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On Subcartesian Spaces Leibniz′ Rule Implies the Chain Rule

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Abstract. We show that derivations of the differential structure of a subcartesian space satisfy the chain rule and have maximal integral curves.

1 Introduction

The structure of a smooth manifold is usually described in terms of its complete atlas. In 1967, Aronszajn [1] applied this description to Hausdorff spaces that are locally diffeomorphic to arbitrary subsets of $\mathbb{R}^n$, which he called subcartesian spaces. In 1973, Walczak [7] showed that that subcartesian spaces of Aronszajn are special cases of the differential spaces introduced by Sikorski [3]. This implied that the geometric structure of a subcartesian space S can be completely described by its ring of smooth functions $C^\infty(S)$.

In recent years, the notions of C^∞-ring and C^∞-ringed space appeared as part of Spivak's definition of derived manifolds [6]. Joyce [2] developed an alternative theory of derived differential geometry going beyond Spivak's derived manifolds.

The definition of derivations of C^∞-rings requires them to satisfy chain rule, while derivations of the differential structure $C^\infty(S)$ of a differential space S are defined algebraically in terms of Leibniz's rule. We show that if S is subcartesian, the derivations of $C^\infty(S)$ also satisfy the chain rule. This ensures that subcartesian spaces do not require the additional assumption that their differential structures are C^∞-rings. In particular, this justifies integration of derivations of differential structures of subcartesian spaces studied in [5].

2 Differential Spaces

A *differential structure* on a topological space S is a family $C^\infty(S)$ of real valued functions on S satisfying the following conditions.

(a) The family $\{f^{-1}(I) \mid f \in C^\infty(S)$, and I is an open interval in $\mathbb{R}\}$ is a sub-basis for the topology of S.
(b) If $f_1, \dots, f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \dots, f_n) \in C^\infty(S)$.

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(c) If $f: S \to \mathbb{R}$ is a function such that for every $x \in S$ there exists an open neighbourhood U of x and a function $f_x \in C^\infty(S)$ satisfying $f_x|_U = f|_U$, then $f \in C^\infty(S)$. Here the vertical bar | denotes restriction.

Functions $f \in C^\infty(S)$ are called *smooth* functions on S. It follows from condition (i) that smooth functions on S are continuous. Condition (ii) with $F(f_1, f_2) = af_1 + bf_2$, where $a, b \in \mathbb{R}$ implies that $C^\infty(S)$ is a vector space. Similarly, taking $F(f_1, f_2) = f_1 f_2$, we conclude that $C^\infty(S)$ is closed under multiplication of functions. A topological space S endowed with a differential structure is called a *differential space*.

In his original definition, Sikorski [4] defined $C^\infty(S)$ to be a family of functions satisfying condition (b). Then he used condition (a) to define topology on S. Finally, he imposed condition (c) as a consistency condition.

A map $\varphi: R \to S$ is *smooth* if $\varphi^* f = f \circ \varphi \in C^\infty(R)$ for every $f \in C^\infty(S)$. A smooth map φ between differential spaces is a *diffeomorphism* if it is invertible and its inverse is smooth.

Proposition 2.1 *A smooth map between differential spaces is continuous.*

Proof See the proof of [5, proposition 2.1.5] ■

A differential space S is *subcartesian* if its topology is Hausdorff and every point $x \in S$ has a neighbourhood U diffeomorphic to a subset V of $\mathbb{R}^n$. It should be noted that V in the definition above can be an arbitrary subset of $\mathbb{R}^n$, and n can depend on $x \in S$. As in the theory of manifolds, diffeomorphisms of open subsets of S onto subsets of $\mathbb{R}^n$ are called *charts on* S. The family of all charts is the complete atlas on S. Aronszajn [1] used the notion of a complete atlas on a Hausdorff topological space in his definition of subcartesian space.

3 Derivations at a Point

Let S be a subcartesian space with differential structure $C^\infty(S)$. A *derivation of* $C^\infty(S)$ *at a point* $x \in S$ is a linear map $v_x: C^\infty(S) \to \mathbb{R}$ such that

$$v_x(f_1 f_2) = v_x(f_1) f_2(x) + f_1(x) v_x(f_2) \tag{3.1}$$

for every $f_1, f_2 \in C^\infty(S)$.

If M is a manifold, then derivations of $C^\infty(M)$ satisfy the chain rule. In other words, for every $v \in TM$, $f_1, \ldots, f_k \in C^\infty(M)$ and $F \in C^\infty(\mathbb{R}^k)$,

$$vF(f_1, \ldots, f_k) = \left[\partial_1 F(f_1, \ldots, f_k)(\tau(v))\right] vf_1 + \cdots + \left[\partial_k F(f_1, \ldots, f_k)(\tau(v))\right] vf_k, \tag{3.2}$$

where $\tau: TM \to M$ is the tangent bundle projection map and $\partial_1, \ldots, \partial_k$ are partial derivatives in $\mathbb{R}^k$. Our aim in this section is to show that the chain rule is also valid for derivations of $C^\infty(S)$, where S is a subcartesian space.

Lemma 3.1 *Let v be a derivation of $C^\infty(S)$ at $x \in S$. For every open neighbourhood U of x and every $f \in C^\infty(S)$, vf depends only on the restriction of f to U.*

Proof Let $f_1, f_2 \in C^\infty(S)$ agree on a neighbourhood U of $x \in S$. By [5, lemma 2.2.1], there exists a function $h \in C^\infty(S)$ satisfying $h_{|V} = 1$ for some neighbourhood V of x contained in U, and $f_{|W} = 0$ for some open set W in S such that $U \cup W = S$. Then $h(f_1 - f_2) = 0$, so that $v[h(f_1 - f_2)] = 0$. Hence,

$$0 = v\big[h(f_1 - f_2)\big] = (vh)(f_1 - f_2)(x) + \big[v(f_1 - f_2)\big]h(x) = vf_1 - vf_2,$$

because $f_1(x) = f_2(x)$ and $h(x) = 1$. This implies $vf_1 = vf_2$, as required. ∎

Let $\varphi\colon S \to R$ be a smooth map between differential spaces with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively.

Lemma 3.2 *The map $\varphi\colon S \to R$ assigns to each derivation v of $C^\infty(S)$ at $x \in S$ a derivation $T\varphi(v)$ of $C^\infty(R)$ at $\varphi(x) \in R$ such that, for every $f \in C^\infty(R)$,*

$$T\varphi(v)f = v(\varphi^* f). \tag{3.3}$$

Proof For every $f \in C^\infty(R)$, $\varphi^* f = f \circ \varphi$ is in $C^\infty(S)$, and we can evaluate the derivation v on $\varphi^* f$. Note that, for $f_1, f_2 \in C^\infty(R)$ and $c_1, c_2 \in \mathbb{R}$,

$$\begin{aligned} T\varphi(v)(c_1 f_1 + c_2 f_2) &= v(\varphi^*(c_1 f_1 + c_2 f_2) = v(c_1\varphi^* f_1) + v(c_2\varphi^* f_2) \\ &= c_1 v(\varphi^* f_1) + c_2 v(\varphi^* f_2) = c_1\big(T\varphi(v)\big)f_1 + c_2\big(T\varphi(v)\big)f_2. \end{aligned}$$

Hence, $f \mapsto T\varphi(v)f$ is a linear mapping of $C^\infty(R)$ into itself. For $f_1, f_2 \in C^\infty(R)$, equation (3.3) yields

$$\begin{aligned} T\varphi(v)(f_1 f_2) &= v\big((\varphi^* f_1)(\varphi^* f_2)\big) = v(\varphi^* f_1)\varphi^* f_2(x) + \varphi^* f_1(x) v(\varphi^* f_2) \\ &= \big[T\varphi(v)f_1\big]\big[f_2(\varphi(x))\big] + \big[f_1(\varphi(x))\big]\big[T\varphi(v)f_2\big]. \end{aligned}$$

Hence, $T\varphi(v)$ is a derivation of $C^\infty(R)$ at $\varphi(x)$. ∎

Theorem 3.3 *For each x in a subcartesian space S, every derivation v of $C^\infty(S)$ at x satisies the chain rule. In other words, for every $k \in \mathbb{N}$, $f_1, \dots, f_k \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^k)$,*

$$v[F(f_1, \dots, f_k)] = [\partial_1 F(f_1, \dots, f_k)(x)]vf_1 + \cdots + [\partial_k F(f_1, \dots, f_k)(x)]vf_k,$$

where $\partial_1, \dots, \partial_k$ are partial derivatives in $\mathbb{R}^k$.

Proof Since S is subcartesian, there exists a diffeomorphism $\varphi\colon W \to \varphi(W) \subseteq \mathbb{R}^n$, where W is an open neighbourhood of x in S. Let $\iota_W\colon W \to S$ be the inclusion map. For every $f \in C^\infty(S)$, $\iota_W^* f = f|_W \in C^\infty(W)$. By Lemma 3.2, vf is completely determined by $f|_W$, and equation (3.3) yields

$$vf = T\iota_W(v)(\iota_W^* f) = T\iota_W(v)(f|_W).$$

Since $\varphi\colon W \to \varphi(W) \subseteq \mathbb{R}^n$ is a diffeomorphism, $h = (\varphi^{-1})^* f|_W \in C^\infty(\varphi(W))$ and

$$\big(T\iota_W(v)\big)f_{|W} = \big[T\varphi(Ti_W(v))\big]h. \tag{3.4}$$

Let $\iota_{\varphi(W)}\colon \varphi(W) \to \mathbb{R}^n$ be the inclusion map. Since $T\varphi(Ti_W(v))$ is a derivation of $C^\infty(\varphi(W))$ at $\varphi(x)$, $T\iota_{\varphi(W)}(T\varphi(Ti_W(v)))$ is a derivation of $C^\infty(\mathbb{R}^n)$ at $\varphi(x)$.

Without loss of generality, we can assume that the function h in equation (3.4) is the restriction to $\varphi(W)$ of a function $H \in C^\infty(\mathbb{R}^n)$. Therefore,

$$\left[T\varphi(Ti_W(v))\right]h = \left[T\iota_{\varphi(W)}(T\varphi(Ti_W(v)))\right]H.$$

Derivations of $C^\infty(\mathbb{R}^n)$ satisfy the chain rule. If $H = F(H_1,\dots,H_k)$, for some $k \in \mathbb{N}$, $H_1,\dots,H_k \in C^\infty(\mathbb{R}^n)$ and $F \in C^\infty(\mathbb{R}^k)$, then equation (3.2) yields

$$\begin{aligned}\left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]F(H_1,\dots,H_k) = \\ \left[\partial_1 F(H_1,\dots,H_k)(\varphi(x))\right]\left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]H_1 + \\ \dots + \left[\partial_k F(H_1,\dots,H_k)(\varphi(x))\right]\left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]H_k.\end{aligned}$$

Therefore,

$$\begin{aligned}vF(f_1,\dots,f_k) &= \left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]F(H_1,\dots,H_k) \\ &= \left[\partial_1 F(H_1,\dots,H_k)(\varphi(x))\right]\left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]H_1 + \\ &\quad \dots + \left[\partial_k F(H_1,\dots,H_k)(\varphi(x))\right]\left[T\iota_{\varphi(W)}(T\varphi(T\iota_W(v)))\right]H_k,\end{aligned}$$

where $H_i|_{\varphi(W)} = (\varphi^{-1})^* f_i|_W$ for $i = 1,\dots,k$. ■

4 The Tangent Bundle

Let T_xS be the set of all derivations of $C^\infty(S)$ at $x \in S$. The set T_xS is a real vector space, which is interpreted to be the *tangent space* to S at x. Let TS be the union of tangent spaces to S at each point x of S. In other words,

$$TS = \bigcup_{x \in S} T_xS$$

The *tangent bundle projection* is the map $\tau\colon TS \to S\colon v = (x, v_x) \to x$, which assigns to each derivation $v_x \in TS$ at x the point $x \in S$. The tangent bundle projection enables us to omit the subscript x in the definition of derivation at a point, and rewrite equation (3.1) in the form

$$v(f_1 f_2) = v(f_1)f_2 + f_1 v(f_2).$$

Each function $f \in C^\infty(S)$ gives rise to two functions on TS, namely,

$$\tau^* f\colon TS \longrightarrow \mathbb{R}\colon v \longmapsto f(\tau(v))$$

and

$$\mathrm{d}f\colon TS \longrightarrow \mathbb{R}\colon v \longmapsto \mathrm{d}f(v) = v(f).$$

The *tangent bundle* of a differential space S is TS with differential structure $C^\infty(TS)$ generated by the family of functions $\{\tau^* f, \mathrm{d}f \mid f \in C^\infty(S)\}$. This definition of $C^\infty(TS)$ ensures that the tangent bundle projection $\tau\colon TS \to S$ is smooth. The *derived map* of a smooth map $\varphi\colon S \to R$ is $T\varphi\colon TS \to TR\colon v \mapsto T\varphi(v)$, where for every $f \in C^\infty(R)$, $[T\varphi(v)]f = v(\varphi^* f)$; see Lemma 3.2. If $\tau_S\colon TS \to S$ and $\tau_R\colon TR \to R$ are tangent bundle projections, then $\tau_R \circ T\varphi = \varphi \circ \tau_R$.

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5 Global Derivations

A *derivation* of $C^\infty(S)$ is a linear map $X\colon C^\infty(S) \to C^\infty(S)\colon f \mapsto X(f)$ satisfying Leibniz's rule

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2)$$

for every $f_1, f_2 \in C^\infty(S)$. Let $\operatorname{Der} C^\infty(S)$ be the space of derivations of $C^\infty(S)$. It has the structure of a Lie algebra with the Lie bracket $[X_1, X_2]$ defined by

$$[X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f))$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty(S)$ and $f \in C^\infty(S)$. Moreover, $\operatorname{Der} C^\infty(S)$ is a module over the ring $C^\infty(S)$ and

$$[f_1 X_1, f_2 X_2] = f_1 f_2 [X_1, X_2] + f_1 X_1(f_2) X_2 - f_2 X_2(f_1) X_1$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty(S)$ and $f_1, f_2 \in C^\infty(S)$. If X is a derivation of $C^\infty(S)$, then for every $x \in S$, we have a derivation $X(x)$ of $C^\infty(S)$ at x given by

(5.1) $$X(x)\colon C^\infty(S) \longrightarrow \mathbb{R}\colon f \longmapsto X(x) f = (Xf)(x).$$

The derivation $X(x)$ (5.1) is called the *value* of X at x. Clearly, the derivation X is uniquely determined by the collection $\{X(x) \mid x \in S\}$ of its values at all points of S. In order to avoid confusion between a derivation of $C^\infty(S)$ and a derivation of $C^\infty(S)$ at a point in S, we often refer to the former as a *global derivation* of $C^\infty(S)$.

Theorem 5.1 *Let S be a differential subspace of $\mathbb{R}^n$ and let X be a derivation of $C^\infty(S)$. For each $x \in S \subseteq \mathbb{R}^n$, there exists a neighbourhood U of x in $\mathbb{R}^n$ and a vector field Y on $\mathbb{R}^n$ such that*

$$X(F|_S)|_{U\cap S} = (Y(F))|_{U\cap S}$$

for every $F \in C^\infty(\mathbb{R}^n)$.

Proof Let Z be a derivation of $C^\infty(S)$ at $x \in S \subseteq \mathbb{R}^n$. For each $F \in C^\infty(\mathbb{R}^n)$, the restriction $F|_S$ of F to S is in $C^\infty(S)$. It is easy to see that the map $C^\infty(\mathbb{R}^n) \to \mathbb{R}\colon F \mapsto Z(F|_S)$ is a derivation at x of $C^\infty(\mathbb{R}^n)$.

We denote the natural coordinate functions on $\mathbb{R}^n$ by $x^1, \ldots, x^n\colon \mathbb{R}^n \to \mathbb{R}$. Every derivation Y of $C^\infty(\mathbb{R}^n)$ is of the form $\sum_{i=1}^n F^i \frac{\partial}{\partial x^i}$, where $F^i = Y(x^i)$ for $i = 1, \ldots, n$. Let X be a derivation of $C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$. For each $x \in S$, the derivation $X(x)$ of $C^\infty(S)$ at x gives a derivation of $C^\infty(\mathbb{R}^n)$ at x. Hence,

$$\begin{aligned} X(F|_S)(x) = X(x)(F|_S) &= \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)\big(X(x)(x^i|_S)\big) \\ &= \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)\big(X(x^i|_S)\big)(x) = \Big(\sum_{i=1}^n X(x^i|_S)\frac{\partial F}{\partial x^i}\Big)(x) \end{aligned}$$

for every $x \in S$. For $i = 1, \ldots, n$, the coefficients $X(x^i|_S)$ are in $C^\infty(S)$. Since S is a differential subspace of $\mathbb{R}^n$, for each $x \in S$, there exists a neighbourhood U of x in $\mathbb{R}^n$ and functions $F^1, \ldots, F^n \in C^\infty(\mathbb{R}^n)$ such that $X(x^i|_S)|_{U\cap S} = F^i|_{U\cap S}$ for each $i = 1, \ldots, n$. Hence,

$$X(F|_S)|_{U\cap S} = \Big(\sum_{i=1}^n F^i \frac{\partial F}{\partial x^i}\Big)\Big|_{U\cap S}.$$

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Since $F^1, \dots, F^n$ are smooth functions on $\mathbb{R}^n$, it follows that $Y = \sum_{i=1}^n F^i \frac{\partial}{\partial x^i}$ is a vector field on $\mathbb{R}^n$. ∎

We can rephrase Theorem 5.1 by saying that every derivation on a differential subspace S of $\mathbb{R}^n$ can be locally extended to a vector field on $\mathbb{R}^n$. Suppose that S is closed. In this case, we can use a partition of unity on $\mathbb{R}^n$ to extend every derivation of $C^\infty(S)$ to a global vector field on $\mathbb{R}^n$. A *section* of the tangent bundle projection $\tau\colon TS \to S$ is a smooth map $\xi\colon S \to TS$ such that $\tau \circ \xi = \mathrm{id}_S$. Let $\mathcal{S}^\infty(TS)$ be the space of sections of the tangent bundle projection $\tau\colon TS \to S$. Since the differential structure $C^\infty(TS)$ is generated by the collection of functions $\{\tau^* f, \mathrm{d}f \mid f \in C^\infty(S)\}$, it follows that a section $\xi\colon S \to TS$ has to satisfy the conditions that $\xi^*(\tau^* f)$ and $\xi^*(\mathrm{d}f)$ are in $C^\infty(S)$ for every $f \in C^\infty(S)$. The first condition holds automatically, because

$$\xi^*(\tau^* f) = (\tau^* f) \circ \xi = f \circ \tau \circ \xi = f \circ \circ \mathrm{id}_S = f.$$

On the other hand, for $x \in S$,

$$\big(\xi^*(\mathrm{d}f)\big)(x) = \big((\mathrm{d}f) \circ \xi\big)(x) = \big\langle \mathrm{d}f \mid \xi(x)\big\rangle = \xi(x)f. \tag{5.2}$$

Proposition 5.2 *Every global derivation X of $C^\infty(S)$ defines a section*

$$X\colon S \longrightarrow TS\colon x \longmapsto X(x), \tag{5.3}$$

where $X(x)f = (Xf)(x)$ for every $f \in C^\infty(S)$ and every $x \in S$.

Proof The section $X\colon S \to TS$, defined by equation (5.3), satisfies equation (5.2), because $X^*(\mathrm{d}f) = X(f) \in \mathrm{Der}\, C^\infty(S)$ by definition of a global derivation. Conversely, if $\xi\colon S \to TS$ is a section, then equation (5.2) implies that $\xi(f) = \xi^*(\mathrm{d}f) \in \mathrm{Der}\, C^\infty(S)$ for every $f \in C^\infty(S)$. Hence, $\xi\colon f \mapsto \xi(f)$ is a global derivation of $C^\infty(S)$. ∎

Equation (5.3) gives a bijection between the space $\mathcal{S}^\infty(TS)$ of sections of the tangent bundle projection and the space $\mathrm{Der}\, C^\infty(S)$. Hence, Proposition 5.2 leads to identification of global derivations of $C^\infty(S)$ with the corresponding sections of the tangent bundle.

Let $c\colon I \to S$ be a smooth map of an interval I in $\mathbb{R}$ containing 0 to a differential space S. We say that c is an *integral curve* of a derivation X of $C^\infty(S)$ *starting at* $x_0 \in S$ if $x_0 = c(0)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) = X(f)(c(t)) \tag{5.4}$$

for every $f \in C^\infty(S)$ and every $t \in I$. In other words, $c\colon I \to S$ is an integral curve of X if $Tc(t) = X \circ c(t)$ for every $t \in I$.

Integral curves of a given derivation X of $C^\infty(S)$ can be ordered by inclusion of their domains. In other words, if $c_1\colon I_1 \to S$ and $c_2\colon I_2 \to S$ are two integral curves of X and $I_1 \subseteq I_2$, then $c_1 \leq c_2$. An integral curve $c_1\colon I \to S$ of X is *maximal* if $c_1 \leq c_2$ implies that $c_1 = c_2$.

Example Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$. Then $C^\infty(\mathbb{Q})$ consists of restrictions to $\mathbb{Q}$ of smooth functions on $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that every function $f \in C^\infty(\mathbb{Q})$ extends to a unique smooth function on $\mathbb{R}$ and every derivation of $C^\infty(\mathbb{R})$ induces a derivation of $C^\infty(\mathbb{Q})$. Let X be the derivation of $C^\infty(\mathbb{Q})$

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induced by the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ on $C^\infty(\mathbb{R})$. In other words, for every $f \in C^\infty(\mathbb{Q})$ and every $x_0 \in \mathbb{Q}$,

$$(Xf)(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

where the limit is taken over $x \in \mathbb{Q}$. On the other hand, no two distinct points in $\mathbb{Q}$ can be connected by a continuous curve.

Definition Let $\tau\colon TS \to S$ be the tangent bundle projection. Let X be a derivation of the differential structure $C^\infty(S)$ of a subcartesian space S. Let x_0 be a point in S and let I be an interval in $\mathbb{R}$ containing $0 \in \mathbb{R}$ or $I = \{0\}$. A *lifted integral curve* of X *starting at* x_0 is a map $\gamma\colon I \to TS$ such that $\gamma(0) = X(x_0)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\tau(\gamma(t))\big) = X(f)\big(\tau(\gamma(t))\big)$$

for every $f \in C^\infty(S)$ and $t \in I$, if $I \neq \{0\}$.

If $I \neq \{0\}$, then setting $c = \tau \circ \gamma$, we recover the definition for an integral curve of a derivation given in equation (5.4). If $I = \{0\}$, then γ is a lifted integral curve of X starting at x_0, because $\gamma(0) = X(c(0)) = X(x_0)$. Our extension of this definition to subcartesian spaces requires lifting the curve $c\colon I \to S$ to the tangent bundle leading to $\gamma\colon I \to TS$, in order to make sense of the condition $\gamma(0) = X(c(0))$.

Theorem 5.3 *Let S be a subcartesian space and let X be a derivation of $C^\infty(S)$. For every $x \in S$, there exists a unique maximal lifted integral curve γ_x of X starting at x.*

Proof

(i) *Local existence.* For $x \in S$, let φ be a diffeomorphism of a neighbourhood V of x in S onto a differential subspace R of $\mathbb{R}^n$. Let $Z = \varphi_* X|_V$ be a derivation of $C^\infty(R)$ obtained by pushing forward the restriction of X to V by φ. In other words,

$$Z(f) \circ \varphi = X|_V(f \circ \varphi)$$

for all $f \in C^\infty(R)$. Without loss of generality, we can assume that there is an extension of Z to a vector field Y on $\mathbb{R}^n$.

Let $z = \varphi(x)$ and let c_0 be a standard integral curve in $\mathbb{R}^n$ of the vector field Y such that $c_0(0) = z$. Let I_x be the connected component of $c_0^{-1}(R)$ containing 0 and let $c\colon I_x \to R$ be the curve in R obtained by the restriction of c_0 to I_x. Clearly, $c(0) = z$.

First, we consider the case when $I_x = \{0\}$, which means that there exists an open neighbourhood U_0 of z such that c_0 intersects $R \cap U$ only at z. In this case, we can consider another extension of Z to a vector field on $\mathbb{R}^n$. If $I_x = \{0\}$ for every extension Y of Z to a vector field on $\mathbb{R}^n$, then the map $\gamma\colon \{0\} \to TR\colon 0 \mapsto Z(z)$ is a maximal lifted integral curve of the vector field Z on R that starts at z. Since V is an open neighboyrhood of x in S and $\varphi\colon V \to R \subseteq \mathbb{R}^n$ is a diffeomorphism, $Z = \varphi_* X$ and $z = \varphi(x)$, it follows that the map $T(\iota_V \circ \varphi^{-1}) \circ \gamma\colon \{0\} \to TS\colon 0 \mapsto TS$ is a maximal lifted integral curve of X starting at x. Here, $\iota_V\colon V \to S$ is the inclusion map.

Suppose now that $I_x \neq \{0\}$. For each $t_0 \in I_x$ and each $f \in C^\infty(R)$, there exists a neighbourhood U of $c(t_0)$ in R and a function $F \in C^\infty(\mathbb{R}^n)$ such that $f|_U = F|_U$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} f\big(c(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} F\big(c(t)\big) = Y(F)\big(c(t_0)\big)\, Y(F)|_U\big(c(t_0)\big) = Z(f)\big(c(t_0)\big).$$

Since $I_x \neq \{0\}$ is a connected subset of $\mathbb{R}$ containing 0, it is an interval. So $c_x = \varphi^{-1} \circ c\colon I_x \to V \subseteq S$ satisfies $c_x(0) = \varphi^{-1}(c(0)) = \varphi^{-1}(z) = x$. Moreover, for every $t \in I_x$ and $h \in C^\infty(S)$, we get $f = h \circ \varphi^{-1} \in C^\infty(R)$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} h\big(c_x(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t} h\big(\varphi^{-1}\big(c(t)\big)\big) = \frac{\mathrm{d}}{\mathrm{d}t}(h \circ \varphi^{-1})\big(c(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) \\ &= Z(f)(c(t)) = Z(h \circ \varphi^{-1})(\varphi \circ c_x(t)) = X(h)\big(c_x(t)\big).\end{aligned}$$

This implies that the map $\gamma_x\colon I_x \to TS\colon t \mapsto X(c_x(t))$ is a lifted integral curve of X starting at x if $I_x \neq \{0\}$. It is also an integral curve of X starting at x when $I_x = \{0\}$, because $\gamma_x(0) = X(x)$.

(ii) *Smoothness.* From the theory of differential equations it follows that the integral curve c_0 in $\mathbb{R}^n$ of a smooth vector field Y is smooth. Hence, $c = c_0|_{I_x}$ is smooth. Since φ is a diffeomorphism of a neighbourhood of x in S to R, its inverse φ^{-1} is smooth and the composition $c_x = \varphi^{-1} \circ c$ is smooth. Since X is a derivation, it gives rise to a smooth section $X\colon S \to TS$ of the tangent bundle projection $\tau\colon TS \to S$. Moreover, the composition $\gamma_x = X \circ c_x$ is smooth.

(iii) *Local uniqueness.* This follows from the local uniqueness of solutions of first order differential equations in $\mathbb{R}^n$.

(iv) *Maximality.* Suppose that $p \leq 0 \leq q$ are the ends of the domain I_x of the integral curve c_x of X starting at x obtained in section (i). If $p = q = 0$ and $\gamma_x = X \circ c_x$ cannot be extended to a larger interval, then γ_x is maximal. If $q > 0$ and $q = \infty$ or $\lim_{t \nearrow q} c_x(t)$ does not exist, then the curve c_x does not extend beyond q. If $x_1 = \lim_{t \nearrow q} c_x(t)$ exists, then x_1 is unique, since the topology of S is Hausdorff. We can repeat the construction of section (i) by starting at the point x_1. In this way, we obtain an integral curve $c^1_{x_1}\colon I_1 \to S$ of X starting at x_1. Let $\widetilde{I_1} = I \cup \{t = q + s \in \mathbb{R} \mid s \in I_1 \cap [0, \infty)\}$ and let $\widetilde{c_1}\colon \widetilde{I_1} \to S$ be given by $\widetilde{c_1}(t) = c_x(t)$ if $t \in I$ and $\widetilde{c_1}(t) = c^1_{x_1}(t - q)$ if $t \in \{q + s \in \mathbb{R} \mid s \in I_1 \cap [0, \infty)\}$. Clearly, the curve $\widetilde{c_1}$ is continuous. Since $x_1 = \lim_{t \nearrow q} c_x(t)$, it follows that the left end p_1 of I_1 is strictly less than zero. Hence, the restriction of the curve c_x to the interval $(\max(p, p_1) + q, q)$ differs from the restriction of $c^1_{x_1}$ to the interval $(\max(p, p_1), 0)$ by the reparametrization $t \mapsto t - q$. Since the curves c_x and $c^1_{x_1}$ are smooth, it follows that the curve $\widetilde{c_1}$ is smooth. Let q_1 be the right end of the interval I_1. If $q_1 \in I$ and either $q_1 = \infty$ or $\lim_{t \nearrow q_1} c^1_x(t)$ does not exist, then the curve curve $\widetilde{c_1}$ does not extend beyond q_1. Otherwise, we can extend $\widetilde{c_1}$ by an integral curve c_2 of X through $x_2 = \lim_{t \nearrow q_1} \widetilde{c_1}(t)$. Continuing this process, we obtain a maximal extension for $t \geq 0$. In a similar way we can construct a maximal extension for $t \leq 0$.

(v) *Global uniqueness.* Let $c\colon I \to S$ and $c'\colon I' \to S$ be maximal integral curves of X starting at x. Let $T^+ = \{t \in I \cap I' \mid t > 0 \text{ and } c(t) \neq c'(t)\}$. Suppose that $T^+ \neq \varnothing$. Since T^+ is bounded from below by 0, there is a greatest lower bound ℓ of T^+. This implies that $c(t) = c'(t)$ for every $0 \leq t \leq \ell$ and for every $\varepsilon > 0$ there is a t_ε with $\ell < t_\varepsilon < \ell + \varepsilon$ such that $c(t_\varepsilon) \neq c'(t)$. Let $x_\ell = c(\ell) = c'(\ell)$ and let $c_\ell\colon I_\ell \to S$ be an integral curve of X starting at x_ℓ as constructed in i). Let q_ℓ be the right end of I_ℓ. If $q_\ell > 0$, the local uniqueness of integral curves implies that $c(t) = c'(t) = c_\ell(t - \ell)$ for all $\ell \le t \le \ell + q_\ell$. This contradicts the fact that ℓ is the greatest lower bound of T^+. If $q_\ell = 0$, then there is no extension of c_ℓ. Let q and q' be the right end of I and I', respectively. Since c and c' are maximal integral curves of X, it follows that $q = q' = \ell$. Hence, the set T^+ is empty, which is a contradiction. A similar argument shows that $T^- = \{t \in I \cap I' \mid t < 0 \text{ and } c(t) \neq c'(t)\} = \varnothing$. Therefore, $c(t) = c'(t)$ for all $t \in I \cap I'$. If $I \neq I'$, then this contradicts the fact that c and c' are maximal. Hence, $I = I'$ and $c = c'$. ∎

6 Vector Fields

Vector fields on a manifold M are not only derivations of $C^\infty(M)$, but they also generate local one-parameter groups of local diffeomorphisms of M. On a subcartesian space S, not all derivations of $C^\infty(S)$ generate local one-parameter groups of local diffeomorphisms of S; see [5, example 3.2.7, p. 37]. We reserve the term vector field for derivations of $C^\infty(S)$ that generate local one-parameter groups of local diffeomorphisms of S. More formally, we adopt the following definition. A *vector field* on a subcartesian space S is a derivation X of $C^\infty(S)$ such that for every $x_0 \in S$, there exists a neighbourhood U_{x_0} of $x_0 \in S$ and $\varepsilon_{x_0} > 0$ such that, for every $x \in U_{x_0}$, the interval $(-\varepsilon_{x_0}, \varepsilon_{x_0})$ is contained in the domain I_x of the lifted integral curve $\gamma_x : I_x \to TS$ of X and the map

$$\mathrm{e}^{tX} : U_{x_0} \longrightarrow S : x \longmapsto \mathrm{e}^{tX}(x) = \tau \circ \gamma_x(t)$$

is defined for every $t \in (-\varepsilon_{x_0}, \varepsilon_{x_0})$ and is a diffeomorphism of U_{x_0} onto an open subset $\mathrm{e}^{tX}(U_{x_0})$ of S.

Note that if X is a vector field on S, for every $x \in S$, the map $c_x : I_x \to S : t \to \mathrm{e}^{tX}(x)$ is an integral curve of S satisfying equation (5.4).[1] Therefore, if we were only interested in vector fields on S, we could use the definition of integral curves given by equation (5.4). We have introduced the notion of lifted integral curves to obtain Theorem 5.3, which ensures the existence and uniquenness of maximal lifted integral curves of derivations of $C^\infty(S)$. Theorem 5.3 replaces [5, theorem 3.2.1], which is incorrect. Our discussion shows that proofs of all results in [5] regarding vector fields on subcartesian spaces are not affected by errors in theorem 3.2.1. In particular, all results in [5, section 3.4 and chapter 4] are valid.

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[1] $\mathrm{e}^{tX}(x)$ is a compact version of the notation $(\exp tX)(x)$ used in [5].

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