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ON THE S-FUNDAMENTAL GROUP SCHEME. II

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Abstract The S-fundamental group scheme is the group scheme corresponding to the Tannaka category of numerically flat vector bundles. We use determinant line bundles to prove that the S-fundamental group of a product of two complete varieties is a product of their S-fundamental groups as conjectured by Mehta and the author. We also compute the abelian part of the S-fundamental group scheme and the S-fundamental group scheme of an abelian variety or a variety with trivial étale fundamental group.

Keywords: fundamental group; complete variety; positive characteristic; numerically flat bundles

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Introduction

Let X be a complete, reduced and connected scheme defined over an algebraically closed field k. A vector bundle E on X is called *numerically flat* if both E and its dual E^* are nef vector bundles.

The category $\mathcal{C}^{\mathrm{nf}}(X)$ of numerically flat vector bundles is a k-linear abelian rigid tensor category. Fixing a closed point $x \in X$ endows $\mathcal{C}^{\mathrm{nf}}(X)$ with a fibre functor $E \to E(x)$ which makes $\mathcal{C}^{\mathrm{nf}}(X)$ into a neutral Tannaka category. Hence there is an equivalence between $\mathcal{C}^{\mathrm{nf}}(X)$ and the category of representations of some affine group scheme $\pi_1^S(X, x)$ that we call an *S-fundamental group scheme* of X with base point x. The S-fundamental group scheme appeared in the curve case in [**1**] and then in general in [**12**] and [**13**].

The strategy of constructing a similar fundamental group scheme using another Tannaka category of essentially finite vector bundles goes back to Nori's influential paper [**19**]. In this paper Nori defined a smaller group scheme, called *Nori's fundamental group scheme*, which is a pro-finite completion of the S-fundamental group scheme. Therefore, the S-fundamental group scheme plays with respect to Nori's fundamental group scheme a similar role as the fundamental group scheme for stratified sheaves with respect to the étale fundamental group scheme.

In [**12**] we exploited another interpretation of the S-fundamental group scheme in case of smooth projective varieties. Namely, numerically flat vector bundles are precisely strongly semistable torsion free sheaves with vanishing Chern classes. Using it one can apply vanishing theorems to establish, e.g. Lefschetz type theorems for the S-fundamental group scheme (see [**12**, Theorems 10.2, 10.4 and 11.3]).

This paper further exploits an observation that all the known properties of Nori's fundamental group scheme should still be valid for the more general S-fundamental group scheme. This allows in particular to obtain interesting properties of numerically flat bundles.

As a main result of this paper we prove that the S-fundamental group of a product of two complete varieties is a product of their S-fundamental groups. This result was conjectured both in [**12**] and [**13**, Remark 5.12]. It implies in particular the corresponding result for Nori's fundamental group that was conjectured by Nori in [**19**] and proven by Mehta and Subramanian in [**15**]. Our proof of Nori's conjecture is completely different from that in [**15**]. On the other hand, our proof also gives the corresponding result for the étale fundamental group scheme that was used in the proof of Mehta and Subramanian.

As a corollary to our theorem we prove that the reduced scheme underlying the torsion component of the identity of the Picard scheme of a product of projective varieties is a product of the corresponding torsion components of its factors (see Corollary 4.7). The author does not know any other proof of this fact.

The proof of the main theorem follows easily from the fact that the push forward of a numerically flat sheaf on the product $X \times Y$ to X is also numerically flat (and in particular locally free). To prove this result we proceed by induction. In the curve case we employ some determinants of line bundles.

In the remaining part of the paper we compute the abelian part of the S-fundamental group scheme (cf. [**3**, Lemma 20 and Theorem 21] for a similar result for the fundamental group scheme for stratified sheaves). This allows us to compute the S-fundamental group scheme for abelian varieties (this can also be done using an earlier result of Mehta and Nori in [**14**] for which we again have a completely different proof).

The last part of the paper is based on [**4**] and [**5**] and it contains computation of the S-fundamental group for varieties with trivial étale fundamental group.

1. Preliminaries

In this section we gather a few auxiliary results.

1.1. Numerical equivalence

Let X be a complete, reduced and connected d-dimensional scheme defined over an algebraically closed field k. We say that a rank r locally free sheaf E on X is *numerically trivial* if for every coherent sheaf F on X we have $\chi(X, F \otimes E) = r\chi(X, F)$.

Now assume that X is smooth. Then we define the *numerical Grothendieck group* $K(X)_{\mathrm{num}}$ as the Grothendieck group (ring) $K(X)$ of coherent sheaves modulo numerical equivalence, i.e. modulo the radical of the quadratic form given by the Euler characteristic

$$(a, b) \mapsto \chi(a \cdot b) = \int_X \mathrm{ch}(a)\,\mathrm{ch}(b)\,\mathrm{td}(X).$$

We say that a coherent sheaf has *numerically trivial Chern classes* if there exists an integer r such that the class $[E] - r[\mathcal{O}_X]$ is zero in $K(X)_{\mathrm{num}}$. By the Riemann–Roch theorem this is equivalent to the vanishing of numerical Chern classes.

1.2. Nefness

Let us recall that a locally free sheaf E on a complete k-scheme X is called *nef* if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on the projectivization $\mathbb{P}(E)$ of E. A locally free sheaf E is nef if and only if for any morphism $f\colon C \to X$ from a smooth projective curve C each quotient of f^*E has a non-negative degree.

We say that E is *numerically flat* if both E and E^* are nef. A locally free sheaf E is numerically flat if and only if for any morphism $f\colon C \to X$ from a smooth projective curve C the pullback f^*E is semistable of degree zero.

If X is projective then any numerically flat sheaf is numerically trivial and a line bundle is numerically flat if and only if it is numerically trivial.

1.3. Picard schemes

Let X be an (integral) variety defined over an algebraically closed field k. Let $\operatorname{Pic} X$ denote the Picard group scheme of X. By $\operatorname{Pic}^0 X$ we denote its connected component of the identity and by $\operatorname{Pic}^\tau X$ we denote the torsion component of the identity. Note that all these group schemes can be non-reduced.

If X is projective then $\operatorname{Pic}^\tau X$ is of finite type and the torsion group $\operatorname{Pic}^\tau X/\operatorname{Pic}^0 X$ is finite. The scheme $\operatorname{Pic}^\tau X$ represents the functor of numerically trivial line bundles. If X is smooth and projective then $\operatorname{Pic}^\tau X$ is the fine moduli space of torsion free rank 1 sheaves with numerically trivial Chern classes (e.g. because by Theorem 2.2 every torsion free rank 1 sheaf with numerically trivial Chern classes is in fact a line bundle).

1.4. Cohomology and base change

Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and let E be a Y-flat coherent $\mathcal{O}_X$-module. Then we say that *cohomology and base change commute for E in degree i* if for every base change diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{\;v\;} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\;u\;} & Y \end{array}$$

the natural map $u^*R^if_*E \to R^ig_*(v^*E)$ is an isomorphism.

If for every point $y \in Y$ the natural map $R^if_*E\otimes k(y) \to H^i(X_y, E_y)$ is an isomorphism then cohomology and base change commute for E in degree i. Indeed, our assumption implies that $u^*R^if_*E \to R^ig_*(v^*E)$ is an isomorphism at every point of Y'.

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1.5. Boundedness of numerically flat sheaves

The following theorem is a corollary of the general boundedness result [**11**, Theorem 4.4].

Theorem 1.1. *The set of numerically flat sheaves on a d-dimensional normal projective variety X is bounded.*

Proof. Let E be a rank r numerically flat sheaf on X. Let us fix a very ample line bundle $\mathcal{O}_X(1)$ on X. Then we can write the Hilbert polynomial of E as

$$P(E)(m) = \chi(X, E(m)) = \sum_{i=0}^{d} \chi(E|_{\bigcap_{j\leqslant i} H_j}) \binom{m+i-1}{i},$$

where $H_1, \dots, H_d \in |\mathcal{O}_X(1)|$ are general hyperplane sections. First let us note that

$$\chi(E|_{\bigcap_{j\leqslant i} H_j}) = r\chi(\mathcal{O}_{\bigcap_{j\leqslant i} H_j}) \quad \text{for } i = d, d-1, d-2.$$

In cases $i = d$ or $d-1$ the assertion is clear since $E|_{\bigcap_{j\leqslant i} H_j}$ is a numerically flat vector bundle on a set of points or on a smooth curve, so the assertion follows from the usual Riemann–Roch theorem.

In the case when $i = d-2$ it is sufficient to show that a rank r numerically flat sheaf F on a normal projective surface Y satisfies equality $\chi(Y, F) = r\chi(Y, \mathcal{O}_Y)$. To prove this let us take a resolution of singularities $f\colon \tilde{Y} \to Y$. Then $\chi(\tilde{Y}, f^*F) = r\chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$, because f^*F is a vector bundle with trivial Chern classes (as follows, for example, from Theorem 2.2). But using the Leray spectral sequence $H^j(\tilde{Y}, R^i f_* F) \Rightarrow H^{i+j}(Y, F)$ and the projection formula we see that

$$\chi(\tilde{Y}, f^*F) = \chi(Y, F) + \chi(Y, F \otimes R^1 f_* \mathcal{O}_{\tilde{Y}}).$$

Similarly, we have

$$\chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \chi(Y, \mathcal{O}_Y) + \chi(Y, R^1 f_* \mathcal{O}_{\tilde{Y}}).$$

Since $R^1 f_* \mathcal{O}_{\tilde{Y}}$ is supported on a finite number of points and F is locally free of rank r we see that $\chi(Y, F \otimes R^1 f_* \mathcal{O}_{\tilde{Y}}) = r\chi(Y, R^1 f_* \mathcal{O}_{\tilde{Y}})$, which proves the required equality.

Now let us write

$$P(E)(m) = \sum_{i=0}^{d} a_i \binom{m+d-i}{d-i}.$$

Then our assertion implies that $a_0(E)$, $a_1(E)$ and $a_2(E)$ depend only on X. But the set of reflexive semistable sheaves with fixed $a_0(E)$, $a_1(E)$ and $a_2(E)$ on a normal projective variety is bounded by [**11**, Theorem 4.4]. □

In case of smooth varieties the above theorem is an immediate corollary of Theorem 2.2 and [**11**, Theorem 4.4].

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2. Fundamental groups in positive characteristic

Let X be a complete connected reduced scheme defined over an algebraically closed field k.

2.1. S-fundamental group scheme

Let $\mathcal{C}^{\mathrm{nf}}(X)$ denote the full subcategory of the category of coherent sheaves on X, which as objects contains all numerically flat (in particular locally free) sheaves. Let us fix a k-point $x \in X$. Then we can define the fibre functor $T_x\colon \mathcal{C}^{\mathrm{nf}}(X) \to k$-mod by sending E to its fibre $E(x)$. One can show that $(\mathcal{C}^{\mathrm{nf}}(X), \otimes, T_x, \mathcal{O}_X)$ is a neutral Tannaka category (see [**12**, § 6]). Therefore, by [**2**, Theorem 2.11] the following definition makes sense.

Definition 2.1. The affine k-group scheme Tannaka dual to this neutral Tannaka category is denoted by $\pi_1^S(X, x)$ and it is called the *S-fundamental group scheme* of X with base point x.

This group scheme was first defined in the curve case by Biswas, Parameswaran and Subramanian in [**1**, § 5], and then independently in [**12**] and [**13**].

The following characterization of numerically flat bundles as semistable sheaves with vanishing Chern classes appears in [**12**, Theorem 4.1 and Proposition 5.1].

Theorem 2.2. *Let X be a smooth projective k-variety of dimension d. Let H be an ample divisor on X and let E be a coherent sheaf on X. Then the following conditions are equivalent.*

(1) *E is a strongly H-semistable torsion free sheaf and its Hilbert polynomial is the same as that of the trivial sheaf of the same rank.*

(2) *E is a strongly H-semistable torsion free sheaf and it has numerically trivial Chern classes.*

(3) *E is a strongly H-semistable reflexive sheaf with $\mathrm{ch}_1(E) \cdot H^{d-1} = 0$ and $\mathrm{ch}_2(E) \cdot H^{d-2} = 0$.*

(4) *E is locally free, nef and $c_1(E)H^{d-1} = 0$.*

(5) *E is numerically flat.*

2.2. Nori's and étale fundamental group schemes

Let us consider the category $\mathcal{C}^N(X)$ of bundles which are trivializable over a principal bundle under a finite group scheme. For a k-point $x \in X$ we can define the fibre functor $T_x\colon \mathcal{C}^N(X) \to k$-mod by sending E to its fibre $E(x)$. This makes $\mathcal{C}^N(X)$ a neutral Tannaka category which is equivalent to the category of representations of an affine group scheme $\pi_1^N(X, x)$ called *Nori's fundamental group scheme*.

If instead of $\mathcal{C}^N(X)$ we consider the category $\mathcal{C}^{\text{ét}}(X)$ of bundles which are trivializable over a principal bundle under a finite étale group scheme then we get an *étale fundamental group* $\pi_1^{\text{ét}}(X, x)$.

Note that both these group schemes can be recovered from $\pi_1^S(X,x)$ as inverse limits of some directed systems (see, for example, [**12**, § 6]). In particular, any theorem proved for the S-fundamental group scheme implies the corresponding theorems for étale and Nori's fundamental group schemes.

2.3. The unipotent part of the S-fundamental group scheme

The largest unipotent quotient of the S-fundamental group scheme of X is called the *unipotent part* of $\pi^S(X,x)$ and denoted by $\pi^U(X,x)$. By the standard Tannakian considerations we know that the category of finite-dimensional k-representations of $\pi^U(X,x)$ is equivalent (as a neutral Tannakian category) to the category $\mathcal{NC}^{\mathrm{nf}}(X)$ defined as the full subcategory of $\mathcal{C}^{\mathrm{nf}}(X)$ whose objects have a filtration with all quotients isomorphic to $\mathcal{O}_X$.

Nori proves that in positive characteristic $\pi^U(X,x)$ is a pro-finite group scheme (see [**19**, Chapter IV, Proposition 3]) and hence $\pi^U(X,x)$ is the unipotent part of $\pi^N(X,x)$. This is no longer true in the characteristic zero case. However, in arbitrary characteristic we know that $\mathrm{Hom}(\pi^U(X,x),\mathbb{G}_a)=H^1(X,\mathcal{O}_X)$. In particular, in characteristic zero the abelian part of $\pi^U(X,x)$ is equal to the group of the dual vector space $H^1(X,\mathcal{O}_X)^*$ (see [**19**, Chapter IV, Proposition 2]).

In positive characteristic, the abelian part of $\pi^U(X,x)$ is the inverse limit of Cartier duals of finite local group subschemes of $\mathrm{Pic}\,X$ (cf. [**19**, Chapter IV, Proposition 6]).

3. Numerically flat sheaves on products of curves

In this section we keep the following notation. Let X and Y be complete k-varieties. Then p, q are the projections of $X\times Y$ onto X and Y, respectively. Let E be a coherent sheaf on $X\times Y$. For a point $y\in Y$ we set $E_y=p_*(E\otimes\mathcal{O}_{X\times\{y\}})$. Similarly, $E_x=q_*(E\otimes\mathcal{O}_{\{x\}\times Y})$ for a point $x\in X$.

Let us consider the following proposition in the special case when X and Y are curves.

Proposition 3.1. *Let X and Y be smooth projective curves. Let F be a locally free sheaf on $X\times Y$, such that F_{x_1} is semistable for some $x_1\in X$. Assume that F is numerically trivial. Then for any closed points $y_1,y_2\in Y$ the corresponding locally free sheaves F_{y_1} and F_{y_2} are isomorphic. Moreover, the bundles F_x are semistable and S-equivalent for all closed points $x\in X$.*

Proof. Let us fix a point $x_1\in X$. By Faltings's theorem [**6**, Theorem I.2] (see also [**21**, Remark 3.2 (b)]) there exists a rank r' vector bundle E on Y such that $H^*(Y,F_{x_1}\otimes E)=0$. Note that this condition implies that E is semistable (see [**21**, Theorem 6.2]). This bundle defines a global section Θ_E of a line bundle $L=\det p_!(F\otimes q^*E)^{-1}$. Set-theoretically, the zero set of this section is equal to $\{x\in X\colon H^*(Y,F_x\otimes E)\neq 0\}$.

But by the Grothendieck–Riemann–Roch theorem (see, for example, [**8**, Appendix A, Theorem 5.3]) for every $u\in K(Y)$ we have

$$\mathrm{ch}(p_!(F\cdot q^*u))=p_*(\mathrm{ch}(F)\cdot q^*(\mathrm{ch}(u)\cdot\mathrm{td}(X)))=rp_*(q^*(\mathrm{ch}(u)\cdot\mathrm{td}(X)))$$

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and the only non-zero part in the last term is in degree zero. Therefore, L has degree zero. Since L has the section Θ_E non-vanishing at x_1 it follows that L is trivial and $H^*(Y, F_x \otimes E) = 0$ for all $x \in X$. By [**21**, Theorem 6.2] this implies that F_x is semistable for every $x \in X$. To prove that F_x are S-equivalent one can use determinant line bundles on the moduli space of semistable vector bundles on X. Since we do not use this fact in the following we omit the proof. An alternative proof can be found in [**21**, Lemma 4.2].

The rest of the proof is similar to part of proof of [**6**, Theorem I.4] (see also Step 5 in the proof of [**9**, Theorem 4.2]). Let us fix a point $y_1 \in Y$ and take any non-trivial extension

$$0 \to E \to E' \to \mathcal{O}_{y_1} \to 0.$$

The Quot-scheme $\mathcal{Q}$ of rank 0 and degree 1 quotients of E' is isomorphic to $\mathbb{P}(E')$. Let us set $E_\pi = \ker \pi$ for a point $[\pi\colon E' \to \mathcal{O}_y] \in \mathcal{Q}$. The set U of points $[\pi] \in \mathcal{Q}$ such that $H^*(Y, F_{x_1} \otimes E_\pi) = 0$ is non-empty and open. The same arguments as before show that $H^*(Y, F_x \otimes E_\pi) = 0$ for all $x \in X$ and $[\pi] \in U$. Applying p_* to the sequence

$$0 \to F \otimes q^* E_\pi \to F \otimes q^* E' \xrightarrow{\mathrm{id}_F \otimes \pi} F \otimes q^* \mathcal{O}_y \to 0$$

for $[\pi\colon E' \to \mathcal{O}_y] \in U$ we see that $p_*(F \otimes q^* E') \simeq F_y$. Therefore, $F_{y_1} \simeq F_y$ for points y in some non-empty open subset of Y. Since similar arguments apply to any other point $y_2 \in Y$ we see that F_{y_1} and F_{y_2} are isomorphic for all points $y_1, y_2 \in Y$. □

The following corollary is analogous to [**7**, Proposition 2.4] in the case of stratified sheaves.

Corollary 3.2. *Let X be a normal complete variety and let Y be a complete variety. Let E be a numerically flat sheaf on $X \times Y$. Then for any closed points $y_1, y_2 \in Y$ the corresponding locally free sheaves E_{y_1} and E_{y_2} are isomorphic. In particular, the sheaf $q_* E$ is locally free.*

Proof. Using Chow's lemma it is easy to see that there exists an irreducible curve on Y containing both y_1 and y_2. Taking its normalization we can replace Y by a smooth projective curve and prove the assertion in this special case. So in the following we assume that Y is a smooth projective curve.

Now we prove the assertion assuming that X is projective. The proof is by induction on the dimension d of X. For $d = 1$ the assertion follows from Proposition 3.1. For $d \geqslant 2$ let us fix a divisor D on X and consider the following exact sequence

$$\mathrm{Hom}(E_{y_1}, E_{y_2}) \xrightarrow{\alpha} \mathrm{Hom}(E_{y_1}|_D, E_{y_2}|_D) \to H^1(\mathcal{H}om(E_{y_1}, E_{y_2}) \otimes \mathcal{O}_X(-D)).$$

Taking a sufficiently ample divisor D, we can assume that D is smooth and

$$H^1(\mathcal{H}om(E_{y_1}, E_{y_2}) \otimes \mathcal{O}_X(-D)) = 0$$

(here we use $d \geqslant 2$; see [**8**, Chapter III, Corollary 7.8]). But then the map α is surjective and the isomorphism $E_{y_1}|_D \simeq E_{y_2}|_D$ (coming from the inductive assumption) can be

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lifted to a homomorphism $\varphi\colon E_{y_1}\to E_{y_2}$. Since φ is injective at the points of D and E_{y_1} is torsion-free, it follows that φ is an injection. But then it must be an isomorphism.

To prove the assertion in case when X is non-projective we can use Chow's lemma. Namely, there exists a normal projective variety $\tilde{X}$ and a birational morphism $f\colon \tilde{X}\to X$. Let us set $g=f\times \mathrm{id}_Y\colon \tilde{X}\times Y\to X\times Y$. By the previous part of the proof we know that $f^*(E_{y_1})=(g^*E)_{y_1}\simeq (g^*E)_{y_2}=f^*(E_{y_2})$. But by Zariski's main theorem we know that $f_*\mathcal{O}_{\tilde{X}}=\mathcal{O}_X$. Hence by the projection formula we have $E_{y_1}\simeq f_*f^*(E_{y_1})\simeq f_*f^*(E_{y_2})\simeq E_{y_2}$.

The last part of the corollary follows from Grauert's theorem (see [**8**, Chapter III, Corollary 12.9]). □

4. S-fundamental group scheme of a product

The following result was conjectured both by the author in [**12**, § 8] and by Mehta in [**13**, Remark 5.12].

Theorem 4.1. *Let X and Y be complete k-varieties. Let us fix k-points $x_0\in X$ and $y_0\in Y$. Then the natural homomorphism*

$$\pi_1^S(X\times_k Y,(x_0,y_0))\to \pi_1^S(X,x_0)\times_k \pi_1^S(Y,y_0)$$

is an isomorphism.

Proof. In characteristic zero the assertion follows from the corresponding fact for topological fundamental groups of complex varieties and the Lefschetz principle. More precisely, in the case of complex varieties the assertion follows from the corresponding isomorphism of topological fundamental groups by passing to a pro-unitary completion. In general, the fact follows from the Lefschetz principle if one notes that Theorem 4.1 follows from Lemma 4.2 and if we use the Lefschetz principle for this special assertion (note that to apply Lefschetz principle we need to reformulate Theorem 4.1 as it involves group schemes which are not of finite type over the field).

So in the following we can assume that the characteristic of k is positive (but we need it only to prove the next lemma which is the main ingredient in proof of Theorem 4.1).

Lemma 4.2. *Let us assume that X and Y are normal and projective. Let E be a numerically flat sheaf on $X\times Y$. Then p_*E is numerically flat.*

Proof. By Corollary 3.2 p_*E is locally free. Let us fix a point $x_0\in X$. Then the sheaf $G_n=p_*((F_X^n\times \mathrm{id}_Y)^*E)=(F_X^n)^*(p_*E)$ is locally free of rank $a=h^0(Y,E_{x_0})$.

Now let us consider the set A of all numerically flat sheaves on $X\times Y$. This set is bounded by Theorem 1.1, so there exist a scheme S of finite type over k and an S-flat sheaf $\mathcal{E}$ on $X\times Y\times S$ such that the set of restrictions $\{\mathcal{E}_s\}_{s\in S}$ contains all the sheaves in the set A. Let us consider a subscheme $S'\subset S$ defined by $S'=\{s\in S\colon h^0(Y,(\mathcal{E}_s)_{x_0})\geqslant a\}$. By semicontinuity of cohomology, S' is a closed subscheme of S. Let us consider an open subset $U\subset S'$ that corresponds to points $s\in S'$ where $h^0(Y,(\mathcal{E}_s)_{x_0})=a$. We consider U

with the reduced scheme structure. By abuse of notation, the restriction of $\mathcal{E}$ to U will be again denoted by $\mathcal{E}$.

We claim that the set $\{p_*(\mathcal{E}_s)\}_{s\in U}$ is a bounded set of sheaves. To prove this let us consider the following diagram

$$\begin{array}{ccccc} & & \overset{j_{(x,s)}}{\frown\!\!\!\longrightarrow} & & \\ Y & \xrightarrow{j_x} & X\times Y & \xrightarrow{\tilde{j}_s} & X\times Y\times U \\ \downarrow{\scriptstyle p_x} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\ \operatorname{Spec} k & \xrightarrow{i_x} & X & \xrightarrow{\tilde{i}_s} & X\times U \\ & & \underset{i_{(x,s)}}{\smile\!\!\!\longrightarrow} & & \end{array}$$

in which the vertical maps are canonical projections and the horizontal maps are embeddings corresponding to fixed points $x\in X$ and $s\in U$. Let us recall that $p_*(\tilde{j}_s^*\mathcal{E})$ is locally free by Corollary 3.2 and the definition of U. Moreover, $\tilde{p}_*\mathcal{E}$ is locally free as $h^0(Y,(\mathcal{E}_s)_x)=h^0(Y,(\mathcal{E}_s)_{x_0})=a$ for every $x\in X$ by Corollary 3.2. Therefore, the above diagram induces the commutative diagram

$$\begin{array}{ccc} i_x^*p_*(\tilde{j}_s^*\mathcal{E}) & \longrightarrow & (p_x)_*j_x^*(\tilde{j}_s^*\mathcal{E}) \\ \uparrow & & \uparrow{\scriptstyle \simeq} \\ i_{(x,s)}^*(\tilde{p}_*\mathcal{E}) & \longrightarrow & (p_x)_*j_{(x,s)}^*\mathcal{E} \end{array}$$

in which the horizontal maps are isomorphisms by Grauert's theorem (see [**8**, Chapter III, Corollary 12.9]). Hence the remaining vertical map is also an isomorphism. This shows that $(\tilde{p}_*\mathcal{E})_s\simeq p_*(\mathcal{E}_s)$, which gives the required claim.

By our assumptions there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of points of U such that $p_*(\mathcal{E}_{s_n})\simeq G_n$. Therefore, the set $\{G_n\}_{n\in\mathbb{N}}$ is bounded.

Now let us take any morphism $f\colon C\to X$ from a smooth projective curve C. Since $(F_C^n)^*(f^*p_*E)\simeq f^*(G_n)$ and the set $\{f^*(G_n)\}_{n\in\mathbb{N}}$ is bounded, we see that f^*p_*E is semistable of degree 0. More precisely, semistability follows from the fact that sequences of slopes of maximal destabilizing subsheaves and minimal destabilizing quotients of Frobenius pullbacks of f^*p_*E are bounded. Similarly, the sheaf f^*p_*E has degree 0 since its Frobenius pullbacks have bounded degree. Therefore, p_*E is numerically flat. □

Now we can go back to the proof of Theorem 4.1.

The first part of proof is the same as the analogous part of proof of [**19**, Chapter IV, Lemma 8]. Namely, the homomorphism

$$\pi_1^S(X\times_k Y,(x_0,y_0))\to\pi_1^S(X,x_0)\times_k\pi_1^S(Y,y_0)$$

is induced by projections $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$. Let $i\colon X\to X\times Y$ be the embedding onto $X\times\{y_0\}$ and let $j\colon Y\to X\times Y$ be the embedding onto $\{x_0\}\times Y$. Since $pi=\mathrm{id}_X$ and qi is constant, the composition

$$\pi_1^S(X,x_0)\to\pi_1^S(X\times_k Y,(x_0,y_0))\to\pi_1^S(X,x_0)\times_k\pi_1^S(Y,y_0)$$

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i s an embedding onto the first component. Similarly $qj = \mathrm{id}_Y$ is the embedding onto the second component, so the homomorphism

$$\pi_1^S(X \times_k Y, (x_0, y_0)) \to \pi_1^S(X, x_0) \times_k \pi_1^S(Y, y_0)$$

can be split and in particular it is faithfully flat.

Hence we only need to prove that it is a closed immersion.

Let us first assume that X and Y are normal and projective. Note that the sheaf $F = \mathcal{H}om(q^*E_{x_0}, E)$ is numerically flat. By our assumptions and Lemma 4.2 the sheaf p_*F is numerically flat. The induced map

$$p^*p_*F \otimes q^*E_{x_0} \to E$$

is surjective as its restriction to $\{x_0\} \times Y$ corresponds to the surjective map

$$\mathrm{Hom}(E_{x_0}, E_{x_0}) \otimes E_{x_0} \to E_{x_0}.$$

Now [**2**, Proposition 2.21 (b)] implies that the natural homomorphism

$$\pi_1^S(X \times_k Y, (x_0, y_0)) \to \pi_1^S(X, x_0) \times_k \pi_1^S(Y, y_0)$$

is a closed immersion and therefore it is an isomorphism.

Now we need the following lemma.

Lemma 4.3. *Let X and Y be complete k-varieties such that the natural map*

$$\pi_1^S(X \times_k Y, (x_0, y_0)) \to \pi_1^S(X, x_0) \times_k \pi_1^S(Y, y_0)$$

is an isomorphism. Let E be a numerically flat sheaf on $X \times Y$. Then for every integer i the sheaf R^ip_*E is numerically flat and cohomology and base change commute for E in all degrees.

Proof. Since $\pi_1^S(X \times_k Y, (x_0, y_0)) \simeq \pi_1^S(X, x_0) \times_k \pi_1^S(Y, y_0)$, E is a subsheaf of a sheaf $p^*E_0 \otimes q^*F_0$ for some numerically flat sheaves E_0 and F_0. This is an easy fact from representation theory as every $G_1 \times G_2$-module is a submodule of the tensor product of G_1 and G_2-modules. Alternatively, [**2**, Proposition 2.21 (b)] implies that every numerically flat sheaf on $X \times Y$ is a quotient of a sheaf of the form $p^*E_0 \otimes q^*F_0$, which implies the required statement by taking the duals.

The quotient $(p^*E_0 \otimes q^*F_0)/E$ is also numerically flat, so it is a subsheaf of a sheaf of the form $p^*E_1 \otimes q^*F_1$ for some numerically flat sheaves E_1 and F_1. Inductively we can therefore construct the following acyclic complex of sheaves on $X \times Y$:

$$0 \to E \to p^*E_0 \otimes q^*F_0 \to p^*E_1 \otimes q^*F_1 \to \cdots \to p^*E_i \otimes q^*F_i \to \cdots \qquad (*)$$

Let us set $\mathcal{C}^i = p^*E_i \otimes q^*F_i$. Note that $R^ip_*\mathcal{C}^j \simeq E_j^{\oplus h^i(Y, F_j)}$ is numerically flat and consider the following spectral sequence

$$E_1^{ij} = R^ip_*\mathcal{C}^j \Longrightarrow R^{i+j}p_*\mathcal{C}^\bullet \simeq R^{i+j}p_*E.$$

Since the category of numerically flat sheaves on X is abelian, kernels and cokernels of objects from this category are also numerically flat. This implies that the limit $R^{i+j}p_*E$ of the above spectral sequence is also numerically flat.

Now note that the complex $(*)$ restricted to $\{x\}\times Y$ remains acyclic, as all the sheaves in this complex are locally free. Therefore, we have a commutative diagram of spectral sequences

$$\begin{array}{ccc} R^ip_*\mathcal{C}^j\otimes k(x) & \Longrightarrow & R^{i+j}p_*E\otimes k(x) \\ \downarrow & & \downarrow \\ H^i(Y,\mathcal{C}^j_x) & \Longrightarrow & H^{i+j}(Y,E_x) \end{array}$$

Since $\mathcal{C}^i = p^*E_i\otimes q^*F_i$, the left vertical map in this diagram is an isomorphism. Therefore, the right vertical map is also an isomorphism. But this implies that cohomology and base change commute for E in all degrees (see § 1.4). □

Now let us return to the proof of Theorem 4.1. Let us first assume that X is normal and projective. By Chow's lemma there exists a projective variety $\tilde{Y}$ and a birational morphism $f\colon \tilde{Y}\to Y$. Passing to the normalization, we can assume that $\tilde{Y}$ is normal. Consider the base change diagram

$$\begin{array}{ccc} X\times\tilde{Y} & \xrightarrow{g} & X\times Y \\ \downarrow{\scriptstyle\tilde{q}} & & \downarrow{\scriptstyle q} \\ \tilde{Y} & \xrightarrow{f} & Y \end{array}$$

Let us take two closed points $y_1, y_2\in Y$. Let us choose closed points $\tilde{y}_1,\tilde{y}_2\in\tilde{Y}$ mapping onto y_1, y_2, respectively. Then $h^0((g^*E)_{\tilde{y}_1}) = h^0((g^*E)_{\tilde{y}_2})$ and hence $h^0(E_{y_1}) = h^0(E_{y_2})$. By Grauert's theorem (and § 1.4) this implies that q_*E is locally free and cohomology and base change commute for E in all degrees. In particular, we have and $f^*(q_*E)\simeq\tilde{q}_*(g^*E)$. But $\tilde{q}_*(g^*E)$ is numerically flat, so q_*E is also numerically flat.

In this case the same proof as in the previous case shows that $\pi_1^S(X\times_k Y,(x_0,y_0))\to \pi_1^S(X,x_0)\times_k\pi_1^S(Y,y_0)$ is an isomorphism.

Now we can again apply Chow's lemma to prove that if X and Y are complete and E is numerically flat on $X\times Y$ then p_*E is numerically flat. As in the previous case this implies that $\pi_1^S(X\times_k Y,(x_0,y_0))\to\pi_1^S(X,x_0)\times_k\pi_1^S(Y,y_0)$ is a closed immersion. □

Remark 4.4. Most of the proof of Theorem 4.1 works in an arbitrary characteristic. But the proof of Lemma 4.2 uses positive characteristic. The characteristic zero version of Theorem 4.1 would follow from the positive characteristic case if one knew that the reduction of a semistable complex bundle for some characteristic is strongly semistable (this is a weak version of Miyaoka's problem; see [**17**, Problem 5.4]). This problem seems to be open even for numerically flat bundles on a product of two curves of genera greater than or equal to 2.

Applying [**12**, Lemma 6.3] as a corollary to Theorem 4.1 we obtain the following result of Mehta and Subramanian (conjectured earlier by Nori in [**19**]).

Corollary 4.5 (see [15, Theorem 2.3]). *Let X and Y be complete k-varieties. Let us fix k-points $x_0 \in X$ and $y_0 \in Y$. Then the natural homomorphism*

$$\pi_1^N(X \times_k Y, (x_0, y_0)) \to \pi_1^N(X, x_0) \times_k \pi_1^N(Y, y_0)$$

is an isomorphism.

Note that Theorem 4.1 and Lemma 4.3 imply the following corollary.

Corollary 4.6. *Let X and Y be complete k-varieties and let E be a numerically flat sheaf on $X \times Y$. Then for every integer i the sheaf $R^i p_* E$ is numerically flat and cohomology and base change commute for E in all degrees.*

Corollary 4.7. *Let X and Y be projective k-varieties. Then*

$$(\mathrm{Pic}^\tau(X \times Y))_{\mathrm{red}} \simeq (\mathrm{Pic}^\tau X)_{\mathrm{red}} \times (\mathrm{Pic}^\tau Y)_{\mathrm{red}}$$

and

$$(\mathrm{Pic}^0(X \times Y))_{\mathrm{red}} \simeq (\mathrm{Pic}^0 X)_{\mathrm{red}} \times (\mathrm{Pic}^0 Y)_{\mathrm{red}}.$$

Proof. The category of representations of the S-fundamental group scheme $\pi_1^S(X, x)$ is equivalent to the category $\mathcal{C}^{\mathrm{nf}}(X)$ of numerically flat sheaves. Since a line bundle is numerically flat if and only if it is numerically trivial, the group $(\mathrm{Pic}^\tau(X))_{\mathrm{red}}$ is the group of characters of $\pi^1(X, x)$ (here we use projectivity of X). Now the first isomorphism follows directly from Theorem 4.1.

The second isomorphism follows immediately from the first one (if X and Y are smooth projective varieties one can also give another proof using comparison of dimensions of $\mathrm{Pic}^0(X \times Y)$ and $\mathrm{Pic}^0 X \times \mathrm{Pic}^0 Y$). □

5. The abelian part of the S-fundamental group scheme

We say that a numerically flat sheaf is *irreducible* if it does not contain any proper numerically flat subsheaves (or, equivalently, if it corresponds to an irreducible representation of $\pi_1^S(X, x)$). In case of projective varieties a numerically flat sheaf is irreducible if and only if it is slope stable (with respect to some fixed polarization; or, equivalently, with respect to all polarizations).

If E is an irreducible numerically flat sheaf then it is simple. This follows from the fact that any endomorphism of such E is either 0 or an isomorphism (otherwise the image would give a proper numerically flat subsheaf).

Theorem 5.1. *Let E be an irreducible numerically flat sheaf on a product $X \times Y$ of complete varieties X and Y. Then there exist irreducible numerically flat sheaves E_1 on X and E_2 on Y such that $E \simeq p^* E_1 \otimes q^* E_2$.*

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Proof. Let us fix a point $x_0 \in X$ and an irreducible numerically flat subsheaf $K \subset E_{x_0}$. Let us set $F = \mathcal{H}om(p^*K, E)$. Since the sheaf F is numerically flat, by Corollary 4.6 we know that q_*F is numerically flat. It is easy to see that the induced map

$$\varphi \colon q^*q_*F \otimes p^*K \to E$$

is injective on $\{x_0\} \times Y$. Since $(q_*F) \otimes k(x_0) = \mathrm{Hom}(K, E_{x_0})$ is non-zero, $q^*q_*F \otimes p^*K$ is numerically flat and E is irreducible, φ is an isomorphism. □

The following lemma follows from the proof of Theorem 5.1 but we give a slightly different proof without using Theorem 4.1 (so in particular the proof is completely algebraic) in a case sufficient for the applications in the next section.

Lemma 5.2. *Let X and Y be complete varieties. Let E be a numerically flat sheaf on $X \times_k Y$. Assume that for some point $y_0 \in Y$, E_{y_0} is simple* (*i.e.* $\mathrm{End}(E_{y_0}) = k$). *Then there exists a numerically trivial line bundle L on Y such that $E \simeq p^*E_{y_0} \otimes q^*L$.*

Proof. For simplicity let us assume that X is smooth and projective. Let us set $F = \mathcal{H}om(p^*E_{y_0}, E)$. Since the sheaf F is numerically flat, by Corollary 3.2 we know that q_*F is locally free. Therefore, the induced map

$$q^*q_*F \otimes p^*E_{y_0} \to E$$

is surjective on all fibres $\{x\} \times Y$ and hence it is surjective. Since E_{y_0} is simple, $L = q_*F$ is a line bundle. Therefore, the above map is a surjective map of locally free sheaves of the same rank and hence it is an isomorphism. This also shows that L is numerically trivial. □

Proposition 5.3. *Let X and Y be complete varieties. Let E be a numerically flat sheaf on $X \times_k Y$. Assume that for some point $x_0 \in X$ and $y_0 \in Y$, E_{x_0} and E_{y_0} are simple. Then both E_{x_0} and E_{y_0} are line bundles and $E \simeq p^*E_{y_0} \otimes q^*E_{x_0}$.*

Proof. By the above lemma we know that $E \simeq p^*E_{y_0} \otimes q^*L$ for some line bundle L. Therefore,

$$E_{x_0} \simeq q_*((p^*E_{y_0} \otimes q^*L) \otimes p^*\mathcal{O}_{x_0}) \simeq L^{\oplus \, rk \, E_{y_0}}.$$

Since E_{x_0} is simple, E_{y_0} is a line bundle and $E_{x_0} \simeq L$. □

The following definition is an analogue of [**7**, Definition 2.5] where the corresponding notion is defined for stratified sheaves.

Definition 5.4. Let E be a numerically flat sheaf on a complete variety X/k. We say that E is *abelian* if there exists a numerically flat sheaf E' on $X \times_k X$ and a closed point $x_0 \in X$ such that E' restricted to both $p^{-1}(x_0)$ and $q^{-1}(x_0)$ is isomorphic to E.

Proposition 5.3 implies that a simple abelian numerically flat sheaf on a complete variety has rank 1. Since every numerically flat sheaf has a filtration with irreducible quotients, we get the following corollary describing the category of representations of $\pi^S_{\mathrm{ab}}(X, x)$.

Corollary 5.5. *Let E be an abelian numerically flat sheaf on a complete variety X. Then E has a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ in which all quotients E_i/E_{i-1} are numerically trivial line bundles. Moreover, if E is simple then it is a line bundle.*

Note that the last part of the corollary is non-trivial as even on complex projective varieties, a simple, semistable bundle need not be stable.

Remark 5.6. In the next section we will see that every numerically flat sheaf on an abelian variety is abelian. Therefore, the above corollary generalizes [**14**, Theorem 2] which describes numerically flat sheaves on abelian varieties assuming boundedness of semistable sheaves with fixed numerical invariants (this is proven in [**11**]). Mehta and Nori proved this theorem using reduction to the case of abelian varieties defined over an algebraic closure of a finite field. Our proof is completely different.

Lemma 5.7. *Let G be an affine k-group scheme. Then the category of representations of the largest abelian quotient G_{ab} of G is isomorphic to the full subcategory of the category of representation of G, whose objects are those G-modules V for which there exists a $G \times G$-module W such that we have isomorphisms of G-modules $W|_{G\times\{e\}} \simeq W|_{\{e\}\times G} \simeq V$.*

Proof. We give a naive proof in the case of algebraic groups. The reader is asked to rewrite it in terms of group schemes using a precise definition of derived subgroup scheme (see [**22**, 10.1]).

Let V be a G_{ab}-module. Since G_{ab} is abelian, the multiplication map $G_{\mathrm{ab}} \times G_{\mathrm{ab}} \to G_{\mathrm{ab}}$ is a homomorphism of affine group schemes. Hence we can treat V as a $G \times G$-module obtaining the required module W.

Now let V be a G-module for which there exists $\rho\colon G \times G \to \mathrm{GL}(W)$ and isomorphisms of G-modules $W|_{G\times\{e\}} \simeq W|_{\{e\}\times G} \simeq V$. The representation $\tilde\rho\colon G \to \mathrm{GL}(V)$ corresponding to V is given by $\tilde\rho(g) = \rho(g, e) = \rho(e, g)$. Since

$$\rho(g, h) = \rho(g, e) \cdot \rho(e, h) = \rho(e, h) \cdot \rho(g, e),$$

we have $\tilde\rho(ghg^{-1}h^{-1}) = e$. Therefore, $\tilde\rho$ vanishes on the derived subgroup of G and hence it defines the required representation of G_{ab}. □

The largest abelian quotient of the S-fundamental group scheme of X is called the *abelian part* of $\pi^S(X, x)$ and denoted by $\pi^S_{\mathrm{ab}}(X, x)$.

Lemma 5.7, together with Theorem 4.1, implies that $\pi^S_{\mathrm{ab}}(X, x)$ is Tannaka dual to the (neutral Tannakian) category of abelian numerically flat sheaves on X. This also explains the name 'abelian' in Definition 5.4.

Remark 5.8. Note that Corollary 5.5 follows immediately from the interpretation of abelian numerically flat bundles given above and the fact that irreducible representations of abelian group schemes are one-dimensional (see [**22**, Theorem 9.4]). In fact, this also proves that all subsheaves E_i in the filtration are abelian.

Let G be an abelian group. Then we define the k-group scheme $\mathrm{Diag}(G)$ by setting

$$(\mathrm{Diag}(G))(\operatorname{Spec} A) = \operatorname{Hom}_{\mathrm{gr}}(G, A^{\times})$$

for a k-algebra A, where $\operatorname{Hom}_{\mathrm{gr}}(G, A^{\times})$ denotes all group homomorphisms from G to the group of units in A. This is the same as taking $\operatorname{Spec} k[G]$ with the natural k-group scheme structure.

Theorem 5.9. *Let X be a smooth projective variety defined over an algebraically closed field k. If the characteristic of k is positive then we have an isomorphism*

$$\pi^S_{\mathrm{ab}}(X, x) \simeq \lim_{\leftarrow} \hat{G} \times \mathrm{Diag}((\operatorname{Pic}^{\tau} X)_{\mathrm{red}}),$$

where $\hat{G}$ denotes the Cartier dual of G and the inverse limit is taken over all finite local group subschemes G of $\operatorname{Pic}^0 X$. If k has characteristic zero then we have

$$\pi^S_{\mathrm{ab}}(X, x) \simeq H^1(X, \mathcal{O}_X)^* \times \mathrm{Diag}(\operatorname{Pic}^{\tau} X).$$

Proof. By [**22**, Theorem 9.5] $\pi^S_{\mathrm{ab}}(X, x)$ is a product of its unipotent and multiplicative parts.

By Nori's results mentioned in § 2.3 (see [**19**, Chapter IV, Proposition 6]) we know that in positive characteristic the abelian part of the unipotent part (or equivalently, the unipotent part of the abelian part) $\pi^U(X, x)$ of the S-fundamental group is isomorphic to the inverse limit of Cartier duals of finite local group subschemes of $\operatorname{Pic} X$. In characteristic zero, $\pi^U_{\mathrm{ab}}(X, x) = H^1(X, \mathcal{O}_X)^*$.

On the other hand, since the character group of $\pi^S(X, x)$ is isomorphic to the reduced scheme underlying $\operatorname{Pic}^{\tau} X$ (in characteristic zero $\operatorname{Pic}^{\tau} X$ is already reduced), the diagonal part of $\pi^S_{\mathrm{ab}}(X, x)$ is given by $\mathrm{Diag}((\operatorname{Pic}^{\tau} X)_{\mathrm{red}})$, which finishes the proof. □

Let us recall that the Neron–Severi group $\mathrm{NS}(X) = (\operatorname{Pic} X)_{\mathrm{red}}/(\operatorname{Pic}^0 X)_{\mathrm{red}}$ is finitely generated. We have a short exact sequence

$$0 \to (\operatorname{Pic}^0 X)_{\mathrm{red}} \to (\operatorname{Pic}^{\tau} X)_{\mathrm{red}} \to \mathrm{NS}(X)_{\mathrm{tors}} \to 0,$$

where $\mathrm{NS}(X)_{\mathrm{tors}}$ is the torsion group of $\mathrm{NS}(X)$ (it is a finite group). Therefore, we have a short exact sequence

$$0 \to \mathrm{Diag}(\mathrm{NS}(X)_{\mathrm{tors}}) \to \mathrm{Diag}((\operatorname{Pic}^{\tau} X)_{\mathrm{red}}) \to \mathrm{Diag}((\operatorname{Pic}^0 X)_{\mathrm{red}}) \to 0.$$

6. Numerically flat sheaves on abelian varieties

Let A be an abelian variety defined over an algebraically closed field of characteristic p and let A_n be the kernel of the multiplication by n map $n_A \colon A \to A$.

Let $A^{\mathrm{r}}_{p^n}$ be the reduced part of A_{p^n}. Then A_{p^n} is a product of $A^{\mathrm{r}}_{p^n}$, its Cartier dual $\hat{A}^{\mathrm{r}}_{p^n}$ (which is a local and diagonalizable group scheme) and a local–local group scheme $A^0_{p^n}$ (see [**18**, § 15]; *local–local* means that both the group scheme and its Cartier dual

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are local). Let r be the p-rank of A. Then $A^{\mathrm{r}}_{p^n} \simeq (\mathbb{Z}/p^n\mathbb{Z})^{\mathrm{r}}$, $\hat{A}^{\mathrm{r}}_{p^n} \simeq (\mu_{p^n})^{\mathrm{r}}$ and $A^0_{p^n}$ is a group scheme of order $p^{2(\dim A - r)n}$.

The *p-adic discrete Tate group* $T^d_p(A)$ is defined as the inverse limit $\varprojlim A^{\mathrm{r}}_{p^n}$ (see [**18**, § 18]). This is a $\mathbb{Z}_p$-module. Similarly, we define *the p-adic local–local Tate group* $T^0_p(A)$ as the inverse limit $\varprojlim A^0_{p^n}$. For an arbitrary prime l (possibly $l = p$) we also define the *l-adic Tate group scheme* as $T_l(A) = \varprojlim A_{l^n}$ (note that for $l = p$ our notation differs from that in [**18**]).

The following theorem is analogous to [**3**, Theorem 21].

Theorem 6.1. *Let A be an abelian variety defined over an algebraically closed field k. Then $\pi^S(A, 0)$ is abelian and it decomposes as a product of its unipotent and diagonal parts. Moreover, if the characteristic of k is positive then*

$$\pi^S(A, 0) \simeq T^0_p(A) \times T^d_p(A) \times \mathrm{Diag}(\mathrm{Pic}^0 A).$$

In characteristic zero we have

$$\pi^S(A, 0) \simeq H^1(A, \mathcal{O}_A)^* \times \mathrm{Diag}(\mathrm{Pic}^0 A).$$

Proof. Let us first remark that every numerically flat bundle E on A is abelian. To show this one can take the addition map $m\colon A \times_k A \to A$. Then $E' = m^*E$ restricted to either $p^{-1}(0)$ or $q^{-1}(0)$ is isomorphic to E. This shows that $\pi^S(A, 0)$ is abelian and we can use Theorem 5.9. In positive characteristic, local group subschemes of the dual abelian variety $\hat{A} = \mathrm{Pic}^0 A$ are of the form $A^0_{p^n} \times \hat{A}^{\mathrm{r}}_{p^n}$. So the inverse limit of their Cartier duals is isomorphic to $T^0_p(A) \times T^d_p(A)$. On the other hand, since $\mathrm{Pic}^\tau A = \mathrm{Pic}^0 A$, the diagonal part of $\pi^S(A, 0)$ is given by $\mathrm{Diag}(\mathrm{Pic}^0 A)$. □

We can also give another proof of the above theorem without using that $\pi^S(A, 0)$ is abelian (which uses the rather difficult Theorem 4.1). Namely, as in the proof of [**3**, Theorem 21] it is sufficient to show that for every indecomposable numerically flat sheaf E on A there exists a unique line bundle $L \in \mathrm{Pic}^0(A)$, such that $L \otimes E$ has a filtration by subbundles with each successive quotient trivial. But by Corollary 5.5 every numerically flat bundle E on A has a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ in which all quotients E_i/E_{i-1} are numerically trivial line bundles (in fact, this filtration is just a Jordan–Hölder filtration for an arbitrary polarization, but Corollary 5.5 says a bit more about this filtration).

Now if E is indecomposable then it is easy to see that all the quotients in the filtration are isomorphic (otherwise one can prove by induction that the filtration would split as $H^1(A, L) = 0$ for a non-trivial line bundle L on the abelian variety A), which finishes the proof.

Corollary 6.2. *Let A be an abelian variety defined over an algebraically closed field of positive characteristic. Then*

$$\pi^N(A, 0) \simeq \varprojlim A_n \simeq \prod_{l \text{ prime}} T_l(A).$$

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The above corollary also follows from [**20**, Remark 3] and [**15**, Theorem 2.3]. It is well known that the above corollary implies the Serre–Lang theorem (see [**18**, § 18]), although this sort of proof is much more complicated than the original one. In [**20**] Nori had to use the Serre–Lang theorem to prove the corollary.

7. The Albanese morphism

Let X be a smooth projective variety and let $x \in X$ be a fixed point. Let $\mathrm{alb}_X \colon X \to \mathrm{Alb}\,X$ be the Albanese morphism mapping x to 0. The variety $\mathrm{Alb}\,X$ is abelian and it is dual to the reduced scheme underlying $\mathrm{Pic}^0 X$. Since the S-fundamental group scheme of an abelian variety is abelian, we have the induced homomorphism $\pi^S_{\mathrm{ab}}(X,x) \to \pi^S_1(\mathrm{Alb}\,X, 0)$. The following proposition proves that this homomorphism is faithfully flat and it describes its kernel.

Theorem 7.1. *We have the following short exact sequence:*

$$0 \to \lim_{G\subset \mathrm{Pic}^0 X} \widehat{G/G_{\mathrm{red}}} \times \mathrm{Diag}(\mathrm{NS}(X)_{\mathrm{tors}}) \to \pi^S_{\mathrm{ab}}(X,x) \to \pi^S_1(\mathrm{Alb}\,X,0) \to 0,$$

where the limit is taken over all local group schemes $G \subset \mathrm{Pic}^0 X$.

Proof. The Albanese morphism induces $\mathrm{alb}^*_X \colon \mathrm{Pic}^0(\mathrm{Alb}\,X) \simeq (\mathrm{Pic}^0 X)_{\mathrm{red}} \to \mathrm{Pic}^0 X$, which is the natural closed embedding. By Theorem 5.9 we have the commutative diagram

$$\begin{array}{ccc} \pi^S_{\mathrm{ab}}(X,x) & \longrightarrow & \pi^S_1(\mathrm{Alb}\,X,0) \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ \lim_{G\subset \mathrm{Pic}^0 X} \hat{G}\times \mathrm{Diag}((\mathrm{Pic}^\tau X)_{\mathrm{red}}) & \longrightarrow & \lim_{G\subset(\mathrm{Pic}^0 X)_{\mathrm{red}}} \hat{G} \times \mathrm{Diag}((\mathrm{Pic}^0 X)_{\mathrm{red}}) \end{array}$$

where the limits are taken over local group schemes G and the lower horizontal map is induced by alb^*_X. In particular, the homomorphism $\pi^S_{\mathrm{ab}}(X,x) \to \pi^S_1(\mathrm{Alb}\,X,0)$ is faithfully flat and one can easily describe its kernel. □

Theorem 7.1 implies in particular that if $\mathrm{Pic}^0 X$ is reduced then the sequence

$$0 \to \mathrm{Diag}(\mathrm{NS}(X)_{\mathrm{tors}}) \to \pi^S_{\mathrm{ab}}(X,x) \to \pi^S_1(\mathrm{Alb}\,X,0) \to 0$$

is exact. Moreover, if $\mathrm{Pic}^\tau X$ is connected and reduced (e.g. if X is a curve or a product of two curves as in proof of Proposition 3.1) then $\pi^S_{\mathrm{ab}}(X,x) \to \pi^S_1(\mathrm{Alb}\,X,0)$ is an isomorphism.

Corollary 7.2 (cf. [16, Chapter III, Corollary 4.19]). *We have*

$$0 \to \lim_{G\subset \mathrm{Pic}^0 X} \widehat{G/G_{\mathrm{red}}} \times \mathrm{Diag}(\mathrm{NS}(X)_{\mathrm{tors}}) \to \pi^N_{\mathrm{ab}}(X,x) \to \pi^N_1(\mathrm{Alb}\,X,0) \to 0$$

and

$$0 \to \Big(\lim_{G\subset \mathrm{Pic}^0 X} \widehat{G/G_{\mathrm{red}}}\Big)_{\mathrm{red}} \times \mathrm{Diag}(\mathrm{NS}(X)_{\mathrm{tors}}) \to \pi^{\text{ét}}_{\mathrm{ab}}(X,x) \to \pi^{\text{ét}}_1(\mathrm{Alb}\,X,0) \to 0.$$

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Proof. For a group scheme H let us consider the directed system of quotients $H \to G$, where G is a finite group scheme (or an étale finite group). Let us consider the functor F (F') from the category of affine group schemes to the category of pro-finite group schemes (pro-finite groups, respectively), which to H associates the inverse limit of the above directed system. Note that these functors are exact, as the directed systems that we consider satisfy the Mittag–Leffler condition. Moreover, the functor F (F') applied to $\pi_1^S(X,x)$ gives $\pi_1^N(X,x)$ ($\pi_1^{\text{ét}}(X,x)$, respectively). Therefore, the required assertions follow by applying the functors F, F' to the sequence from Theorem 7.1. □

8. Varieties with trivial étale fundamental group

This section contains computation of the S-fundamental group scheme for varieties with trivial étale fundamental group. It contains a generalization of the main result of [**5**] and it is based on the method of [**4**].

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$. Let M_r be the moduli space of rank r slope stable bundles with numerically trivial Chern classes. It is known that it is a quasi-projective scheme.

A closed point $[E] \in M_r$ is called *torsion* if there exists a positive integer n such that $(F_X^n)^*E \simeq E$.

Lemma 8.1. *Assume that $\pi^{\text{ét}}(X,x) = 0$. If E is a strongly stable numerically flat vector bundle then its rank is equal to 1 and there exists n such that $(F_X^n)^*E \simeq \mathcal{O}_X$.*

Proof. By assumption all vector bundles $E_n = (F_X^n)^*E$ are stable. Let N be the Zariski closure of the set $\{[E_0], [E_1], \dots, \}$ in M_r. If N has dimension 0 then some Frobenius pull back $E' = (F_X^n)^*E$ is a torsion point of $M = M_r$. In this case by the Lange–Stuhler theorem (see [**10**, Satz 1.4]) there exists a finite étale covering $f\colon Y \to X$ such that f^*E' is trivial. But by assumption there are no non-trivial étale coverings so E' is trivial.

Therefore, we can assume that N has dimension at least 1. Note that the set N' of irreducible components of N of dimension greater than or equal to 1 is Verschiebung divisible (see [**4**, Definition 3.6]), since $V|_N$ is defined at points E_n for $n \geqslant 1$. Now we can proceed exactly as in proof of [**4**, Theorem 3.15] to conclude that the trivial bundle is dense in N', a contradiction. □

Proposition 8.2. *Assume that $\pi^{\text{ét}}(X,x) = 0$. Let E be a rank r numerically flat vector bundle on X. Then there exists some integer $n \geqslant 0$ such that $(F_X^n)^*E \simeq \mathcal{O}_X^r$.*

Proof. Proof is by induction on the rank r of E. When $r = 1$ then the assertion follows from the above lemma.

Let us recall that there exists an integer n such that $(F_X^n)^*E$ has a Jordan Hölder filtration $E_0 = 0 \subset E_1 \subset \cdots \subset E_m = (F_X^n)^*E$ in which all quotients $E^i = E_i/E_{i-1}$ are strongly stable numerically flat vector bundles (see [**12**, Theorem 4.1]). By taking further Frobenius pull backs and using the above lemma we can also assume that $E^i \simeq \mathcal{O}_X$.

By our induction assumption taking further Frobenius pull backs we can also assume that $E_m/E_1 \simeq \mathcal{O}_X^{r-1}$. Now we need to show that there exists some integer s such that the extension

$$0 \to (F_X^s)^* E_1 \to (F_X^s)^* E \to (F_X^s)^*(E_m/E_1) \to 0$$

splits. To prove that it is sufficient to note that the endomorphism $F^*\colon H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is nilpotent. But we know that F^* induces the Fitting decomposition $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)_s \oplus H^1(X, \mathcal{O}_X)_n$ into stable and nilpotent parts and the assertion follows from equality

$$H^1(X, \mathcal{O}_X)_s = \mathrm{Hom}(\pi_1^{\text{ét}}(X, x), \mathbb{Z}/p) \otimes_{\mathbb{F}_p} k = 0.$$

□

Corollary 8.3. *If $\pi^{\text{ét}}(X, x) = 0$ then $\pi^S(X, x) \simeq \pi^N(X, x)$.*

As a special case we get [**5**, Theorem 1.2]: if $\pi^N(X, x) = 0$ then $\pi^S(X, x) = 0$.

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References

1. I. Biswas, A. J. Parameswaran and S. Subramanian, Monodromy group for a strongly semistable principal bundle over a curve, *Duke Math. J.* **132** (2006), 1–48.
2. P. Deligne and J. S. Milne, Tannakian categories, in *Hodge cycles, motives, and Shimura varieties* (ed. P. Deligne, J. S. Milne, A. Ogus and K. Shih), Lecture Notes in Mathematics, Volume 900, pp. 101–228 (Springer, 1982).
3. J. P. P. dos Santos, Fundamental group schemes for stratified sheaves, *J. Alg.* **317** (2007), 691–713.
4. H. Esnault and V. Mehta, Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles, *Invent. Math.* **181** (2010), 449–465.
5. H. Esnault and V. Mehta, Weak density of the fundamental group scheme, *Int. Math. Res. Not.* **2010** (2010), 3071–3081.
6. G. Faltings, Stable G-bundles and projective connections, *J. Alg. Geom.* **2** (1993), 507–568.
7. D. Gieseker, Flat vector bundles and the fundamental group in non-zero characteristics, *Annali Scuola Norm. Sup. Pisa* **2** (1975), 1–31.
8. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Volume 52 (Springer, 1977).
9. G. Hein, Generalized Albanese morphisms, *Compositio Math.* **142** (2006), 719–733.
10. H. Lange and U. Stuhler, Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, *Math. Z.* **156** (1977), 73–83.
11. A. Langer, Semistable sheaves in positive characteristic, *Annals Math.* **159** (2004), 251–276 (addendum: *Annals Math.* **160** (2004), 1211–1213).
12. A. Langer, On the S-fundamental group scheme, *Annales Inst. Fourier*, in press.
13. V. B. Mehta, Some remarks on the local fundamental group scheme and the big fundamental group scheme, unpublished manuscript.
14. V. B. Mehta and M. V. Nori, Semistable sheaves on homogeneous spaces and abelian varieties, *Proc. Indian Acad. Sci. Math. Sci.* **93** (1984), 1–12.
15. V. B. Mehta and S. Subramanian, On the fundamental group scheme, *Invent. Math.* **148** (2002), 143–150.

16. J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, Volume 33 (Princeton University Press, 1980).
17. Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, in *Algebraic Geometry, Sendai 85*, Advanced Studies in Pure Mathematics, Volume 10, pp. 449–476 (American Mathematical Association, Providence, RI, 1987).
18. D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, Volume 5 (Oxford University Press, 1970).
19. M. V. Nori, The fundamental group-scheme, *Proc. Indian Acad. Sci. Math. Sci.* **91** (1982), 73–122.
20. M. V. Nori, The fundamental group-scheme of an abelian variety, *Math. Annalen* **263** (1983), 263–266.
21. C. S. Seshadri, Vector bundles on curves, *Contemp. Math.* **153** (1993), 163–200.
22. W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, Volume 66 (Springer, 1979.)