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# Extremal Numbers for Odd Cycles

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We describe the  $C_{2k+1}$ -free graphs on  $n$  vertices with maximum number of edges. The extremal graphs are unique for  $n \notin \{3k-1, 3k, 4k-2, 4k-1\}$ . The value of  $\text{ex}(n, C_{2k+1})$  can be read out from the works of Bondy [3], Woodall [14], and Bollobás [1], but here we give a new streamlined proof. The complete determination of the extremal graphs is also new.

We obtain that the bound for  $n_0(C_{2k+1})$  is  $4k$  in the classical theorem of Simonovits, from which the unique extremal graph is the bipartite Turán graph.

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## 1. Introduction, exact Turán numbers

Given a class of simple graphs  $\mathcal{F}$ , let us call a graph  $\mathcal{F}$ -free if it contains no copy of  $F$  as a (not necessarily induced) subgraph for each  $F \in \mathcal{F}$ . Let  $\text{ex}(n, \mathcal{F})$  denote the maximal number of edges in an  $\mathcal{F}$ -free graph on  $n$  vertices. If the class of graphs  $\mathcal{F} = \{F_1, F_2, \dots\}$  consists of a single graph then we write  $\text{ex}(n, F)$  instead of  $\text{ex}(n, \{F\})$ .

Let  $T_{n,p}$  denote the *Turán graph*, the complete equi-partite graph,  $K_{n_1, n_2, \dots, n_p}$  where  $\sum_i n_i = n$  and  $\lfloor n/p \rfloor \leq n_i \leq \lceil n/p \rceil$ . By Turán's theorem [12, 13] we have  $\text{ex}(n, K_{p+1}) = e(T_{n,p})$ ; furthermore,  $T_{n,p}$  is the unique  $K_{p+1}$ -free graph that attains the extremal number. The case  $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$  was shown earlier by Mantel [10].

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There are very few cases when the Turán number  $\text{ex}(n, \mathcal{F})$  is known exactly for all  $n$ . One can mention the case when  $F = M_{v+1}$  is a matching of a given size,  $v + 1$ . Erdős and Gallai [6] showed that

$$\text{ex}(n, M_{v+1}) = \max \left\{ \binom{2v+1}{2}, \binom{v}{2} + v(n-v) \right\}.$$

For the path on  $k$  vertices Erdős and Gallai [6] determined an asymptotic value, and  $\text{ex}(n, P_k)$  was determined for all  $n$  and  $k$  by Faudree and Schelp [7] and independently by Kopylov [9]. Erdős and Gallai [6] determined an asymptotic value for the class of long cycles

$$\mathcal{C}_{\geq \ell} := \{C_\ell, C_{\ell+1}, C_{\ell+2}, \dots\}.$$

The exact value of the Turán number  $\text{ex}(n, \mathcal{C}_{\geq \ell})$  was determined by Woodall [15] and independently and at the same time by Kopylov [9].

There is one outstanding result which gives infinitely many exact Turán numbers: Simonovits' chromatic critical edge theorem [11]. It states that if  $\min\{\chi(F) : F \in \mathcal{F}\} = p + 1 \geq 3$  and there exists an  $F \in \mathcal{F}$  with an edge  $e \in E(F)$  such that by removing this edge we have  $\chi(F - e) \leq p$ , then there exists an  $n_0(\mathcal{F})$  such that  $T_{n,p}$  is the only extremal graph for  $\mathcal{F}$  for  $n \geq n_0$ . The authors are not aware of any (non-trivial) further result when  $\text{ex}(n, \mathcal{F})$  is known for all  $n$ , nor any  $F$  for which the value of  $n_0(F)$  had been determined, except the case of odd cycles discussed below.

## 2. The result; the extremal graphs without $C_{2k+1}$

The aim of this paper is to determine the Turán number of odd cycles for all  $n$  together with the extremal graphs. The value of  $\text{ex}(n, C_{2k+1})$  can be extracted from the works of Bondy [2, 3], Woodall [14], and Bollobás [1, pp. 147–156] concerning (weakly) pancyclic graphs. For a recent presentation see Dzido [5], who also considered the Turán number of wheels. But here we give a new streamlined proof and a complete description of the extremal graphs.

Since  $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  contains no odd cycles, for any  $k \geq 1$  we have  $\text{ex}(n, C_{2k+1}) \geq \lfloor n^2/4 \rfloor$ . For  $C_3$  equality holds for all  $n$ , with the only extremal graph being  $T_{n,2}$  by the Turán–Mantel theorem. From now on, we suppose that  $2k + 1 \geq 5$ . Also, for  $n \leq 2k$  obviously  $\text{ex}(n, C_{2k+1}) = \binom{n}{2}$ , so we may suppose that  $n \geq 2k + 1$ .

Every edge of an odd cycle is colour-critical, so Simonovits' theorem implies that the complete bipartite graph is the only extremal graph, and

$$\text{ex}(n, C_{2k+1}) = e(T_{n,2}) = \lfloor n^2/4 \rfloor \quad \text{for } n \geq n_0(C_{2k+1}).$$

After choosing the right tools we present a streamlined proof, and show that  $n_0(C_{2k+1}) = 4k$  when  $2k + 1 \geq 5$ .

We define two classes of  $C_{2k+1}$ -free graphs which could have at least as many edges as  $T_{n,2}$  for  $n \leq 4k - 1$ . A *cactus*  $B(n; n_1, \dots, n_s)$  (for  $n \geq 2, s \geq 1$  with  $\sum_i (n_i - 1) = n - 1$ ) is a connected graph where the 2-connected blocks are complete graphs of sizes  $n_1, \dots, n_s$ . Let  $g(n, k)$  denote the largest size of an  $n$ -vertex cactus avoiding  $C_{2k+1}$ . For this maximum all

block sizes should be exactly  $2k$  but at most one, which is smaller. Write  $n$  in the form  $n = (s - 1)(2k - 1) + r$ , where  $s \geq 1$ ,  $2 \leq r \leq 2k$  are integers. Then

$$g(n, k) = (s - 1) \binom{2k}{2} + \binom{r}{2}. \tag{2.1}$$

Note that  $g(n, k) > \lfloor n^2/4 \rfloor$  for  $3 \leq n \leq 4k - 3$  and we have  $g(n, k) = e(T_{n,2}) = \lfloor n^2/4 \rfloor$  if  $n \in \{4k - 2, 4k - 1\}$ . Thus the Simonovits threshold  $n_0(C_{2k+1})$  is at least  $4k$ .

For  $n \geq k$ , define the graph  $H_1(n, k)$  on  $n$  vertices by its degree sequence; it has  $k$  vertices of degree  $n - 1$  and all other vertices have degree  $k$ . Then  $H_1(n, k)$  is a complete bipartite graph  $K_{k, n-k}$ , together with all possible edges added in the first partite set. This graph does not contain the cycle  $C_{2k+1}$ . Letting  $h_1(n, k)$  denote the size of  $H_1(n, k)$ ,

$$h_1(n, k) = \binom{k}{2} + k(n - k). \tag{2.2}$$

Note that  $h_1(n, k) \leq g(n, k)$  for all  $k \leq n$  and here equality holds if  $n$  is in the form  $n = (s - 1)(2k - 1) + r$ , where  $s \geq 1$  and  $r \in \{k, k + 1\}$ .

**Theorem 2.1.** *For any  $n \geq 1$  and  $2k + 1 \geq 5$ ,*

$$\text{ex}(n, C_{2k+1}) = \begin{cases} \binom{n}{2} & \text{for } n \leq 2k, \\ g(n, k) & \text{for } 2k + 1 \leq n \leq 4k - 1, \text{ and} \\ \lfloor n^2/4 \rfloor & \text{for } n \geq 4k - 2. \end{cases}$$

*Furthermore, the only extremal graphs are  $K_n$  for  $n \leq 2k$ ;  $B(n; 2k, n - 2k + 1)$  for  $2k + 1 \leq n \leq 4k - 1$ ;  $H_1(n, k)$  for  $n \in \{3k - 1, 3k\}$ ; and the complete bipartite graph  $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  for  $n \geq 4k - 2$ .*

### 3. A lemma on 2-connected graphs without $C_{2k+1}$

**Lemma 3.1.** *Suppose that  $n \geq 2k + 1 \geq 5$ , and  $G$  is a 2-connected,  $C_{2k+1}$ -free, non-bipartite graph with at least  $\lfloor n^2/4 \rfloor$  edges. Then  $e(G) \leq \text{ex}(n, C_{2k+1})$ , and here equality holds only if  $n \in \{3k - 1, 3k\}$  and  $G = H_1(n, k)$ .*

For  $5 \leq 2k + 1 \leq n$ , define the graph  $H_2(n, k)$  on  $n$  vertices and

$$h_2(n, k) := \binom{2k - 1}{2} + 2(n - 2k + 1)$$

edges, consisting of a complete graph  $K_{2k-1}$  containing two special vertices which are connected to all other vertices. Then  $H_2(n, k)$  is a 2-connected  $C_{2k+1}$ -free graph. For  $k = 2$  the graphs  $H_1(n, k)$  and  $H_2(n, k)$  are isomorphic. Recall a result of Kopylov [9] in the following form. Suppose that the 2-connected graph  $G$  on  $n$  vertices contains no cycles of length  $2k + 1$  or larger and  $n \geq 2k + 1 \geq 5$ . Then

$$e(G) \leq \max\{h_1(n, k), h_2(n, k)\}, \tag{3.1}$$

and this bound is the best possible. Moreover, only the graphs  $H_1(n, k)$  and  $H_2(n, k)$  could be extremal. For further explanation and background see the recent survey [8].

The other result we need is due to Brandt [4]. Let  $G$  be a non-bipartite graph of order  $n$  and suppose that

$$e(G) > (n - 1)^2/4 + 1. \tag{3.2}$$

Then  $G$  contains cycles of every length between 3 and the length of its longest cycle.

**Proof of Lemma 3.1.** The inequality  $e(G) \leq \text{ex}(n, C_{2k+1})$  follows from the definition. Suppose that equality holds. Apply Brandt’s theorem (3.2). We obtain that  $G$  contains cycles of all lengths  $3, 4, \dots, \ell$ , where  $\ell$  stands for the longest cycle length in  $G$ . It follows that  $\ell \leq 2k$ . Kopylov’s theorem (3.1) implies that

$$\max\{g(n, k), \lfloor n^2/4 \rfloor\} \leq \text{ex}(n, C_{2k+1}) = e(G) \leq \max\{h_1(n, k), h_2(n, k)\}.$$

Since  $g(n, k) > h_2(n, k)$  except for  $(n, k) \in \{(5, 2), (6, 2)\}$ , and  $g(n, k) > h_1(n, k)$  unless  $n$  is in the form  $n = (s - 1)(2k - 1) + r$ , where  $s \geq 2$  and  $r \in \{k, k + 1\}$ , we obtain that  $e(G) = h_1(n, k)$ ,  $n$  should be in this form, and  $G = H_1(n, k)$ .

Finally,  $h_1(n, k) < \lfloor n^2/4 \rfloor$  for  $n \geq 4k$  so we obtain that indeed  $n \in \{3k - 1, 3k\}$ . □

#### 4. The proof of Theorem 2.1

Suppose that  $G$  is an extremal  $C_{2k+1}$ -free graph, i.e.,  $e(G) = \text{ex}(n, C_{2k+1})$ . Then  $G$  is connected. Consider the cactus-like block-decomposition of  $G$ ,  $V(G) = V_1 \cup V_2 \cup \dots \cup V_s$ , where the induced subgraphs  $G[V_i]$  are either edges or maximal 2-connected subgraphs of  $G$ . Let  $n_i := |V_i|$ . Then we have  $n - 1 = \sum_i (n_i - 1)$ , and each  $n_i \geq 2$ . We have  $e(G[V_i]) = \text{ex}(n_i, C_{2k+1})$ ; otherwise one can replace  $G[V_i]$  with an extremal graph of the same order  $n_i$  and obtain another  $C_{2k+1}$ -free graph of size larger than  $e(G)$ . Therefore  $e(G[V_i]) \geq \lfloor n_i^2/4 \rfloor$ , and there are three types of blocks:

- complete graphs (if  $n_i \leq 2k$ ),
- bipartite blocks with  $e(G[V_i]) = \lfloor n_i^2/4 \rfloor$ .

Finally,

- if  $n_i \geq 2k + 1$  and  $G[V_i]$  is not bipartite, then Lemma 3.1 implies that  $n_i \in \{3k - 1, 3k\}$  and  $G[V_i] = H_1(n_i, k)$ .

We may rearrange the graphs  $G[V_i]$  and the sets  $V_i$  in such a way that they share a common vertex  $v \in \cap V_i$  and otherwise the sets  $V_i \setminus \{v\}$  are pairwise disjoint. The new graph  $G^*$  obtained is also  $C_{2k+1}$ -free and extremal, and it has the same size and order as  $G$ .

If  $s = 1$  then we are done. Suppose  $s \geq 2$ . If all blocks are complete graphs, then  $e(G) \leq g(n, k)$ . Since  $g(n, k) < e(T_{n,2})$  for  $n > 4k - 1$ , we get that  $n \leq 4k - 1$  and  $G^*$  (and  $G$ ) has only two blocks and at least one of them is of size  $2k$ .

Finally, suppose that there are two blocks  $V_i$  and  $V_j$ ,  $|V_i| = a$  and  $|V_j| = b$ , such that  $G[V_i]$  and  $G[V_j]$  are not both complete subgraphs. We claim that in this case one can remove the edges of  $G[V_i]$  and  $G[V_j]$  from  $G^*$  and place a copy of  $T_{a+b-1,2}$  or some other graph  $L$  onto  $V_i \cup V_j$  such that the new graph obtained is  $C_{2k+1}$ -free and it has more edges than  $e(G)$ , a contradiction.

Indeed, if  $G[V_i]$  is a large bipartite graph,  $a := n_i \geq 2k + 1$ ,  $G[V_i] = T_{a,2}$  and  $G[V_j]$  is a complete bipartite graph, too, then we can increase  $e(G^*)$  since

$$e(T_{a,2}) + e(T_{b,2}) \leq \frac{1}{4}a^2 + \frac{1}{4}b^2 < \left\lfloor \frac{1}{4}(a+b-1)^2 \right\rfloor = e(T_{a+b-1,2}). \quad (4.1)$$

In the remaining cases the inequalities concerning the number of edges of  $e(L)$  are just elementary algebra. If  $G[V_i] = T_{a,2}$  and  $G[V_j] = H_1(b,k)$  or  $K_b$  then we can replace them again with a complete bipartite graph  $T_{a+b-1,2}$ . From now on, we may suppose that each block is either a complete graph (of size at most  $2k$ ) or an  $H_1(a,k)$ . If  $G[V_i] = H_1(a,k)$  for some  $a \in \{3k-1, 3k\}$  and  $G[V_j] = H_1(b,k)$  (with  $b \in \{3k-1, 3k\}$ ) or  $G[V_j] = K_b$  with  $k \leq b \leq 2k$ , then we replace  $G[V_i] \cup G[V_j]$  again with a  $T_{a+b-1,2}$ . Finally, if  $G[V_i] = H_1(a,k)$  for some  $a \in \{3k-1, 3k\}$  and  $G[V_j] = K_b$  with  $2 \leq b \leq k$ , then we replace  $G[V_i] \cup G[V_j]$  with two complete graphs of sizes  $2k$  and  $a+b-2k$  and use

$$e(H_1(a,k)) + e(K_b) < e(B(a+b-1; 2k, a+b-2k))$$

to get a contradiction. This completes the proof of the claim and Theorem 2.1.  $\square$

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