Robotica (2016) volume 34, pp. 584–600. © Cambridge University Press 2014
doi:10.1017/S0263574714001714

Characteristics analysis and stabilization of a planar 2R underactuated manipulator

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(Accepted June 4, 2014. First published online: July 9, 2014)

SUMMARY
The weightless planar 2R underactuated manipulators with passive last joint are considered in this paper for investigating a feasible method to stabilize the system, which is a second-order nonholonomic-constraint mechanical system with drifts. The characteristics including the controllability of the linear approximation model, the minimum phase property, the Small Time Local Controllability (STLC), the differential flatness, and the exactly nilpotentizable properties, are analyzed. Unfortunately, these negative characteristics indicate that the simplest underactuated mechanical system is difficult to design a stable closed-loop control system. In this paper, nilpotent approximation and iterative steering methods are utilized to solve the problem. A globally effective nilpotent approximation model is developed and the parameterized polynomial input is adopted to stabilize the system to its non-singularity equilibrium configuration. In accordance with this scheme, it is shown that designing a stable closed-loop control system for the underactuated mechanical system can be ascribed to solving a set of nonlinear algebraic equations. If the nonlinear algebraic equations are solvable, then the controller is asymptotically stable. Some numerical simulations demonstrate the effectiveness of the presented approach.

KEYWORDS: Underactuation; Manipulators; Nilpotent approximation; Iterative steering; Control.

1. Introduction

The underactuated mechanical systems (UMSs) have been a class of important research objects in robotics and nonlinear control theory in the last two decades. The UMSs are defined to be a class of mechanical systems that the number of the independent inputs is less than the degrees of freedom (DoFs) of the mechanism. According to this definition, many robot systems can be classified to this class of systems, for instance the wheeled robots,[1] hopping robots,[2] inverted pendulums,[3] Acrobots,[4] spherical robots,[5] the multi-fingers hands,[6] etc. The UMSs with potential forces, for instance, the inverted pendulums[3] and Acrobots,[4] of which the tangent linearization at the equilibrium configuration is controllable. However, the UMSs without weight or elastic forces, such as the horizontal planar manipulators with free-swing passive joints, of which the linear approximate models are not controllable,[7] and generally show the second-order nonholonomic constraints systems.[8] Thus the UMSs without potential forces cannot be stabilized by any smooth time-invariant pure state feedbacks,[8] and attract some scholars.

Existing methods associated with the nonholonomic motion planning and control issues primarily depend on two special properties, namely exactly nilpotentizable or differentially flat properties. For instance, Murray *et al.*[1,2] proposed the sinusoidal input methods for the chained form (a special nilpotent form) systems. De Luca *et al.*[9] proposed the dynamic feedback linearization methods for

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planar three links manipulator with a last passive joint, where the position of the center of percussion (CP) of the passive links is used to be the location of the dynamic output. By selecting the Cartesian position of the CP of the last passive link as the flat output, De Luca showed that the system is linearizable by dynamic feedback. In ref. [10] they extended this method to stabilize a manipulator system with two last passive joints. Analogous results can also be found in the literatures.[11,12] In ref. [12] Shiroma *et al*. extended the approach to stabilize the underactuated manipulators with n last passive joints. It is noteworthy that, in these relevant researches, the UMSs have two inputs, at least.

For the weightless planar underactuated manipulator with single input, De Luca *et al*.[7,10,13–15] pointed out that the linearization models of this class systems are not controllable, and even the original systems do not satisfy the sufficient conditions of the Small Time Local Controllability (STLC) theorem. They used the approximate nilpotentization[16] and iterative steering paradigm[17] approaches to stabilize the underactuated systems. For the same system under consideration, Arai *et al*.[18] proposed a time-scaling control method based on the bi-directional motion planning, Hong *et al*.[19] proposed an oscillatory inputs method based on the average theory, Alfredo Rosas-Flores *et al*.[20] adopted the sliding mode control, while Mahindrakar *et al*.[21] suggested that the friction of the passive joints can be used to stabilize the underactuated manipulator. In the relevant researches, the approaches presented by the group of De Luca showed better theoretical foundation, and the approach can be applied to other underactuated systems.[22–25] Although the control scheme was suggested at first by Lafferriere and Sussmann[26] implicitly, the explicit algorithm for the underactuated manipulators was proposed by De Luca *et al*.

In this paper, the stabilization of the planar 2R underactuated manipulators is considered. The main contributions of the paper include the following: (1) the characteristics of the weightless planar 2R underactuated manipulator are analyzed in detail; (2) the approximate nilpotentization algorithm for affined nonlinear systems with drift has been presented, and the globally valid basis is used to construct the accessible matrix for developing the globally valid approximate nilpotentization model; and (3) a parameterized polynomial is applied to construct the iterative inputs, such that the stabilization of the UMSs can be ascribed to solve a set of nonlinear algebraic equations. If the nonlinear algebraic equations are solvable, then the closed-loop system will be asymptotically stable.

2. Dynamic Model

A horizontal planar underactuated manipulator considered in this paper is shown in Fig. 1. The first joint hinged to base is actuated, while the second joint is passive. Let l_1, l_2 denote the lengths of the two links, m_1, m_2 denote the masses of the two links respectively, l_{c1}, l_{c2} denote the lengths between the mass center of the link and the corresponding axis of the joints respectively, and θ_1, θ_2 be the generalized coordinates of the system. Then the dynamic model of the system can be expressed as

$$m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + h_1 = \tau_u, \tag{1a}$$

$$m_{21}\ddot{\theta}_1 + m_{22}\ddot{\theta}_2 + h_2 = 0, \tag{1b}$$

where τ_u is the torque of the actuated joint, and

$$m_{11} = I_1 + I_2 + m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + 2m_2 l_1 l_{c2}\cos\theta_2, \quad m_{12} = I_2 + m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\theta_2,$$

$$m_{21} = m_{12}, \quad m_{22} = I_2 + m_2 l_{c2}^2, \quad h_1 = -m_2 l_1 l_{c2}\sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2), \quad h_2 = m_2 l_1 l_{c2}\sin\theta_2\dot{\theta}_1^2.$$

Let $u = \ddot{\theta}_1$ be a new input, the partial feedback linearization[27] of system (1) can be expressed by

$$\ddot{\theta}_1 = u, \tag{2a}$$

$$\ddot{\theta}_2 = -m_{22}^{-1}m_{21}\ddot{\theta}_1 - m_{22}^{-1}h_2. \tag{2b}$$

The state space equations can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{3}$$

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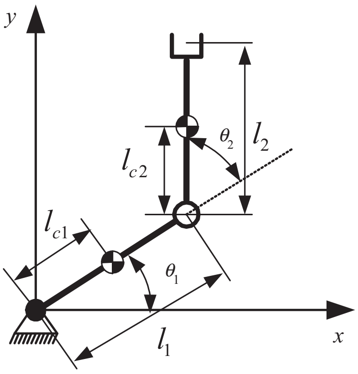

Fig. 1. The model of planar 2R underactuated manipulator.

where $\mathbf{x} = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^{\mathrm{T}} \in \mathbf{R}^4$ is considered, and the smooth vector fields are expressed as follows: $\mathbf{f}(\mathbf{x}) = [\dot{\theta}_1 \ \dot{\theta}_2 \ 0 \ -H\sin\theta_2\dot{\theta}_1^2]^{\mathrm{T}}$, $\mathbf{g}(\mathbf{x}) = [0 \ 0 \ 1 \ -(1 + H\cos\theta_2)]^{\mathrm{T}}$, where $H = m_2 l_1 l_{c2} \big/ (I_2 + m_2 l_{c2}^2)$.

3. System Analysis

3.1. The controllability of the linear approximation model

For a given equilibrium point $x = [\theta_1^* \ \theta_2^* \ 0 \ 0]$, the linear approximation of system (3) is

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}\mathbf{x} + \tilde{\mathbf{g}}u, \tag{4}$$

where $\tilde{\mathbf{f}} = 0 \in \mathbf{R}^{4\times4}$, $\tilde{\mathbf{g}} = [0 \ 0 \ 1 \ -(1+H)]^{\mathrm{T}}$. Obviously $\mathrm{rank}[\tilde{\mathbf{g}} \ \tilde{\mathbf{f}}\tilde{\mathbf{g}} \ \tilde{\mathbf{f}}^2\tilde{\mathbf{g}} \ \tilde{\mathbf{f}}^3\tilde{\mathbf{g}}] = 1 \neq 4$, hence approximation system (4) is not controllable.

3.2. The property of the non-minimum phase

The minimum-phase or non-minimum-phase nature of a system is an input–output characteristic that depends on the selected outputs.[27] As to the 2R manipulator considered in this paper, the nature outputs can be defined to be $y = \theta_i$, $i = 1, 2$, or $\mathbf{y} = [\theta_1 \ \theta_2]^{\mathrm{T}}$. System (3) with defining an output can be expressed by the following input–output system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \\ \mathbf{y} &= \mathbf{y}(\mathbf{x}). \end{aligned} \tag{5}$$

For the single output $y = \theta_1$, Eq. (2a) is the linearized part of the input–output system, while Eq. (2b) becomes the *Internal Dynamics* of the system. When the linearized part is controlled to its equilibrium $\ddot{y} = \dot{y} = 0$, the internal dynamics is defined to be the *Zero Dynamics*, i.e. $\ddot{\theta}_2 = 0$, which indicates that the motion of the passive joint of the underactuated manipulator is $\theta_2(t) = \theta_2(0) + \dot{\theta}_2(0)t$. If $\dot{\theta}_2(0) \neq 0$, the zero dynamics is unstable. If the zero dynamics is unstable, then the system is defined to be the *non-minimum-phase* system.

If the output is selected as $y = \theta_2$, to investigate the same problem, the first-order time-derivative of the output can be written as

$$\dot{y} = \frac{\partial y}{\partial \mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} = L_f y(\mathbf{x}) + L_g y(\mathbf{x})u, \tag{6}$$

where $L_f y(\mathbf{x}) = (\partial y \big/ \partial \mathbf{x})(\mathbf{x})\mathbf{f}(\mathbf{x})$ denotes the Lie derivative of y along the smooth vector field $\mathbf{f}(\mathbf{x})$, and $L_g y(\mathbf{x})$ is defined analogously. A simple calculation reveals $L_f y(\mathbf{x}) = \dot{\theta}_2$ and $L_g y(\mathbf{x}) = 0$. The

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second-order derivative of output y is expressed by

$$\ddot{y} = \frac{\partial L_f y}{\partial \mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} = L_f^2 y(\mathbf{x}) + L_g L_f y(\mathbf{x})u, \tag{7}$$

where $L_f^2 y(\mathbf{x}) = H \sin\theta_2 \dot{\theta}_1^2$ and $L_g L_f y(\mathbf{x}) = 1 + H\cos\theta_2$. If $1 + H\cos\theta_2 \neq 0$, select feedback

$$u = \ddot{\theta}_1 = -\frac{L_f^2 y}{L_g L_f y}(x) = -\frac{H\sin\theta_2\dot{\theta}_1^2}{1 + H\cos\theta_2} \tag{8}$$

such that $\ddot{y} = \dot{y} = 0$; then the zero dynamics is expressed by

$$\ddot{\theta}_1 = -\frac{H\sin\theta_2}{1 + H\cos\theta_2}\dot{\theta}_1^2. \tag{9}$$

When $\ddot{y} = \dot{y} = 0$, θ_2 is constant. If $\theta_2 \neq \pm k\pi$, $k = 0, 1, 2, \ldots$, and $\theta_2 \neq \arccos(-1/H)$, the solution of Eq. (11) can be written to

$$\theta_1(t) = \theta_1(0) + \frac{1}{H^*}\log(1 + \dot{\theta}_1(0)H^* t), \tag{10}$$

where $H^* = H\sin\theta_2/(1 + H\cos\theta_2)$. When $\dot{\theta}_1(0)H^* > 0$, $\theta_1(t)|_{t\to\infty} \to \infty$, whereas $\dot{\theta}_1(0)H^* < 0$, $\left(\log(1 + \dot{\theta}_1(0)H^* t)\right)\big|_{t \geq -\frac{1}{\dot{\theta}_1(0)H^*}} = 0$. Obviously, zero dynamics (9) is unstable, thus system (5) with output $y = \theta_2$ is non-minimum phase.

When the output is selected to be $\mathbf{y} = [\,\theta_1\ \theta_2\,]^{\mathrm{T}}$, the explicit relationship between the input and the output is given by Eq. (2), the maximal *relation degree* $r = 2 < n = 4$. The strict less relation indicates that system (5) with output $\mathbf{y} = [\,\theta_1\ \theta_2\,]^{\mathrm{T}}$ cannot be linearized by any feedback, thus the relevant internal dynamics, zero dynamics, minimum phase, or non-minimum phase do not exist.

3.3. *The small-time local controllability*

To investigate the controllability of a general affine system (5), the only systematic way to this end so far is to use the famous STLC theorem proposed by Sussmann.[28] The STLC is just a sufficient condition. Define $[\mathbf{f}, \mathbf{g}]$ to be the Lie bracket generated by $\mathbf{f}, \mathbf{g}$, and referring to Eq. (3), part of Lie brackets generated by $\mathbf{f}, \mathbf{g}$ can be calculated as

$$\begin{aligned}
\mathbf{g}_0 &= \mathbf{f}(\mathbf{x}) = \left[\,\dot{\theta}_1\ \dot{\theta}_2\ 0\ -H\sin\theta_2\dot{\theta}_1^2\,\right]^{\mathrm{T}},\\
\mathbf{g}_1 &= \mathbf{g}(\mathbf{x}) = \left[\,0\ 0\ 1\ -(1 + H\cos\theta_2)\,\right]^{\mathrm{T}},\\
\mathbf{g}_2 &= [\mathbf{f}, \mathbf{g}]\,(\mathbf{x}) = \left[\,-1\ 1 + H\cos\theta_2\ 0\ H(\dot{\theta}_2 + 2\dot{\theta}_1)\sin\theta_2\,\right]^{\mathrm{T}},\\
\mathbf{g}_3 &= [\mathbf{f}, [\mathbf{f}, \mathbf{g}]]\,(\mathbf{x}) = \left[\,0\ -2H\sin\theta_2(\dot{\theta}_1 + \dot{\theta}_2)\ 0\ H\cos\theta_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + H^2\dot{\theta}_1^2\cos(2\theta_2)\,\right]^{\mathrm{T}},\\
\mathbf{g}_4 &= [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]\,(\mathbf{x}) = \left[\,0\ 0\ 0\ -2H^2\sin\theta_2\cos\theta_2\,\right]^{\mathrm{T}},\\
\mathbf{g}_5 &= [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]\,(\mathbf{x}) = \left[\,0\ 2H^2\sin\theta_2\cos\theta_2\ 0\ -2H^2\dot{\theta}_2\cos(2\theta_2)\,\right]^{\mathrm{T}}.
\end{aligned} \tag{11}$$

It is obvious that the vector fields satisfy $[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_4, \mathbf{g}_5] \in R^4$ for all equilibriums $\mathbf{x}^* = [\,\theta_1^*\ \theta_2^*\ 0\ 0\,]^{\mathrm{T}}$ with $\theta_2^* \neq \pm k\pi/2$, $k = 0, 1, \ldots$, thus system (3) is globally accessible except for a few of intrinsic singular configurations. The vector field $\mathbf{g}_4 = [\mathbf{g}, [\mathbf{f}, \mathbf{g}]](\mathbf{x})$ has degrees $\delta(\mathbf{g}_4) = \delta^0(\mathbf{g}_4) + \delta^1(\mathbf{g}_4) = 1 + 2$, thus it is a bad bracket.[28] Whereas the bad bracket $\mathbf{g}_4$ cannot be expressed by the linear combinations of the good brackets[28] of $\mathbf{g}_1, \mathbf{g}_2$ with less degrees, therefore system (3) does not satisfy the second condition of the STLC theorem. As stated above, the STLC theorem is only a sufficient condition; violating the sufficient condition cannot conclude that system (3) is uncontrollable. In fact Alessandro *et al.*[10] pointed out that the underactuated 2R manipulator with first actuated joint is controllable (while the inverse case is not.[8])

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For conclusive result of the STLC for system (3), a necessary condition of STLC for single input system was given in ref. [30]. Refer to expression (11), the Lie brackets with degrees greater than three are generated by multiple vectors of **f** and single vector **g**, and contain the generalized speed $\dot{\theta}_i, i = 1, 2$ and the generalized coordinates $\theta_i, i = 1, 2$. For $\forall \mathbf{x}^* = [\theta_1^* \ \theta_2^* \ 0 \ 0]^{\mathrm{T}}$ with $\theta_2^* \neq \pm k\pi/2, k = 0, 1, \ldots,$ one can verify that $\dim(\chi(\mathbf{x}^*)) = 2 < n = 4$ while $\dim \operatorname{span}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_4, \mathbf{g}_5\} = 4 = n$. Because of violating the necessary condition, system (3) is not STLC at any equilibrium $\forall \mathbf{x}^* = [\theta_1^* \ \theta_2^* \ 0 \ 0]^{\mathrm{T}}$ with $\theta_2^* \neq \pm k\pi/2, k = 0, 1, \ldots$.

3.4. *The property of differential flatness*

Differential flatness is an appealing property of some special affine nonlinear systems. If a system is differentially flat, the dynamic feedback linearization or backstepping method can be used to stabilize the original system (5). For the weightless planar underactuated manipulators, if the control inputs satisfy $\mathbf{u} \in \mathbf{R}^n, n \geq 2$, and the serial passive links are hinged to their CP, then the Cartesian coordinates of CP of the last passive link can be selected to be the flat output. If we can find a flat output for a system, then the system is differentially flat. The differential flatness and feedback linearization are equivalent for single input system.[29,30] As shown in Section 3.2., system (3) has relative degrees $r = 2 < n = 4$, which indicates system (3) is not linearizable by any feedback, thus the single input system (3) is not differential flat.

For strictly proving a system is not flat, the ruled-manifold criterion[31] provides a simple necessary condition. The criterion means that eliminating **u** from $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ yields a set of equations $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) = 0$, which holds the following property: for all $(\mathbf{x}, \mathbf{p})$ that satisfy $\mathbf{F}(\mathbf{x}, \mathbf{p}) = 0$, there exist $\mathbf{a} \in \mathbf{R}^n$ and $\mathbf{a} \neq 0$ such that $\forall \lambda \in R$, $\mathbf{F}(\mathbf{x}, \mathbf{p} + \lambda\mathbf{a}) = 0$. $\mathbf{F}(\mathbf{x}, \mathbf{p}) = 0$ is thus a ruled manifold containing straight lines of direction $\boldsymbol{a}$.

As to the object considered in this paper, referring to Eq. (2), if one eliminates $u = \ddot{\theta}_1$ from the equation, then it follows that

$$q = \dot{\theta}_2, \quad \dot{q} = -H \sin\theta_2 \dot{\theta}_1^2. \tag{12}$$

Substituting $(\dot{\theta}_2, \dot{q}, \dot{\theta}_1)$ for $(\dot{\theta}_2 + \lambda a_1, \dot{q} + \lambda a_2, \dot{\theta}_1 + \lambda a_3)$ into Eq. (12) yields $q = (\dot{\theta}_2 + \lambda a_1)$, $(\dot{q} + \lambda a_2) = -H \sin\theta_2 (\dot{\theta}_1 + \lambda a_3)^2$. It is obvious that above equations hold for all $\lambda \in R$ if and only if $(a_1, a_2, a_3) = (0, 0, 0)$, which means $(\dot{\theta}_2, \dot{q}, \dot{\theta}_1)$ does not define a ruled sub-manifold for any (θ_2, q, θ_1). Hence system (2) is not flat.

3.5. *The property of exact nilpotentization*

Nilpotent or exactly nilpotentizable by feedback transformation is another appealing property of some nonlinear affine system, especially for nonholonomic systems.[32] For the nilpotent or exactly nilpotentizable systems, there are some feasible motion planning and control methods such as in refs. [1], [2], [16], [22–24], and [33–36]. A general affine system (5) is nilpotent if there exists an integer k such that all Lie products with the length greater than k generated by smooth vector fields **f**, **g** are zero. k is called the order of nilpotency. If the original system is not nilpotent but it can be exactly nilpotentizable by invertible feedback transformations, then the motion planning and control problem can be dealt with by a similar procedure of the nilpotent systems. Due to the special Lie algebra structure, the motion planning of the nilpotent affine systems can be solved by applying explicit quadratures.

The general issue of finding a nilpotent basis for a distribution has been studied by Hermes *et al.*[34] They presented a set of necessary conditions for the existence of a local nilpotent basis and gave two sets of sufficient conditions. However, the necessary condition is only used in this paper, and the necessary condition is presented for more general systems.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{k} \mathbf{g}_i(\mathbf{x}) u_i. \tag{13}$$

Let V denote the real vector space of the real analytic vector fields on R^n. $\Delta^k(\mathbf{x}) = \operatorname{span}\{\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_k(\mathbf{x})\}$ locally defines a $k-$ dimensional distribution. Consider system (3), part of Lie products are shown by Eq. (11), of which the distribution $\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5\}$ spans R^4 at all points

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except $\theta_2^* = \pm k\pi/2, k = 0, 1, \ldots$, whereas an invariant of distribution $\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5\}, \mathbf{g}_3 = 0$, is satisfied if and only if $\dot{\theta}_1 = \dot{\theta}_2 = 0$. This contradiction implies distribution $\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5\}$ could not admit a nilpotent basis. System (3) is not nilpotent. By applying the necessary condition of exact nilpotentization presented in ref. [34], it is obvious that the system is not exactly nilpotentizable by feedback transformation.

4. Nilpotent Approximations

The negative results of the characteristics for weightless planar 2R underactuated manipulator indicate that there does not exist a straightforward approach of synthesizing the control laws for the sake of stabilizing the second-order nonholonomic system.[8] Existing methods associated with nonholonomic motion planning and control issues are primarily considered the exactly nilpotentizable or differentially flat systems. For nonholonomic systems that do not hold these properties, the approximate steering techniques are also a valuable approach towards the solution. In recent years, the utilization of the nilpotent approximations for solving the nonholonomic motion planning and control issues have been given a lot of attention by Oriolo,[23] Vendittelli,[24,25,35] De Luca,[7,10,13–15] Bellaïche,[33] Sussmann,[36] and Hermes *et al*.[32,34] As pointed out by Oriolo *et al*. in ref. [23], the nilpotent approximation is a higher order approximation with increased adherence to the original dynamics, especially useful for tangent linearization without retaining the controllability as in the nonholonomic systems. At the same time, closed-form integration of the approximate model under parameterized inputs allows to design the steering controls. These merits brought by nilpotent approximation interest us to investigate its application in the control of second-order nonholonomic constraints mechanical systems.

Hermes *et al*.[16,32] and Bellaïche *et al*.[33] had studied the algorithms for nilpotent approximation. In this section, we use the results proposed in ref. [33] and refined in ref. [35] by Vendittelli *et al*., to develop the nilpotent approximation model for system (5). Similar to the work given by De Luca *et al*. in refs. [7], [13], and [14] in which the procedure was very summarized and a non-global basis was selected to construct the accessible matrix so that the motion planning has to be separated to multi-phases, alignment phase, contraction phase, and if needed, transition phase, and the special cyclic open-loop control command is also far from intuition.

The nilpotent approximation algorithm presented by Bellaïche[33] is applicable for driftless system. We extend this result to drift system (3). Let $\mathbf{g}_0(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ and $u_0 = 1$, then system (13) can be expressed as the following driftless system:

$$\dot{\mathbf{x}} = \sum_{i=0}^{k} \mathbf{g}_i(\mathbf{x}) u_i, \tag{14}$$

where $\mathbf{u} \in R^{k+1}$ is a new vector of the control input. Suppose system (14) satisfies the Lie Algebra Rank Condition (LARC),[36] i.e. the system is locally accessible. Fix a point $\mathbf{x}_0 \in \mathbf{R}^n$ and let $L^s(\mathbf{x}_0)$ be the vector space generated by the values at $\mathbf{x}_0$ of the Lie brackets of the smooth vector fields $\mathbf{g}_i(\mathbf{x})$, $i = 0, 1, 2, \ldots, k$, with length $\leq s, s = 1, 2, \ldots$. The input vector fields are brackets with length one. The accessibility of system (14) guarantees that there exists the smallest integer $r = r(\mathbf{x}_0)$ such that $\dim\{L^r(\mathbf{x}_0)\} = n$. The integer $r(\mathbf{x}_0)$ is called the degree of nonholonomy of system (14) at $\mathbf{x}_0$. Let $n_s(\mathbf{x}_0) = \dim\{L^s(\mathbf{x}_0)\}, s = 1, 2, \ldots, r$, and define the growth vector of the distribution $\Delta^s = L^s(\mathbf{x}_0)$ at $\mathbf{x}_0$ to be $(n_1(\mathbf{x}_0), n_2(\mathbf{x}_0), \ldots, n_r(\mathbf{x}_0))$.

According to the definition of the function's order presented in ref. [35], the input vector fields $\mathbf{g}_0(\mathbf{x}), \mathbf{g}_1(\mathbf{x}), \cdots, \mathbf{g}_k(\mathbf{x})$ have order ≥ -1.

The order of functions and vector fields expressed in privileged coordinates[35] can be computed in an algebraic way, i.e.:

1. The order of monomial $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ is equal to its weighted degree;
2. The order of a function $h(z)$ at $z = 0$ (the image of $\mathbf{x}_0$) is the least-weighted degree of the monomials actually appearing in the Taylor expansion of $h(z)$ at $z = 0$;

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3. The order of a vector field $\mathbf{g}(z) = \sum_{j=1}^{n} g_j(z)\partial_{z_j}$ at $z = 0$ is the least-weighted degree of the monomials actually appearing in the Taylor expansion of $\mathbf{g}(z)$ at $z = 0$, i.e. $\mathbf{g}(z) \sim \sum_{\alpha,j} \beta_{\alpha,j} \mathrm{z}_1^{\alpha_1} \cdots \mathrm{z}_\mathrm{n}^{\alpha_\mathrm{n}} \partial_{\mathrm{z_j}}$;

   with considering the term $\beta_{\alpha,j} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_{z_j}$ as a monomial and assigning to ∂_{z_j} the weight $-w_j$.

Consider system (14) and an approximation point $\mathbf{x}_0 \in R^n$, the algorithm[35] for computing a set of privileged coordinates and a nilpotent approximation at $\mathbf{x}_0$ is recalled here, and the procedure for computing the nilpotent approximation model of system (3) is going step by step:

1. Compute the growth vector $(n_1(\mathbf{x}_0), n_2(\mathbf{x}_0), \ldots, n_r(\mathbf{x}_0))$ and the weights $w_1, \ldots, w_n$ at $\mathbf{x}_0$.
   Refer to the Lie products expressed by (11), $\dim \operatorname{span}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_4, \mathbf{g}_5\} = 4 = n$ at all points $\mathbf{x} \in R^4$ except the intrinsic singular point $\theta_2 = \pm k\pi/2$, $k = 0, 1, \ldots$. In this paper the intrinsic singular points in configuration space of system (3) are not considered. The length of $\mathbf{g}_5 = [\mathbf{f}, [\mathbf{g}, [\mathbf{f}, \mathbf{g}]]]$ is the least-length of Lie bracket such that distribution Δ^4 spans the full state space R^4. Thus the degree of nonholonomy of system (3) is $r = 4$. The growth vector is $(n_1, n_2, n_3, n_4) = (1, 2, 3, 4)$. The weights are $(w_1, w_2, w_3, w_4) = (1, 2, 3, 4)$.
2. Choose vector fields $\gamma_1(\mathbf{x}_0), \ldots, \gamma_n(\mathbf{x}_0)$ such that their values at $\mathbf{x}_0$ form a basis of $L^r(\mathbf{x}_0)$ with $\dim(L^r(\mathbf{x}_0)) = n$ and such that $\gamma_{n_{s-1}+1}(\mathbf{x}), \ldots, \gamma_{n_s}(\mathbf{x}) \in L^s(\mathbf{x})$, $s = 1, \ldots, r$ for any $\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$, with $n_0 = 0$. Construct accessible matrix $\mathbf{A} = [\,\gamma_1(\mathbf{x}_0), \cdots, \gamma_n(\mathbf{x}_0)\,] \in R^{n\times n}$.
   For system (3), the vector fields $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_4, \mathbf{g}_5$ are selected as the basis for R^4. We noticed that this basis is valid for $\forall \mathbf{x}_0 = [\,\theta_{10}\ \theta_{20}\ \dot{\theta}_{10}\ \dot{\theta}_{20}\,]^{\mathrm{T}}$ except the singular point. In ref. [10] De Luca adopted the vector fields $\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ as the basis that is not valid at points with $\dot{\theta}_{10} = \dot{\theta}_{20} = 0$. This causes that the motion planning for the 2R underactuated manipulator has to be separated to multi-phases. As we will show in Section 5, the new selection can avoid this bother. The accessible matrix can be given by

$$
\begin{aligned}
\mathbf{A} &= \left[\,\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_4\ \mathbf{g}_5\,\right](\mathbf{x}_0) \\
&= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 + Hc_{20} & 0 & 2H^2 s_{20} c_{20} \\ 1 & 0 & 0 & 0 \\ -(1 + Hc_{20}) & H(\dot{\theta}_{20} + 2\dot{\theta}_{10}) s_{20} & -2H^2 s_{20} c_{20} & -2H^2 \dot{\theta}_{20} \cos(2\theta_{20}) \end{bmatrix},
\end{aligned}
$$

   where $s_{20} = \sin\theta_{20}$, $c_{20} = \cos\theta_{20}$.
3. From the original coordinates $\mathbf{x}$, compute local coordinates $\mathbf{y}$ as $\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x} - \mathbf{x}_0)$.
   Since

$$
\mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ \zeta & \frac{\dot{\theta}_{20}\cos(2\theta_{20})}{-2H^2(s_2 c_2)^2} & \frac{1+Hc_{20}}{-2H^2 s_{20} c_{20}} & \frac{-1}{2H^2 s_{20} c_{20}} \\ \frac{1+Hc_{20}}{2H^2 s_{20} c_{20}} & \frac{1}{2H^2 s_{20} c_{20}} & 0 & 0 \end{bmatrix},
$$

   where $\zeta = \frac{H(\dot{\theta}_{20}+2\dot{\theta}_{10})(s_{20})^2 c_{20} + (1+Hc_{20})\dot{\theta}_{20}\cos(2\theta_{20})}{-2H^2(s_{20}c_{20})^2}$, one gets that

$$
\begin{aligned}
y_1 &= \dot{\theta}_1 - \dot{\theta}_{10}, \\
y_2 &= -(\theta_1 - \theta_{10}), \\
y_3 &= \zeta\,(\theta_1 - \theta_{10}) - \frac{\dot{\theta}_{20}\cos(2\theta_{20})}{2H^2\,(s_{20}c_{20})^2}\,(\theta_2 - \theta_{20}) - \frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}(\dot{\theta}_1 - \dot{\theta}_{10}) - \frac{1}{2H^2 s_{20} c_{20}}(\dot{\theta}_2 - \dot{\theta}_{20}), \\
y_4 &= \frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}\,(\theta_1 - \theta_{10}) + \frac{1}{2H^2 s_{20} c_{20}}\,(\theta_2 - \theta_{20})\,.
\end{aligned}
\tag{15}
$$

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For all equilibriums $\mathbf{x}_e = [\theta_{10}\ \theta_{20}\ 0\ 0]^{\mathrm{T}}$ with $\theta_{20} \neq \pm\frac{k\pi}{2}$, $k = 0, 1, 2, \ldots$, the local coordinates (15) can be rewritten as

$$\begin{aligned} y_1 &= \dot{\theta}_1, \\ y_2 &= -(\theta_1 - \theta_{10}), \\ y_3 &= -\frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}\dot{\theta}_1 - \frac{1}{2H^2 s_{20} c_{20}}\dot{\theta}_2, \\ y_4 &= \frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}(\theta_1 - \theta_{10}) + \frac{1}{2H^2 s_{20} c_{20}}(\theta_2 - \theta_{20}). \end{aligned} \tag{16}$$

4. Build the privileged coordinates $z_1, \ldots, z_n$ around $\mathbf{x}_0$ via the recursive formula[33,35]

$$z_j = y_j + \sum_{k=2}^{w_j - 1} h_k(y_1, \ldots, y_{j-1}), \quad j = 1, \ldots, n, \tag{17}$$

where

$$h_k(y_1, \ldots, y_{j-1}) = -\sum_{\substack{|\alpha| = k \\ w(\alpha) < w_j}} \beta_j \gamma_1^{\alpha_1} \cdots \gamma_{j-1}^{\alpha_{j-1}} \left(y_j + \sum_{p=2}^{k-1} h_p \right)(\mathbf{x}_0), \tag{18}$$

with $\beta_j = \prod_{i=1}^{j-1}\left(y_i^{\alpha_i}/\alpha_i!\right)$, $|\alpha| = \sum_{i=1}^{n}\alpha_i$, $w(\alpha) = \sum_{i=1}^{n} w_i\alpha_i$, where α_i, $i = 1, 2, \ldots, n$, are positive integers. Obviously, the monomials in $\sum h_k$ have weight ≥ 2 and $< w_j$. Thus the polynomial $\sum h_k$ occurs in the privileged coordinates z_j if and only if weight $w_j \geq 3$.
Let $\mathbf{x}_0 = \mathbf{x}_e$, we interest the local privileged coordinates at equilibrium. Refer to Eqs. (16) and (17), the first two privileged coordinates can be computed as

$$z_1 = y_1, \quad z_2 = y_2.$$

According to Eq. (17), we have the privileged coordinate $z_3 = y_3 + h_2(y_1, y_2)(\mathbf{x}_e)$, where

$$h_2(y_1, y_2)(\mathbf{x}_e) = -\sum_{\substack{\alpha_1 + \alpha_2 = 2 \\ w_1\alpha_1 + w_2\alpha_2 < w_3 = 3}} \beta_3 \left(\gamma_1^{\alpha_1}\gamma_2^{\alpha_2}\right) y_3(\mathbf{x}_e). \tag{19}$$

Since $w_1 = 1$, $w_2 = 2$, and α_1, α_2 are positive, the unique solution satisfying the in equations $\begin{cases} \alpha_1 + \alpha_2 = 2 \\ \alpha_1 + 2\alpha_2 < 3 \end{cases}$ is $(\alpha_1, \alpha_2) = (2, 0)$. Substituting $(\alpha_1, \alpha_2) = (2, 0)$ into Eq. (18), one can find the second-order Lie derivate $\left(\gamma_1^2\right) y_3(\mathbf{x}_e) = 0$, thus $h_2(y_1, y_2)(\mathbf{x}_e) = 0$. This indicates $z_3 = y_3$.
According to Eq. (17), the fourth privileged coordinate is given by

$$z_4 = y_4 + h_2(y_1, y_2, y_3) + h_3(y_1, y_2, y_3), \tag{20}$$

where

$$h_2(y_1, y_2, y_3)(\mathbf{x}_e) = -\sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = 2 \\ w_1\alpha_1 + w_2\alpha_2 + w_3\alpha_3 < w_4 = 4}} \beta_4 \left(\gamma_1^{\alpha_1}\gamma_2^{\alpha_2}\gamma_3^{\alpha_3}\right) y_4(\mathbf{x}_e). \tag{21}$$

Since $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, and $\alpha_1, \alpha_2, \alpha_3$ are positive, the solutions of the in equations

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 < 4 \end{cases}$$

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are $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$ or $(2, 0, 0)$. However, the second-order Lie derivative $(\gamma_1\gamma_2)\, y_4(\mathbf{x}_e) = \left(\gamma_1^2\right) y_4(\mathbf{x}_e) = 0$, thus $h_2(y_1, y_2, y_3)(\mathbf{x}_e) = 0$. In Eq. (20), the third term

$$h_3(y_1, y_2, y_3)(\mathbf{x}_e) = -\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=3\\ w_1\alpha_1+w_2\alpha_2+w_3\alpha_3<w_4=4}} \beta_4\left(\gamma_1^{\alpha_1}\gamma_2^{\alpha_2}\gamma_3^{\alpha_3}\right) y_4(\mathbf{x}_e). \tag{22}$$

Similarly, since $w_1 = 1$, $w_2 = 2$, $w_3 = 3$ and $\alpha_1, \alpha_2, \alpha_3$ are positive, the unique solution of the in equations

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 3 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 < 4 \end{cases}$$

is $(\alpha_1, \alpha_2, \alpha_3) = (3, 0, 0)$. The third-order Lie derivative $\left(\gamma_1^3\right) y_4(\mathbf{x}_e) = 0$, thus $h_3(y_1, y_2, y_3)(\mathbf{x}_e) = 0$, the privileged coordinates z_4 hold $z_4 = y_4$.

The detailed computations given above show that the local coordinates y_j happen to be the privileged coordinate z_j. This can also be confirmed by computing the order of the local coordinates $y_j, j = 1, \ldots, 4$. By definition 1, one can verify that the order of y_j exactly equals to w_j, $j = 1, \ldots, 4$. For the purpose of clarity, the privileged coordinates are given by

$$\begin{aligned} z_1 &= \dot{\theta}_1, \\ z_2 &= -\left(\theta_1 - \theta_{10}\right), \\ z_3 &= -\frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}\dot{\theta}_1 - \frac{1}{2H^2 s_{20} c_{20}}\dot{\theta}_2, \\ z_4 &= \frac{1 + Hc_{20}}{2H^2 s_{20} c_{20}}\left(\theta_1 - \theta_{10}\right) + \frac{1}{2H^2 s_{20} c_{20}}\left(\theta_2 - \theta_{20}\right). \end{aligned} \tag{23}$$

5. Express the dynamics of the original system in privileged coordinates $\dot{\mathbf{z}} = \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{z}) u_i$. Referring to Eqs. (3) and (23), one obtains $\dot{z}_1 = u_1$, $\dot{z}_2 = -z_1$, $\dot{z}_3 = \frac{1}{2Hc_{20}} z_1^2$, and $\dot{z}_4 = -z_3$. Then the dynamics in privileged coordinates can be concisely expressed as

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 \\ -z_1 \\ z_1^2/(2Hc_{20}) \\ -z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1. \tag{24}$$

6. Expand the vector fields $\mathbf{g}_i(\mathbf{z})$ in Taylor series at $z = 0$ and express them in terms of vector fields that are homogeneous with respect to the weighted degree $\mathbf{g}_i(\mathbf{z}) = \mathbf{g}_i^{(-1)}(\mathbf{z}) + \mathbf{g}_i^{(0)}(\mathbf{z}) + \mathbf{g}_i^{(1)}(\mathbf{z}) + \ldots$. Let $\hat{\mathbf{g}}_i(\mathbf{z}) = \mathbf{g}_i^{(-1)}(\mathbf{z})$, where $\mathbf{g}_i^{(-1)}(\mathbf{z})$ is the principal component of $\mathbf{g}_i(\mathbf{z})$. The nilpotent approximation system of system (14) can be defined as

$$\dot{z}_i = \sum_{j=0}^{k} \hat{\mathbf{g}}_{ji} u_j, \quad i = 1, \ldots, v, \tag{25}$$

$$\dot{z}_s = \sum_{j=0}^{k} \hat{\mathbf{g}}_{js}\left(z_1, \ldots, z_{s-1}\right) u_j, \quad s = v + 1, \ldots, n, \tag{26}$$

where v is the dimension of distribution $\Delta = \text{span}\left\{\mathbf{g}_0, \ldots, \mathbf{g}_k\right\}(\mathbf{x}_e)$. For $i = 1, \ldots, v$, it is straightforward that $\hat{\mathbf{g}}_{0i}, \ldots, \hat{\mathbf{g}}_{ki}$ are constants. For $s = v + 1, \ldots, n$, $\hat{\mathbf{g}}_{js}\left(z_1, \ldots, z_{s-1}\right)$ are polynomial functions of homogeneous degree $w_s - 1$.

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By the approximating procedure given above, one can find that the nilpotent approximation of system (24) is itself, i.e.

$$\dot{\mathbf{z}} = \hat{\mathbf{g}}_0(\mathbf{z}) + \hat{\mathbf{g}}_1(\mathbf{z})u_1 = \begin{bmatrix} 0 \\ -z_1 \\ z_1^2/(2Hc_{20}) \\ -z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1. \tag{27}$$

It is easy to show that the Lie products of the approximation system are

$$\begin{aligned} \hat{\mathbf{g}}_1 &= \left[1\ 0\ 0\ 0\right]^{\mathrm{T}}, \\ [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1] &= \left[0\ 1\ \mathrm{z}_1/(Hc_{20})\ 0\right]^{\mathrm{T}}, \\ [\hat{\mathbf{g}}_0, [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]] &= \left[0\ 0\ 0\ \mathrm{z}_1/(Hc_{20})\right]^{\mathrm{T}}, \\ [\hat{\mathbf{g}}_1, [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]] &= \left[0\ 0\ -1/(Hc_{20})\ 0\right]^{\mathrm{T}}, \\ [\hat{\mathbf{g}}_0, [\hat{\mathbf{g}}_1, [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]]] &= \left[0\ 0\ 0\ 1/(Hc_{20})\right]^{\mathrm{T}}. \end{aligned}$$

The vector fields $\hat{\mathbf{g}}_1$, $[\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]$, $[\hat{\mathbf{g}}_1, [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]]$, $[\hat{\mathbf{g}}_0, [\hat{\mathbf{g}}_1, [\hat{\mathbf{g}}_0, \hat{\mathbf{g}}_1]]]$ span the full state space R^4, thus the approximate system (27) is accessible. By further calculations it is shown that the Lie products with length greater than 4 are zero, hence the approximate system (27) is nilpotent of order 4. As shown in Eq. (27), the nilpotent approximation shows a triangular polynomial structure. Therefore, if we apply a parameterized input, the nilpotent system (27) can be integrated as the closed form as presented in the next section. For an exactly nilpotentizable system, the motion planning is solved accurately by this method. For the nilpotent approximate system, this leads to an iterative steering procedure[7,10–13,23–25,37] because of the errors between the original system and the nilpotent approximate system.

5. Designing the Controller

Similar to the method presented in the works of De Luca *et al.*,[7,10,13–15] we also investigate the feasible control method for weightless planar 2R underactuated manipulator by iterative steering. However, a multi-phases steering scheme was presented in ref. [7]. The method proposed in this paper does not need separate the motion to different phases, the stability of the controller is guaranteed by the existence of the solution of a set of nonlinear algebra equations.

Due to the triangular structure of system (27), a parameterized polynomial is used to synthesize the closed-loop controller. By simple inspection, the control must satisfy four sets of boundary conditions, namely

$$\ddot{\theta}_1(0) = \ddot{\theta}_1(1) = 0, \quad \dot{\theta}_1(0) = \dot{\theta}_1(1) = 0, \quad \ddot{\theta}_2(0) = \ddot{\theta}_2(1) = 0, \quad \dot{\theta}_2(0) = \dot{\theta}_2(1) = 0. \tag{28}$$

To stabilize the system to a target configuration, the configuration errors must contract after every cyclic of the input, viz.

$$e_i = \theta_{id} - \theta_i(1) = \eta_i\left(\theta_{id} - \theta_i(0)\right), \quad i = 1, 2, \tag{29}$$

where $\eta_i \in (0, 1)$ is the ratio of contraction of the position errors. To simplify the formulations, at start of every cyclic the time reset to zero is considered in (29). Due to the four sets of boundary conditions (28) and the contraction condition of errors (29), let us select a five-order polynomial as the cyclic input

$$u_1(\tau) = \ddot{\theta}_1(\tau) = a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5, \tag{30}$$

where $a_i, i = 1, \ldots, 5$ are free coefficients, and $\tau = \frac{t}{T} \in [0, 1]$.

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From the boundary condition $\ddot{\theta}_1(0) = \ddot{\theta}_1(1) = 0$, one obtains $a_5 = -(a_1 + a_2 + a_3 + a_4)$, therefore the input becomes

$$u_1(\tau) = \ddot{\theta}_1(\tau) = a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 - (a_1 + a_2 + a_3 + a_4)\,\tau^5. \tag{31}$$

Refer to Eq. (27), and by considering (31), we have

$$z_1 = \dot{\theta}_1 = \int_0^\tau u_1(\sigma)\mathrm{d}\sigma = \frac{1}{2}a_1\tau^2 + \frac{1}{3}a_2\tau^3 + \frac{1}{4}a_3\tau^4 + \frac{1}{5}a_4\tau^5 - \frac{1}{6}(a_1 + a_2 + a_3 + a_4)\,\tau^6. \tag{32}$$

Equation (32) should agree with the boundary condition $\dot{\theta}_1(0) = \dot{\theta}_1(1) = 0$, viz.

$$a_4 = -\left(10a_1 + 5a_2 + 5/2a_3\right).$$

Therefore, Eq. (31) has the form

$$u_1(\tau) = a_1\tau + a_2\tau^2 + a_3\tau^3 - (10a_1 + 5a_2 + 5/2a_3)\tau^4 + \left(9a_1 + 4a_2 + 3/2a_3\right)\tau^5, \tag{33}$$

then Eq. (32) becomes

$$z_1 = a_1\left(\frac{1}{2}\tau^2 - 2\tau^5 + \frac{3}{2}\tau^6\right) + a_2\left(\frac{1}{3}\tau^3 - \tau^5 + \frac{2}{3}\tau^6\right) + a_3\left(\frac{1}{4}\tau^4 - \frac{1}{2}\tau^5 + \frac{1}{4}\tau^6\right). \tag{34}$$

From Eqs. (23), (27), and (34) we obtain

$$z_2 = -\int_0^1 z_1(\tau)\mathrm{d}\tau = -\frac{1}{21}a_1 - \frac{1}{84}a_2 - \frac{1}{420}a_3 = -(\theta_1(1) - \theta_{10}). \tag{35}$$

From Eq. (27), we also have

$$z_3 = \frac{1}{2Hc_{20}}\int_0^\tau (z_1(\sigma))^2\mathrm{d}\sigma = \frac{1}{2Hc_{20}}\int_0^\tau\left(\int_0^\sigma u(\rho)\mathrm{d}\rho\right)^2\mathrm{d}\sigma. \tag{36}$$

By considering the boundary condition $\dot{\theta}_2(0) = \dot{\theta}_2(1) = 0$, from Eq. (36) one gets

$$b_1a_1^2 + b_2a_2^2 + b_3a_3^2 + b_4a_1a_2 + b_5a_1a_3 + b_6a_2a_3 = 0. \tag{37}$$

Once more, from Eq. (27), we have

$$z_4 = -\int_0^1 z_3(\tau)\mathrm{d}\tau = D, \tag{38}$$

where $D = \frac{1}{2Hc_{20}}\left(c_1a_1^2 + c_2a_2^2 + c_3a_3^2 + c_4a_1a_2 + c_5a_1a_3 + c_6a_2a_3\right)$. In Eqs. (37) and (38), the coefficients $b_i, i = 1, \ldots, 6$ and $c_i, i = 1, \ldots, 6$ are constants, which are listed in the Appendix. It is obvious that Eqs. (34) and (37) guarantee input (33) to satisfy the speed boundary conditions $\dot{\theta}_1(0) = \dot{\theta}_1(1) = 0$ and $\dot{\theta}_2(0) = \dot{\theta}_2(1) = 0$, which means the system is periodically stable. From the fourth equation of (24) and Eq. (38), the following equation can be obtained:

$$\psi_1(\theta_1(1) - \theta_{10}) + \psi_2(\theta_2(1) - \theta_{20}) = D, \tag{39}$$

where $\psi_1 = \frac{1+Hc_{20}}{2H^2s_{20}c_{20}}$, $\psi_2 = \frac{1}{2H^2s_{20}c_{20}}$.

The error contraction condition for θ_1 can be deduced from Eqs. (35) and (29),

$$\xi_1(a_1, a_2, a_3) = (1 - \eta_1)e_1, \tag{40}$$

where $e_1 = \theta_{1d} - \theta_1(0)$, $\xi_1(a_1, a_2, a_3) = \frac{1}{21}a_1 + \frac{1}{84}a_2 + \frac{1}{420}a_3$.

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The error contraction condition for θ_2 can be deduced from Eqs. (29) and (39):

$$\xi_2(a_1, a_2, a_3) = \psi_2(1 - \eta_2)e_2 + \psi_1\xi_1, \tag{41}$$

where $e_2 = \theta_{2d} - \theta_2(0)$, $\xi_2(a_1, a_2, a_3) = D$.

Up to now, the parameterized polynomial cyclic command (33) can be defined by Eqs. (37), (40), and (41), of which Eq. (37) is induced by speed boundary $\dot{\theta}_2(1) = 0$ while the latter two are contraction conditions of the position errors of the two joints. To simplify the formulations of stabilization conditions (37), (40), and (41), we solve a_2 from Eq. (40), and then we have

$$a_2 = -(d_1 a_1 + d_3 a_3 + d_4), \tag{42}$$

where $d_1 = 4$, $d_3 = \frac{1}{5}$, $d_4 = -84(1 - \eta_1)e_1$. Substituting (42) into (37), one gets

$$p_{11}a_1^2 + p_{12}a_3^2 + p_{13}a_1a_3 + p_{14}a_1 + p_{15}a_3 + p_{16} = 0. \tag{43}$$

Substituting (42) into (41), it follows that

$$p_{21}a_1^2 + p_{22}a_3^2 + p_{23}a_1a_3 + p_{24}a_1 + p_{25}a_3 + p_{26} = 0. \tag{44}$$

In Eqs. (43) and (44), coefficients $p_{1i}, i = 1, \ldots, 6$ and $p_{2i}, i = 1, \ldots, 6$ are constants, which are listed in the Appendix. Newton-Raphson method[37] can be used to solve the nonlinear algebraic Eqs. (43) and (44). If the solution is obtainable, the cyclic input (33) defined by (42)–(44) can stabilize system (3) from any initial configuration to final configuration except a few of singular points.

It is worth mentioning that the presented controller could be utilized to track a continuous path by point-to-point (PTP) control approach if the given path does not contain any singular configurations (i.e. $\theta_2^* \neq \pm k\pi/2$). However, the presented stabilization approach is hard to be applied to track given trajectories due to the long settling time of the controller, and this is also shown in the next section.

6. Numerical Simulations

To test the feasibility of the cyclic input (33) with the stabilizable conditions (42)–(44), the position stabilization is simulated numerically in this section. We define $e_i = \theta_{id} - \theta_i(t)$, $i = 1, 2$; then three different initial conditions are considered for testing the validity of the controller:

1. $e_1 \cdot e_2 < 0$, i.e. the initial position errors of the two joints have different sign;
2. $e_1 < 0$, $e_2 < 0$, the initial position errors are negative;
3. $e_1 > 0$, $e_2 > 0$, the initial position errors are positive.

Typically, the simulation results correspond to the three cases illustrated in Figs. 2–4, respectively. Fig. 5(a) and (b) show the data of initial 0–2 s from Fig. 2(b) and (c) respectively. The period of the cyclic input is $T = 1$ s in the simulations. One can find the speeds and the accelerations of the joints satisfying the boundary conditions (28) from Fig. 5. Referring to Figs. 2–4, we notice that the first case has relatively better convergence rate than the others. This phenomenon can be explained by the dynamics coupling Eq. (2b), which indicates that the active joint and the passive joint always have the accelerations with contrary directions. Thus if the initial position errors of the two joints have different signs, the system is easier to be stabilized to final position relative to the others cases. Apparently, in Figs. 3 and 4, the input has to steer the errors to a state that is coincident with the first case during the initial moment, whereafter converge to zero.

7. Conclusions

We show that the weightless planar 2R underactuated manipulator with first active joint has the following properties: the linear approximation model is not controllable, non-minimum phase, not STLC, not differential flat, and not exactly nilpotentizable. These negative characteristics considerably complicate the controller synthesis issues. Similar to the works presented by De Luca and his

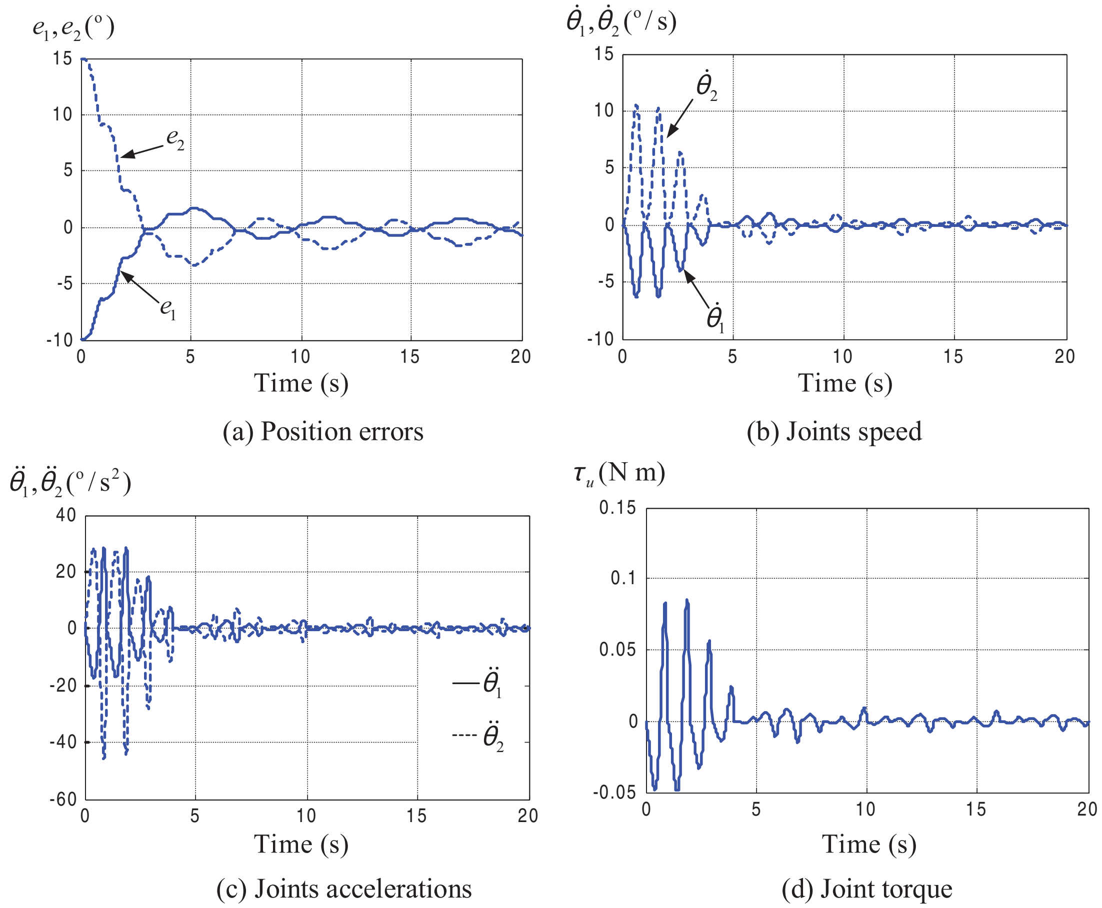

Fig. 2. Position control with initial position error relationship, $e_1 \cdot e_2 < 0$: (a) position errors; (b) joints speed; (c) joints accelerations; (d) joint torque.

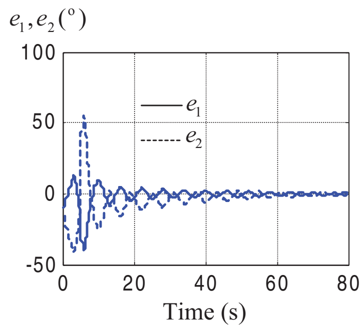

Fig. 3. Position control with initial position errors, $e_1 < 0,\ e_2 < 0$.

coworkers, we adopt the nilpotent approximation and iterative steering scheme to stabilize the second-order nonholonomic underactuated 2R manipulator. However, as distinguished from the relevant approaches, it is shown in the present paper that using a set of global valid basis to construct the accessible matrix and whereafter to deduce the nilpotent approximation model can remarkably simplify the motion planning for the sake of stabilizing the underactuated manipulator. On the other hand, it is shown that the synthesis of a stabilizable controller by parameterized polynomial input is ascribed to solve a set of nonlinear algebraic equations. If the nonlinear algebraic equations are solvable, the controller is asymptotically stable. Thus the control strategy presented in the paper shows more generality and can be applied to stabilize other UMSs without controllable linearization.

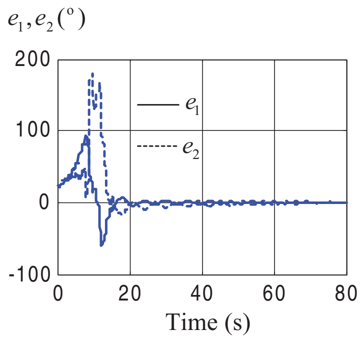

Fig. 4. Position control with initial position errors, $e_1 > 0$, $e_2 > 0$.

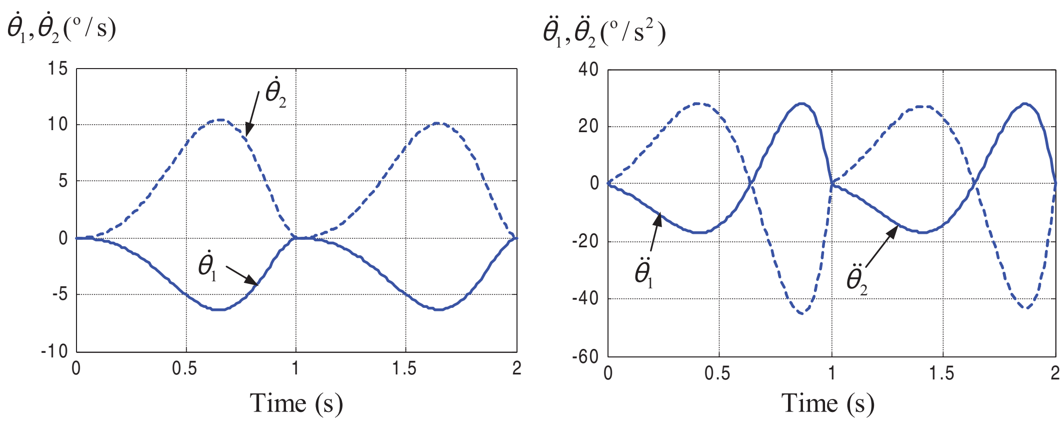

Fig. 5. Joints speed and acceleration during 0–2 s from Fig. 2: (a) joints speed with 0–2 s data from Fig. 2(b); (b) joints acceleration with 0–2 s data from Fig. 2(c).

Acknowledgements

The authors would like to thank the anonymous reviewers for giving many helpful suggestions, such that the presentation of the paper is improved. This work is supported by the Nature Science Foundation of China (Nos. 50975004, 51375016) and Beijing Nature Science Foundation (No. 3062009).

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Appendix

The coefficients in Eq. (37) are given by:

$$
\begin{aligned}
b_1 &= \frac{1}{20}+\frac{4}{11}+\frac{9}{52}-\frac{1}{4}+\frac{1}{6}-\frac{1}{2},\\
b_2 &= \frac{1}{63}+\frac{1}{11}+\frac{4}{117}-\frac{2}{27}+\frac{1}{45}-\frac{1}{9},\\
b_3 &= \frac{1}{144}+\frac{1}{44}+\frac{1}{208}-\frac{1}{40}+\frac{1}{88}-\frac{1}{48},\\
b_4 &= \frac{1}{18}+\frac{4}{27}+\frac{1}{10}-\frac{1}{8}+\frac{4}{11}-\frac{1}{4}+\frac{2}{27}-\frac{2}{9}+\frac{2}{13},\\
b_5 &= \frac{1}{28}-\frac{1}{10}+\frac{3}{44}-\frac{1}{16}+\frac{2}{11}-\frac{1}{8}+\frac{1}{36}-\frac{1}{12}+\frac{3}{52},\\
b_6 &= \frac{1}{48}-\frac{1}{20}+\frac{1}{33}-\frac{1}{27}+\frac{1}{11}-\frac{1}{18}+\frac{1}{60}-\frac{1}{24}+\frac{1}{39}.
\end{aligned}
$$

The coefficients in Eq. (38) can be given as follows:

$$
\begin{aligned}
c_1 &= \frac{1}{120}+\frac{4}{132}+\frac{9}{728}-\frac{1}{36}+\frac{1}{60}-\frac{1}{26},\\
c_2 &= \frac{1}{504}+\frac{1}{132}+\frac{4}{1638}-\frac{1}{135}+\frac{1}{495}-\frac{1}{117},\\
c_3 &= \frac{1}{1440}+\frac{1}{528}+\frac{1}{2912}-\frac{1}{440}+\frac{1}{1056}-\frac{1}{624},\\
c_4 &= \frac{1}{126}-\frac{2}{135}+\frac{1}{110}-\frac{1}{72}+\frac{1}{33}-\frac{1}{26}+\frac{1}{135}-\frac{2}{117}+\frac{1}{91},\\
c_5 &= \frac{1}{224}-\frac{1}{110}+\frac{1}{176}-\frac{1}{144}+\frac{1}{66}-\frac{1}{104}+\frac{1}{360}-\frac{1}{156}+\frac{1}{728},\\
c_6 &= \frac{1}{432}-\frac{1}{220}+\frac{1}{396}-\frac{1}{270}+\frac{1}{132}-\frac{1}{234}+\frac{1}{660}-\frac{1}{312}+\frac{1}{546}.
\end{aligned}
$$

The coefficients in Eq. (43) are given by:

$$
\begin{aligned}
p_{11} &= b_2 d_1^2 - b_4 d_1 + b_1,\\
p_{12} &= b_2 d_3^2 - b_4 d_3 + b_3,\\
p_{13} &= 2b_2 d_1 d_3 - b_4 d_1 - b_4 d_3 + b_5,\\
p_{14} &= 2b_2 d_1 d_4 - b_4 d_4,\\
p_{15} &= 2b_2 d_1 d_3 - b_4 d_4,\\
p_{16} &= b_2 d_4^2,
\end{aligned}
$$

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The coefficients in Eq. (44) can be given as follows:

$$
\begin{aligned}
p_{21} &= c_2 d_1^2 - c_4 d_1 + c_1, \\
p_{22} &= c_2 d_3^2 - c_4 d_3 + c_3, \\
p_{23} &= 2c_2 d_1 d_3 - c_4 d_1 - c_4 d_3 + c_5, \\
p_{24} &= 2c_2 d_1 d_4 - c_4 d_4, \\
p_{25} &= 2c_2 d_1 d_3 - c_4 d_4, \\
p_{26} &= c_2 d_4^2 - \frac{\psi_2}{\psi_3}(1 - \eta_2)\, e_2 - \frac{\psi_2}{\psi_3}(1 - \eta_1)\, e_1,
\end{aligned}
$$

where $\psi_3 = \frac{1}{2Hc_{20}}$.

https://doi.org/10.1017/S0263574714001714 Published online by Cambridge University Press