k$, we sometimes write $T_{\mathcal{M}}$ instead of T_k.
Naturally, $sl(FM(\mathbb{N})^Y) = \{\varphi : \ FM(\mathbb{N})^Y \models_{sl} \varphi\}$. ⊣

It is worth noting that the family $FM(\mathbb{N})^Y$ is not a proper potentially-infinite domain in the sense defined by Mostowski (2016), i.e. for any natural numbers $r < s$ $\mathbb{N}_r^Y$ is not a submodel of $\mathbb{N}_s^Y$, since the interpretation of the truth predicate Tr varies between subsequent models from the domain.

Observation 1. For any class $\mathcal{K}$, if $\mathcal{K} \models_{sl} YD$, then for sufficiently large $\mathcal{M} \in \mathcal{K}$, there is exactly one $n \in \omega$ s.t. $\ulcorner Y(n)\urcorner \in T_{\mathcal{M}}$.

Theorem 11. $\forall n \in \omega \ \ FM(\mathbb{N})^Y \models_{sl} \neg Y(n)$.

Proof. We claim that for any n there exists m such that for any $k > m$:

$$\mathbb{N}_k^Y \models \neg Y(n).$$

Indeed, let us fix n and take m such that $\ulcorner Y(n+1)\urcorner \in |\mathbb{N}_m^Y|$ and for any $x < n+1$ and any $k > m$ we have:

$$\mathbb{N}_k^Y \models \ Y(x) \ \equiv \ \forall w > x \ \neg Tr(\ulcorner Y(\dot{w})\urcorner).$$

Let $k > m$. Then obviously $\ulcorner Y(n+1)\urcorner \in |\mathbb{N}_k^Y|$. Let j be the greatest number such that $\ulcorner Y(j)\urcorner \in |\mathbb{N}_k^Y|$. Such a number exists since our FM-domain is infinite and every

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model in it is finite. Obviously, $n < j$. From the definition of the class $FM(\mathbb{N})^Y$ and by the choice of j we obtain:

$$\mathbb{N}_k^Y \models Tr(\ulcorner Y(j) \urcorner),$$

and for any $x < j$ we get:

$$\mathbb{N}_k^Y \models \exists w(w > x \, Tr(\ulcorner Y(\dot{w}) \urcorner)).$$

Thus, since $n < j$, we obtain:

$$\mathbb{N}_k^Y \models \neg Y(n),$$

which ends the proof. □

Corollary 4. $sl(FM(\mathbb{N})^Y)$ is ω-inconsistent.

Proof. We have just shown that

$$\forall n \in \omega \ (\neg Y(n) \in sl(FM(\mathbb{N})^Y)).$$

Intuitively speaking, for each sufficiently large finite segment, the last Yablo sentence whose code exists in it will be satisfied, and so the existential claim *some Yablo sentence is true* will be satisfied in all sufficiently large segments, without *the n-th Yablo sentence is true* being satisfied in all sufficiently large segments (because in different segments different objects will be the last Yablo sentences). The argument for ω-inconsistency relies on this tension.

We obviously also have that for all sufficiently large k there exists $j < k$ such that by the definition of $FM(\mathbb{N})^Y)$ we have $\mathbb{N}_k^Y \models Y(j)$, so by existential generalization we obtain that there is m such that for all $k > m$ $\mathbb{N}_k^Y \models \exists x Y(x)$ and thus:

$$\exists x Y(x) \in sl(FM(\mathbb{N})^Y).$$

□

So, it seems, there is a finitistic approach to arithmetic, according to which which all Yablo sentences are false. The cost of this move, however, isn't negligible: the set of arithmetical formulae true in the intended model is ω-inconsistent. Therefore, we pursue the topic further, looking at another way to think finitistically about the issue.

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4 Modal Interpretation of Quantifiers in Potentially Infinite Domains

Definition 14 (Accessibility relation in FM-domains). Let $\mathcal{K}$ be an FM-domain. For any $M, N \in \mathcal{K}$ N is accessible from M ($R(M,N)$) if $M \subseteq N$. For $m, n \in \omega$ and elements $\mathbb{N}_m$, $\mathbb{N}_n$ of the FM-domain $FM(\mathbb{N})$ this boils down to the condition $m \leq n$.

Definition 15 (Modal semantics for FM-domains (m-semantics)). Let $\mathcal{K}$ be an FM-domain over some structure $\mathbb{A}$ (i.e. $\mathcal{K} = FM(\mathbb{A})$) and $M \in \mathcal{K}$:

- If φ is atomic, then $(\mathcal{K}, M) \models_m \varphi$, if $M \models \varphi$.
- Satisfaction clauses for boolean connectives and negation are standard.
- $(\mathcal{K}, M) \models_m \exists x \varphi(x)$ iff there are $N \in \mathcal{K}$ and $a \in N$ s.t.
  $R(M,N)$ and $(\mathcal{K}, N) \models_m \varphi[a]$.

Thus we also have that $(\mathcal{K}, M) \models_m \forall x \varphi(x)$ iff for all $N \in \mathcal{K}$ s.t. $R(M,N)$ and for all $a \in N$ $(\mathcal{K}, N) \models_m \varphi[a]$.

The intuition behind this semantics is as follows. '$\exists x\, \varphi(x)$' reads 'there could be enough objects so that for some a, $\varphi(a)$' and '$\forall x\, \varphi(x)$' reads 'however many more objects there could be, it still would be the case that for any a, $\varphi(a)$'.

Now, let $msl(FM(\mathbb{N})) = \{\varphi : \exists n\, \forall k\; k \geq n \Rightarrow (FM(\mathbb{N}), \mathbb{N}_k) \models_m \varphi\}$. That is, intuitively, $msl(FM(\mathbb{N}))$ is the set of those formulas, which are true in all sufficiently large models, where the notion of truth involves the modal reading of quantifiers.

As an example, let $\varphi = \exists x\, \forall y\; x \geq y$ — that is, φ says: *there exists a maximal element*. While, as we remember, $\varphi \in sl(FM(\mathbb{N}))$, things are different under m-semantics — φ is false in every possible world of the FM-domain.

Before we move on, let us emphasize that just as with sl-semantics, we work with a relational arithmetical language. While in the case of sl-semantics this wasn't too important, it becomes crucial when we turn to msl-semantics.

For otherwise, we need to treat, say, addition and successor as total functions. This being the case, for each finite initial segment we'd need to identify the candidates for the values of functions which intuitively should surpass the capabilities of that segment. The least unnatural way to do this would be to plug in loops at ends of segments, so that $s(max(\mathbb{N}_k)) = max(\mathbb{N}_k)$ etc. But then we would run into problems. For instance, take $\varphi := \exists x\; x + x = x \wedge x \neq 0$. If we evaluate atomic sentences in the elements of our

FM-domain, then for any k we have:

$$\mathbb{N}_k \models_m \varphi \text{ iff } \exists j \geq k \, \exists a < j \, \mathbb{N}_j \models a + a = a \wedge a \neq 0.$$

However, the above would come out true. After all, let $a = max(\mathbb{N}_j)$. Then, $a + a = \underbrace{s \dots s}_{a} a = a$ and we have $\mathbb{N}_j \models a \neq 0 \wedge a + a = a$ and so $\mathbb{N}_k \models_m \varphi$ for any k. Thus, we would have $\varphi \in msl(FM(\mathbb{N}))$, while $\texttt{PA} \vdash \neg\varphi$.

The underlying cause of the issue is that when we work with a functional language we cannot think about the initial segments as submodels of larger initial segments, because the functions are not preserved when we move to superstructures. The problem disappears when we abandon function symbols and use a relational language instead. So this is what we'll do in what follows.

In particular, we'll be working towards a theorem according to which the resulting arithmetic is the classical arithmetic, unlike in the previous case. We'll start with two lemmata.

Lemma 3. For any $k > 0$, $\mathbb{N}_k$ is a submodel of $\mathbb{N}$.

Proof. This holds because our language is relational, and for any r-ary relation symbol R^r we have $(R^r)^{\mathbb{N}} \cap \mathbb{N}_k^r = (R^r)^{\mathbb{N}_k}$. □

Lemma 4. For any quantifier-free $\varphi(x_1, \dots, x_n)$ and for any choice of parameters $a_1, \dots a_n \in \mathbb{N}$, if we have $a_1, \dots, a_n \in \mathbb{N}_k$, then it holds that:

$$\mathbb{N}_k \models \varphi[a_1, \dots, a_n] \text{ iff } \mathbb{N} \models \varphi[a_1, \dots, a_n]$$

Proof. Immediate by Lemma 3. □

Theorem 12. Let $msl(FM(\mathbb{N}))$ denote the msl theory (i.e. the sl-theory of the FM-domain of natural numbers with the modal interpretation of quantifiers). Then we have: $msl(FM(\mathbb{N})) = Th(\mathbb{N})$.[4]

Proof. The proof is by induction on formula complexity.

For the basic case of quantifier-free formulae, the claim holds by Lemma 4. For boolean connectives, the equivalence is trivial. The only interesting case is for $\varphi := \exists x \psi(x)$.

[4]Here, we're concerned with initial segments of $\mathbb{N}$. The result clearly generalizes to collections of models, as long as accessibility correlates with the submodel relation; not even the assumption of finitude is required. Such collections are studied for instance in Hamkins and Linnebo (2017).

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$\supseteq$: Suppose $\varphi \in Th(\mathbb{N})$, that is, $\mathbb{N} \models \exists x \psi(x)$. Then there is a witness $a \in \mathbb{N}$, such that $\psi[a] \in Th(\mathbb{N})$. By IH, $\psi[a] \in msl(FM(\mathbb{N}))$. This means:

$$\exists k \, \forall l \geq k \, \mathbb{N}_l \models_m \psi[a] \tag{10}$$

and from this it follows that:

$$\exists k \forall l \geq k \, \exists j \geq l \, \exists a < j \, \mathbb{N}_j \models_m \psi[a]. \tag{11}$$

which means that $\varphi \in msl(FM(\mathbb{N}))$.

$\subseteq$: Say $\varphi := \exists x \, \psi(x) \in msl(FM(\mathbb{N}))$. So (11) holds as well and there is an a such that (10) also holds. This step essentially depends on the language being relational; thanks to this assumption for any $k_1 < k_2$ we have $\mathbb{N}_{k_1} \subseteq \mathbb{N}_{k_2} \subseteq \mathbb{N}$ (we mean the submodel relation here) and by the m-semantics witnesses of existential claims remain such witnesses in supermodels). This means $\psi[a] \in msl(FM(\mathbb{N}))$, and so by the IH, $\psi[a] \in Th(\mathbb{N})$, and therefore $\varphi \in Th(\mathbb{N})$. □

5 Yablo Sequences in Potentially Infinite Domains under the Modal Interpretation of Quantifiers

The semantics in sufficiently large models in potentially infinite domains under modal interpretation of quantifiers presented above entails that Yablo sentences stay paradoxical even for the finitist, if she interprets the quantifiers in the modal manner. That is, we'll be arguing that not even local Yablo Disquotation can be included in an msl-theory. Let's start with a lemma.

Lemma 5. For any $FM(\mathbb{N})^Y$-domain with $YD \subseteq msl(FM(\mathbb{N})^Y)$ it holds that:
$\forall n \in \omega \, Y(n) \notin msl(FM(\mathbb{N})^Y)$.

Proof. First note that the Diagonal Lemma holds for the extended language in the msl-theory (because its arithmetical part is $Th(\mathbb{N})$). So Yablo sentences belong to it. Now, suppose some Yablo sentence is in the msl-theory, that is

$$\exists n \, \left[Y(n) \in msl(FM(\mathbb{N}^Y)) \right].$$

This means:

$$\exists l \, \forall k \geq l \, \mathbb{N}_k \models_m Y(n).$$

Pick an l witnessing this. By the definition of Yablo sentences this entails:

$$\forall k \geq l \, \mathbb{N}_k \models_m \forall x \, (x > n \to \neg Tr(Y(x))).$$

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By the semantics, this means:

$$\forall k \geq l \forall p \geq k \forall a < p \; \mathbb{N}_p \models_m a > n \to \neg Tr(Y(a)).$$

But then:

$$\forall p \geq l \, \forall a \in (n, p) \, \mathbb{N}_p \models_m \neg Tr(Y(a)). \tag{12}$$

So, by Yablo Disquotation, if we take a p sufficiently large to satisfy Tarski biconditionals for Yablo sentences $Y(a)$ for $n < a < p$ (such p exist by the fact that $YD \subseteq msl(FM(\mathbb{N})^Y)$), we have that models of size p fail to satisfy Yablo sentences for numbers between n and p:

$$\forall p \geq l \, \forall a \in (n, p) \, \mathbb{N}_p \models_m \neg Y(a).$$

Now, fix p and a. By the definition of $Y(x)$ the above means:

$$\mathbb{N}_p \models_m \exists x > a \, Tr(Y(x)).$$

By our definition of $\models_m$ this is equivalent to:

$$\exists q \geq p \, \exists b < q \, \mathbb{N}_q \models_m b > a \wedge Tr(Y(b)).$$

Hence:

$$\exists q \geq p \, \exists b \in (a, q) \, \mathbb{N}_q \models_m Tr(Y(b)).$$

This, however, contradicts (12), which completes the argument. □

With this lemma at hand, we can proceed to the theorem which tells us that not only Yablo sentences are not in any *msl*-theory, but also that no *msl*-domain can (modally) satisfy the local Yablo Disquotation principles either.

Theorem 13. There is no $FM(\mathbb{N})^Y$-domain such that $YD \subseteq msl(FM(\mathbb{N})^Y)$.

Proof. Suppose there is an $msl(FM(\mathbb{N})^Y)$ which contains all Local Yablo Disquotation sentences.

By Lemma 5, we know that $\forall n \, Y(n) \notin msl(FM(\mathbb{N})^Y)$. We therefore have:

$$\forall n \, \forall l \, \exists k \geq l \, \mathbb{N}_k \not\models_m Y(n).$$

By the definition of the Yablo sentences we infer:

$$\forall n \forall l \exists p \geq l \exists a > n \, \mathbb{N}_p \models_m Tr(Y(a)).$$

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Which, by Yablo Disquotation, yields:

$$\forall n \forall l \exists p \geq l \exists a > n \, \mathbb{N}_p \models_m Y(a).$$

The claim holds for any n and l. For us, it is enough to look at $n = l = 0$. By the definition of $Y(a)$ and m-semantics we obtain:

$$\exists p, a > 0 \forall q \geq p \, \mathbb{N}_q \models_m \forall x > a \, \neg Tr(Y(x)).$$

Pick an $a > 0$ witnessing the above claim. By the definition of msl-theory we now have

$$Y(a) \in msl(FM(\mathbb{N})^Y),$$

which is impossible by Lemma 5. □

So, when we consider Yablo sequences with different treatments of infinity in the background, the following observations come to mind:

1. In the standard setting, without potential infinity, Local Arithmetical Disquotation and Local Yablo Disquotation are consistent, yet ω-inconsistent with the background arithmetical theory. Once ω-rule or Uniform Yablo Disquotation are introduced, the theory is inconsistent.

2. Under sl-semantics, Yablo sentences are all false (in the limit), yet the sl-theory of a given FM-domain is consistent, but ω-inconsistent. This is a particular case of a general flaw of sl-semantics, because $sl(FM(\mathbb{N}))$ itself is ω-inconsistent.

3. Under msl-semantics, i.e. semantics in sufficiently large models in potentially infinite domains under the modal interpretation of quantifiers, even adding only the Local Arithmetical Disquotation and Yablo Disquotation results in an inconsistent theory. Uniform Yablo Disquotation or ω-rule is not needed to ensure this.

6 A remark on semantics & ontology

One question that might come to mind in this context is this. If for the first-order relational language in the limit we simply get the true arithmetic, what's the point? Why isn't this just syntactic sugar? Doesn't this mean that the finitistic and potentialist parlance is a bit of cheating and that the entire framework gives rise to actualism with respect to the structure that we approximate?

Well, while the semantics—as desired—legitimates the (*prima facie* infinitistic) theory of the standard model of arithmetic, at the ontological level the difference remains: the ontology is indeed finitistic and we evaluate formulae in finite models. While it is hard to capture this idea in terms of a first-order language, it becomes more transparent when we look at a second-order language.

If we take a relational second-order relational language, where in the standard model quantifiers range over the powerset of $\mathbb{N}$ and in m-semantics at each segment they only range over its powerset, the set of formulae belonging to the msl-theory will be quite different from those true in the standard model. Namely, it will be those formulae which are true in the standard model where second-order quantifiers are taken to range over *finite sets*.

Formally, let us extend the definition of m-satisfaction for the standard FM-domain, including the second order domain of quantification. Let $\mathcal{P}(\mathbb{N}_k)$ denote the powerset of the set $|\mathbb{N}_k| = \{0, ..., k-1\}$, i.e.

$$\mathcal{P}(\mathbb{N}_k) = \mathcal{P}(\{0, ..., k-1\}).$$

Now add the following clause for the existential second-order quantification to the definition of m-semantics. $(\mathbb{N}_k, \mathcal{P}(\mathbb{N}_k)) \models_m \exists X \phi(X)$ iff

$$\exists n \geq k \exists A \in P(N_n)(\mathbb{N}_n, \mathcal{P}(\mathbb{N}_n)) \models_m \phi[A].$$

Further, we extend the definition of the msl-theory of the FM-domain for the standard second-order model of arithmetic, i.e. to the structure $\mathbb{N}_2 := (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ in the obvious way. Observe, that for the expanded models, it still holds that if $k < n$, then $(\mathbb{N}_k, \mathcal{P}(\mathbb{N}_k))$ is a submodel of $(\mathbb{N}_n, \mathcal{P}(\mathbb{N}_n))$.

Interestingly, the following now holds:

Theorem 14. For any second-order arithmetical formula ψ the following are equivalent:

1. $\psi \in msl(FM((\mathbb{N}_2)))$,
2. $(\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N})) \models \psi$,

where $\mathcal{P}_{fin}(\mathbb{N})$ denotes the family of all finite subsets of natural numbers.

Proof. We just need to extend the proof for the formulae containing second-order quantifiers. The case of atomic second-order formulae is trivial. Suppose $\psi = \exists X \varphi(X)$.

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(⇒) By assumption, for sufficiently large k ψ holds under m-semantics in every $(\mathbb{N}_k, \mathcal{P}(\mathbb{N}_k))$. So, by definition, for each such k there is an $n > k$ and $A \subset |\mathbb{N}_n|$ with

$$(\mathbb{N}_n, \mathcal{P}(\mathbb{N}_n)) \models \varphi[A].$$

This means that we almost always (i.e. for all but finitely many finite models – the ones given by the sufficiently large k) have *finite* witnesses for existential second-order claims. By the extended m-semantics, witnesses of existential claims remain such witnesses in supermodels, thus $\varphi[A] \in msl(FM((\mathbb{N}_2)))$. Since $A \in \mathcal{P}_{fin}(\mathbb{N})$, by the Inductive Hypothesis,

$$(\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N})) \models \varphi[A],$$

so by existential generalization

$$(\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N})) \models \exists X \varphi(X).$$

(⇐) Assume $(\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N})) \models \exists X \varphi(X)$, which means that there is a finite set of natural number A such that

$$\mathbb{N} \models \varphi[A].$$[5]

By the inductive assumption, for sufficiently large k we have

$$(\mathbb{N}_k, \mathcal{P}(\mathbb{N}_k)) \models_m \varphi[A],$$

thus giving that for all but finitely many k:

$$\exists A \in \mathcal{P}(\mathbb{N}_k)\ (\mathbb{N}_k, \mathcal{P}(\mathbb{N}_k)) \models_m \varphi[A],$$

and therefore (since $k < n$ and $A \subseteq |\mathbb{N}_k|$, then also $A \subseteq |\mathbb{N}_n|$), we have that for all but finitely many k there is an $n > k$ with

$$\exists A \in \mathcal{P}(\mathbb{N}_n)\ (\mathbb{N}_n, \mathcal{P}(\mathbb{N}_n)) \models_m \varphi[A].$$

This, by the extended definition of m-semantics, means that $\exists X \varphi(X) \in msl(FM((\mathbb{N}_2)))$.

□

This fact is best understood as: in our potentialist-finitistic framework, the finitist ontology is hidden in the realm of the domain of second-order quantification, not in the semantics of first-order formulas, which – as desired – gives rise to true arithmetic,

[5] Formally this holds with A as a parameter, but all finite sets are definable, so we do not have to worry about that.

and thus vindicates the infinitistic idealization for arithmetic on potentialist-finitistic grounds. We can adopt a purely instrumentalist approach to the extension of m-semantics to the level of subsets of natural numbers and treat it just as an explanatory tool needed to answer a question about *location* of finitism in the potentialist framework that we propose. The answer then should be that although within the potentialist semantics for first-order arithmetic we essentially quantify over all natural numbers (with the modal understanding of this!), when it comes to collections of numbers, we can only express and recognize the truths concerning finite sets. This is exactly as it should be. The ontology of FM-domains is finitistic in the sense that the domain does not provide a single infinite universe, but a family of its approximations — finite initial segments, the universes of which (and subsets of these) are precisely the finite sets of natural numbers, so if we allow quantification over not only numbers, but also of their collections, i.e. if we turn to second-order semantics, it is quite natural that we will be able to capture the properties of finite sets only. What seems to require an explanation then is why — under modal potentialist semantics — can we capture all the properties of individual natural numbers (i.e. in the first-order setting)? The answer lies in the question itself — if one shares finitistic intuitions, formalizes them in a model-theoretic manner and adopts correct potentialist semantics, then there is indeed no difference from the perspective of first-order number theory. The reason why this is the case is that — in a sense specified by the modal semantics — a position of an extremely *local* object such as an individual number in a finite structure that can be indefinitely extended is not distinct from the position of such object in the entire structure, provided we can make references to individual objects only. In a very loose sense, it does not matter whether we look at a number form the perspective of an infinite set or from the perspective of a finite, yet unbounded one. Consequently, finitism isn't too revolutionary in the context of first-order arithmetic, since the modal finitistic semantics vindicates the corresponding infinitistic idealization.

However, finitistically available sets of numbers, provided we can refer to such sets using quantification (i.e. in the second-order setting), can have very different properties in an infinite structure and in its finite approximations, even if allow the finite segments to be indefinitely extendable, since in the system of approximations there simply are no infinite sets — they are never objects in the FM-domains framework (whereas individual numbers are always finite objects, irregardless of whether they are located in $\mathbb{N}$ or in the elements of $FM(\mathbb{N})$).

We claim that this is the correct (ontological) commitment of (arithmetical) finitism. Any (finite) number of individual objects can be added to a given finite universe and

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it will still be finite, but there is no way to finitely extend the universe of a finite model so that the model becomes infinite, and this affects the properties of subsets of a given structure rather than individual elements of the structure. To illustrate the consequences of this, observe that for each particular number n one can always extend a finite initial segment A of $\mathbb{N}$ to B so that n is not the maximal element of B even if it was a maximal element of A. However, for no subset X of A can we find an extension to B so that X was not injectable into its proper subset in A, but is injectable in its proper subset in B.[6] So while it seems that for first-order context extreme finitism is untenable, and one might thus think that our potentialist account of arithmetic ideologically reduces to actualism, it is our results concerning the second-order framework that suggest that there is an open door for an argument against the claim that arithmetical potentialism is actualism in sheep's clothing. There still can be a difference between the family of approximations of a structure and the structure itself, but semantically it has to be captured by higher-order means. Whether these means are allowed, is a completely different question that we do not engage with here.

Interestingly, our approach somehow contrasts with the so-called predicativist approach to arithmetic, under which reference to *all subsets* of an infinite set is not legitimate, and only some infinite subsets of natural numbers are *accepted* if they can be referred to, where what type of predicativism we're dealing with largely depends on what means of reference are admissible.[7] Under the modal semantics, the theory of sets of natural numbers seems to be more restrictive, since, in a sense, only finite sets are allowed. Consequently, there will be sentences of second-order arithmetic true under msl-semantics that are false in predicative second-order arithmetic. This, however, does not present a problem, since msl-semantics is provided to give account not of predicativism in arithmetic, but of finitism and potentialism therein. The contrast between predicativism and our formalism thus plays its role in reflecting on the difference between these philosophical approaches to arithmetic, and illustrating yet another (apart from the vindication of infinitistic idealization above) possible weak point in finitism — at the second-order level it is a position even more restrictive than predicativism. We leave it for further work to investigate if this is an accurate argument.

All in all, our reading of the results concerning the first and second-order theories of arithmetic under msl-semantics provide the following philosophical standpoints:

[6]We thank the anonymuous Referee for this example - it illustrates one of our points.

[7]Some authors have claimed that the arithmetical theory ATR_0 captures the idea of predicativism, however the details of this debate are not important to us here.

- It is second-order potentialist semantics (rather than the first-order one) that captures the finitistic ontology of FM-domains.

- At the first-order level, the potentialist semantics vindicates the infinistic idealization of natural numbers,

- For this reason, a revisionary finitistic approach to first-order arithmetic is unjustified.

- Nevertheless, the difference between second-order *msl* theory of natural numbers and standard second-order true arithmetic allow for arguing that that potentialist systems of approximations to a given structure does not reduce to hidden actualism.

- However, the second-order *msl*-theory of $\mathbb{N}$ is very restrictive.

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