J. Plasma Physics (2011), *vol.* 77, *part* 2, *pp.* 231–236. © Cambridge University Press 2010
doi:10.1017/S0022377810000140

On the possibility of refraction of dust acoustic waves

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(Received 3 December 2009 and accepted 10 February 2010, first published online 18 March 2010)

Abstract. We theoretically investigate conditions for the refraction of long-wavelength dust acoustic waves by arrays of periodic cylinders in a dusty plasma. This is based on a recent analysis of the refraction of shallow water waves by periodic cylinder arrays (Hu and Chan, *Phys. Rev. Lett.* **95** (2005), 154501). In the dusty plasma case, however, the boundary conditions involve the formation of voids around the cylinders. Possible experimental conditions are discussed.

Dust acoustic waves (DAWs) are the very ultra low-frequency waves, where the inertia is provided by massive highly charged dust grains, while the restoring force comes from the pressures of Boltzmann distributed inertia less electrons and ions [1]. Recently, the diffraction of DAWs by a circular cylinder placed in a dusty plasma was studied experimentally by Kim et al. [2], who pointed out the similarity between the equations describing DAWs and those describing sound waves in a gas. That means, the dynamics of the long wavelength (in comparison with the dusty plasma Debye radius [3]) waves can be described by the same type of linearized continuity and momentum equations as sound waves [2]. For the DAWs, the governing equations are

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{u}_1 = 0, \tag{1a}$$

and

$$\frac{\partial \mathbf{u}_1}{\partial t} + \frac{c_d^2}{n_0} \nabla n_1 = 0, \tag{1b}$$

where n_1 is the small dust number density perturbation in the equilibrium dust number density n_0, $\mathbf{u}_1$ is the dust fluid velocity, and the dust acoustic speed (for cold dust) is $c_d = \omega_{pd}\lambda_D$, where $\omega_{pd} = (4\pi Z^2 e^2 n_0/m)^{1/2}$ is the dust plasma frequency, with Z and m being the charge state and the mass of a dust grain, respectively. The dusty plasma Debye radius λ_D is given by [3]

$$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2},$$

where $\lambda_{Dj} = (T_j/4\pi n_j e^2)^{1/2}$, with $j = e, i$ denoting the electrons and ions, respectively. Assuming that all perturbed quantities vary with time as $\exp(-i\omega t)$, where ω is the frequency, we can Fourier transform (1a) and (1b) and combine the resultant equations to obtain

$$\nabla \cdot \left(\frac{c_d^2}{n_0} \nabla n_1 \right) + \frac{\omega^2}{n_0} n_1 = 0, \tag{2}$$

which governs the propagation of the DAWs in an inhomogenous plasma. In a uniform dusty plasma, (2) gives the dust acoustic wave frequency $\omega = kc_d$, where k is the wavenumber [1].

Recently, Hu and Chan [4] have considered the refraction and focusing of shallow water waves by an array of periodic bottom-mounted vertical cylinders. Shallow water waves are characterized by $kh \ll 1$, where h is the constant depth of the water. The shallow water wave equation reads [4]

$$\nabla \cdot (h \nabla \eta) + \frac{\omega^2}{g} \eta = 0, \tag{3}$$

where η is the vertical displacement of the water surface and g is the gravitational acceleration. Equation (3) gives the linear dispersion relation for the shallow water waves, viz. $\omega = \sqrt{gh}k$ [4]. Thus, the dispersion law for the shallow water waves is similar to that of long-wavelength DAWs, and also to sound waves in air. Indeed, it was pointed out by Hu and Chan [4] that the system they studied has an acoustic analog, that being an array of rigid cylinders in air, because the shallow water wave equation has the same form similar to the governing equation for sound waves in air (see also [5]).

By using (3) Hu and Chan [4] investigated the refraction of shallow water waves by a square array of cylinders of radius r and spacing a, where a is the distance between the centers of neighboring cylinders, for long wavelengths, viz. $ka \ll 1$. Hu and Chan [4] modeled the system in a cylindrical co-ordinate system with origin at the center of the cylinder. Each cylinder is surrounded by a circular water column, which in turn is surrounded by the effective medium of the cylinder array. Using homogenization techniques, it was found that the shallow water waves propagate through the cylinder array as if the water had an effective depth h_e and an effective gravitational constant g_e, which are dependent on the filling fraction of the cylinders given by $\pi r^2/a^2$ [4].

The wave equation (3) for the shallow water waves becomes the same as (2) for the DAWs, after the replacements $\eta \rightarrow n_1$, $g \rightarrow n_0$ and $h \rightarrow c_d^2/n_0$. Thus, we expect that there could be a similar behavior for the DAWs as for the shallow water waves (and sound waves in air). One possible difference between the dust acoustic wave case and the shallow water wave case, analyzed in [4], is the effective radius of the cylinder. When an object is placed in a dusty plasma, a dust-free region around the obstacle (the so-called dust "void") can be formed around the object, as demonstrated experimentally and theoretically in [6]. When the obstacle is biased negatively with respect to the plasma potential, the void is formed because the negatively charged dust is pushed away from the obstacle by the electrostatic force, and pulled toward the obstacle by the ion drag force due to ions flowing toward the obstacle [6]. In contrast to the case of water waves where there is a boundary condition of no flow through the cylinder walls, for the case of the DAWs there would be no dust flow through the void; thus the gradient of the perturbed density normal to the cylinder surface would go to zero at a radial distance from the cylinder given approximately by the size of the void formed around each cylinder. We note that if the cylinder is biased near the plasma potential, the size of the void could theoretically become quite small [6].

Thus, we consider that a square array of cylinders is immersed in a dusty plasma, and a dust acoustic wave is incident on the array. The radius of each cylinder is r_c and the spacing between the centers of the neighboring cylinders is a, which is much larger than λ_D. We assume $ka \ll 1$. As discussed above, since the dust acoustic wave equation (2) has the same form as the equation for the shallow water waves, the results of [4] for the refraction of the shallow water waves from a cylinder array, in the limit $ka \ll 1$, rolls over to the dust acoustic wave case with the appropriate replacement of the physical quantities as discussed previously.

Simple analytic expressions were found in the work of Hu and Chan [4], relating the filling fraction of the cylinders to an effective depth and an effective gravitational constant, leading to an effective refractive index for the medium that can be used with Snell's law to obtain reflection and transmission coefficients for the shallow water waves from the effective water medium. For the dust acoustic wave case, there is an effective $(c_d^2)_{ef}$ and $(n_0)_{ef}$ in the effective medium composed of cylinder array immersed in a dusty plasma, which is given by

$$\frac{(n_0)_{ef}}{n_0} = \frac{1}{1 - f_s}, \tag{4a}$$

$$\frac{(c_d^2)_{ef}}{c_d^2} = \frac{1}{1 + f_s}, \tag{4b}$$

where f_s is the filling fraction of the cylinders. Denoting by r_v the distance of the dust void edge from the center of each cylinder, we have $f_s = \pi r_v^2/a^2$. These renormalizations can perhaps be interpreted roughly in the following way. As the volume/unit length of the dusty plasma is reduced due to the presence of the cylinders by a factor $1 - f_s$, this leads to an increase in the dust density, as given by (4a). In addition, the cylinders displace a mass/unit length proportional to $\pi r_v^2 N \sim f_s$, where N is the number of cylinders [7] (see also [8]), so the quantity $c_d^2 \propto 1/m$ would effectively increase as $1/(1 + f_s)$, as given by (4b). Note that the latter assumes that $Z^2\lambda_D^2 \propto (1 - f_s)$, which may be related to the increase of the background plasma number density due to the volume/unit length taken up by the cylinders. In spite of these rough plausible arguments, we believe that further work is needed to explore various physics issues related to charged cylinders or wires in a plasma. This includes issues such as the difference between the dust, ion and electron number density changes due to the presence of the cylinders and the different forces operating on these particles. For example, the boundary condition of no flow through the cylinder walls (in contrast to no flow through the void for dust grains) might be appropriate.

At any rate, because the speed of the DAWs in the effective dusty plasma medium is given by $(c_d)_{ef}$, the index of refraction of the effective medium is

$$N_{ef} = \frac{c_d}{(c_d)_{ef}} = \sqrt{1 + f_s}. \tag{5}$$

Following Hu and Chan [4], Snell's law for refraction of an incident plane long-wavelength dust acoustic wave on the effective medium is given by

$$\sin\theta = N_{ef} \sin\chi, \tag{6}$$

where θ and χ are the angles of incidence and refraction, respectively (measured with respect to the normal to the surface of the effective medium). Similarly, following [4], the amplitude reflection coefficient R_A and amplitude transmission coefficient T_A at the interface between the dusty plasma and the effective medium are

$$R_A = \frac{\cos\theta - K\cos\chi}{\cos\theta + K\cos\chi}, \quad T_A = \frac{2\cos\theta}{\cos\theta + K\cos\chi}, \tag{7}$$

where

$$K = \frac{n_0}{(n_0)_{ef}}\frac{1}{N_{ef}}.$$

The incidence angle, where reflection is zero (Brewster angle), is

$$\theta = \theta_0 = \arccos\frac{1-f_s}{2}, \tag{8}$$

while complete reflection occurs at $\theta = \pi/2$ where $R_A = -1$ [4].

We consider a set of possible experimental parameters. Our main experimental constraints are that (1) $ka \ll 1$ in order for the analysis of Hu and Chan [4] to apply, and (2) $kc_d \gg v_{dn}$ in order for the DA waves not to be collisionally damped by dust-neutral collisions (here $v_{dn} \sim 4r_d^2 n_n v_n m_n/m_d$ is the dust-neutral collision frequency, where n_n, v_n and m_n are the density, thermal speed and mass of neutrals, respectively [9]). These two conditions yield

$$1 > ka > \frac{v_{dn}}{\omega_{pd}}\frac{a}{\lambda_D}. \tag{9}$$

From (9) we see that we require a system with very low ratio of the dust collision frequency to the dust plasma frequency, since the spacing a between cylinders has to be much larger than the dusty plasma Debye radius, in order for the voids around the probes not to overlap.

We discuss the possible size of the void around each wire/cylinder. To do this, we consider that in experimental studies of the formation of a void around probe wires [6,10], it was found that the void boundary was of the order of about 1−4 mm, depending on the plasma and probe conditions. A stable void is maintained by the balance of electric and ion drag forces on the dust particles. As the radial electric field of a small cylindrical probe should decrease roughly with distance ρ from the probe as $(\phi_s/\rho)\exp(-\rho/\lambda_{De})$, where ϕ_s is the magnitude of the surface potential, this balance may occur outside the sheath boundary where ions enter the sheath with the Bohm speed. Klindworth et al [10] studied voids formed around probes in argon (pressure $\sim$ 40 Pa) dusty plasmas under microgravity, and found that the balance of the electric field and ion drag forces is established in the pre-sheath around the probe. Klindworth et al. [10] estimated the area of the sheath surface balancing the electron and ion currents to the floating probe, finding that the sheath area scales as the probe area; the void radius at a probe tip of diameter 50 μm was found to be about 1 mm. Thomas et al. [6] studied voids around probes in a laboratory argon (pressure $\sim$ 16 Pa) dusty plasma. The authors of [6] investigated the effect of biasing the probe, showing that the void size decreases as the absolute value of the probe potential minus plasma potential gets smaller in magnitude. Thus, it appears that using wires with small radii and biasing the wire could tend to result in smaller void sizes around the wire cylinders.

For the dusty plasma, we consider the following set of parameters including a lower pressure gas in order to reduce the ratio ν_{dn}/ω_{pd}. We consider a weakly ionized argon plasma, with gas pressure of about 2 Pa, the ion number density $n_i \sim 10^9\,\mathrm{cm}^{-3}$, the electron temperature $T_e \sim 2$ eV and the ion temperature $T_i \sim 0.05$ eV (these plasma values being similar to those discussed in [11]. The dusty plasma Debye radius in the system is then close to the ion Debye radius, which is roughly 50 μm; the electron Debye radius is about 0.3 mm. For the dust, we consider grains of radius $r_d \sim 3$ μm, mass density $\sim 4\,\mathrm{g/cm}^3$ (*e.g.* alumina) and the dust number density $n \sim 3 \times 10^4\,\mathrm{cm}^{-3}$ (average inter-grain spacing of about 0.2 mm). In this case, the negative charge state of the grain can be estimated as $Z \sim 2.5 r_d T_e/e^2 \sim 10^4$. For these dusty plasma parameters, $\omega_{pd} \sim 139$ rad/s while $\nu_{dn} \sim 0.8\ \mathrm{s}^{-1}$, so that the ratio $\nu_{dn}/\omega_{pd} \sim 0.006$. Thus, in order to satisfy the constraint (9), we require $a/\lambda_{Di} \leqslant 160$. If we assume that the void size can be made to be about 1 mm, and that a is about 4 mm, we have $a/\lambda_{Di} \sim 80$, so that $\nu_{dn} a/\omega_{pd}\lambda_{Di} \sim 0.5$. In this case, the constraint (9) may be marginally satisfied. We, however, note that the analytic results of Hu and Chan [4] for shallow water waves were found to be valid for wavelengths as short as $ka < 0.5\pi$ via numerical analysis, so the constraint of (9) may be somewhat less severe. We also note that because $\nu_{dn}/\omega_{pd} \propto 1/\sqrt{m}$, and $\lambda_{Di} \propto \sqrt{n_i}$, it could be easier to satisfy (9) for more massive dust and lower ion number density.

For the case given above, the filling factor $f_s = \pi(r_v^2)/a^2 \sim \pi/16 \sim 0.2$. Thus, from (4b), we expect the dust acoustic wave speed to decrease by about 10% in the effective dusty plasma medium. The Brewster angle is $\theta_0 \sim 66.4^o$.

The question arises as how large a wire array can be accommodated in a plasma, because the electrons are attracted to the wires, as well as to the dust grains. Suppose we consider a relatively large cylindrical plasma chamber volume of size 3 cm in radius and 50 cm in length. If we consider a wire array composed of 20 rows of 10 wires each of radius 0.1 mm and length of 4 cm, the cumulative surface area of the wires would be about $50\,\mathrm{cm}^2$. This can be compared with the surface area of the chamber, which is about $940\,\mathrm{cm}^2$. Thus, we expect the collection of electrons by the wire array to be much less important than the collection by the chamber walls.

An important question is how to excite DAWs that have wavelengths larger than the inter-wire spacing (which is larger than the dusty plasma Debye radius). One possibility is an ion-dust streaming instability, but this leads to the excitation of a spectrum of DAWs including the required long wavelengths where $k\lambda_D\omega_{pd} > \nu_{dn}$. Perhaps, a better possibility is to launch the ultra low frequency long-wavelength waves, perhaps electrically or via radiation pressure. We note that the refraction of the dust ion-acoustic wave [12] by a cylinder/wire array may also be possible. The scenario may be somewhat more similar to shallow water waves, because we would not expect the formation of ion voids around the cylinders. On the other hand, the dust ion-acoustic wave could not be visualized by laser scattering, while the DAWs, of course, can be.

To summarize, first we have pointed out that the wave equation for the long-wavelength DAWs has the same form as the wave equation for the shallow water waves (as well as the wave equation for sound waves in air). Then, we theoretically investigated conditions for the refraction of the long-wavelength DAWs by arrays of periodic cylinders or wires in a dusty plasma. This was based on a recent analysis of Hu and Chan [4], who investigated the refraction of the shallow water waves by an array of periodic bottom-mounted vertical cylinders. In the dusty plasma case, however, the boundary conditions involve the formation of dust voids around the wires. Following the analysis of the shallow water wave case [4], we showed that as a function of the filling fraction of the wire/void system, the dust acoustic speed decreases in the effective medium and there is a Brewster angle where no reflection occurs. Some differences between the physics of the shallow water wave case and the dusty plasma case were pointed out, and theoretical issues for further study were discussed. Possible experimental conditions were also considered.

Acknowledgements

The work of MR was supported by NSF grant No. PHY-0903808. The work of RLM was supported by Department of Energy grant No. DE-FG02-04ER54795.

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