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A Comment on Ergodic Theorem for Amenable Groups

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Abstract. We prove a version of the ergodic theorem for an action of an amenable group, where a Følner sequence need not be tempered. Instead, it is assumed that a function satisfies certain mixing conditions.

In [5], E. Lindenstrauss proved the ergodic theorem for actions of amenable groups, which is commonly used. The Følner sequence along which ergodic averages converge to an invariant function must satisfy the condition of being tempered. Since every Følner sequence has a subsequence that is tempered, the theorem is sufficient for many applications. In [3] it was shown that for a Bernoulli group shift the assumption that the Følner sequence is tempered may be relaxed if one considers frequency of visits in a cylinder set. The aim of the current paper is to push further in this direction and investigate in what circumstances temperedness is not necessary.

Let G be an amenable group and $(F_n)_{n\in\mathbb{N}}$ a Følner sequence in G. Assume that for every $\alpha \in [0,1)$ the series $\sum_{n=1}^{\infty} \alpha^{|F_n|}$ converges. Since it is already satisfied if $|F_n|$ strictly increases, the assumption is much weaker than temperedness. Let G act via measure preserving transformations on a probability space (X,μ). To simplify the notation we will identify G with the related group of automorphisms, and we write gx for the outcome of the action of an automorphism associated to g on x, and $f \circ g$ for the composition of a function f and the automorphism. According to [1], for any action of $\mathbb{Z}$, one can find a Følner sequence (with cardinalities of sets increasing slowly) such that the ergodic averages with respect to that sequence fail to converge for some function. Therefore, some constraints must be put on the function whose ergodic averages we study.

Definition 1 We will say that f is *ε-independent from a sub-σ-algebra* Σ_0 if for every $B \in \Sigma_0$ of positive measure it holds that

$$\left| \int_B f \, d\mu_B - \int f \, d\mu \right| < \varepsilon,$$

where μ_B is the conditional measure on B.

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A set A is ε-independent from Σ_0 if its characteristic function $\mathbb{1}_A$ is, *i.e.*, for all $B \in \Sigma_0$ such that $\mu(B) > 0$,

$$|\mu_B(A) - \mu(A)| < \varepsilon$$

or, in other words,

$$|\mu(A \cap B) - \mu(A)\mu(B)| < \varepsilon\mu(B).$$

For $\mathcal{F} \subset L^1(\mu)$ let $\sigma(\mathcal{F})$ denote the smallest sub-σ-algebra with respect to which every $f \in \mathcal{F}$ is measurable.

Theorem 2 *Let $G = \{g_1, g_2, \ldots\}$ be an amenable group acting on a probability space (X, μ). Let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence in G, such that for every $\alpha \in [0,1)$ the series $\sum_{n=1}^{\infty} \alpha^{|F_n|}$ converges. Let $f \in L^\infty(\mu)$ be such that for every $\varepsilon > 0$ there exists a finite set $K \subset G$ such that f is ε-independent from $\sigma(\{f \circ g : g \notin K\})$. Then*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int f\, d\mu \quad \mu\text{-almost everywhere.}$$

Note that the neutral element of G belongs to K for non-constant f.

For a finite nonempty set $F \subset G$ and any set $S \subset G$ we denote

$$\underline{d}_F(S) = \inf_{g\in G} \frac{|S \cap Fg|}{|F|} \quad \text{and} \quad \overline{d}_F(S) = \sup_{g\in G} \frac{|S \cap Fg|}{|F|}.$$

The *lower* and *upper Banach densities* of S are defined by formulas:

$$\underline{d}(S) = \sup\{\underline{d}_F(S) : F \subset G, F \text{ is finite}\},$$
$$\overline{d}(S) = \inf\{\overline{d}_F(S) : F \subset G, F \text{ is finite}\}.$$

If (F_n) is a Følner sequence then we also have

$$\underline{d}(S) = \lim_{n\to\infty} \underline{d}_{F_n}(S) \quad \text{and} \quad \overline{d}(S) = \lim_{n\to\infty} \overline{d}_{F_n}(S).$$

The following lemma was proved in [3].

Lemma 3 *For every finite set $K \subset G$ and $\delta > 0$ there exists a partition $D_0, D_1, \ldots, D_r$ of G such that*

(i) $\overline{d}(D_0) \le \delta$,
(ii) $\underline{d}(D_i) > 0$ *for every* $i = 1, \ldots, r$,
(iii) *for every* $i = 1, \ldots, r$, *if* $g, h \in D_i$ *then* $Kg \cap Kh = \varnothing$.

Before the proof, let us recall the following concentration inequality, which was proved in [4] for sums of independent variables and generalized in [2] for the case of martingales with bounded differences.

Theorem 4 (Azuma–Hoeffding inequality) *Suppose that $(M_n)_{n\in\mathbb{N}}$ is a martingale on (X, μ) such that $M_0 = 0$, $EM_n = 0$ for all $n \in \mathbb{N}$ and $|M_k - M_{k-1}| \le d_k$ almost surely for some constants d_k. Then, for every $\varepsilon > 0$*

$$\mu\big(\{x : |M_n(x)| > \varepsilon\}\big) \le 2\exp\Big(\frac{-\varepsilon^2}{2\sum_{k=1}^{n} d_k^2}\Big).$$

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Proof of Theorem 2 Fix a function $f \in L^\infty(\mu)$ and a number $\varepsilon > 0$ and let K be as in the assumption, chosen for $\varepsilon/2$. Using Lemma 3, choose a partition $D_0, D_1, \dots, D_r$ for $\delta = \varepsilon/(7\|f\|_\infty)$. For $H \subset G$, $|H| < \infty$, define

$$Y_H = \sum_{g \in H} f \circ g \qquad \overline{Y}_H = \frac{1}{|H|} Y_H.$$

Then $\int f\, d\mu = E\overline{Y}_H$.

Let $H_k^{(n,i)} = \{g_{i_1}, \dots, g_{i_k}\}$ be a set of k first elements of G belonging to $H^{(n,i)} = F_n \cap D_i$. Let $\mathcal{F}_k^{(n,i)} = \sigma(\{f \circ g : g \in H_k^{(n,i)}\})$. Clearly, $\{\mathcal{F}_k^{(n,i)}\}$ is a finite filtration and for each pair (n,i) the process $Y_k^{(n,i)} = Y_{H_k^{(n,i)}}$ is adapted to it. Let $\mathcal{F}_0^{(n,i)}$ be the trivial σ-algebra in X and let $Y_0^{(n,i)} = 0$ a.s. By Doob's decomposition,

$$Y_k^{(n,i)} = M_k^{(n,i)} + N_k^{(n,i)},$$

where

$$M_k^{(n,i)} = \sum_{j=1}^{k} \big(Y_j^{(n,i)} - E(Y_j^{(n,i)} | \mathcal{F}_{j-1}^{(n,i)})\big)$$

is a martingale and

$$N_k^{(n,i)} = \sum_{j=1}^{k} \big(E(Y_j^{(n,i)} | \mathcal{F}_{j-1}^{(n,i)}) - Y_{j-1}^{(n,i)}\big) = \sum_{j=1}^{k} E(f \circ g_{i_j} | \mathcal{F}_{j-1}^{(n,i)})$$

is a predictable process, *i.e.*, each $N_k^{(n,i)}$ is $\mathcal{F}_{k-1}^{(n,i)}$-measurable. Then $EM_k^{(n,i)} = 0$ and $EN_k^{(n,i)} = k \int f\, d\mu$. Note also, that

$$\begin{aligned} |M_k^{(n,i)} - M_{k-1}^{(n,i)}| &\le |Y_k^{(n,i)} - Y_{k-1}^{(n,i)}| + |N_k^{(n,i)} - N_{k-1}^{(n,i)}| \\ &= |f \circ g_{i_k}| + |E(f \circ g_{i_k} | \mathcal{F}_{k-1}^{(n,i)})| \le 2\|f\|_\infty. \end{aligned}$$

Similarly, we write $Y_{H^{(n,i)}} = M^{(n,i)} + N^{(n,i)}$ and we denote:

$$\overline{M}^{(n,i)} = \frac{1}{|H^{(n,i)}|} M^{(n,i)}, \quad \overline{N}^{(n,i)} = \frac{1}{|H^{(n,i)}|} N^{(n,i)}.$$

For a given n we will estimate the measure μ of the set

$$\left\{x : \left| \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) - \int f\, d\mu \right| > \varepsilon \right\} = \left\{x : |\overline{Y}_{F_n}(x) - E\overline{Y}_{F_n}| > \varepsilon \right\}.$$

It is a subset of the set

$$\mathcal{A}_n = \left\{x : \sum_{i=0}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{Y}_{H^{(n,i)}}(x) - E\overline{Y}_{H^{(n,i)}}| > \varepsilon \right\},$$

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which in turn is contained in the union of

$$\mathcal{M}_n = \Big\{ x : \frac{|H^{(n,0)}|}{|F_n|} \left|\overline{Y}_{H^{(n,0)}}(x) - E\overline{Y}_{H^{(n,0)}}\right| + \sum_{i=1}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| > \frac{\varepsilon}{2} \Big\}$$

and

$$\mathcal{N}_n = \Big\{ x : \sum_{i=1}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{N}^{(n,i)}(x) - E\overline{N}^{(n,i)}| > \frac{\varepsilon}{2} \Big\}.$$

For each j,

$$\left| E(f \circ g_{i_j} | \mathcal{F}_{j-1}^{(n,i)}) - \int f \, d\mu \right| < \frac{\varepsilon}{2}$$

almost surely, because $f \circ g_{i_j}$ is $\varepsilon/2$-independent from $\mathcal{F}_{j-1}^{(n,i)}$. Then

$$\left| \left(\overline{N}^{(n,i)}(x) - E\overline{N}^{(n,i)} \right) \right| \leq \frac{1}{H^{(n,i)}} \sum_{j \in H^{(n,i)}} \left| E(f \circ g_{i_j} | \mathcal{F}_{j-1}^{(n,i)}) - \int f \, d\mu \right| < \frac{\varepsilon}{2}$$

almost surely. The average also satisfies

$$\sum_{i=1}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{N}^{(n,i)}(x) - E\overline{N}^{(n,i)}| < \frac{\varepsilon}{2},$$

so $\mu(\mathcal{N}_n) = 0$ and the measure of $\mathcal{A}_n$ is equal to the measure of $\mathcal{M}_n$.

Now we will estimate $\mu(\mathcal{M}_n)$. We have $\overline{d}(D_0) \leq \varepsilon/7 \, \|f\|_\infty$, so for large n,

$$\frac{|H^{(n,0)}|}{|F_n|} < \varepsilon/6 \, \|f\|_\infty .$$

Clearly, $Y_{H^{(n,0)}}$ is a sum of functions with bounded range, hence

$$\frac{|H^{(n,0)}|}{|F_n|} \left| \overline{Y}_{H^{(n,0)}}(x) - E\overline{Y}_{H^{(n,0)}} \right| \leq \frac{|H^{(n,0)}|}{|F_n|} \cdot 2 \, \|f\|_\infty < \frac{\varepsilon}{3}$$

for large n.

By Azuma–Hoeffding inequality, for each $i = 1, \ldots, r$ it holds that

$$\mu\Big(\Big\{ x : |\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| > \frac{\varepsilon}{6} \Big\}\Big) = \mu\Big(\Big\{ x : |M^{(n,i)}(x) - EM^{(n,i)}| > \frac{\varepsilon}{6} |H^{(n,i)}| \Big\}\Big) \leq 2\gamma^{|H^{(n,i)}|}$$

for $\gamma = \exp(-\frac{\varepsilon^2}{8\|f\|_\infty^2}) < 1$ (recall that $|M_k^{(n,i)} - M_{k-1}^{(n,i)}|$ is bounded almost surely by $2 \, \|f\|_\infty$). Letting

$$X_\varepsilon = \Big\{ x : \exists \, i = 1, \ldots, r \, |\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| > \frac{\varepsilon}{6} \Big\},$$

we obtain

$$\mu(X_\varepsilon) \leq 2r \cdot \gamma^{\min_i |H^{(n,i)}|}.$$

Thus, outside X_ε not only $|\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| \leq \varepsilon/6$, but also the weighted average satisfies the same inequality

$$\sum_{i=1}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| \leq \frac{\varepsilon}{6},$$

so it is only the set X_ε on which it may happen that

$$\frac{|H^{(n,0)}|}{|F_n|} \left|\overline{Y}_{H^{(n,0)}}(x) - E\overline{Y}_{H^{(n,0)}}\right| + \sum_{i=1}^{r} \frac{|H^{(n,i)}|}{|F_n|} |\overline{M}^{(n,i)}(x) - E\overline{M}^{(n,i)}| > \frac{\varepsilon}{2}.$$

By positive density of each D_i, there is a positive number β such that $\min_i |H^{(n,i)}| > \beta |F_n|$ for large n. We obtain

$$\mu(\mathcal{A}_n) = \mu(\mathcal{M}_n) \leq 2r \cdot \gamma^{\min_i |H^{(n,i)}|} \leq 2r(\gamma^\beta)^{|F_n|}.$$

By the assumption on the Følner sequence, the series $\sum_n \alpha^{|F_n|}$ converges for $\alpha = \gamma^\beta$. The Borel–Cantelli lemma yields $\mu(\limsup_n \mathcal{A}_n) = 0$ and, consequently,

$$\mu\left(\left\{x : \forall N\ \exists\ n \geq N \left| \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) - \int f\, d\mu \right| > \varepsilon \right\}\right) = 0.$$

Finally, this implies that the convergence

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int f$$

may fail only on a set of measure 0. ∎

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