

SPECTRAL ANALYSIS OF MORSE–SMALE FLOWS I: CONSTRUCTION OF THE ANISOTROPIC SPACES

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(Received 5 April 2017; revised 23 August 2018; accepted 28 August 2018;
first published online 13 November 2018)

Abstract We prove the existence of a discrete correlation spectrum for Morse–Smale flows acting on smooth forms on a compact manifold. This is done by constructing spaces of currents with anisotropic Sobolev regularity on which the Lie derivative has a discrete spectrum.

Keywords: transfer operators; Pollicott–Ruelle resonances; Morse–Smale flows; Microlocal analysis

2010 *Mathematics subject classification:* Primary 37D15
Secondary 58J50

1. Introduction

Given a smooth (\mathcal{C}^∞), compact, oriented, boundaryless manifold M of dimension $n \geq 1$ and a smooth flow $\varphi^t : M \rightarrow M$, a basic question from dynamical systems is to understand the long time behavior of the flow. There are many ways to approach this problem. For instance, one can define the *correlation function*:

$$C_{\psi_1, \psi_2}(t) := \int_M \varphi^{-t*}(\psi_1) \wedge \psi_2, \quad (1)$$

with $\psi_1 \in \Omega^k(M)$ and $\psi_2 \in \Omega^{n-k}(M)$. Then, if one can describe the limit of this quantity as $t \rightarrow +\infty$, then it gives some information on the weak limit of $\varphi^{-t*}(\psi_1)$ in the sense of currents. Studying directly the limit of $C_{\psi_1, \psi_2}(t)$ as $t \rightarrow +\infty$ is not often possible and one may first introduce its Laplace transform:

$$\hat{C}_{\psi_1, \psi_2}(z) := \int_0^{+\infty} e^{-tz} C_{\psi_1, \psi_2}(t) dt. \quad (2)$$

Note that this is well defined for $\operatorname{Re}(z) > c$ with $c > 0$ depending only on the flow φ^t . Instead of studying the long time limit, one could then try to understand if this

holomorphic function has a meromorphic extension to a larger half-plane. If so, the poles and their residues also give some informations on the long time dynamics of the flow. Therefore, in the sequel, the set of poles and residues of the Laplace transformed correlation functions will be called *correlation (or Pollicott–Ruelle) spectrum* of the flow. These kinds of questions were for instance considered by Pollicott [35] and Ruelle [37] in the framework of Axiom A flows that we shall now discuss.

In fact, this type of problem is very hard at this level of generality and some assumptions on the nature of the flow should be made to obtain some nontrivial results. A natural situation where one may expect some answer is when some hyperbolicity is involved in the system, e.g., for Axiom A flows in the sense of Smale [39]. In that framework, one can decompose the nonwandering set of the flow into finitely many invariant hyperbolic subsets $(\Lambda_j)_{j=1}^K$, which are called the basic sets of the flow. Examples of such flows are geodesic flows on negatively curved manifolds or gradient flows associated with a Morse function. As most of the time on a given orbit is spent in some neighborhood of these basic sets, it is natural to first restrict to test forms which are supported in a small neighborhood of a given Λ_j . For a slightly different correlation function associated with a Gibbs measure of Λ_j , Pollicott [35] and Ruelle [37] proved the meromorphic extension of the Laplace transform to some half-plane $\text{Re}(z) \geq -\delta$ with $\delta > 0$. Their proof relies on the symbolic coding by Markov partitions of such flows that was constructed by Bowen [4]. In the last fifteen years, many progresses have been made toward this problem by adopting a slightly different point of view. Namely, one can observe that, for a k -form ψ_1 , its pull-back $\varphi^{-t*}(\psi_1)$ by the flow solves the following partial differential equation:

$$\partial_t \psi = -\mathcal{L}_V^{(k)} \psi, \quad \psi(t = 0) = \psi_1,$$

where $\mathcal{L}_V^{(k)} = (d + \iota_V)^2$ is the Lie derivative along the vector field V associated with the flow φ^t . In particular, if one can find an appropriate Banach space on which $-\mathcal{L}_V$ has a discrete spectrum (with finite multiplicity) on the half-plane $\text{Re}(z) \geq -\delta$ for some positive δ , then one can verify that $\hat{C}_{\psi_1, \psi_2}(z)$ has a meromorphic extension to the same half-plane. This approach has been initiated by Liverani [28] in the context of contact Anosov flows and it was further developed in [6, 21] where it is proved, among other things, that $\hat{C}_{\psi_1, \psi_2}(z)$ has a meromorphic extension to the entire complex plane for Anosov flows. In these references, one of the key ingredients is the construction of Banach spaces with anisotropic Hölder regularity on which $-\mathcal{L}_V$ has good spectral properties. Alternative spaces based on microlocal tools were developed by Dyatlov, Faure, Sjöstrand, Tsujii and Zworski [12, 16, 17, 42, 43]. This complementary approach allowed to bring new perspectives on the fine structure of this correlation spectrum in the Anosov case. Coming back to the case of Axiom A flows, Dyatlov and Guillarmou proved that $\hat{C}_{\psi_1, \psi_2}(z)$ admits a meromorphic extension to \mathbb{C} provided that we only consider test forms which are compactly supported in a neighborhood of a fixed basic set Λ_j [11]. We should point that progress for flows follow from earlier results for hyperbolic diffeomorphisms that we will not discuss here. We refer the reader to the book of Baladi for a recent detailed account on that case [1].

If we look at the general framework of Axiom A flows, we already noticed that these different results do not say at first sight much things on the *global dynamics* of the flow.

In fact, in all the results we mentioned so far, it is important that we restrict ourselves to test forms which are compactly supported near a fixed basic set Λ_j of the flow. In the case of geodesic flows on negatively curved manifolds, this restriction is of course artificial as there is only one basic set which covers the entire manifold. However, in general, there may be several basic sets that are far from covering the entire manifold and, if we remember that Axiom A flows arise as far-reaching generalizations of gradient flows associated with a Morse function [39], then understanding the global dynamics sounds also important as it provides some information on the topology of the manifold [18, 38]. Several difficulties appear if we want to consider this global question and let us mention at least two of them: (1) the flow is not topologically transitive on M (while it is on a fixed Λ_j), (2) hyperbolicity only holds on the basic sets.

From Morse–Smale gradient flows to Morse–Smale flows. In the case of certain Morse–Smale gradient flows, the fact that $C_{\psi_1, \psi_2}(t)$ admits a limit as $t \rightarrow +\infty$ for any choice of ψ_1 and ψ_2 was proved by Harvey and Lawson [23, 24] – see also [27] for earlier related results. While the proof of Harvey and Lawson was based on the theory of currents à la Federer, we recently showed how to develop an appropriate *global* spectral theory for such gradient flows [8] and to derive a complete asymptotic expansion of the correlation function – see also [20] for related results in the context of quantum field theory. In particular, this shows that \hat{C}_{ψ_1, ψ_2} has a meromorphic extension to \mathbb{C} . Moreover, as a byproduct of our spectral analysis, we obtained a new spectral interpretation of the Thom–Smale–Witten complex as the kernel of the operator $-\mathcal{L}_V$ acting on certain anisotropic spaces of currents from which one can easily deduce the finiteness of Betti numbers, the Poincaré duality and the classical Morse inequalities.

The goal of the present work is to continue to *explore the global dynamics* of Axiom A flows by focusing on the particular case of Morse–Smale flows [33, 38] for which the basic sets are either closed orbits or fixed points. These flows also satisfy some transversality assumptions necessary to develop proper topological applications [38] – see § 3 for more details. These flows are more general than the families of gradient flows we considered in [8] as they may have closed periodic orbits. They also form a natural subfamily of simple Axiom A flows [39] where one may expect to develop a proper *global* spectral theory. Recall that Peixoto proved that, in dimension 2, these flows form an open and dense (in the C^∞ topology) family of all smooth vector fields [34] while in higher dimension, Palis showed that they form an open subset of all smooth vector fields [32].

This article is the first in a series. Here, we develop a convenient *global* functional framework for Morse–Smale flows in order to prove that the Laplace transformed correlator $\hat{C}_{\psi_1, \psi_2}(z)$ defined by equation (2) has a meromorphic extension to the entire complex plane. In [9], we will elaborate more on Morse–Smale flows and show how to give an explicit description of the poles and of residues of $\hat{C}_{\psi_1, \psi_2}(z)$ provided certain nonresonance assumptions are satisfied. Finally, in [10], we will explain how to give topological interpretations for the correlation spectrum of a class of flows which have a proper *global* spectral theory that we call microlocally tame. This class contains Anosov and Morse–Smale flows. We will show how to extract Morse inequalities for Pollicott–Ruelle resonant states in the kernel of \mathcal{L}_V and we will also give some new

identities relating regularized products of Pollicott–Ruelle resonances on the imaginary axis with a torsion function introduced by Fried which coincides with Reidemeister torsion when V is nonsingular [19].

2. Statement of the main results

In the article, M will denote a smooth (C^∞), compact, oriented manifold without boundary and of dimension $n \geq 1$. We will also endow this manifold with a smooth (C^∞) Riemannian metric that will be denoted by g .

2.1. Discrete correlation spectrum

Our main result shows the existence of a meromorphic extension to \mathbb{C} of $\hat{C}_{\psi_1, \psi_2}(z)$:

Theorem 2.1 (Resonances). *Let φ^t be a C^∞ Morse–Smale flow which is C^1 linearizable. Denote by V the corresponding vector field and let $0 \leq k \leq n$. Then, there exists a minimal^a discrete subset $\mathcal{R}_k(V) \subset \mathbb{C}$ such that, given any $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$, the map*

$$z \mapsto \hat{C}_{\psi_1, \psi_2}(z)$$

has a meromorphic extension from $\{\operatorname{Re}(z) > c\}$ to \mathbb{C} whose poles are of finite order and contained inside $\mathcal{R}_k(V)$.

^aWe just mean that, for every $z_0 \in \mathcal{R}_k(V)$, one can find (ψ_1, ψ_2) such that there is indeed a pole at z_0 .

We refer to §3 for a precise definition of a Morse–Smale flow. By discrete, we mean that $\mathcal{R}_k(V)$ has no accumulation points. In particular, it is at most countable. Elements inside $\mathcal{R}_k(V)$ are often referred as *Pollicott–Ruelle resonances* or as the *correlation spectrum* of the flow. Besides the fact that the flow is Morse–Smale, we need to make an assumption on the fact that the flow is C^1 -linearizable – see §3.3 for the precise definition. Roughly speaking, it means that the flow is C^1 -conjugated to a linear flow in a neighborhood of each basic set of the flow. This may sound like a big constraint. Yet, thanks to the Sternberg–Chen theorem [7, 31, 44], it is satisfied as soon as a certain number of nonresonance assumptions are made on the Lyapunov exponents of the basic set. In particular, they are satisfied for a generic choice of Morse–Smale flow. We refer to Appendix A.5 for a brief account on these nonresonance hypotheses. This assumption of being C^1 -linearizable may be artificial at this stage of our analysis but it does not look obvious to us how to remove it in an easy manner.

Note that, if ψ_1 and ψ_2 were supported in a small neighborhood of some given basic set, then the existence of this discrete correlation spectrum could be deduced near critical points from [2, 3, 22] and near closed orbits from [11]. Here, the main novelty is that the result holds *globally* on the manifold, i.e., without any restriction on the supports of ψ_1 and ψ_2 . It also generalizes our previous results from [8] which were only valid for Morse–Smale *gradient flows which are not allowed to have periodic orbits*. Observe that, even if the nonwandering set is the union of finitely many basic sets, it is not

obvious that the *global correlation spectrum* should be the union of the correlation spectra associated with each individual basic set. We shall see in [9] that this is indeed the case if enough nonresonance conditions are satisfied by the Lyapunov exponents. Without these assumptions, it is not completely obvious if some unexpected phenomenon may occur in the correlation spectrum.

In fact, thanks to its spectral nature, our proof will not only give the meromorphic extension of $\hat{C}_{\psi_1, \psi_2}(z)$ but also some information on its residues:

Theorem 2.2 (Resonant states). *Let φ^t be a Morse–Smale flow which is \mathcal{C}^1 linearizable. Denote by V the corresponding vector field and let $0 \leq k \leq n$.*

Then, for every $z_0 \in \mathcal{R}_k(V)$, there exists an integer $m_k(z_0) \geq 1$ and a linear map of finite rank

$$\pi_{z_0}^{(k)} : \Omega^k(M) \rightarrow \mathcal{D}^k(M)$$

such that, given any $(\psi_1, \psi_2) \in \Omega^k(M) \times \Omega^{n-k}(M)$, one has, in a small neighborhood of z_0 ,

$$\hat{C}_{\psi_1, \psi_2}(z) = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{\langle (\mathcal{L}_V^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}(\psi_1, \psi_2) \rangle}{(z - z_0)^l} + R_{\psi_1, \psi_2}(z),$$

where $R_{\psi_1, \psi_2}(z)$ is a holomorphic function. Moreover, any element u in the range of $\pi_{z_0}^{(k)}$ satisfies the generalized eigenvalue equation

$$(\mathcal{L}_V^{(k)} + z_0)^{m_k(z_0)}(u) = 0.$$

Here, $\mathcal{D}^k(M)$ denotes the De Rham currents of degree k , i.e., the topological dual of $\Omega^{n-k}(M)$. Elements inside the range of $\pi_{z_0}^{(k)}$ are called the *Pollicott–Ruelle resonant states*. Our proof will say more on the Sobolev regularity of these currents along stable and unstable directions. In [9], we will show how to exploit this Sobolev regularity to give a rather precise description of these resonant states in a neighborhood of the basic sets of the flow. In [10], we will show that some of these states have a deep topological meaning related to the De Rham complex [36] and to its Reidemeister torsion [19]. Finally, we emphasize that Dyatlov and Guillarmou characterized elements inside the range of $\pi_{z_0}^{(k)}$ in terms of currents solving the generalized eigenvalue equation and verifying some support and wavefront assumptions [11]. This was valid for general Axiom A flows but only near a fixed basic set. It is plausible that a similar property holds globally for Morse–Smale flows but we shall not discuss this question here.

2.2. About the proofs of Theorems 2.1 and 2.2

We will consider a slightly more general framework than the one we described so far. Fix a complex vector bundle $\mathcal{E} \rightarrow M$ of rank N and some connection $\nabla : \Omega^0(M, \mathcal{E}) \rightarrow \Omega^1(M, \mathcal{E})$ [29] – see also §9.1 for a brief reminder. Then, one can define a covariant derivative $d^\nabla : \Omega^\bullet(M, \mathcal{E}) \rightarrow \Omega^{\bullet+1}(M, \mathcal{E})$ and introduce the operator

$$\forall 0 \leq k \leq n, \quad \mathcal{L}_{V, \nabla}^{(k)} := d^\nabla \circ \iota_V + \iota_V \circ d^\nabla : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^k(M, \mathcal{E}).$$

Note that, in the present article, we will not make the assumption that ∇ is flat, i.e., that $d^\nabla \circ d^\nabla = 0$. Our goal is to introduce anisotropic Sobolev spaces of currents adapted to the dynamics of the Morse–Smale vector field V in the sense that $-\mathcal{L}_{V,\nabla}^{(k)}$ has a discrete spectrum on this space at least for $\text{Re}(z) \geq -\sigma$ with $\sigma > 0$. Introducing this algebraic framework is in fact important for the applications to topology we have in mind [10]. In particular, it will allow us to formulate generalizations of the Morse inequalities for general vector bundles and to relate the Pollicott–Ruelle spectrum to the Reidemeister torsion appearing in the works of Fried [19].

The proofs of Theorems 2.1 and 2.2 will follow the microlocal approach of Faure and Sjöstrand [16]. Recall that their construction is based on the fact that the operator¹ $-\mathcal{L}_{V,\nabla}^{(k)}$ is a differential operator whose principal symbol is $H(x; \xi)\text{Id}_{\Lambda^k(T^*M) \otimes \mathcal{E}}$ where

$$\forall(x; \xi) \in T^*M, \quad H(x; \xi) := \xi(V(x)). \tag{3}$$

Then, they show that the spectrum of the operator $\mathcal{L}_{V,\nabla}^{(k)}$ (hence the correlation spectrum) can be obtained in a similar manner as in the theory of semiclassical resonances [13, 25]. In particular, this requires to understand the dynamical properties of the Hamiltonian flow induced by H on T^*M , equivalently of the symplectic lift of φ^t to the cotangent space. Precisely, it requires to describe the topological and dynamical properties of the set of points which are trapped by the Hamiltonian dynamics either in the future or in the past. Due to the fact that we want to deal with the global correlation spectrum, we need to analyze precisely the global Hamiltonian dynamics which is the content of § 4 and which is probably the main new difficulty compared to the Anosov case treated in [16]. We emphasize that this part of the proof is implicitly related to the classical results of Smale [38] on Morse–Smale flows – see also [45] for a formulation closer to ours in the case of gradient flows. The major difference is that we are interested here in the Hamiltonian dynamics on T^*M rather than the dynamics on the base space M . Note that we already had to deal with similar difficulties in the context of gradient flows [8] and we give here a more systematic approach which allows to deal with closed orbits. We should also point out that the results we obtain in that direction are in some sense related to some results of Laudenbach who gave a very precise description of the closure of unstable manifolds for Morse–Smale gradient flows [27]. Once this is well understood, we construct in Lemma 8.1 an appropriate escape (or Lyapunov) function for the Hamiltonian dynamics. This construction combines our analysis of the global dynamics with some results due to Meyer on the existence of energy functions for Morse–Smale flows [30]. Given this escape function, we can follow the spectral construction of Faure and Sjöstrand whose crucial ingredient is in fact the existence of such a function – see § 9 for more details.

2.3. Organization of the article

In § 3, we review some classical facts on Morse–Smale flows and introduce some conventions that we will use all along the article. Then, in § 4, we explain how to define the symplectic lift of a flow and we introduce coordinate systems adapted to our

¹They do not actually deal with this generalized geometric framework but their construction can be adapted as the operators under consideration have scalar principal symbols.

problem. Along the way, we collect a few facts from Floquet theory. In §5, we introduce the conormal bundle to the unstable manifolds and we state the main new dynamical results of the article, namely Theorems 5.2 and 5.4. Sections 6 and 7 are devoted to the proof of these results. In §8, we use these results to construct an escape function for the symplectic lift of our flow. Finally, we prove Theorems 2.1 and 2.2 by constructing anisotropic Sobolev spaces adapted to the vector field V . In the Appendix A, we collect some classical results on hyperbolic fixed points and closed orbits that we use in our proofs and, in Appendix B, we briefly recall the proofs of some results due to Smale [38] which may be helpful to understand the proofs of our results on the Hamiltonian dynamics.

3. Review on Morse–Smale flows

The purpose of this preliminary section is to collect some well-known facts on the so-called Morse–Smale flows which were introduced by Smale in [38] as a generalization of gradient flows induced by a Morse function. Besides the seminal work of Smale, good references on the subject are [18, Chapter 8] and [33, Chapter 4].

Remark 3.1. All along the article, we implicitly assume that M is endowed with a C^∞ Riemannian structure g which plays an auxiliary role, all the results being independent of the choice of g .

3.1. Definition and examples

We say that $\Lambda \subset M$ is an elementary critical element if Λ is either a fixed point or a closed orbit of φ^t . Such an element is said to be hyperbolic if the fixed point or the closed orbit is hyperbolic – see Appendix A for a brief reminder. Following [39, p. 798]:

Definition 3.2. A flow φ^t is called a *Morse–Smale flow* if the following properties hold:

- (1) the nonwandering set $NW(\varphi^t)$ is the union of finitely many elementary critical elements $\Lambda_1, \dots, \Lambda_K$ which are hyperbolic;
- (2) for every $(i, j) \in \{1, \dots, K\}^2$ and for every x in $W^u(\Lambda_i) \cap W^s(\Lambda_j)$, one has

$$T_x M = T_x W^u(\Lambda_i) + T_x W^s(\Lambda_j) \quad (1)$$

where $W^u(\Lambda_i)$ (resp. $W^s(\Lambda_j)$) denotes the unstable (resp. stable) manifold of Λ_* .^a

^aSee Appendix A for the definition of the stable/unstable manifolds $W^{s/u}(\Lambda)$.

The second assumption should be understood as a transversality property between stable and unstable manifolds. This hypothesis is crucial in our analysis as it was already the case in the works of Smale [38]. Note that these two submanifolds may in fact be equal at some points. For instance, if we consider the gradient flow associated with the height function on \mathbb{S}^2 endowed with the canonical induced metric, any points outside the poles \mathbf{n} and \mathbf{s} belong to $W^u(\mathbf{s})$ and $W^s(\mathbf{n})$ which are both two-dimensional.

As was already mentioned, these flows generalize the so-called Morse–Smale gradient

flows, and they are the simplest examples of Axiom A flows in the sense of Smale [39, p. 803]. Before giving some remarkable properties of these flows, let us start with the following observation (which follows only from the first part of the definition):

Lemma 3.3. *For every x in M , there exists a unique pair $(i, j) \in \{1, \dots, K\}^2$ such that $x \in W^u(\Lambda_i) \cap W^s(\Lambda_j)$.*

In particular, we have:

Lemma 3.4. *The unstable manifolds $(W^u(\Lambda_j))_{j=1, \dots, K}$ form a partition of M , i.e.,*

$$M = \bigcup_{j=1}^K W^u(\Lambda_j), \quad \text{and} \quad \forall i \neq j, \quad W^u(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset.$$

The same property holds for stable manifolds.

Proof. Fix some element x in M . We aim at proving that there exists a unique i such that $x \in W^u(\Lambda_i)$. By reversing the time, we would get the same conclusion for stable manifolds. Note that the hyperbolicity assumption ensures that the sets $(\Lambda_j)_j$ are disjoint. Hence, we can assume that $x \notin \bigcup_{j=1}^K \Lambda_j$. Set J to be the subset of $\{1, \dots, K\}$ such that $\forall j \in J, x \in W^u(\Lambda_j)$ and $\forall j \notin J, x \notin W^u(\Lambda_j)$. Assume that J contains more than two elements. Then, we pick a small enough open neighborhood O of $\bigcup_{j \in J} \Lambda_j$ having at least two connected components. By continuity of $\varphi^{-t}(x)$ in t , we can extract a sequence $(t_n)_n$ of times tending to $+\infty$ such that $\varphi^{-t_n}(x) \in M \setminus \tilde{O}$ ($\forall n$) where $\tilde{O} \supset O$ is some open neighborhood of the compact set $NW(\varphi^t)$. Therefore, by extracting again, we get a sequence which converges to some point $y \in M$ belonging to $NW(\varphi^t)$ as a limit point of $\varphi^{-t_n}(x)$. On the other hand, $y \notin \tilde{O}$ by construction which yields the contradiction. Hence, $|J| = 1$. □

3.2. Classical results on Morse–Smale flows

We shall now collect some other useful facts on Morse–Smale flows following the seminal article of Smale [38]. As the proofs are not very long and as they are instructive for the arguments of the upcoming sections, we briefly recall how to prove most of them in Appendix B.

3.2.1. First properties. We start with the following direct consequence from the definition – see Appendix B for details.

Lemma 3.5. *Let x be an element in $W^u(\Lambda_i) \cap W^s(\Lambda_j)$. Then, one has*

$$\dim W^u(\Lambda_i) \geq \dim W^u(\Lambda_j).$$

Moreover, if $x \notin \Lambda_j$, equality can occur only if Λ_j is a closed orbit.

It roughly means that the dimension of the unstable manifolds must decrease along the limit sets of the flow. We continue our description of the properties of Morse–Smale flow with the no-cycle property.

Lemma 3.6 (No-cycle). *If x belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$, then $x \in \Lambda_j$.*

Note that, in his original article [38], Smale took this property as one of the axioms satisfied by his flows. Yet, in [39], this assumption was removed as it can be deduced from the other axioms using the so-called λ -lemma (or inclination lemma) recalled in the Appendix B.

3.2.2. Ordering unstable manifolds. Let us now turn to the main feature of these flows which was proved by Smale in [38] – see also [39, p. 752]:

Theorem 3.7 (Smale). *Suppose that φ^t is a Morse–Smale flow. Then, for every $1 \leq j \leq K$, the closure of $W^u(\Lambda_j)$ is the union of certain $W^u(\Lambda_{j'})$. Moreover, if we define a relation \leq as follows: $W^u(\Lambda_{j'}) \leq W^u(\Lambda_j)$ whenever $W^u(\Lambda_{j'})$ is contained in the closure of $W^u(\Lambda_j)$, then \leq is a partial order on the set $W^u(\Lambda_j)_{j=1}^K$. Finally, if $W^u(\Lambda_{j'}) \leq W^u(\Lambda_j)$, then $\dim W^u(\Lambda_{j'}) \leq \dim W^u(\Lambda_j)$.*

We shall also denote $W^u(\Lambda_{j'}) < W^u(\Lambda_j)$ if $W^u(\Lambda_{j'}) \leq W^u(\Lambda_j)$ and $j' \neq j$. We refer to Appendix B for a reminder on the proof of this result. This partial order relation was crucial for Smale to construct a filtration of the manifold in order to prove his Morse inequalities [38] and it is related to the concept of topological stratification. A stratum is less than a bigger stratum if it lies in the closure of the bigger stratum. Following [39, p. 753] – see also [33, Chapter 4], we can set

Definition 3.8 (Smale quiver). The partial order relation on the collection of subsets $W^u(\Lambda_j)_{j=1}^K$ defined above is called *Smale causality relation*. We define an oriented graph D , called *Smale quiver*, whose K vertices are given by $W^u(\Lambda_j)_{j=1}^K$. Two vertices $W^u(\Lambda_j), W^u(\Lambda_i)$ are connected by an oriented edge starting at $W^u(\Lambda_j)$ and ending at $W^u(\Lambda_i)$ iff $W^u(\Lambda_j) > W^u(\Lambda_i)$.

The set $(W^u(\Lambda_j)_{j=1}^K, \leq)$ being partially ordered by Smale’s theorem, the oriented graph D is sometimes referred to as the Hasse diagram of this partially ordered set.

3.3. C^1 Linearizable flows

In order to construct anisotropic Sobolev spaces adapted to a Morse–Smale flow, we will need to make some extra assumption that will roughly say that the flow is linearizable in a C^1 chart near every critical element Λ_j . Such an assumption may sound quite restrictive. Yet, thanks to Sternberg–Chen’s theorem (see Appendix A.5 for a brief reminder), it is automatically satisfied as soon as certain (generic) nonresonance assumptions on the Lyapunov exponents are satisfied.

For a hyperbolic fixed point Λ , we say that φ^t is *C^1 -linearizable near Λ* if there exists a C^1 diffeomorphism $h : B_n(0, r) \rightarrow O$ (where O is a small open neighborhood of Λ and $B_n(0, r)$ is a small ball of radius r centered at 0 in \mathbb{R}^n) s.t. $h(0) = \Lambda$ and $h^*(V)(x) = D_x h(L(x))$ where $L(x)$ is a vector field on $B_n(0, r)$ defined as

$$L(x) := Ax \cdot \partial_x,$$

for some $A \in M_n(\mathbb{R})$.

For a hyperbolic closed orbit Λ of minimal period \mathcal{P}_Λ , we say that φ^t is \mathcal{C}^1 -linearizable near Λ if there exists a \mathcal{C}^1 diffeomorphism $h : (z, \theta) \in B_{n-1}(0, r) \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}) \rightarrow \mathcal{O}$ (where \mathcal{O} is a small open neighborhood of Λ and $r > 0$ is small) and a \mathcal{C}^1 map $A : \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z} \rightarrow M_{n-1}(\mathbb{R})$ such that

$$h(\{0\} \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})) = \Lambda,$$

and such that $h^*(V) = d_{z,\theta}h(L(z, \theta))$ where $L(z, \theta)$ is a vector field on $B_{n-1}(0, r) \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$ defined as

$$L(z, \theta) = A(\theta)z \cdot \partial_z + \partial_\theta.$$

Finally, we say that a Morse–Smale flow is \mathcal{C}^1 -linearizable if it is \mathcal{C}^1 -linearizable near each critical element $(\Lambda_j)_{j=1,\dots,K}$. More generally, for any $k \geq 1$, we will say that it is \mathcal{C}^k -linearizable if the ‘linearizing’ diffeomorphism can be chosen of class \mathcal{C}^k for every critical element.

3.4. Existence of an energy function

To conclude our short review on Morse–Smale flows, we mention the following result due to Meyer [30, p. 1034] that will be central in our construction of anisotropic Sobolev spaces:

Theorem 3.9 (Meyer). *Let V be a vector field generating a \mathcal{C}^∞ Morse–Smale flow. Then, there exists a smooth function $E : M \rightarrow \mathbb{R}$ such that*

$$\mathcal{L}_V E \geq 0 \quad \text{on } M, \quad \text{and} \quad \mathcal{L}_V E > 0 \quad \text{on } M \setminus \left(\bigcup_{i=1}^K \Lambda_i \right).$$

Moreover, E is constant on every connected component of $\bigcup_{j=1}^K \Lambda_j$ and one can choose E such that $E|_{\Lambda_j} > E|_{\Lambda_i}$ whenever $W^u(\Lambda_j) < W^u(\Lambda_i)$.

In the articles of Meyer [30] and Smale [39], such a function is called an energy function for the flow. It is also quite common to call E a Lyapunov function for the flow.

4. Lifted dynamics on the cotangent space

The results of Smale described the structure of the closure of unstable manifolds for Morse–Smale flows. In this article, we aim at applying microlocal methods for the study of such flows following for instance the strategy developed in [16]. For that purpose, we need to understand more things about these flows, more specifically we need to describe the properties of their symplectic lift to T^*M in order to be able to construct the anisotropic Sobolev spaces.

Following the strategy initiated in [8] for Morse–Smale gradient flows, this includes describing the closure of the conormals to unstable manifolds. Before entering this delicate issue, we recall how to define this symplectic lift and we collect in this section a few tools from Floquet theory that will be used in our proofs. Our main emphasis here is on giving explicit coordinate representation of the flow near critical elements.

4.1. Hamiltonian lift

A flow on M can be lifted to the cotangent space T^*M as follows. In this paragraph, we shall denote by $(x; \xi)$ the elements of T^*M where $x \in M$, $\xi \in T_x^*M$. We associate to the vector field V a Hamiltonian function,

$$\forall (x; \xi) \in T^*M, \quad H(x; \xi) := \xi(V(x)).$$

This Hamiltonian function also induces a Hamiltonian flow that we denote by $\Phi^t : T^*M \rightarrow T^*M$ and whose vector field will be denoted by X_H . We note that, by construction,

$$\Phi^t(x; \xi) := \left(\varphi^t(x); \left(d\varphi^t(x)^T \right)^{-1} \xi \right), \quad (5)$$

and that *this flow induces a diffeomorphism between $T^*M \setminus 0$ and $T^*M \setminus 0$* . Observe that we denoted by φ^{t*} the action of the flow by pull-back on smooth differential form in the introduction. This lifted flow also induces a smooth flow on the unit cotangent bundle S^*M , i.e.,

$$\forall t \in \mathbb{R}, \quad \forall (x; \xi) \in S^*M, \quad \tilde{\Phi}^t(x; \xi) = \left(\varphi^t(x); \frac{(d\varphi^t(x)^T)^{-1} \xi}{\| (d\varphi^t(x)^T)^{-1} \xi \|_x} \right).$$

Here, $\|\cdot\|_x$ represents the metric induced on T_x^*M by g_x . Note that this is an auxiliary datum in our analysis: introducing it allows us to work with compact subsets of T^*M rather than with conical subsets. We denote by \tilde{X}_H the induced smooth vector field on S^*M .

4.2. Writing the flow in local coordinates near critical points

Let us rewrite the Morse–Smale flow and its Hamiltonian lift near the critical elements of φ^t in some well-chosen local coordinates. For that purpose, we now make the assumption that φ^t is a Morse–Smale flow which is \mathcal{C}^1 -linearizable. Near a *fixed point* Λ , we can choose local \mathcal{C}^1 coordinates $(\tilde{x}, \tilde{y}) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$ such that

$$\varphi^t(\tilde{x}, \tilde{y}) = (e^{-t\Omega_s} \tilde{x}, e^{t\Omega_u} \tilde{y}), \quad (6)$$

with Ω_s (resp. Ω_u) an element in $M_{n_s}(\mathbb{R})$ (resp. $M_{n_u}(\mathbb{R})$) all of whose eigenvalues have positive real part from the hyperbolicity assumption. In particular, there exist some positive constants $0 < C_1 < C_2$ and $0 < \chi_- < \chi_+$ such that, for every $t \geq 0$ and for every (\tilde{x}, \tilde{y}) ,

$$C_1 e^{-t\chi_+} \|(\tilde{x}, \tilde{y})\| \leq \|(e^{-t\Omega_s} \tilde{x}, e^{-t\Omega_u} \tilde{y})\| \leq C_2 e^{-t\chi_-} \|(\tilde{x}, \tilde{y})\| \quad (7)$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n . As the chart is of class \mathcal{C}^1 , by equation (5) we can also write the corresponding Hamiltonian flow (in the induced local coordinates $(\tilde{x}, \tilde{y}; \tilde{\xi}, \tilde{\eta})$ on T^*M) under the form:

$$\Phi^t(\tilde{x}, \tilde{y}; \tilde{\xi}, \tilde{\eta}) = (e^{-t\Omega_s} \tilde{x}, e^{t\Omega_u} \tilde{y}; e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta}). \quad (8)$$

4.3. ODE with periodic coefficients

We now turn to the case where Λ is a *closed orbit* of minimal period $\mathcal{P}_\Lambda > 0$. For that purpose, we shall first recall a few facts from Floquet theory [41, p. 91]. For every $\theta_0 \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}$, we consider the matrix valued ordinary differential equation:

$$\frac{dU(\theta, \theta_0)}{d\theta} = A(\theta)U(\theta, \theta_0), \quad U(\theta_0, \theta_0) = \text{Id}, \tag{9}$$

where $A(\theta)$ is given by the \mathcal{C}^1 -linearization hypothesis near Λ . The solution to this ODE satisfies the periodicity condition $U(\theta + \mathcal{P}_\Lambda, \theta_0 + \mathcal{P}_\Lambda) = U(\theta, \theta_0)$. The matrix $M(\theta_0) = U(\theta_0 + \mathcal{P}_\Lambda, \theta_0)$ is called the *monodromy matrix* of the system. Recall that $M(\theta_1) = U(\theta_1, \theta_0)M(\theta_0)U(\theta_1, \theta_0)^{-1}$. In particular, its eigenvalues are independent of θ_0 . We denote by $(\lambda_1, \dots, \lambda_n)$ the eigenvalues of $M = M(0)$. We write these eigenvalues under an exponential form, i.e., $\lambda_j := e^{\mathcal{P}_\Lambda \mu_j(\Lambda)}$. The $\mu_j(\Lambda)$ are called the *Floquet exponents of the closed orbit* $\{0\} \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$ while their real parts are the so-called *Lyapunov exponents*.

4.4. Results from Floquet theory

According to [41, Lemma 3.34, Corollary 3.16], the following Proposition gives one of the main results from Floquet theory concerning the reduction of the solutions to the ODE with periodic coefficients (9):

Proposition 4.1. *With the conventions of §4.3, the squared matrix $M(\theta_0)^2$ can be put under the real exponential form:*

$$M(\theta_0)^2 = e^{2\mathcal{P}_\Lambda \Omega(\theta_0)}$$

with $\Omega(\theta_0)$ real valued. Moreover, there exists a real valued matrix $P(\theta_0 + \theta, \theta_0)$ which is $2\mathcal{P}_\Lambda$ -periodic in θ and such that, for every θ in $[0, 2\mathcal{P}_\Lambda]$,

$$U(\theta, \theta_0) = P(\theta, \theta_0)e^{(\theta - \theta_0)\Omega(\theta_0)}. \tag{10}$$

Note that we have to take the square of $M(\theta_0)$ in order to take into account the fact that $M(\theta_0)$ may have negative eigenvalues. We next use this Proposition to find some explicit coordinate representation of the flow near periodic orbits.

4.5. Writing the flow in local coordinates near closed orbits

By the assumption that the periodic orbit Λ is hyperbolic and using the \mathcal{C}^1 -linearization assumption, we can fix from Proposition 4.1 a system of \mathcal{C}^1 local coordinates $(z, \theta) \in \mathbb{R}^{n-1} \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$ such that

$$\varphi^t(z, \theta) = \left(P(\theta + t, 0)e^{t\Omega_\Lambda} P(\theta, 0)^{-1}z, \theta + t \bmod (\mathcal{P}_\Lambda) \right), \tag{11}$$

where $t \mapsto P(\theta + t, 0)$ is $2\mathcal{P}_\Lambda$ -periodic and where the eigenvalues of the matrix $\Omega_\Lambda = \Omega(0)$ have a nonzero real part. Equivalently, one has

$$\varphi^t(z, \theta) = (U(t + \theta, \theta)z, t + \theta \bmod (\mathcal{P}_\Lambda)), \tag{12}$$

where

$$U(t + \theta, \theta) = P(\theta + t, \theta)e^{t\Omega(\theta)} = P(\theta + t, 0)e^{t\Omega(0)}P(\theta, 0)^{-1} \tag{13}$$

is the fundamental solution for the Floquet problem associated with $A(t)$.

Furthermore, we can split $\mathbb{R}^{n-1} = E_{ss} \oplus E_{uu}$ where E_{uu} (resp. E_{ss}) corresponds to the eigenvalues of $\Omega_\Lambda = \Omega(0)$ with positive (resp. negative) real part. This allows to rewrite the matrix Ω_Λ under a block diagonal form, i.e., $\Omega_\Lambda = \text{diag}(-\Omega_s, \Omega_u) \in M_{n_s+n_u}(\mathbb{R})$ with $\Omega_{u/s}$ of size $n_{u/s}$ and having eigenvalues with positive real parts. We set $z = (\tilde{x}, \tilde{y})$ for the system of coordinates adapted to the above decomposition and $(\tilde{\xi}, \tilde{\eta})$ the corresponding dual coordinates. In particular, there are some positive constants $0 < C_1 < C_2$ and $0 < \chi_- < \chi_+$ such that, for every $t \geq 0$,

$$C_1 e^{-t\chi_+} \|(\tilde{x}, \tilde{y})\| \leq \|(e^{-t\Omega_s} \tilde{x}, e^{-t\Omega_u} \tilde{y})\| \leq C_2 e^{-t\chi_-} \|(\tilde{x}, \tilde{y})\|. \tag{14}$$

In the next paragraph, we use the above Floquet coordinates so that, for each $\theta \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}$, \mathbb{R}^{n-1} decomposes as a sum of stable and unstable subspaces of $M(\theta)$ which gives some coordinates representation of the stable and unstable manifolds near the periodic orbits.

4.6. Coordinate representation of stable and unstable manifolds near periodic orbits

From this expression, one can verify that the unstable and stable manifolds are given in local coordinates near the closed orbit Λ , by

$$W^u(\Lambda) := \{(P(\theta, 0)(0, \tilde{y}), \theta) : \theta \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}, \tilde{y} \in \mathbb{R}^{n_u}\} \tag{15}$$

and

$$W^s(\Lambda) := \{(P(\theta, 0)(\tilde{x}, 0), \theta) : \theta \in \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}, \tilde{x} \in \mathbb{R}^{n_s}\}. \tag{16}$$

Similarly, we can verify that the strong unstable and stable manifolds at a given point $(0, 0, \theta)$ of the closed orbit Λ are

$$W^{uu}(0, 0, \theta) := \{(P(\theta, 0)(0, \tilde{y}), \theta) : \tilde{y} \in \mathbb{R}^{n_u}\} \tag{17}$$

and

$$W^{ss}(0, 0, \theta) := \{(P(\theta, 0)(\tilde{x}, 0), \theta) : \tilde{x} \in \mathbb{R}^{n_s}\}. \tag{18}$$

We next lift the Floquet representation to T^*M .

4.7. The Hamiltonian flow near closed orbits

We used Floquet theory to write the flow in nice coordinates near closed orbits. The purpose of the present paragraph is to lift the coordinate representation to cotangent space. Following the notations of the previous paragraph, we consider the system of coordinates $(z, \theta; \zeta, \Theta)$ in $T^*(\mathbb{R}^{n-1} \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z}))$. From equation (5), we need to compute the element $(d\varphi^t(x)^T)^{-1}(\zeta, \theta)$ in our coordinate system. A straightforward calculation in the Floquet coordinates (z, θ) yields

$$d\varphi^t(z, \theta) = \begin{pmatrix} U(t + \theta, \theta) & \partial_\theta U(t + \theta, \theta)z \\ 0 & 1 \end{pmatrix} \implies d\varphi^t(z, \theta)^T = \begin{pmatrix} U(t + \theta, \theta)^T & 0 \\ (\partial_\theta U(t + \theta, \theta)z)^T & 1 \end{pmatrix}$$

where we wrote the $((n - 1) + 1) \times ((n - 1) + 1)$ block decomposition of the differential $d\varphi^t$. This tells us that

$$((d\varphi^t)^T(z, \theta))^{-1} = \begin{pmatrix} (U(\theta + t, \theta)^{-1})^T & 0 \\ -(U(\theta + t, \theta)^{-1}\partial_\theta(U(\theta + t, \theta)z))^T & 1 \end{pmatrix}.$$

Fix now a point $(z, \theta; \zeta, \Theta) = (P(\theta, 0)(\tilde{x}, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta)$ in $T^*(\mathbb{R}^{n-1} \times \mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$. Applying the previous formula and (13) to calculate $(d\varphi^t)^T(z, \theta)^{-1}(\zeta, \Theta)$, we find that:

$$\begin{aligned} & \begin{pmatrix} (U(\theta + t, \theta)^{-1})^T & 0 \\ -(U(\theta + t, \theta)^{-1}\partial_\theta(U(\theta + t, \theta)z))^T & 1 \end{pmatrix}_{(z=P(\theta,0)(\tilde{x},\tilde{y}),\theta)} \quad ((P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta) \\ &= \begin{pmatrix} ((P(\theta + t, 0)^{-1})^T(e^{-t\Omega_\Lambda})^T P(\theta, 0)^T & 0 \\ -(\partial_\theta(U(\theta + t, \theta)z))^T (U(\theta + t, \theta)^{-1})^T & 1 \end{pmatrix}_{(z=P(\theta,0)(\tilde{x},\tilde{y}),\theta)} \quad ((P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta). \end{aligned}$$

Using the block decomposition of $\Omega_\Lambda = \begin{pmatrix} -\Omega_s & 0 \\ 0 & \Omega_u \end{pmatrix}$ with equation (13), this yields:

$$\begin{aligned} & \left((d_{P(\theta,0)(\tilde{x},\tilde{y}),\theta}\varphi^t)^T \right)^{-1} \left((P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta \right) \\ &= \left((P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T}\tilde{\xi}, e^{-t\Omega_u^T}\tilde{\eta}), \Theta + R(t, \tilde{x}, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta}) \right), \end{aligned} \tag{19}$$

where

$$R(t, \tilde{x}, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta}) = - \left\langle \partial_\theta(U(\theta + t, \theta))P(\theta, 0)(\tilde{x}, \tilde{y}), (P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T}\tilde{\xi}, e^{-t\Omega_u^T}\tilde{\eta}) \right\rangle \tag{20}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural duality pairing between \mathbb{R}^{n-1} and \mathbb{R}^{n-1*} . We can now write the corresponding Hamiltonian flow:

$$\Phi^t(z, \theta; \zeta, \Theta) = \left(\varphi^t(z, \theta); (U(t + \theta, \theta)^T)^{-1}\zeta, \Theta + R(t, z, \theta, \zeta) \right), \tag{21}$$

with $R(t, z; \theta, \zeta)$ which is defined above. Equivalently, one has

$$\Phi^t(z, \theta; \zeta, \Theta) = \left(\varphi^t(z, \theta); (P(\theta + t, 0)^T)^{-1}e^{-t\Omega_\Lambda^T}P(\theta, 0)^T(\zeta), \Theta + R(t, z, \theta, \zeta) \right), \tag{22}$$

where R can be split as follows:

$$\begin{aligned} R(t, \tilde{x}, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta}) &= - \left\langle \partial_\theta P(\theta + t, 0)(e^{-t\Omega_s}\tilde{x}, e^{t\Omega_u}\tilde{y}), (P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T}\tilde{\xi}, e^{-t\Omega_u^T}\tilde{\eta}) \right\rangle \\ &\quad + \left\langle \partial_\theta P(\theta, 0)(\tilde{x}, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}) \right\rangle. \end{aligned}$$

5. Conormals of stable and unstable manifolds

Smale’s theorem 3.7 describes the closure of any unstable manifold $W^u(\Lambda)$ of a Morse–Smale flow. In particular, this theorem shows that each closure $\overline{W^u(\Lambda)}$ has a stratified structure whose strata are given by unstable manifolds. In the following,

we need to understand the fine properties of the conormal $N^*(W^u(\Lambda))$, in particular we need to describe precisely its closure $\overline{N^*(W^u(\Lambda))}$ inside T^*M . Analyzing the conormals of unstable manifolds is the core of this article and this is the content of this section. The main results of this section are Theorems 5.2 and 5.4. Note that these results were already obtained in the case of Morse–Smale gradient flows in [8]. The main novelty here is that we allow closed orbits in our analysis.

5.1. Conormals of the unstable/stable manifolds

Given a smooth submanifold S inside M , one can define its *conormal* (bundle) as follows:

$$N^*S := \{(x; \xi) \in T^*M : x \in S, \xi \neq 0, \text{ and } \forall v \in T_x S, \xi(v) = 0\}.$$

Observe that, for a submanifold of dimension n , the conormal is empty. For instance, this is the case for the unstable manifold of an expanding fixed point. Here, the relevant sets for the lifted dynamics will be

$$\Sigma_{uu} := \bigcup_{j=1}^K N^*(W^s(\Lambda_j)) \cap S^*M \quad \text{and} \quad \Sigma_{ss} := \bigcup_{j=1}^K N^*(W^u(\Lambda_j)) \cap S^*M.$$

In the forthcoming Lemma 5.1, we will see that the set $\bigcup_{j=1}^K N^*(W^u(\Lambda_j)) \cap S^*M$ is in fact an attractor for the lifted dynamics $(\tilde{\Phi}^t)_t$, this is why it is denoted by Σ_{ss} to emphasize it is the stable set for the flow $(\tilde{\Phi}^t)_t$ in S^*M . From Appendix A, we also know that the unstable (resp. stable) manifolds are invariantly fibered by smooth submanifolds, i.e.,

$$\forall 1 \leq j \leq K, \quad W^u(\Lambda_j) := \bigcup_{x \in \Lambda_j} W^{uu}(x), \quad \text{and} \quad W^s(\Lambda_j) := \bigcup_{x \in \Lambda_j} W^{ss}(x).$$

We then define

$$\Sigma_u := \bigcup_{j=1}^K \bigcup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M \quad \text{and} \quad \Sigma_s := \bigcup_{j=1}^K \bigcup_{x \in \Lambda_j} N^*(W^{uu}(x)) \cap S^*M.$$

Note that, in the case of a fixed point $\Lambda_j = \{x\}$, $W^{ss/uu}(x)$ coincides with $W^{s/u}(\Lambda_j)$. Finally, we can translate the Smale transversality assumption from Definition 3.2 in this microlocal setting:

$$\Sigma_{uu} \cap \Sigma_{ss} = \Sigma_{uu} \cap \Sigma_s = \Sigma_{ss} \cap \Sigma_u = \emptyset. \quad (23)$$

5.2. Coordinate representation of $N^*(W^u(\Lambda_j))$ near Λ_j

We previously defined the set Σ_{ss} as the union of conormals of unstable manifolds intersected with S^*M . In order to have a concrete representation of these sets, let us write the conormal bundle $N^*(W^u(\Lambda_j))$ in the local coordinates we have introduced in §§ 4.2, 4.4 and 4.5. Note already that this does not represent the whole set Σ_{ss} near Λ_j as one can also find points inside $N^*(W^u(\Lambda_i))$ with $i \neq j$ arbitrarily close to Λ_j . Yet, we will only need to use the exact representation of $N^*(W^u(\Lambda_j))$ near Λ_j in our proofs.

In the (\tilde{x}, \tilde{y}) coordinates near a fixed point Λ_j , using the coordinate system from § 4.2, the set $N^*(W^u(\Lambda_j))$ can be represented as:

$$\left\{ (0, \tilde{y}; \tilde{\xi}, 0) : \tilde{y} \in \mathbb{R}^{n_u}, \tilde{\xi} \in \mathbb{R}^{n_s} \setminus \{0\} \right\}. \tag{24}$$

Near a closed orbit Λ_j , using the coordinate system from § 4.5, $N^*(W^u(\Lambda_j))$ can be written as:

$$\left\{ \left(P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0), -\langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle \right) : (*) \right\}, \tag{25}$$

where $(*)$ means $\tilde{y} \in \mathbb{R}^{n_u}$, $\theta \in \mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z}$ and $\tilde{\xi} \in \mathbb{R}^{n_s} \setminus \{0\}$. To see this, recall that the weak unstable manifold reads:

$$W^u(\Lambda_j) = \{ (P(\theta, 0)(0, \tilde{y}), \theta) : \tilde{y} \in \mathbb{R}^{n_u}, \theta \in \mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z} \}.$$

Hence tangent vectors to $W^u(\Lambda_j)$ are generated by vectors of the form $(P(\theta, 0)(0, Y), 0)$ and $(\partial_\theta P(\theta, 0)(0, \tilde{y}), 1)$ for some $Y \in \mathbb{R}^{n_u}$. So $(P(\theta, 0)(0, \tilde{y}), \theta; \zeta, \Theta) \in N^*(W^u(\Lambda_j))$ iff, for every $Y \in \mathbb{R}^{n_u}$,

$$\begin{aligned} \langle \partial_\theta P(\theta, 0)(0, \tilde{y}), \zeta \rangle + \Theta &= 0 \\ \langle P(\theta, 0)(0, Y), \zeta \rangle &= 0. \end{aligned}$$

So $\zeta = \left((P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \right)$ by the second equation and the first equation implies Θ satisfies $\langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle + \Theta = 0$.

5.3. Coordinate representation of $\bigcup_{x \in \Lambda_j} N^*(W^{uu}(x))$ near Λ_j

It is also useful to represent these objects in local coordinates. In the case of a fixed point, the expression is the same as in § 5.2. Yet, in the case of a closed orbit, there is a small difference. Namely, $\bigcup_\theta N^*(W^{uu}(0, 0, \theta))$ can be represented near Λ_j as:

$$\left\{ \left(P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0), \Theta \right) : \tilde{y} \in \mathbb{R}^{n_u}, \theta \in \mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z}, (\tilde{\xi}, \Theta) \in \mathbb{R}^{n_s+1} \setminus \{0\} \right\}.$$

Again, it only represents the contribution of Λ as there may be points associated to the unstable manifold of Λ_i (with $i \neq j$) which are arbitrarily close to Λ_j . Finally, observe that one has $\Sigma_{uu} \subset \Sigma_u$ and $\Sigma_{ss} \subset \Sigma_s$.

5.4. The sets Σ_u, Σ_{uu} as attractors for the backward flow on the unit cotangent bundle

Let us start our description of the dynamics in the cotangent bundle with the following lemma:

Lemma 5.1. *Let φ^t be a Morse–Smale flow which is \mathcal{C}^1 -linearizable. Then, one has:*

- (1) $\forall (x; \xi) \in S^*M \setminus \Sigma_{ss}, \lim_{t \rightarrow -\infty} d_{S^*M}(\tilde{\Phi}^t(x; \xi), \Sigma_u) = 0;$
- (2) $\forall (x; \xi) \in S^*M \setminus \Sigma_s, \lim_{t \rightarrow -\infty} d_{S^*M}(\tilde{\Phi}^t(x; \xi), \Sigma_{uu}) = 0.$

By reversing the time, one can verify that the same properties hold if we intertwine the roles of s and u . We shall divide the proof of this lemma in two parts and make use of the coordinate representation of §§ 5.2 and 5.3.

5.4.1. Proof of part 1 of Lemma 5.1. Let ρ be an element in $S^*M \setminus \Sigma_{ss}$. According to Lemma 3.3, there exists a unique $1 \leq i \leq K$ such that the projection of ρ on M belongs to $W^u(\Lambda_i)$. Up to applying the flow in backward times, we can suppose that $\rho \in S^*M \setminus \Sigma_{ss}$ belongs to the linearizing chart near Λ_i which was defined in §4. Let us start with the case where Λ_i is a fixed point of the flow. Using our local coordinates, we can then write $\rho = (0, \tilde{y}; \tilde{\xi}, \tilde{\eta})$ with $\tilde{\eta} \neq 0$ as $\rho \notin N^*(W^u(\Lambda_i))$ by definition. Recall now that the flow Φ^t reads in these local coordinates:

$$\Phi^t(0, \tilde{y}; \tilde{\xi}, \tilde{\eta}) = (0, e^{t\Omega_u} \tilde{y}; e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta}),$$

where all the eigenvalues of $\Omega^{u/s}$ have positive real parts. In order to conclude, we need to normalize the cotangent vector, i.e., consider $(e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta}) / \|(e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta})\|$. All the norms on \mathbb{R}^n being equivalent and by the inequality (7) and the fact that $\tilde{\eta} \neq 0$, we find that the $\tilde{\xi}$ component of this normalized covector tends to 0 as $t \rightarrow -\infty$. In particular, this implies that any accumulation point of $\Phi^t(\rho)$ (as $t \rightarrow -\infty$) is of the form $(0, 0; 0, \tilde{\eta}') \neq 0$. This implies that any accumulation point of $\Phi^t(\rho)$ belongs to $N^*(W^s(\Lambda_i)) \subset \Sigma_u$ by the results from §5.2.

Consider now the case where Λ_i is a closed orbit. Again, up to applying the flow in backward times, we may suppose that ρ belongs to the neighborhood of Λ_i where we have our linearizing coordinates. In this chart, ρ can be represented as

$$\rho = (P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta),$$

with either $\tilde{\eta} \neq 0$, or $\tilde{\eta} = 0$ and $\Theta + \langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle \neq 0$ as ρ does not belong to Σ_{ss} – see equation (25).

We begin with the case $\tilde{\eta} \neq 0$. What we have to understand is the asymptotic behavior of

$$\left((d_{P(\theta, 0)(0, \tilde{y}), \theta} \varphi^t)^T \right)^{-1} \left((P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta \right)$$

as $t \rightarrow -\infty$. This quantity corresponds to the evolution of the cotangent component of ρ under the Hamiltonian flow Φ^t . Recall that an explicit expression for this quantity was given in equation (19) which we reproduce here in the case where $\tilde{x} = 0$:

$$\begin{aligned} & \left((d_{P(\theta, 0)(0, \tilde{y}), \theta} \varphi^t)^T \right)^{-1} \left((P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta \right) \\ &= \left((P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta}), \Theta + R(t, 0, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta}) \right). \end{aligned}$$

Combined with the explicit expression of the remainder R given in (22), one finds that there exist some $C, \chi_0 > 0$ such that

$$\forall t \leq 0, \quad |R(t, 0, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta})| \leq C \left(1 + e^{t\chi_0} \|\tilde{y}\| \|e^{-t\Omega_u^T} \tilde{\eta}\| \right).$$

This expression tells us that the cotangent component is in fact of the form

$$\left((P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T} \tilde{\xi}, e^{-t\Omega_u^T} \tilde{\eta}), \mathcal{O}(1) + o(1) \|e^{-t\Omega_u^T} \tilde{\eta}\| \right), \quad \text{as } t \rightarrow -\infty.$$

According to (14), one knows that $\|e^{-t\Omega_u^T} \tilde{\eta}\|$ is exponentially large as $t \rightarrow -\infty$ since $\tilde{\eta} \neq 0$ by assumption. From this explicit expression and from the fact that all the matrices have

eigenvalues with positive real parts, we find that the leading contribution comes from the term $(P(\theta + t, 0)^T)^{-1}(0, e^{-t\Omega_u^T} \tilde{\eta})$ as t tends to $-\infty$. Hence, the normalized cotangent component of $\tilde{\Phi}^t(\rho)$ will approach the set $\bigcup_{\theta} \{((P(\theta, 0)^T)^{-1}(0, \tilde{\eta}), 0) : \tilde{\eta} \neq 0\}$ as t tends to $-\infty$. It implies that any accumulation point of $\tilde{\Phi}^t(\rho)$ (as $t \rightarrow -\infty$) is a point inside $\bigcup_{x \in \Lambda_i} N^*(W^{ss}(x)) \cap S^*M \subset \Sigma_u$ by § 5.3.

In the case $\tilde{\eta} = 0$, we must have $\Theta \neq -\langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle$ and the cotangent component of $\Phi^t(\rho)$ can be expressed in local coordinates as follows, as $t \rightarrow -\infty$:

$$\left((P(\theta + t, 0)^T)^{-1}(e^{t\Omega_s^T} \tilde{\xi}, 0), \Theta + \langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle + o(1)\|e^{t\Omega_s^T} \tilde{\xi}\| \right),$$

where we used the expression given in (22) one more time. Since all the eigenvalues of Ω_s^T have eigenvalues with positive real parts, we find that any accumulation point (as $t \rightarrow -\infty$) of the cotangent component will be of the form $(0, \Theta') \neq 0$. Again, if we consider the normalized version, the limit vector will be $(0, 1)$. Using one more time the coordinate representation of Σ_u from § 5.3, we can conclude that $\tilde{\Phi}^t(\rho)$ tends to Σ_u as $t \rightarrow -\infty$.

5.4.2. Proof of part 2 of Lemma 5.1. In order to prove the second part of the lemma, we follow the same strategy. We fix a point $\rho \in S^*M \setminus \Sigma_s$ whose projection on M belongs to some unstable manifold $W^u(\Lambda_i)$. Up to applying the flow in backward times, we can one more time suppose that ρ belongs to the linearizing chart near Λ_i . In the case where Λ_i is a fixed point, the proof is basically the same. In the case of a closed orbit, we now write ρ as

$$\rho = (P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta),$$

with $\tilde{\eta} \neq 0$ and not only $(\tilde{\eta}, \Theta + \langle \partial_\theta P(\theta, 0)(0, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0) \rangle) \neq 0$. Hence, the same argument as above allows to conclude one more time.

5.5. Main results on compactness and stable neighborhoods

We now state the main results on the geometric and topological properties of the sets we have just defined. The proofs will be given in the next two sections. The first theorem deals with the compactness of the sets Σ_* previously defined:

Theorem 5.2 (Compactness theorem). *Let φ^t be a Morse–Smale flow which is C^1 -linearizable. Then $\Sigma_u, \Sigma_{uu}, \Sigma_s$ and Σ_{ss} are compact subsets of S^*M .*

The proof of this result was already given in [8, Lemma 3.6] in the particular case of Morse–Smale gradient flows satisfying a certain (generic) linearization property. We shall give in § 6 a proof which is valid for *any Morse–Smale flow* (including of course the case of gradient flows) satisfying also certain (generic) linearization property. The property of being C^1 -linearizable seems crucial in our proof as it allows us to control the asymptotic behavior of cotangent vectors under the flow near a critical element Λ_j . In fact, having a C^1 -chart enables us to use the local expressions of §§ 4.2, 4.4 and 4.5 for the Hamiltonian

flow. Using only the hyperbolicity² at Λ_j does not seem to be enough to control the asymptotic behavior of cotangent vectors outside Λ_j in our proof of compactness.

Note that the statement of this theorem is in certain cases trivial. Take for instance the gradient flow associated with the height function on the 2-sphere endowed with its canonical metric. In that case, the flow has two critical points: the north pole \mathbf{n} and the south pole \mathbf{s} . Then, $\Sigma_{uu} = \Sigma_u$ is equal to the set $\{(\mathbf{s}, \xi) \in S^*M : \|\xi\|_{\mathbf{s}} = 1\}$ which is obviously compact. Coming back to the general case, the situation may be subtle as illustrated by the following example on the plane \mathbb{R}^2 .

Example 5.3. In the (x, y) -plane, consider the curve $Y = \{(x, \sin(\frac{1}{x})); x \in \mathbb{R}_{\geq 0}\}$ which is the graph of the function $\sin(\frac{1}{x})$ and the vertical line $X = \{x = 0\}$. Then $X \cup Y$ is closed since $\overline{Y} \setminus Y \subset X$, yet the conormal $N^*X = \{(0, y; \xi, 0); y \in \mathbb{R}, \xi \in \mathbb{R}\}$ is not contained in $\overline{N^*Y}$ since the curve Y will oscillate near $x = 0$, hence conormal covectors to Y will not converge to a fixed codirection $(\xi, 0)$. Therefore $N^*X \cup N^*Y$ is not a closed, conical subset in $T^*\mathbb{R}^2$ although $X \cup Y$ was closed. This shows that taking the union of the conormals of two submanifolds may not give rise to a closed subset even if the union of the two submanifolds is closed itself.

Theorem 5.2 described the topological properties of the subsets Σ_* and we will now turn to more dynamical properties. More precisely, our next result is that the sets Σ_* are attractors or repellers of the flow $\tilde{\Phi}^t$:

Theorem 5.4 (Stable neighborhood theorem). *Let φ^t be a Morse–Smale flow which is C^1 -linearizable. Let $\epsilon > 0$. Then, there exists an open neighborhood V^{ss} (resp. V^s) of Σ_{ss} (resp. Σ_s) inside S^*M all of whose points are at a distance $\leq \epsilon$ of Σ_{ss} (resp. Σ_s) and such that $\forall t \geq 0, \tilde{\Phi}^t(V^{ss}) \subset V^{ss}$, (resp. $\forall t \geq 0, \tilde{\Phi}^t(V^s) \subset V^s$).*

If we replace s by u , the same conclusion holds except that we have to replace positive times by negative ones. This theorem may be thought of as a ‘monotonic’ version of Lemma 5.1. Like for the property of compactness, this result was already proved in [8] in the particular case of Morse–Smale *gradient* flows satisfying certain linearization properties given for instance by the Sternberg–Chen theorem. Again, we prove that the extension to more general Morse–Smale flows is still true.

We shall now devote the next two sections to the proofs of these two theorems.

6. Proof of the compactness Theorem 5.2

It is sufficient to prove that Σ_s and Σ_{ss} are compact. The other cases follow by reversing the time. The proof will proceed by a contradiction argument and it is based on an important technical lemma that we will present in the next paragraph.

²For instance, hyperbolicity allows us to use the Grobman–Hartman theorem but it only provides a C^0 -chart.

6.1. A technical lemma

The following result generalizes to the cotangent framework earlier results of Smale [38]:

Lemma 6.1 (Key technical lemma). *Let $(z_\infty; \zeta_\infty)$ be some element of S^*M such that $(z_\infty; \zeta_\infty) \notin \Sigma_{ss}$ (resp. Σ_s) and $z_\infty \in W^u(\Lambda_j)$ for some elementary critical element Λ_j . Let $(z_m; \zeta_m) \rightarrow (z_\infty; \zeta_\infty)$ be a sequence in S^*M such that, for every $m \geq 0$, $z_m \in W^u(\Lambda_i)$ for some fixed Λ_i . Then, one has:*

- either $i = j$ and $(z_m; \zeta_m)$ does not belong to Σ_{ss} (resp. Σ_s) for m large enough;
- or $i \neq j$ and there exists a convergent subsequence $(z_{\phi(m)}^{(1)}; \zeta_{\phi(m)}^{(1)})_{m \geq 0}$, $\phi : \mathbb{Z}_+ \mapsto \mathbb{Z}_+$ injective, with the following properties:
 - (1) $(z_{\phi(m)}^{(1)}; \zeta_{\phi(m)}^{(1)})$ belongs to the integral curve $\{\tilde{\Phi}^t(z_{\phi(m)}; \zeta_{\phi(m)}) | t \in \mathbb{R}\}$, in particular $z_{\phi(m)}^{(1)} \in W^u(\Lambda_i)$ for every $m \in \mathbb{N}$;
 - (2) $\lim_{m \rightarrow +\infty} (z_{\phi(m)}^{(1)}; \zeta_{\phi(m)}^{(1)}) = (z_\infty^{(1)}; \zeta_\infty^{(1)}) \in \bigcup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M$ (resp. $N^*(W^s(\Lambda_j))$);
 - (3) $z_\infty^{(1)} \in W^s(\Lambda_j) \setminus \Lambda_j$.

In particular, in the second case, we can conclude from (23) that the new limit point does not belong to Σ_{ss} (resp. Σ_s).

Let us first observe that the case $i = j$ is easy to deal with. Indeed, we can fix $T > 0$ large enough to ensure that $\varphi^{-T}(z_\infty)$ belongs to the linearizing chart near Λ_i . By continuity of $\varphi^{-T}(\cdot)$ for fixed $T \in \mathbb{R}$, we know that for $m \geq 1$ large enough, $\varphi^{-T}(z_m)$ also belongs to this chart. Then, the explicit expressions of the conormals in these local coordinates from §§ 5.2 and 5.3 give the conclusion as the limit point $\tilde{\Phi}^{-T}(z_\infty; \zeta_\infty)$ does not belong to Σ_{ss} (resp. Σ_s).

Hence, the main difficulty lies in the case where $i \neq j$ that will be divided in two subcases: (1) Λ_j is a fixed point and (2) Λ_j is a periodic orbit.

6.1.1. Λ_j is a fixed point. Up to applying the flow φ^t in backward times, we can suppose that z_∞ belongs to the linearizing chart near Λ_j . Moreover, by continuity of the flow, we can suppose that for m large enough, z_m also belongs to this linearizing chart. Fix two small enough $\delta_1, \delta > 0$ with $\delta_1 \gg \delta$. As $(z_\infty; \zeta_\infty)$ does not belong to Σ_{ss} (hence $(z_\infty; \zeta_\infty) \notin N^*(W^u(\Lambda_j))$), Lemma 5.1 tells us that $\tilde{\Phi}^t(z_\infty; \zeta_\infty)$ will be attracted by Σ_u when $t \rightarrow -\infty$. We also have $\lim_{t \rightarrow -\infty} \varphi^{-t}(z_\infty) = \Lambda_j$ since $z_\infty \in W^u(\Lambda_j)$. Therefore, without loss of generality, we may apply the flow $\tilde{\Phi}^t$ in backward times to $(z_\infty; \zeta_\infty)$ to ensure that $(z_\infty; \zeta_\infty)$ is at a distance $\leq \delta$ of $N^*(W^s(\Lambda_j)) \cap S^*M$.

Again, by continuity of $\tilde{\Phi}^{-t}(\cdot)$ acting on S^*M , we find that, for m large enough, we may assume that $(z_m; \zeta_m)$ is also at a distance less than 2δ from $N^*(W^s(\Lambda_j)) \cap S^*M$. Write now the expression of these points in local coordinates:

$$(z_m; \zeta_m) = (\tilde{x}_m, \tilde{y}_m; \tilde{\xi}_m, \tilde{\eta}_m) \quad \text{and} \quad (z_\infty; \zeta_\infty) = (0, \tilde{y}_\infty; \tilde{\xi}_\infty, \tilde{\eta}_\infty).$$

Observe that, as $i \neq j$ and $z_m \in W^u(\Lambda_i)$, one necessarily has $\tilde{x}_m \neq 0$ for every m large enough. Moreover, as $(z_m; \zeta_m)$ is within a distance δ from $N^*(W^s(\Lambda_j)) \cap S^*M$, we know that $\|\tilde{\eta}_m\|$ is uniformly bounded from below by a positive constant. Apply now the flow

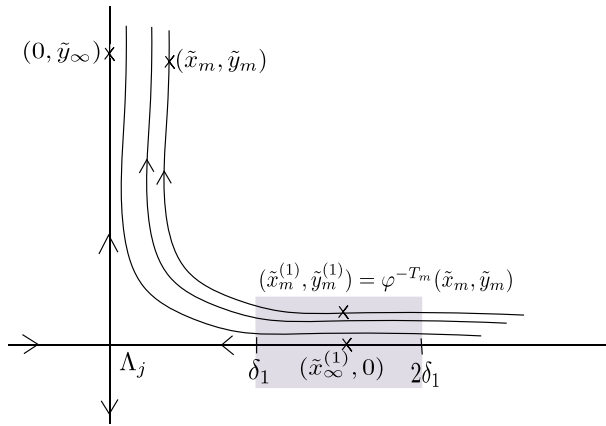


Figure 1. Construction of the new sequence.

$\tilde{\Phi}^{-t}$ for some positive t to the sequence $(z_m; \zeta_m)$. In local coordinates, this reads

$$\tilde{\Phi}^{-t}(z_m; \zeta_m) = \left(e^{t\Omega_s} \tilde{x}_m, e^{-t\Omega_u} \tilde{y}_m; \frac{(e^{-t\Omega_s^T} \tilde{\xi}_m, e^{t\Omega_u^T} \tilde{\eta}_m)}{\|(e^{-t\Omega_s^T} \tilde{\xi}_m, e^{t\Omega_u^T} \tilde{\eta}_m)\|} \right).$$

For all $m \in \mathbb{N}$, we choose some $T_m \in \mathbb{R}$ large enough to ensure that $\delta_1 \leq \|e^{T_m \Omega_s} \tilde{x}_m\| \leq 2\delta_1$. Precisely, it means that one has to take T_m of order $|\log \|\tilde{x}_m\||$.

We define a new sequence (see Figure 1):

$$(z_m^{(1)}; \zeta_m^{(1)}) = \tilde{\Phi}^{-T_m}(z_m; \zeta_m) = \left(e^{T_m \Omega_s} \tilde{x}_m, e^{-T_m \Omega_u} \tilde{y}_m; \frac{(e^{-T_m \Omega_s^T} \tilde{\xi}_m, e^{T_m \Omega_u^T} \tilde{\eta}_m)}{\|(e^{-T_m \Omega_s^T} \tilde{\xi}_m, e^{T_m \Omega_u^T} \tilde{\eta}_m)\|} \right).$$

For every m , $z_m^{(1)} = \varphi^{-T_m}(z_m)$ belongs to $W^u(\Lambda_j)$ by assumption. We know that $\|\tilde{x}_m\|$ goes to 0 as m tends to $+\infty$. In particular, we will have $e^{-T_m \Omega_u} \tilde{y}_m \rightarrow 0$ as m tends to $+\infty$. Since $\|e^{T_m \Omega_s} \tilde{x}_m\|$ is constrained to be in the interval $[\delta_1, 2\delta_1]$, we can extract a subsequence so that $e^{T_m \Omega_s} \tilde{x}_m$ converges to some value $\tilde{x}_\infty^{(1)} \neq 0$. Moreover, up to another extraction, $\frac{(e^{-T_m \Omega_s^T} \tilde{\xi}_m, e^{T_m \Omega_u^T} \tilde{\eta}_m)}{\|(e^{-T_m \Omega_s^T} \tilde{\xi}_m, e^{T_m \Omega_u^T} \tilde{\eta}_m)\|}$ converges to some element $(0, \tilde{\eta}_\infty^{(1)})$ since $\|\tilde{\eta}_m\|$ is uniformly bounded from below by a positive constant. By construction, the limit $(z_\infty^{(1)}; \zeta_\infty^{(1)})$ reads $(\tilde{x}_\infty^{(1)}, 0; 0, \tilde{\eta}_\infty^{(1)})$ where $\tilde{\eta}_\infty^{(1)} \neq 0, \tilde{x}_\infty^{(1)} \neq 0$ hence the limit $(z_\infty^{(1)}; \zeta_\infty^{(1)})$ belongs to $N^*(W^s(\Lambda_j) \setminus \Lambda_j) \cap S^*M$ which proves our claim in the case of fixed points.

6.1.2. Λ_j is a closed orbit. We proceed to the case of a closed orbit and we start with the case of Σ_{ss} . As before, we may apply the flow φ^t in backward times in order to ensure that, for every m large enough, z_m belongs to the linearizing chart near the closed orbit Λ_j . As $(z_\infty; \zeta_\infty)$ does not belong to Σ_{ss} , we know from Lemma 5.1 that it will converge to a point in Σ_u under the action of the lifted flow $\tilde{\Phi}^t$. Hence, up to applying the lifted flow in backward times, we can suppose again that for every m large enough, $(z_m; \zeta_m)$ is within a distance δ from the component induced by Λ_j of Σ_u

for some fixed $\delta > 0$ small enough. We write these points in local coordinates near Λ_j :

$$(z_m; \zeta_m) = (P(\theta_m, 0)(\tilde{x}_m, \tilde{y}_m), \theta_m; (P(\theta_m, 0)^T)^{-1}(\tilde{\xi}_m, \tilde{\eta}_m), \Theta_m)$$

and

$$(z_\infty; \zeta_\infty) = (P(\theta_\infty, 0)(0, \tilde{y}_\infty), \theta_\infty; (P(\theta_\infty, 0)^T)^{-1}(\tilde{\xi}_\infty, \tilde{\eta}_\infty), \Theta_\infty).$$

As z_m belongs to $W^u(\Lambda_i)$ with $i \neq j$, we have one more time $\tilde{x}_m \neq 0$. From the expression³ of the Λ_j component of Σ_u near Λ_j given in § 5.3, we also know that $\|(\tilde{\eta}_\infty, \Theta_\infty)\| \neq 0$, hence, for m large enough, $\|(\tilde{\eta}_m, \Theta_m)\|$ is uniformly bounded from below by some positive constant. In these local coordinates, the flow reads

$$\varphi^{-t}(z_m) = (P(\theta_m - t, 0)(e^{t\Omega_s} \tilde{x}_m, e^{-t\Omega_u} \tilde{y}_m), \theta_m - t),$$

while the cotangent component evolves as

$$\frac{((P(\theta_m - t, 0)^T)^{-1}(e^{-t\Omega_s^T} \tilde{\xi}_m, e^{t\Omega_u^T} \tilde{\eta}_m), \Theta_m + R(-t, \tilde{x}_m, \tilde{y}_m, \theta_m, \xi_m, \eta_m))}{\|((P(\theta_m - t, 0)^T)^{-1}(e^{-t\Omega_s^T} \tilde{\xi}_m, e^{t\Omega_u^T} \tilde{\eta}_m), \Theta_m + R(-t, \tilde{x}_m, \tilde{y}_m, \theta_m, \xi_m, \eta_m))\|}, \tag{26}$$

where $R(-t, \tilde{x}_m, \tilde{y}_m, \theta_m, \xi_m, \eta_m)$ was defined precisely in § 4 – see equations (20) and (22). As in the case of a critical point, we fix $\delta_1 > 0$ and, for every m large enough, we may choose some time T_m large enough to ensure that $\delta_1 \leq \|e^{T_m\Omega_s} \tilde{x}_m\| \leq 2\delta_1$. Again, T_m will be of order $\|\log\|x_m\|\|$ thanks to property (14) and we define our new sequence as $(z_m^{(1)}; \zeta_m^{(1)}) = \tilde{\Phi}^{-T_m}(z_m; \zeta_m)$. As above, we can extract some subsequence which converges to a limit point $(z_\infty^{(1)}; \zeta_\infty^{(1)})$. Let us verify that this new sequence satisfies the claim of the lemma.

Again, property (1) is directly verified from our construction. Moreover, as $\|\tilde{x}_m\|$ tends to 0, we can verify that $z_\infty^{(1)} = (P(\theta_\infty^{(1)}, 0)(\tilde{x}_\infty^{(1)}, 0), \theta_\infty^{(1)})$ for some $\tilde{x}_\infty^{(1)} \neq 0$. Hence, $z_\infty^{(1)}$ belongs to $W^s(\Lambda_j) \setminus \Lambda_j$ which is property (3) we are looking for. It remains to show that the limit point $(z_\infty^{(1)}; \zeta_\infty^{(1)})$ belongs to $N^*(W^s(\Lambda_j)) \cap S^*M$. For that purpose, we may distinguish two cases: $\|e^{T_m\Omega_u^T} \tilde{\eta}_m\|$ tends to $+\infty$, or $\|e^{T_m\Omega_u^T} \tilde{\eta}_m\|$ remains bounded.

In the first case, using (22), one can find some constant $C > 0$ depending only on the flow near the closed orbit such that

$$\begin{aligned} C_m &:= \|((P(\theta_m - T_m, 0)^T)^{-1}(e^{-T_m\Omega_s^T} \tilde{\xi}_m, e^{T_m\Omega_u^T} \tilde{\eta}_m), \Theta_m + R(-T_m, \tilde{x}_m, \tilde{y}_m, \theta_m, \xi_m, \eta_m))\| \\ &\geq C(1 - C\delta_1)\|e^{T_m\Omega_u^T} \tilde{\eta}_m\| \geq \frac{C}{2}\|e^{T_m\Omega_u^T} \tilde{\eta}_m\|, \end{aligned}$$

if $\delta_1 > 0$ is small enough. Similarly, one gets an upper bound on C_m which is of order $\|e^{T_m\Omega_u^T} \tilde{\eta}_m\|$. In particular, up to an extraction, $e^{T_m\Omega_u^T} \tilde{\eta}_m / C_m \rightarrow \tilde{\eta}_\infty^{(1)} \neq 0$. Hence, one has, using one more time (22), that the limit covector will be of the form

$$\left((P(\theta_\infty^{(1)}, 0)^T)^{-1}(0, \tilde{\eta}_\infty^{(1)}), -\langle \partial_\theta P(\theta_\infty^{(1)}, 0)(\tilde{x}_\infty^{(1)}, 0), (P(\theta_\infty^{(1)}, 0)^T)^{-1}(0, \tilde{\eta}_\infty^{(1)}) \rangle \right).$$

Hence, $(z_\infty^{(1)}, \zeta_\infty^{(1)})$ belongs to $N^*(W^s(\Lambda_j))$. In particular, it belongs to $\bigcup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M$.

³Observe that we wrote the expression for Σ_s and that the expression for Σ_u is analogous.

Suppose now that $\|e^{T_m \Omega_u^T} \tilde{\eta}_m\|$ remains bounded. In particular, this implies that $\tilde{\eta}_\infty = 0$ and that $\Theta_\infty \neq -(\partial_\theta P(\theta_\infty, 0)(0, \tilde{y}_\infty), (P(\theta, 0)^T)^{-1}(\tilde{\xi}_\infty, 0))$. Observe also that $e^{-T_m \Omega_s^T} \tilde{\xi}_m$ still goes to 0 as $m \rightarrow +\infty$. Hence, in that case, one can verify, using (22) that the limit covector $\zeta_\infty^{(1)}$ is of the form $((P(\theta_\infty^{(1)}, 0)^T)^{-1}(0, \tilde{\eta}_\infty^{(1)}), \Theta_\infty^{(1)})$ (note that $\tilde{\eta}_\infty^{(1)} = 0$ iff $\|e^{T_m \Omega_u^T} \tilde{\eta}_m\| \rightarrow 0^+$), i.e., $(z_\infty^{(1)}, \zeta_\infty^{(1)})$ belongs to $\bigcup_{x \in \Lambda_j} N^*(W^{ss}(x)) \cap S^*M$.

In the case of Σ_s , the situation is slightly simpler as the fact that the limit point does not belong to Σ_s implies that the component $\|\tilde{\eta}_m\|$ has to be uniformly bounded from below by a positive constant. In particular, $\|e^{T_m \Omega_u^T} \tilde{\eta}_m\|$ tends to $+\infty$, and the $\tilde{\eta}$ component of the limit point does not vanish. We already discussed that case above and we saw that the new limit point does not belong to $N^*(W^u(\Lambda_j))$.

6.2. Conclusion of the proof of Theorem 5.2

Using the technical lemma, we can give a proof of Theorem 5.2 by contradiction. Suppose that there exists a sequence $(z_m; \zeta_m)$ in Σ_{ss} (resp. Σ_s) which converges to a point $(z_\infty; \zeta_\infty)$ that does not belong to Σ_{ss} (resp. Σ_s). Without loss of generality, we may assume that all elements $(z_m; \zeta_m)$ belong to $N^*(W^u(\Lambda))$ (resp. $\bigcup_{x \in \Lambda} N^*(W^{uu}(\Lambda))$) for some $\Lambda \in NW(\varphi^t)$. We have two situations for the sequence $(z_m; \zeta_m)$ and its limit $(z_\infty; \zeta_\infty)$:

- either $z_\infty \in W^u(\Lambda)$ and we get the contradiction from the first part of Lemma 6.1;
- or $z_\infty \in W^u(\Lambda_{i_1})$ for some $\Lambda_{i_1} \neq \Lambda$ and Lemma 6.1 gives us a new sequence $(z_m^{(1)}; \zeta_m^{(1)})$ in Σ_{ss} (resp. Σ_s) which converges to $(z_\infty^{(1)}; \zeta_\infty^{(1)}) \in \bigcup_{x \in \Lambda_{i_1}} W^{ss}(x)$ (resp. $N^*(W^s(\Lambda_{i_1}))$) with $z_m^{(1)} \in W^u(\Lambda)$ for all m . Moreover, $(z_\infty^{(1)}; \zeta_\infty^{(1)}) \notin \Sigma_{ss}$ (resp. Σ_s) with $z_\infty^{(1)} \in W^s(\Lambda_{i_1}) \setminus \Lambda_{i_1}$.

In the second case, $z_\infty^{(1)} \in W^u(\Lambda_{i_2})$ with $i_2 \neq i_1$ and we can repeat the same argument. In fact, we can reproduce this procedure as long as $\Lambda_{i_p} \neq \Lambda$ and we obtain some sequences $(z_m^{(1)}; \zeta_m^{(1)}), (z_m^{(2)}; \zeta_m^{(2)}), \dots$, together with their respective limits $(z_\infty^{(1)}; \zeta_\infty^{(1)}) \notin \Sigma_{ss}, (z_\infty^{(2)}; \zeta_\infty^{(2)}) \notin \Sigma_{ss}, \dots$ (resp. $\notin \Sigma_s$) and some sequence of critical elements $(\Lambda_{i_1}, \Lambda_{i_2}, \dots)$ which are *two by two distinct* such that $z_\infty^{(1)} \in W^s(\Lambda_{i_1}) \cap W^u(\Lambda_{i_2}), z_\infty^{(2)} \in W^s(\Lambda_{i_2}) \cap W^u(\Lambda_{i_3}), \dots$. Indeed, if we had $\Lambda_{i_p} = \Lambda_{i_q}$ for $q > p$, then, by Lemma B.2, we would necessarily have the full set of equalities $\Lambda_{i_p} = \dots = \Lambda_{i_q}$. But we would also have that $z_\infty^{(k)} \in W^u(\Lambda_{i_{k+1}} = \Lambda_{i_k}) \cap W^s(\Lambda_{i_k})$ and $z_\infty^{(k)} \notin \Lambda_{i_k}$ by construction of the sequence $(z_m^{(k)})_m$. This would contradict the property that the intersection $W^u(\Lambda_i) \cap W^s(\Lambda_i)$ is reduced to Λ_i from Lemma 3.6.

Since the number of critical elements is finite and the critical elements $\Lambda_{i_1}, \Lambda_{i_2}, \dots$ produced by our procedure are two by two distinct, this algorithm must terminate at some $\Lambda_{i_p} = \Lambda$. This leads us to situation 1 for the sequence $(z_m^{(p)}; \zeta_m^{(p)})$ and its limit $(z_\infty^{(p)}; \zeta_\infty^{(p)})$ and we would get the result that infinitely many terms in the sequence $(z_m^{(p)}; \zeta_m^{(p)})$ do not belong to Σ_{ss} (resp. Σ_s) contradicting the initial assumption. This concludes the proof of compactness.

6.3. Proof of Lemma 6.2 about the closure of unstable manifolds

Note that the proof we just gave was independent of Smale’s Theorem 3.7 as it only used Lemma B.2 from the appendix. In fact, Lemma 6.1 can also be used to recover a result due to Smale [38, Lemmas 3.6 and 3.7] (see also [45]):

Lemma 6.2. *Suppose that $W^u(\Lambda_j) \cap \overline{W^u(\Lambda_i)} \neq \emptyset$. Then, there exists a sequence $j = i_1, \dots, i_q = i$ such that $W^s(\Lambda_{i_p}) \cap W^u(\Lambda_{i_{p+1}}) \neq \emptyset$ for every $1 \leq p \leq q - 1$. In particular, from Lemmas B.1 and B.2, $W^u(\Lambda_j) \subset \overline{W^u(\Lambda_i)}$.*

This lemma is part of Smale’s proof of Theorem 3.7 and it could in fact be derived without the C^1 -linearization property. We briefly recall how this result could be deduced from Lemma 6.1.

Proof. The proof follows a similar algorithm as in § 6.2 but working only on the base M . Consider some sequence $(z_m)_{m \in \mathbb{N}}$ in $W^u(\Lambda_i)$ such that $z_m \xrightarrow{m \rightarrow \infty} z_\infty \in W^u(\Lambda_j)$. Then we have two situations:

- (1) either $\Lambda_i = \Lambda_j$ and we are done with $i_1 = i = j$.
- (2) or $\Lambda_i \neq \Lambda_j$ and flowing backwards by the flow, we can find, up to extraction, some subsequence $(z_m^{(1)})_m$ in $W^u(\Lambda_i)$ which converges to $z_\infty^{(1)} \in W^s(\Lambda_j) \setminus \Lambda_j$. Hence, there exists i_2 such that $z_\infty^{(1)} \in W^u(\Lambda_{i_2})$.

Then, either $i_2 = i$ and we are done or $i_2 \neq i$. In the latter case, we can apply one more time Lemma 6.1 to produce a new sequence. Iterating this argument, we will be given a sequence of elementary critical elements $(\Lambda_{i_1}, \Lambda_{i_2}, \dots)$ which are two by two distinct by the arguments of § 6.2. Hence, the procedure will end at some step i_q where $i_q = i$. □

6.4. Further comments

Actually, reproducing the same argument as for the proof of Theorem 5.2 allows to prove something slightly stronger:

Theorem 6.3. *Fix $J \subset \{1, \dots, K\}$. Then,*

$$\Sigma_{ss}^J := \bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} N^*(W^u(\Lambda_i)) \cap S^*M,$$

and

$$\Sigma_s^J := \bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} \bigcup_{x \in \Lambda_j} N^*(W^{uu}(x)) \cap S^*M$$

are compact subsets of S^*M . By considering negative times of the flow, the same of course holds for the stable manifolds with the associated partial order relation.

7. Proof of the stable neighborhood Theorem 5.4

We now turn to the proof of Theorem 5.4 in the case of Σ_{ss} . Here, it will somehow be more convenient to work with conical neighborhoods rather than neighborhoods in the unit cotangent bundle. More precisely we define

$$\Sigma_{ss} := \bigcup_{j=1}^K N^*(W^u(\Lambda_j)).$$

Let us state a precised version of Theorem 5.4.

Theorem 7.1 (Conical stable neighborhood). *For every $\epsilon > 0$, we will construct an open conical neighborhood \mathbf{V}_{ss} of Σ_{ss} in $T^*M \setminus 0$ such that:*

- (1) $\forall t \geq 0, \Phi^t(\mathbf{V}_{ss}) \subset \mathbf{V}_{ss}$;
- (2) *for every $(z; \zeta)$ in \mathbf{V}_{ss} , $(z; \zeta / \|\zeta\|_z)$ is within a distance ϵ from some element in Σ_{ss} .*

We will focus on the case of Σ_{ss} and we will explain at each step how the proof has to be adapted for Σ_s . Once this conical neighborhood is constructed, one can conclude the proof of Theorem 5.4 by relating the flow $\tilde{\Phi}^t$ to Φ^t .

We note that we used so far an auxiliary metric g to define the distance. In the upcoming proofs, near every elementary critical element Λ_j , we shall use a norm denoted by \mathbf{N}_j to define our neighborhood in every linearizing chart near the fixed Λ_j . This norm has *a priori* nothing to do with the norm induced by g on the local chart. For the sake of simplicity, we shall start with the Euclidean metric $\|\cdot\|$ in the chart and then show how to adapt it to the dynamics. In order to construct the neighborhood, we will also introduce three small parameters:

- (1) $\epsilon_i > 0$ which controls the distance of base points to $W^u(\Lambda_i)$;
- (2) $\epsilon'_i > 0$ which controls the distance of base points to $W^s(\Lambda_i)$;
- (3) $\epsilon''_i > 0$ which controls the aperture of some cone in the cotangent fiber.

This triple of parameters will be adjusted in the inductive construction of the stable neighborhood. In order to clarify the upcoming statements, $\epsilon_i, \epsilon'_i, \epsilon''_i$ will be adjusted in terms of ϵ and of the $\epsilon_j, \epsilon'_j, \epsilon''_j$ with $W^u(\Lambda_j) \leq W^u(\Lambda_i)$. Moreover, ϵ''_i will be adjusted in terms of ϵ'_i and ϵ_i in terms of ϵ''_i .

We now fix $\epsilon > 0$ (small enough) and proceed to the construction of \mathbf{V}_{ss} by induction on Smale's partial order relation.

7.1. Construction near minimal elements of Smale's partial ordering

We start by setting

$$J_0 := \{1 \leq j \leq K : W^u(\Lambda_j) \geq W^u(\Lambda_{j'}) \implies j = j'\}.$$

These are the minimal elements for Smale's partial order relation. Recall that such points are attracting for the flow. Fix $j \in J_0$ and some small parameter $\epsilon_j > 0$ that we will adjust with respect to the value of ϵ . Suppose first that Λ_j is a fixed point and let us explain how to construct \mathbf{V}_{ss} near this point. Without loss of generality, we can assume that we are in a neighborhood of Λ_j where we can use the linearizing chart of § 4.2. Following classical ideas for hyperbolic dynamical systems [5, Proposition 5.2.2], we introduce the conical neighborhood:

$$V_j := T^*B_j \setminus \{0\} \quad \text{for } B_j = \left\{ \tilde{x} : \int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_j \right\}, \quad (27)$$

where the integral is convergent thanks to (7). Recall that, for any fixed point which is minimal, $N^*(W^u(\Lambda_j)) = T_{\Lambda_j}^*M$. By construction, this set is invariant under the forward flow Φ^t . Obviously, if $(\tilde{x}; \tilde{\xi}) \in \Sigma_{ss}$ is such that $\int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_j$, then $(\tilde{x}; \tilde{\xi})$

belongs to V_j . Moreover, recalling the expression of $N^*(W^u(\Lambda_j))$ given in §5.2, by choosing $\epsilon_j > 0$ small enough, and every point in V_j is ϵ -close to Σ_{ss} in the sense of the second claim of Theorem 7.1.

Let us discuss the case where Λ_j is a closed orbit. In that case, we fix two small parameters $\epsilon_j > 0$ and $\epsilon''_j > 0$ that will both depend on ϵ . Moreover, we will fix ϵ_j in terms of ϵ''_j . First of all, we introduce a new norm on \mathbb{R}^{n-1} :

$$\mathbf{N}_j(\tilde{\xi}) := \int_0^{+\infty} \|e^{-t\Omega_s^T} \tilde{\xi}\| e^{t\lambda_j} dt,$$

which is well defined for $\lambda_j > 0$ small enough⁴ thanks to the inequality (14). With this norm, one has $\mathbf{N}_j(e^{-t_0\Omega_s^T} \tilde{\xi}) \leq e^{-t_0\lambda_j} \mathbf{N}_j(\tilde{\xi})$ for every $t_0 \geq 0$ and every $\tilde{\xi} \in \mathbb{R}^{n-1}$. Then, using the notational conventions of §4.4, we set

$$B_j := \left\{ (P(\theta, 0)\tilde{x}, \theta) : \int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_j \right\},$$

and

$$V_j := \left\{ (P(\theta, 0)\tilde{x}, \theta, (P(\theta, 0)^T)^{-1}\tilde{\xi}), \Theta \in T^*B_j \setminus 0 : (*) \text{ holds} \right\},$$

where (*) means that

$$\epsilon''_j \mathbf{N}_j(\tilde{\xi}) > \left| \Theta + \langle \partial_\theta P(\theta, 0)\tilde{x}, (P(\theta, 0)^T)^{-1}\tilde{\xi} \rangle \right|, \tag{28}$$

which is a conical set in $T^*M \setminus 0$. This condition on the cotangent component follows from the exact expression given in (25).

Remark 7.2. In the case of Σ_s , the situation is slightly simpler at this step as we just need to impose $(\tilde{\xi}, \Theta) \neq 0$.

Let us verify that V_j is invariant under the flow in positive time. We fix a point in V_j . For the variable on M , this follows from the definition. For the cotangent component, we write using (22) for $t_0 \geq 0$:

$$\begin{aligned} & \left| \Theta + R(t_0, \tilde{x}, \theta, \tilde{\xi}) + \langle \partial_\theta P(\theta + t_0, 0)e^{-t_0\Omega_s} \tilde{x}, (P(\theta + t_0, 0)^T)^{-1} e^{t_0\Omega_s^T} \tilde{\xi} \rangle \right| \\ &= \left| \Theta + \langle \partial_\theta P(\theta, 0)\tilde{x}, (P(\theta, 0)^T)^{-1}\tilde{\xi} \rangle \right| < \epsilon''_j \mathbf{N}_j(\tilde{\xi}) \leq \epsilon''_j \mathbf{N}_j(e^{t_0\Omega_s^T} \tilde{\xi}) \end{aligned}$$

Hence, the set we have just defined is invariant under the action of the flow Φ^{t_0} , $t_0 \geq 0$. Let us now verify that it is ϵ -close to Σ_{ss} which is the second claim of Theorem 7.1. For $\tilde{x} = 0$, this is immediate as the condition on $\tilde{\xi}$ reads $\epsilon''_j \mathbf{N}_j(\tilde{\xi}) > |\Theta|$. Hence, if we choose ϵ_j, ϵ''_j small enough (with respect to ϵ), we are done. In fact, as our conditions are continuous with respect to the different variables, we can verify that this remains true for \tilde{x} small enough.

⁴We just choose $\lambda_j < \chi_+$.

7.2. Adjusting the neighborhood property to Σ_{ss}

Finally, we would like to show that, for small enough ϵ_j and ϵ''_j , any element $(P(\theta, 0)\tilde{x}, \theta; (P(\theta, 0)^T)^{-1}\tilde{\xi}), \Theta) \in \Sigma_{ss}$ such that $(P(\theta, 0)\tilde{x}, \theta)$ belongs to B_j is in fact inside V_j . This is not *a priori* obvious for the following reason. Since $\Sigma_{ss} = \bigcup_{\Lambda \in NW(\varphi^t)} N^*(W^u(\Lambda)) \cap S^*M$, there are two kinds of elements in $\Sigma_{ss} \cap S^*B_j$ near the periodic orbit Λ_j :

- the elements in $N^*(W^u(\Lambda_j)) \cap S^*M$ contained in V_j by construction;
- the points coming from the conormals $N^*(W^u(\Lambda)) \cap S^*M$ for critical elements Λ such that $\overline{W^u(\Lambda)} \cap W^u(\Lambda_j) \neq \emptyset$.

Up to this point, what is obvious is the fact that V_j forms a neighborhood of $N^*(W^u(\Lambda_j))$ by construction and we would like to show V_j also contains the other points. For that purpose, we fix $\epsilon''_j > 0$ and we argue by contradiction. Precisely, we suppose that for every $\epsilon_j > 0$, we can find some element in $S^*B_j \cap \Sigma_{ss}$ not belonging to V_j . Consider some sequence $(\epsilon_{j,p})_{p \in \mathbb{N}}$ such that $\lim_{p \rightarrow \infty} \epsilon_{j,p} = 0$, set $B_{j,p} = \{(P(\theta, 0)\tilde{x}, \theta) : \int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_{j,p}\}$ and $V_{j,p} = \{(P(\theta, 0)\tilde{x}, \theta; (P(\theta, 0)^T)^{-1}\tilde{\xi}), \Theta) \in T^*B_{j,p} \setminus 0 : (*) \text{ holds}\}$ where $(*)$ is given by equation (28) for the fixed parameter ϵ''_j . We thus obtain some sequence $(z_p; \zeta_p) \in S^*B_{j,p} \cap \Sigma_{ss}$ such that $\text{dist}(z_p, \Lambda_j) \rightarrow 0$ and ζ_p fails to satisfy the inequality (28) for all p :

$$\epsilon''_j \mathbf{N}_j \left(\tilde{\xi}_p \right) \leq \left| \Theta_p + \langle \partial_\theta P(\theta_p, 0)\tilde{x}_p, (P(\theta_p, 0)^T)^{-1}\tilde{\xi}_p \rangle \right|. \tag{29}$$

By Theorem 5.2 the set $S^*B_j \cap \Sigma_{ss}$ is compact. Hence, up to extraction, we can find a subsequence $(z_p; \eta_p)$ which converges to $(z; \zeta) \in S^*B_j \cap \Sigma_{ss}$ with $z \in \Lambda_j$. But, since Σ_{ss} is partitioned as $\bigcup_{\Lambda \in NW(\varphi^t)} N^*(W^u(\Lambda)) \cap S^*M$, the element $(z; \zeta)$ actually belongs to $N^*(W^u(\Lambda_j)) \cap S^*M$. Therefore, by the coordinate representation of $N^*(W^u(\Lambda_j))$ from § 5.2, $(z; \zeta)$ is of the form $(0, \theta; (P(\theta, 0)^T)^{-1}\tilde{\xi}, 0)$, and $(z_p; \zeta_p) = (\tilde{x}_p, \theta_p; (P(\theta_p, 0)^T)^{-1}\tilde{\xi}_p, \Theta_p)$ where $\tilde{x}_p \rightarrow 0, \Theta_p \rightarrow 0, \tilde{\xi}_p \rightarrow \tilde{\xi} \neq 0$. But this means that, for $p \rightarrow +\infty$,

$$\epsilon''_j \mathbf{N}_j \left(\tilde{\xi}_p \right) \rightarrow \epsilon''_j \mathbf{N}_j \left(\tilde{\xi} \right) \neq 0,$$

and

$$\left| \Theta_p + \langle \partial_\theta P(\theta_p, 0)\tilde{x}_p, (P(\theta_p, 0)^T)^{-1}\tilde{\xi}_p \rangle \right| \rightarrow 0.$$

This contradicts the inequality (29).

We note that the importance of being able to adjust the parameter ϵ_j will play an important role at each step of our construction. This concludes the construction of the neighborhood near minimal elements for Smale’s partial order relation.

7.3. Induction on Smale’s partial ordering: the case of fixed points

We now proceed to the second step of the induction and set

$$J_1 := \{j \notin J_0 : W^u(\Lambda_j) \supseteq W^u(\Lambda_{j'}) \implies j = j' \text{ or } j' \in J_0\}.$$

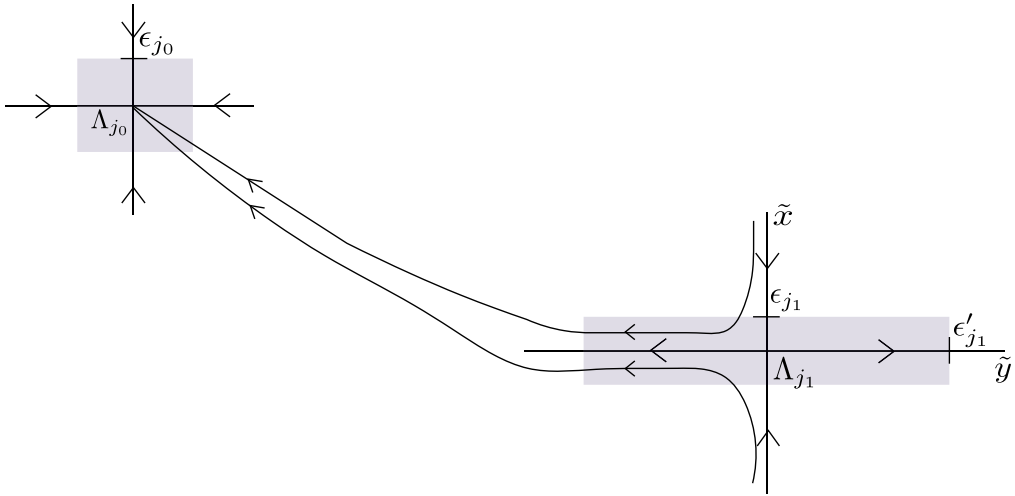


Figure 2. Construction of the neighborhood inside M .

We will now construct an adapted neighborhood near every Λ_j such that $j \in J_1$. Again, we start with the case of a fixed point Λ_j and we fix several small parameters $\epsilon_j, \epsilon'_j, \epsilon''_j > 0$ that will be determined.

Recall that in the linearizing chart, $N^*(W^u(\Lambda_j))$ can be written as

$$\left\{ (0, \tilde{y}, \tilde{\xi}, 0) : \tilde{y} \in \mathbb{R}^{n_u}, \tilde{\xi} \in \mathbb{R}^{n_s} \setminus \{0\} \right\}.$$

First of all, we define a small neighborhood of Λ_j inside M as follows (see Figure 2):

$$B_j := \left\{ (\tilde{x}, \tilde{y}) : \int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_j \text{ and } \|\tilde{y}\| < \epsilon'_j \right\}.$$

Recall that the integrals converge thanks to (7).

7.3.1. Reaching previous boxes in finite time. Let us first verify that there exists a uniform $T_j > 0$ such that, for every point of the form $(0, \tilde{y})$ with $\|\tilde{y}\| = \epsilon'_j$ which are elements on the boundary of the box B_j , $\varphi^{T_j}(0, \tilde{y})$ belongs to one of the previously constructed box B_i with $i \in J_0$. Indeed, suppose by contradiction that it is not true. Then, one can construct a point $(0, \tilde{y}')$ with $\|\tilde{y}'\| = \epsilon'_j$ such that $\varphi^t(0, \tilde{y}')$ reaches these neighborhoods in infinite time and this would contradict the fact that $\varphi^t(0, \tilde{y}')$ must converge to some Λ_i with $i \in J_0$. By uniform continuity of the flow and up to decreasing the value of ϵ_j a little bit (in a way that depends on ϵ'_j), we can thus assume that every point $(\tilde{x}, \tilde{y}) \in \overline{B_j}$ such that $\|\tilde{y}\| = \epsilon'_j$ will reach one of the neighborhoods B_i (with $i \in J_0$) in a uniform time T_j .

7.3.2. Cotangent vectors. Now we take care of the cotangent part of the elements in B_j . Precisely, as before, we first define a new norm on \mathbb{R}^{n_s} :

$$\mathbf{N}_j(\tilde{\xi}) := \int_0^{+\infty} \|e^{-t\Omega_s^T} \tilde{\xi}\| e^{t\lambda_j} dt,$$

which converges for small enough $\lambda_j > 0$ thanks to (7). Then, one has $\mathbf{N}_j(e^{-t_0\Omega_s^T} \tilde{\xi}) \leq e^{-t_0\lambda_j} \mathbf{N}_j(\tilde{\xi})$ for every $t_0 \geq 0$ and every $\tilde{\xi} \in \mathbb{R}^{n_s}$. We now set

$$V_j := \left\{ (\tilde{x}, \tilde{y}; \tilde{\xi}, \tilde{\eta}) \in T^*B_j \setminus 0 : (*) \text{ holds} \right\},$$

where (*) means that

$$\epsilon_j'' \mathbf{N}_j(\tilde{\xi}) > \int_0^{+\infty} \|e^{-t\Omega_u^T} \tilde{\eta}\| dt.$$

Again this integral converges thanks to (7). This defines clearly an open conical subset of T^*B_j , and we have, for every $t_0 \geq 0$,

$$\begin{aligned} \int_0^{+\infty} \|e^{-t\Omega_u^T} e^{-t_0\Omega_u^T} \tilde{\eta}\| dt &\leq \int_0^{+\infty} \|e^{-t\Omega_u^T} \tilde{\eta}\| dt < \epsilon_j'' \mathbf{N}_j(\tilde{\xi}) \\ &\leq e^{-t_0\lambda_j} \epsilon_j'' \mathbf{N}_j(e^{t_0\Omega_s^T} \tilde{\xi}) \leq \epsilon_j'' \mathbf{N}_j(e^{t_0\Omega_s^T} \tilde{\xi}). \end{aligned}$$

In other words, it means that the conical condition on the cotangent vectors is preserved under the action of the forward flow Φ^{t_0} on T^*M . All the norms on \mathbb{R}^{n_s} being equivalent, we can also verify that every point in V_j is ϵ -close to $N^*(W^u(\Lambda_j))$ in the sense of condition (2) of Theorem 7.1 (at least if the ϵ_j^* are chosen small enough). Up to decreasing the value of ϵ_j and ϵ_j'' (in a way that depends only on ϵ_j' and ϵ), we may also suppose that for every $(z; \zeta) \in V_j$ and for every $0 \leq t \leq T_j$, the point $\Phi^t(z; \zeta)$ remains ϵ -close to $N^*(W^u(\Lambda_j))$ in the sense of (2).

We now introduce the subsets

$$B_j := \bigcup_{t \geq 0} \varphi^t \left(\bigcup_{i \in J_0} B_i \cup B_j \right) \subset M,$$

and

$$V_j := \bigcup_{t \geq 0} \Phi^t \left(\bigcup_{i \in J_0} V_i \cup V_j \right) \subset T^*M \setminus 0.$$

These are invariant subsets under the forward flow by definition. From our construction, all the points inside V_j are also ϵ -close to Σ_{ss} .

7.3.3. Adjusting the constants to Σ_{ss} . Let us come back to the important observation made in § 7.2. Recall that Σ_{ss} is the union of all the conormals $N^*(W^u(\Lambda_i))$ where $1 \leq i \leq K$ and we aim at constructing a neighborhood of Σ_{ss} . In this second step of the induction, we have indeed constructed a neighborhood V_j of $N^*(W^u(\Lambda_j))$ for $j \in J_1$ because we know from the first step of the induction that every element $(z; \zeta) \in \Sigma_{ss}$ with

$z \in \bigcup_{i \in J_0} B_i$ belongs to $\bigcup_{i \in J_0} V_i$. In order to continue the procedure, we need to ensure the same property near Λ_j for $j \in J_1$.

Hence, the last thing we want to impose is that, if $(z; \zeta)$ belongs to $T^*B_j \cap \Sigma_{ss}$, then necessarily $(z; \zeta)$ belongs to \mathcal{V}_j which means we have really constructed some neighborhood of Σ_{ss} . Again, we fix $\epsilon''_j > 0$ and we argue by contradiction as in § 7.2. Again, we suppose that, for every $\epsilon_j > 0$, we can find point of Σ_{ss} near Λ_j not lying in this neighborhood. Then, up to an extraction, we can find a sequence of points inside⁵ Σ_{ss} that would converge to a point not belonging to $N^*(W^u(\Lambda_j)) \cup \left(\bigcup_{i \in J_0} V_i\right)$. As in § 7.2, this would contradict the fact that Σ_{ss} is compact.

7.4. Induction on Smale’s partial ordering: the case of closed orbits

We now treat the case where Λ_j is a closed orbit. The procedure is more or less the same but we repeat it to take into account the effects on the θ variable and its dual variable Θ . Recall that, in this case, $N^*(W^u(\Lambda_j))$ can be written in the linearizing chart as:

$$\left\{ \left(P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0), \Theta(\theta, \tilde{y}, \tilde{\xi}) \right) : \tilde{y} \in \mathbb{R}^{n_u}, \theta \in \mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z}, \tilde{\xi} \in \mathbb{R}^{n_s} \setminus \{0\} \right\},$$

where $\Theta(\theta, \tilde{y}, \tilde{\xi}) = -(\partial_\theta P(\theta, 0)\tilde{x}, (P(\theta, 0)^T)^{-1}\tilde{\xi})$. As above, we fix three small parameters $\epsilon_j, \epsilon'_j, \epsilon''_j > 0$ that will be adjusted in terms of the dynamics and we define first a neighborhood inside M :

$$B_j := \left\{ (P(\theta, 0)(\tilde{x}, \tilde{y}), \theta) \in \mathbb{R}^{n-1} \times (\mathbb{R}/\mathcal{P}_{\Lambda_j} \mathbb{Z}) : \int_0^{+\infty} \|e^{-t\Omega_s} \tilde{x}\| dt < \epsilon_j \text{ and } \|\tilde{y}\| < \epsilon'_j \right\}.$$

Arguing as with fixed points, we can find a uniform $T_j > 0$ such that every point inside $\overline{B_j}$ (with $\|\tilde{y}\| = \epsilon'_j$) will belong to one of the neighborhoods B_i with $i \in J_0$ at time T_j .

We now need to have a look at the way we lift this neighborhood in the cotangent space. As before, we define a new norm \mathbf{N}_j on \mathbb{R}^{n_s} such that $\mathbf{N}_j(e^{-t_0\Omega_s^T} \tilde{\xi}) \leq e^{-t_0\lambda_j} \mathbf{N}_j(\tilde{\xi})$ for every $t_0 \geq 0$ and every $\tilde{\xi} \in \mathbb{R}^{n_s}$. Here, $\lambda_j > 0$ is one more time a small enough parameter. Mimicking what we have done before, we then define

$$V_j := \left\{ \left(P(\theta, 0)(\tilde{x}, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}), \Theta \right) \in T^*B_j \setminus 0 : (*) \text{ holds} \right\},$$

where $(*)$ now means that

$$\epsilon''_j \mathbf{N}_j(\tilde{\xi}) > \int_0^{+\infty} \|e^{-t\Omega_u^T} \tilde{\eta}\| dt + \left| \Theta + (\partial_\theta P(\theta, 0)(\tilde{x}, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta})) \right|.$$

This defines an open conical set inside T^*V_j , and we want to check that property $(*)$ is preserved under the action of the forward flow Φ^{t_0} for $t_0 \geq 0$:

$$\int_0^{+\infty} \|e^{-(t+t_0)\Omega_u^T} \tilde{\eta}\| dt \leq \int_0^{+\infty} \|e^{-t\Omega_u^T} \tilde{\eta}\| dt, \mathbf{N}_j(\tilde{\xi}) \leq \epsilon''_j \mathbf{N}_j(e^{t_0(\Omega_s)^T} \tilde{\xi})$$

⁵Note, from the first step of the induction, that such a sequence cannot be contained inside $\bigcup_{i \in J_0} V_i$.

and, thanks to (22),

$$\begin{aligned} & \left| \Theta + \langle \partial_\theta P(\theta, 0)(\tilde{x}, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}) \rangle \right| \\ &= \left| \Theta + R(t_0, \tilde{x}, \tilde{y}, \theta, \tilde{\xi}, \tilde{\eta}) + \langle \partial_\theta P(\theta + t_0, 0)(e^{-t_0\Omega_s} \tilde{x}, e^{t_0\Omega_u} \tilde{y}), \right. \\ & \quad \left. (P(\theta + t_0, 0)^T)^{-1}(e^{t_0\Omega_s^T} \tilde{\xi}, e^{-t_0\Omega_u^T} \tilde{\eta}) \rangle \right|. \end{aligned}$$

Combining these three inequalities, we find that the property of the cotangent component is preserved under the action of the forward flow. Arguing as for fixed points, we can ensure that up to decreasing the value of ϵ_j and ϵ'_j (in a way that depends only on ϵ'_j and on ϵ), we can verify that, for every $(z; \zeta) \in V_j$ and for every $0 \leq t \leq T_j$, the point $\Phi^t(z; \zeta)$ remains ϵ -close to $N^*(W^u(\Lambda_j))$ in the sense of § 7.1.

We now connect these two neighborhoods with the ones constructed for $j' \in J_0$. Again, we define the (forward) invariant sets:

$$\mathcal{B}_j := \bigcup_{t \geq 0} \varphi^t \left(\left(\bigcup_{i \in J_0} B_i \right) \cup B_j \right) \subset M,$$

and

$$\mathcal{V}_j := \bigcup_{t \geq 0} \Phi^t \left(\left(\bigcup_{i \in J_0} V_i \right) \cup V_j \right) \subset T^*M \setminus 0.$$

By construction, any point $(z; \zeta)$ in \mathcal{V}_j is ϵ -close to Σ_{ss} . Once again, we can make use of the compactness of Σ_{ss} to verify that, for $\epsilon_j > 0$ small enough, the following holds. If $(z; \zeta)$ belongs to $T^*\mathcal{B}_j \cap \Sigma_{ss}$, then necessarily $(z; \zeta)$ belongs to \mathcal{V}_j – see § 7.2.

7.5. The case of Σ_s

We now have to discuss what has to be modified in the case of Σ_s . Recall that, in this case, $\bigcup_{x \in \Lambda_j} N^*(W^{uu}(x))$ can be written in the linearizing chart as

$$\left\{ \left(P(\theta, 0)(0, \tilde{y}), \theta; (P(\theta, 0)^T)^{-1}(\tilde{\xi}, 0), \Theta \right) : \tilde{y} \in \mathbb{R}^{n_u}, \theta \in \mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z}, (\tilde{\xi}, \Theta) \in \mathbb{R}^{n_s+1} \setminus \{0\} \right\}.$$

The argument is exactly the same and the only point that needs to be modified is the condition (*) appearing in the definition of V_j . Precisely, we set (*) to be the condition:

$$\epsilon''_j \left(\mathbf{N}_j \left(\tilde{\xi} \right) + \left| \Theta + \langle \partial_\theta P(\theta, 0)(\tilde{x}, \tilde{y}), (P(\theta, 0)^T)^{-1}(\tilde{\xi}, \tilde{\eta}) \rangle \right| \right) > \int_0^{+\infty} \|e^{-t\Omega_u^T} \tilde{\eta}\| dt.$$

Again, this is an open conical condition. The same calculation as before shows that the condition (*) is preserved under the action of the forward flow (as soon as we remain close to Λ_j).

7.5.1. Conclusion of the proof. In order to conclude the construction, we continue this induction procedure up to the point where we have exhausted all the critical elements $(\Lambda_j)_{j=1, \dots, K}$. In the case of Σ_{ss} , this means that we exhaust all the critical points such

that $\dim(W^u(\Lambda_j)) > 0$ and all closed orbits such that $\dim(W^u(\Lambda_j)) > 1$. In the case of Σ_s , we have to go one step further and include closed orbits such that $\dim(W^u(\Lambda_j)) > 0$.

Actually, our procedure allows more than just constructing neighborhoods of Σ_s and Σ_{ss} . Indeed, for any subfamily $(\Lambda_j)_{j \in J}$ of critical elements, we could repeat the same construction inductively for the Smale partial order relation, and we would prove the following theorem that will be used in [9]:

Theorem 7.3. *Suppose that φ^t is a C^1 -linearizable Morse–Smale flow. Fix a subfamily $(\Lambda_j)_{j \in J}$ of critical elements. Then for every $\epsilon > 0$, there exists an ϵ -neighborhood of*

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} W^u(\Lambda_i)$$

which is invariant under application of the flow in forward times. Similarly, there exist forward invariant small neighborhoods of the following subsets of S^*M :

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} N^*(W^u(\Lambda_i)) \cap S^*M,$$

and

$$\bigcup_{j \in J} \bigcup_{W^u(\Lambda_i) \leq W^u(\Lambda_j)} \bigcup_{x \in \Lambda_i} N^*(W^{uu}(x)) \cap S^*M.$$

8. Construction of an escape function

We now proceed to the proof of our last dynamical statement before explaining the spectral construction that follows from such analysis. Precisely, one of the key ingredients in the spectral construction of Faure–Sjöstrand is the construction of a nice enough ‘escape function’. In order to state the existence of an escape function in our framework, we introduce the following subset inside T^*M :

$$\Gamma_0 := \{(x; \xi) \in T^*M \setminus 0 : \exists 1 \leq j \leq K, x \in \Lambda_j \text{ and } (x; \xi) \in N^*(W^{uu}(x)) \cap N^*(W^{ss}(x))\}.$$

This set corresponds to the neutral direction of the flow above closed orbits of φ^t . In terms of local coordinates near a closed orbit Λ , it reads

$$\{(0, 0, \theta; 0, 0, \Theta) : (\theta, \Theta) \in (\mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z}) \times (\mathbb{R} - \{0\})\}.$$

The main dynamical ingredient to start our spectral construction is the following lemma:

Lemma 8.1 (Escape function). *Let V be a C^∞ vector field inducing a C^1 linearizable Morse–Smale flow. Let (u, n_0, s) be elements in \mathbb{R} with $u < -2\|E\|_\infty < 2\|E\|_\infty < n_0 < s$ where E is the energy function of Theorem 3.9. Let N_0 be an arbitrarily small conic neighborhood of Γ_0 inside $T^*M \setminus 0$.*

*Then, there exists a smooth metric $\|\cdot\|_x$, a smooth function $m(x; \xi) \in C^\infty(T^*M)$ called an order function, taking values in $[u, s]$, and an escape function on T^*M defined by*

$$G_m(x; \xi) := \frac{1}{2}m(x; \xi) \log(1 + f(x; \xi)^2),$$

where $f(x; \xi) \in C^\infty(T^*M)$. Moreover, $f(x; \xi)$ is positive and positively homogeneous of degree 1 for $\|\xi\|_x \geq 1$, $f(x; \xi) = \|\xi\|_x$ outside a small neighborhood of N_0 and $f(x; \xi) = |\xi(V(x))| = |H(x; \xi)|$ in a small neighborhood of N_0 . Finally, one has:

- (1) For $\|\xi\|_x \geq 1$, $m(x; \xi)$ depends only on $\xi/\|\xi\|_x$ and it takes values $\leq \frac{u}{4}$ (resp. $\geq \frac{n_0}{4}$ and $\geq \frac{s}{4}$) in a small neighborhood of $\bigcup_{j=1}^K N^*(W^u(\Lambda_j))$ (resp. Γ_0 and $\bigcup_{j=1}^K N^*(W^s(\Lambda_j))$). It also takes value $\geq \frac{n_0}{10} - \frac{u}{2}$ outside a slightly larger neighborhood of $\bigcup_{j=1}^K N^*(W^u(\Lambda_j))$.
- (2) There exists $R > 0$ such that, for every $(x; \xi)$ in T^*M satisfying $\|\xi\|_x \geq R$, one has

$$X_H G_m(x; \xi) \leq 0.$$

- (3) If in addition $(x; \xi) \notin N_0$, then

$$X_H G_m(x; \xi) \leq -C_m < 0,$$

with

$$C_m := c \min(|u|, s),$$

for some constant $c > 0$ independent of u , n_0 and s .

In fact, the metric in this lemma is rather arbitrary. As we shall see, we only need to fix its value on the Λ_j in a manner that depends on the hyperbolic dynamics. Note that, as $\xi(V(x))$ does not vanish in a neighborhood of Γ_0 , the function $f(x; \xi)$ does not vanish on S^*M . The same result was proved in the Anosov framework in [16, Lemma 1.2]. The main input compared with that case is Meyer's Theorem 3.9 on the one hand and Theorems 5.2 and 5.4 on the other. We now proceed to the construction of the escape function.

8.1. A preliminary lemma

The main lines of the argument are very close to the one given for Anosov vector fields by Faure and Sjöstrand in [16]. Yet, several steps need to be adapted in order to fit into our dynamical framework. To begin with, we recall a dynamical lemma from [16, Lemma 2.1]:

Lemma 8.2. *Let V^{uu} and V^s be small open neighborhoods of Σ_{uu} and Σ_s respectively, and let $\epsilon > 0$. Then, there exist smaller open neighborhoods $\mathcal{O}^{uu} \subset V^{uu}$ and $\mathcal{O}^s \subset V^s$ of Σ_{uu} and Σ_s respectively, \tilde{m}_1 in $C^\infty(S^*M, [0, 1])$, $\eta_1 > 0$ such that $\tilde{X}_H \tilde{m}_1 \geq 0$ on S^*M , $\tilde{X}_H \tilde{m}_1 \geq \eta_1 > 0$ on $S^*M - (\mathcal{O}^{uu} \cup \mathcal{O}^s)$, $\tilde{m}_1(x; \xi) > 1 - \epsilon$ for $(x; \xi) \in \mathcal{O}^s$ and $\tilde{m}_1(x; \xi) < \epsilon$ for $(x; \xi) \in \mathcal{O}^{uu}$.*

Recall that \tilde{X}_H is the vector field induced by the Hamiltonian $H(x; \xi) = \xi(V(x))$ on S^*M . This lemma was proved in [16, Lemma 8] for Anosov flows, and we can follow the same strategy now that we have properly settled the dynamical properties of the symplectic lift of Morse–Smale flows. For the sake of completeness, we briefly recall the proof as this is where we will crucially use Theorems 5.2 and 5.4.

Proof. As Σ_{uu} and Σ_s are compact from Theorem 5.2 and as their intersection is empty thanks to (23), we may suppose without loss of generality that V^{uu} and V^s have empty

intersection. Thanks to Theorem 5.4, we may also assume that

$$\forall t \geq 0, \quad \tilde{\Phi}^t(V^s) \subset V^s \quad \text{and} \quad \forall t \leq 0, \quad \tilde{\Phi}^t(V^{uu}) \subset V^{uu}.$$

We are now in position to (briefly) repeat the argument from [16, Lemma 8]. Before that, note that, while these properties were easy to check in the Anosov case, they constitute the core of our construction in the Morse–Smale case.

The argument is as follows. Fix $T > 0$ and set $\mathcal{O}^s = \tilde{\Phi}^T(S^*M \setminus V^{uu})$ and $\mathcal{O}^{uu} = \tilde{\Phi}^{-T}(S^*M \setminus V^s)$. Using Lemma 5.1 and Theorem 5.4, we know that for $T > 0$ large enough $\mathcal{O}^{uu} \subset V^{uu}$ and $\mathcal{O}^s \subset V^s$. We now fix a smooth function m_0 in $C^\infty(S^*M, [0, 1])$ such that $m_0 = 1$ on V^s and $m_0 = 0$ on V^{uu} . We then set:

$$\tilde{m}_1(x; \xi) := \frac{1}{2T} \int_{-T}^T m_0 \circ \tilde{\Phi}^t(x; \xi) dt,$$

and we have:

$$\tilde{X}_H \tilde{m}_1(x; \xi) = \frac{1}{2T} \left(m_0 \circ \tilde{\Phi}^T(x; \xi) - m_0 \circ \tilde{\Phi}^{-T}(x; \xi) \right) \geq 0.$$

In particular, for $(x; \xi) \notin \mathcal{O}^{uu} \cup \mathcal{O}^s$, one has $\tilde{X}_H \tilde{m}_1(x; \xi) = \frac{1}{2T} > 0$. It now remains to prove the statements on the value of \tilde{m}_1 in $\mathcal{O}^{uu} \cup \mathcal{O}^s$. For that purpose, recall from our assumptions on V^{uu} and V^s that

$$\forall (x; \xi) \in S^*M, \quad \mathcal{I}(x; \xi) := \left\{ t \in \mathbb{R} : \tilde{\Phi}^t(x; \xi) \notin V^{uu} \cup V^s \right\}$$

is a closed connected interval whose length is bounded by some constant $\tau > 0$ (that does not depend on T). Then, one can verify that

$$(x; \xi) \in \mathcal{O}^s \Rightarrow \tilde{m}_1(x; \xi) \geq \frac{2T - \tau}{2T} \quad \text{and} \quad (x; \xi) \in \mathcal{O}^{uu} \Rightarrow \tilde{m}_1(x; \xi) < \frac{\tau}{2T},$$

which concludes the proof of the preliminary lemma. Note that, similarly, $\tilde{m}_1(x; \xi) \geq \frac{1}{2} - \frac{\tau}{2T}$ if $(x, \xi) \in V^s$ and $\tilde{m}_1(x; \xi) \leq \frac{1}{2} + \frac{\tau}{2T}$ if $(x, \xi) \in V^{uu}$. □

Remark 8.3. We remark that by inverting the sense of time, we can obtain a similar result for Σ_{ss} and Σ_u . More precisely, we let V^{ss} and V^u be small open neighborhoods of Σ_{ss} and Σ_u respectively, and we let $\epsilon > 0$. Then, there exist open neighborhoods $\mathcal{O}^{ss} \subset V^{ss}$ and $\mathcal{O}^u \subset V^u$, \tilde{m}_2 in $C^\infty(S^*M, [0, 1])$, $\eta_2 > 0$ such that $\tilde{X}_H \tilde{m}_2 \geq 0$ on S^*M , $\tilde{X}_H \tilde{m}_2 \geq \eta_2 > 0$ on $S^*M - (\mathcal{O}^{ss} \cup \mathcal{O}^u)$, $\tilde{m}_2(x; \xi) > 1 - \epsilon$ for $(x; \xi) \in \mathcal{O}^{ss}$ and $\tilde{m}_2(x; \xi) < \epsilon$ for $(x; \xi) \in \mathcal{O}^u$.

8.2. Proof of Lemma 8.1

We can now start the construction of the escape function. Again, the proof follows closely what is done in the Anosov case except that a couple of steps need to be revisited. By homogeneity, it is sufficient to perform the construction of m inside S^*M and then extend the definition by homogeneity for $\|\xi\|_x \geq 1$.

We keep the constants $u < 0 < n_0 < s$ and the conic neighborhood N_0 of Γ_0 exactly as in the statement of Lemma 8.1. Let us also consider the constant $\epsilon > 0$

and the collection $\mathcal{O}^{uu} \subset V^u, \mathcal{O}^s \subset V^s, \mathcal{O}^{ss} \subset V^{ss}, \mathcal{O}^u \subset V^u$ of neighborhoods from the statements of Lemma 8.2 and Remark 8.3. Following [16, §3.3.1] and [8, Appendix A], we set

$$\tilde{m}(x; \xi) := -E(x) + s + (n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2, \tag{30}$$

where E is the energy function of Theorem 3.9. We already observe that

$$\forall(x; \xi) \in S^*M, \quad \tilde{X}_H \tilde{m}(x; \xi) \leq 0 \tag{31}$$

since the function $-E$ decreases along the flow, $u - n_0 < 0, n_0 - s < 0$ and $\tilde{X}_H \tilde{m}_i \geq 0$, for $i \in \{1, 2\}$.

8.2.1. First properties. We start by collecting a few properties of the function \tilde{m} on S^*M . First of all, one knows from Meyer’s Theorem 3.9 that, if we fix $\delta > 0$ and if x does not belong to a δ -neighborhood \mathcal{V}_δ of $NW(\varphi')$, then there exists some constant $c(\delta) > 0$ such that

$$\forall(x; \xi) \notin S^*\mathcal{V}_\delta, \quad \tilde{X}_H \tilde{m}(x; \xi) \leq -c(\delta) < 0. \tag{32}$$

This shows that away from the critical elements of the flow, \tilde{m} is strictly decaying with a control in terms of the distance to the critical elements.

We now analyze more precisely the properties of \tilde{m} near the critical elements. For that purpose, we set $\eta := \min(\eta_1, \eta_2)$. Using the same convention as in [16] (in order to facilitate the comparison), we now define several open sets as follows:

$$\tilde{N}_s := \mathcal{O}^{uu} \cap \mathcal{O}^u, \quad \tilde{N}_0 := \mathcal{O}^u \cap \mathcal{O}^s, \quad \tilde{N}_u := \mathcal{O}^{ss} \cap \mathcal{O}^s.$$

We already observe that $(\tilde{N}_s, \tilde{N}_0, \tilde{N}_u)$ define disjoint open neighborhoods inside S^*M of $(\Sigma_{uu}, \Gamma_0 \cap S^*M, \Sigma_{ss})$ respectively. Then, as in [16, p. 338] we can verify the following properties:

- On $S^*M - (\tilde{N}_s \cup \tilde{N}_0 \cup \tilde{N}_u) = (S^*M - (\mathcal{O}^{uu} \cup \mathcal{O}^s)) \cup (S^*M - (\mathcal{O}^{ss} \cup \mathcal{O}^u))$, one has

$$\tilde{X}_H \tilde{m}(x; \xi) \leq -\eta \min(|n_0 - s|, |u - n_0|), \tag{33}$$

with $\eta = \min\{\eta_1, \eta_2\}$ since, for $(x; \xi)$ in this region, either $\tilde{X}_H \tilde{m}_1(x; \xi) \geq \eta_1 \geq \eta$ or $\tilde{X}_H \tilde{m}_2(x; \xi) \geq \eta_2 \geq \eta$.

- On $\tilde{N}_s = \mathcal{O}^{uu} \cap \mathcal{O}^u$, one has both $\tilde{m}_1 \leq \epsilon, \tilde{m}_2 \leq \epsilon$ therefore

$$\tilde{m}(x; \xi) = \underbrace{-E(x)}_{\geq -\|E\|_\infty} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\geq \epsilon(u-s)}$$

hence by $\|E\|_\infty < \frac{s}{2}$, we find that

$$\tilde{m}(x; \xi) \geq -\|E\|_\infty + u\epsilon + (1 - \epsilon)s \geq \left(\frac{1}{2} - \epsilon\right)s + u\epsilon \geq \frac{s}{4}, \tag{34}$$

where the last inequality holds provided we take $\epsilon < \frac{s}{4(s-u)}$.

- Similarly, on $\tilde{N}_u = \mathcal{O}^{ss} \cap \mathcal{O}^s$, one has both $\tilde{m}_1 \geq 1 - \epsilon, \tilde{m}_2 \geq 1 - \epsilon$ therefore

$$\tilde{m}(x; \xi) = \underbrace{-E(x)}_{\leq -\min E} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\leq (1-\epsilon)(u-s)}$$

hence by $\frac{u}{2} < -\|E\|_\infty$, we find that

$$\tilde{m}(x; \xi) \leq -\min E + u(1 - \epsilon) + s\epsilon \leq \left(\frac{1}{2} - \epsilon\right)u + s\epsilon \leq \frac{u}{4}, \tag{35}$$

where the last inequality holds provided we take $\epsilon < \frac{u}{4(u-s)}$.

- On $\tilde{N}_0 = \mathcal{O}^u \cap \mathcal{O}^s$, one has both $1 - \epsilon < \tilde{m}_1 \leq 1$ and $\tilde{m}_2 \leq \epsilon$ hence

$$\begin{aligned} \tilde{m}(x; \xi) &= \underbrace{-E(x)}_{\geq -\|E\|_\infty} + s + \underbrace{(n_0 - s)\tilde{m}_1 + (u - n_0)\tilde{m}_2}_{\geq (n_0 - s) + (u - n_0)\epsilon} \\ \tilde{m}(x; \xi) &\geq -\max E + (1 - \epsilon)n_0 + (u - n_0)\epsilon \geq \left(\frac{1}{2} - 2\epsilon\right)n_0 + u\epsilon \geq \frac{n_0}{4}, \end{aligned} \tag{36}$$

where the last inequality holds provided $\epsilon < \frac{n_0}{4(2n_0 - u)}$.

- Finally, by similar arguments and using the final remark in the proof of Lemma 8.2, we can verify that $\tilde{m}(x; \xi) \geq \frac{s}{10} - \frac{u}{2}$ outside a slightly bigger set than \tilde{N}_u .

We now extend \tilde{m} into a smooth function m defined on T^*M by setting $m(x; \xi) = 0$ for $\|\xi\|_x \leq 1/2$ and

$$\forall (x; \xi) \in T^*M \text{ s.t. } \|\xi\|_x \geq 1, \quad m(x; \xi) = \tilde{m}\left(x; \frac{\xi}{\|\xi\|_x}\right).$$

From (34), (35) and (36), we remark that for $\|\xi\|_x \geq 1$, $m(x; \xi)$ depends only on $\xi/\|\xi\|_x$ and it takes values $\leq \frac{u}{4}$ (resp. $\geq \frac{n_0}{4}$ and $\geq \frac{s}{4}$) in a small neighborhood of $\bigcup_{j=1}^K N^*(W^u(\Lambda_j))$ (resp. Γ_0 and $\bigcup_{j=1}^K N^*(W^s(\Lambda_j))$). Hence point (1) of Lemma 8.1 is proved.

8.2.2. Decay of G_m along the flow. We start by defining $f(x; \xi)$ according to the neighborhoods that have already been introduced. Recall that \mathcal{V}_δ was some δ -neighborhood of the nonwandering set. We set f to be a smooth function which is positive and positively homogeneous of degree 1 for $\|\xi\|_x \geq 1$. Moreover, we suppose that for $\xi \in N_0 \cap T^*\mathcal{V}_\delta$, one has $f(x; \xi) = |H(x; \xi)|$ and $f(x; \xi) = \|\xi\|_x$ when ξ does not belong to a small open neighborhood of N_0 . Without loss of generality, we can assume that $f(x; \xi) \geq C_0\|\xi\|$ for $\|\xi\| \geq 1$ and for some positive constant $C_0 > 0$ depending on N_0 .

Remark 8.4. We observe that, up to choosing a smaller value of δ and smaller neighborhoods in the statements of Lemma 8.2 and Remark 8.3, we can always suppose that $T^*\mathcal{V}_\delta \cap \tilde{N}_0 \subset S^*M \cap N_0$.

Let us now discuss the decay properties of

$$G_m := \frac{1}{2}m(x; \xi) \log\left(1 + f^2(x; \xi)\right)$$

and prove properties (2) and (3) of Lemma 8.1. It suffices to show that, in some conic neighborhood of N_0 , we have the bound $X_H G_m \leq 0$ and outside some conic neighborhood

of N_0 , we have a uniform negative bound on $X_H G_m$. We decompose $X_H G_m$ as the sum of two terms:

$$2X_H G_m(x; \xi) = (X_H m) (\log(1 + f(x; \xi)^2)) + m(x; \xi) X_H (\log(1 + f(x; \xi)^2)). \quad (37)$$

We will split the discussion between points $x \in M$ which are close to the nonwandering set and those which are far from it.

Remark 8.5. Before discussing this, we observe that we have a crude upper bound on the term $m(x; \xi) X_H \log(1 + f(x; \xi)^2)$ which is uniform for $(x; \xi) \in T^*M$ satisfying $\|\xi\|_x$ larger than some $R > 0$. For that purpose, we first remark that

$$X_H \log(1 + f(x; \xi)^2) = \frac{X_H f(x; \xi)^2}{1 + f(x; \xi)^2}, \quad (38)$$

defines a bounded function on T^*M thanks to the homogeneity properties of f . Then, we just note that \tilde{m} is uniformly bounded by $\mathcal{O}(|u| + s + \|E\|_\infty)$.

The element x does not belong to the δ neighborhood \mathcal{V}_δ of the nonwandering set. Fix an element $(x; \xi) \notin T^*\mathcal{V}_\delta$ satisfying $\|\xi\|_x \geq 1$. Recall that we have the estimates

$$X_H m(x; \xi) \leq -c(\delta) \quad \text{and} \quad \log(1 + f(x; \xi)^2) \geq \log(1 + C_0|\xi|^2).$$

Combined with (37) and with Remark 8.5, this yields the bound

$$2X_H G_m(x; \xi) \leq -c(\delta) \log(1 + C_0^2\|\xi\|^2) + C_1(\|E\|_\infty + s + |u|), \quad (39)$$

for some positive constant $C_1 > 0$ depending only on N_0 . Hence, provided R is large enough (in terms of $(C_i)_{i=0,1}$, δ , u and s) in the statement of Lemma 8.1, parts (2) and (3) of the lemma are satisfied for these points of phase space.

The element $x \in \mathcal{V}_\delta$ and $(x; \xi) \in N_0$. It now remains to analyze the situation near the nonwandering set (and thus fix a small enough value of δ). Here, the situation follows closely what was done in [16, p. 339]. We note that as $X_H H = \{H, H\} = 0$, one can show that for $(x; \xi) \in N_0$ satisfying $\|\xi\|_x \geq 1$, one has

$$X_H G_m(x; \xi) = X_H m(x; \xi) \log(1 + H(x; \xi)^2)^{\frac{1}{2}} + m(x; \xi) X_H \log(1 + H(x; \xi)^2) \leq 0. \quad (40)$$

This proves part (2) of the lemma in this region of phase space.

The element $x \in \mathcal{V}_\delta$ and $(x; \xi) \notin N_0$. Thanks to Remark 8.4, we can cover $S^*\mathcal{V}_\delta \setminus N_0$ by three regions where we will prove uniform decay of G_m along the Hamiltonian flow:

$$S^*\mathcal{V}_\delta \setminus N_0 \subset \left(\tilde{N}_u \cap S^*\mathcal{V}_\delta \right) \cup \left(\tilde{N}_s \cap S^*\mathcal{V}_\delta \right) \cup \left(S^*\mathcal{V}_\delta \setminus (\tilde{N}_0 \cup \tilde{N}_u \cup \tilde{N}_s) \right).$$

- We begin with the case where $(x; \tilde{\xi}) \in S^*M \setminus (\tilde{N}_0 \cup \tilde{N}_u \cup \tilde{N}_s)$. In that case, $f(x; \xi) \geq C_0\|\xi\|_x$ by construction (at least for $\|\xi\|_x$ large enough). Hence, it follows

from (33) that

$$2X_H G_m(x; \xi) = (X_H m)(x; \xi) \log(1 + f(x; \xi)^2) + m(x; \xi) X_H \log(1 + f(x; \xi)^2) \\ \leq -\eta \min(|n_0 - s|, |u - n_0|) \log(1 + C_0^2 \|\xi\|_x^2) + C_1 (\|E\|_{C^0} + s + |u|).$$

Hence, as for the case $x \notin \mathcal{V}_\delta$, we can ensure that parts (2) and (3) of the lemma are satisfied by picking $R > 0$ large enough (in a way that depends on η, u, s and N_0) so that the negative term $-\eta \min(|n_0 - s|, |u - n_0|) \log(1 + C_0^2 \|\xi\|_x^2)$ predominates over the positive term $C_1 (\|E\|_{C^0} + s + |u|)$ and makes sure the r.h.s. is uniformly negative as long as $\|\xi\|_x \geq R$.

- Assume now that $(x; \xi / \|\xi\|_x) \in \tilde{N}_u \cap S^* \mathcal{V}_\delta$. We show that we can pick $\delta > 0$ small enough to ensure that the bound holds for $(x; \xi / \|\xi\|_x) \in \tilde{N}_u \cap S^* \mathcal{V}_\delta$. In this case $f(x; \xi) = \|\xi\|_x$, hence

$$2X_H G_m(x; \xi) = \underbrace{(X_H m)}_{\leq 0} \log \underbrace{\left(1 + \|\xi\|_x^2\right)}_{\geq 0} + m(x; \xi) X_H \log \left(1 + \|\xi\|_x^2\right) \\ \leq m(x; \xi) \frac{X_H (\|\xi\|_x^2)}{2(1 + \|\xi\|_x^2)}.$$

We will now make use of the hyperbolicity of the flow one more time in order to control the term $X_H (\|\xi\|_x^2)$.⁶ We assume without loss of generality (periodic orbits are treated similarly) that we are near a critical point Λ_j . For $(x; \xi)$ in $T_{\Lambda_j}^* M \cap N^*(W^u(\Lambda_j))$, one can deduce from the hyperbolicity bound (7) that

$$X_H \|\xi\|_x^2 \geq c_0 \|\xi\|_x^2,$$

for some positive constant c_0 depending only on (M, g) and φ^t . This can be achieved by choosing the value of $\|\cdot\|_x$ for $x \in \Lambda_j$ in a way to have the constant equal to 1 in (7) – see [5, Proposition 5.2.2]. By compactness of Σ_{ss} , we can ensure that this inequality remains true in $\tilde{N}_u \cap S^* \mathcal{V}_\delta$, where \tilde{N}_u is a conical neighborhood of $N^*(W^u(\Lambda_j))$, provided that we replace c_0 by $c_0/2$ and that $\delta > 0$ and all neighborhoods $(V^{ss}, V^{uu}, V^s, V^u)$ from Lemma 8.2 are chosen small enough. Thus, thanks to inequality (35) that yields $\tilde{m}(x; \xi) \leq \frac{u}{4} < 0$, one finds that

$$\left(x, \frac{\xi}{\|\xi\|_x}\right) \in \tilde{N}_u \cap S^* \mathcal{V}_\delta \implies X_H G_m(x; \xi) \leq \frac{c_0 u}{8},$$

where $c_0 > 0$ is a geometric constant.

- The case where $(x; \tilde{\xi}) \in \tilde{N}_s \cap S^* \mathcal{V}_\delta$ is treated similarly thanks to (34). Then we can derive similar bounds of the form

$$\left(x, \frac{\tilde{\xi}}{\|\tilde{\xi}\|_x}\right) \in \tilde{N}_s \cap S^* \mathcal{V}_\delta \implies X_H G_m(x; \tilde{\xi}) \leq -\frac{c_0 s}{8},$$

which concludes the construction of the escape function from Lemma 8.1.

⁶Observe that hyperbolicity was already used in Meyer’s theorem and in the proof of Lemma 8.2 which relied on Theorems 5.2 and 5.4.

9. Anisotropic Sobolev spaces

In this section, we aim at proving Theorems 2.1 and 2.2 or more precisely their generalization to vector bundles (\mathcal{E}, ∇) equipped with a connection ∇ . More precisely, we will consider the Lie derivative $-\mathcal{L}_{V, \nabla}$ acting on currents on M valued in the bundle \mathcal{E} and the connection ∇ allows to lift the action of the flow from the base M to the total space of \mathcal{E} . The reason why we need to introduce such algebra is that we plan to discuss in [10] the relationship between the Pollicott–Ruelle spectrum and the Reidemeister torsion. To define the Reidemeister torsion, we need to fix a unitary representation of the fundamental group $\pi_1(M)$ and then consider geometric complexes on M twisted by the representation. Geometrically, the twisting operation corresponds to considering some flat bundle (\mathcal{E}, ∇) over M , such that the monodromy of the flat connection ∇ gives the corresponding representation of $\pi_1(M)$. In that framework, the De Rham complex of differential forms $(\Omega^\bullet(M), d)$ is replaced by the twisted De Rham complex of differential forms valued in \mathcal{E} , $\Omega^\bullet(M, \mathcal{E})$ with twisted differential d^∇ .

The purpose of this final section is to recall how one can construct anisotropic Sobolev spaces adapted to the operator $-\mathcal{L}_{V, \nabla}^{(k)}$ for every $0 \leq k \leq n$ starting from the escape function of Lemma 8.1. More precisely, for every $\sigma > 0$, we let $m_\sigma \in C^\infty(T^*M)$ be the corresponding order function from Lemma 8.1 with $|u|, s \gg \sigma$, and for every $0 \leq k \leq n$, we will construct Hilbert spaces of currents $\mathcal{H}_k^{m_\sigma}(M, \mathcal{E})$ such that

$$-\mathcal{L}_{V, \nabla}^{(k)} : \mathcal{H}_k^{m_\sigma}(M, \mathcal{E}) \longrightarrow \mathcal{H}_k^{m_\sigma}(M, \mathcal{E})$$

has discrete spectrum on $\operatorname{Re}(z) > -\sigma$ – see Proposition 9.3. Once this spectral framework is properly settled, in §9.4 we will see how one can easily deduce our main theorems on the existence of a discrete set of Pollicott–Ruelle resonances. Hence, everything boils down to constructing appropriate Hilbert spaces, and now that we are given a nice escape function m adapted to the Morse–Smale dynamics, we just need to follow step by step the (microlocal) strategy initiated by Faure and Sjöstrand in [16]. Recall that this consists in showing that $-\mathcal{L}_{V, \nabla}^{(k)}$ has nice spectral properties on that space via microlocal arguments and analytic Fredholm theory. This kind of strategy is natural in the context of the study of semiclassical resonances of Schrödinger operators [25] and we refer to [16] for a more detailed discussion on that aspect.

The proof in [16] only deals with the scalar case for $k = 0$ and $\mathcal{E} = M \times \mathbb{C}$. Yet, the argument can be adapted to any $0 \leq k \leq n$ as it only relies on the construction of a proper escape function for the flow as in the scalar case. We emphasize that the case of vector bundles for Anosov flows was considered via microlocal methods by Dyatlov and Zworski in order to prove the meromorphic continuation of the zeta function [12]. We chose to use the microlocal techniques from these references but it is most likely that the methods developed in [6, 21] could also be developed to define a proper spectral framework.

Remark 9.1. Keeping in mind the applications to differential topology [10], it may be convenient to choose a different m_σ for every $0 \leq k \leq n$. Typically, we may pick in degree k an order function of the form $m(x; \xi) + n - k$ where $m(x; \xi)$ is given by Lemma 8.1.

The following analysis would not be affected by this choice of (shifted) order function for large enough choices of $|u|$ and s in this lemma.

9.1. Vector bundles and connection

Before defining the Sobolev spaces, we collect a few definitions and properties and we refer to [29, Chapter 6–8–12] for more details. For every integer $k \in \{0, \dots, n\}$, recall the bundle $\Lambda^k(T^*M)$ of k -forms on M has all its elements of the form $\sum_{0 \leq i_1 < \dots < i_k \leq n} \xi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ in local coordinates where (dx^1, \dots, dx^n) forms a basis of T^*M in local coordinates. Smooth sections of $\Lambda^k(T^*M)$ are denoted by $\Omega^k(M)$. If we are given a metric g on M , this induces some inner product $\langle \cdot, \cdot \rangle_{g^*}^{(k)}$ on $\Lambda^k(T^*M)$. Then, the Hodge star operator is the unique isomorphism $\star_k : \Lambda^k(T^*M) \rightarrow \Lambda^{n-k}(T^*M)$ such that for every $\psi_1 \in \Omega^k(M) = \Omega^k(M, \mathbb{C})$ and $\psi_2 \in \Omega^{n-k}(M) = \Omega^{n-k}(M, \mathbb{C})$,

$$\int_M \psi_1 \wedge \psi_2 = \int_M \langle \psi_1, \star_k^{-1} \psi_2 \rangle_{g^*}^{(k)} \omega_g,$$

where ω_g is the volume form induced by the Riemannian metric g on M . Given a smooth Hermitian vector bundle $\pi : \mathcal{E} \rightarrow M$ [29, Definition 6.21] of rank N with inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ on the fibers, the space $\Omega^k(M, \mathcal{E}), k = 1, \dots, n$ of k -forms on M valued in \mathcal{E} is defined as the tensor product $\Omega^k(M) \otimes_{C^\infty(M)} \Omega^0(M, \mathcal{E})$ where $\Omega^0(M, \mathcal{E})$ denotes the smooth sections of \mathcal{E} . Using the Hermitian structure on \mathcal{E} , we can identify \mathcal{E} with its dual bundle and we can also introduce the following fiberwise pairing, for every $0 \leq k, l \leq n$,

$$\langle \cdot \wedge \cdot \rangle_{\mathcal{E}} : \Omega^k(M, \mathcal{E}) \times \Omega^l(M, \mathcal{E}) \rightarrow \Omega^{k+l}(M).$$

This induces a map $\star_k : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^{n-k}(M, \mathcal{E})$ which acts trivially on the \mathcal{E} -coefficients. Then, we can define the positive definite Hodge inner product on $\Omega^k(M, \mathcal{E})$ as

$$(\psi_1, \psi_2) \in \Omega^k(M, \mathcal{E}) \times \Omega^k(M, \mathcal{E}) \mapsto \int_M \langle \psi_1 \wedge \star_k(\psi_2) \rangle_{\mathcal{E}}.$$

In particular, we can define $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$ as the completion of $\Omega^k(M, \mathcal{E})$ for this scalar product.

We would now like to define an analogue of the Lie derivative along a vector field V in the context of differential forms with values in a vector bundle \mathcal{E} . For that purpose, we observe that the contraction operator ι_V is well defined and we have to introduce a proper substitute for the De Rham differential d . For this, we fix a smooth (Koszul) connection [29, Definition 12.1 and p. 505], i.e., a linear map

$$\nabla : \Omega^0(M, \mathcal{E}) \rightarrow \Omega^1(M, \mathcal{E}),$$

satisfying, for every $\psi \in C^\infty(M)$ and for every u in $\Omega^0(M, \mathcal{E})$,

$$\nabla(\psi u) = \psi \nabla u + (d\psi) u,$$

where d is the usual De Rham differential on M . According to [29, Theorem 12.57], this map uniquely extends to a map d^∇ such that, for every $0 \leq k, l \leq n$:

- $d^\nabla : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^{k+1}(M, \mathcal{E});$

- for every (ψ, u) in $\Omega^k(M) \times \Omega^l(M, \mathcal{E})$,

$$d^\nabla(\psi \wedge u) = d\psi \wedge u + (-1)^k \psi \wedge d^\nabla u \quad \text{and} \quad d^\nabla(u \wedge \psi) = d^\nabla u \wedge \psi + (-1)^l u \wedge d\psi;$$

- for every u in $\Omega^0(M, \mathcal{E})$, $d^\nabla u = \nabla u$.

The operator d^∇ is the coboundary operator associated with (\mathcal{E}, ∇) , and we define the corresponding *Lie derivative* along the vector field V as

$$\mathcal{L}_{V, \nabla} = d^\nabla \circ \iota_V + \iota_V \circ d^\nabla. \tag{41}$$

Remark 9.2. We define the *curvature* of the vector bundle (\mathcal{E}, ∇) as the unique map F_∇ in $\Omega^2(M, \text{End}(\mathcal{E}))$ such that, for every $0 \leq k \leq n$ and for every u in $\Omega^k(M, \mathcal{E})$,

$$d^\nabla \circ d^\nabla u = F_\nabla \wedge u.$$

We say (\mathcal{E}, ∇) is a *flat vector bundle* if $F_\nabla = 0$, equivalently, if $d^\nabla \circ d^\nabla = 0$ which means that $(\Omega^\bullet(M, \mathcal{E}), d^\nabla)$ is a cochain complex.

In the following, $\mathcal{E} \rightarrow M$ is always a smooth complex vector bundle endowed with a Hermitian structure $\langle, \rangle_{\mathcal{E}}$ and a connection ∇ (not necessarily flat).

9.2. Definition of the anisotropic Sobolev spaces

Now, we set

$$A_m(x; \xi) := \exp(G_m(x; \xi)) \in C^\infty(T^*M), \tag{42}$$

where $G_m(x; \xi)$ is given by Lemma 8.1. Let now $0 \leq k \leq n$ and $\mathcal{E} \rightarrow M$ be a smooth vector bundle equipped with a Hermitian structure. We consider the vector bundle $\Lambda^k(T^*M) \otimes \mathcal{E}$. Consider some finite cover $(U_i)_{i \in I}$ of M by contractible open subsets. For every U_i , choose some local trivialization $\kappa : \mathcal{E} \otimes \Lambda^k(T^*M)|_{U_i} \mapsto \tilde{U} \times \mathbb{R}^N \times \mathbb{R}^{\frac{n!}{k!(n-k)!}}$ of $\mathcal{E} \otimes \Lambda^k(T^*M)$ where \tilde{U} is some open subset of \mathbb{R}^n . Set $(\mathbf{e}_j)_{j=1}^N$ to be the canonical basis of \mathbb{R}^N . Then we define some operator $\mathbf{A}_i : \Omega_c^k(U_i, \mathcal{E}) \mapsto \Omega^k(M, \mathcal{E})$ explicitly in local coordinates as follows:

$$\kappa \circ \mathbf{A}_i \circ \kappa^{-1}(\mathbf{e}_j f dx^{i_1} \wedge \dots \wedge dx^{i_k})|_{\tilde{U}} = \mathbf{Op}^W(A_m)(f) \mathbf{e}_j dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

for every $f \in C_c^\infty(U_i)$, $j \in \{1, \dots, r\}$, $1 \leq i_1 < \dots < i_k \leq n$ and where $\mathbf{Op}^W(A_m)$ is the usual Weyl quantization defined on \mathbb{R}^n [40, equation (14.5) p. 68]. Choose some partition of unity $(\chi_i)_{i \in I}$, $\sum_{i \in I} \chi_i = 1$ subordinated to the cover $(U_i)_{i \in I}$ and for every $i \in I$, choose $\Psi_i \in C_c^\infty(U_i)$ such that $\Psi_i = 1$ on $\text{supp}(\chi_i)$. Then define a global linear operator $\Omega^k(M, \mathcal{E}) \mapsto \Omega^k(M, \mathcal{E})$ as:

$$\mathbf{A}_m^{(k)} = \frac{1}{2} \sum_{i \in I} \Psi_i \mathbf{A}_i \chi_i + \frac{1}{2} \left(\sum_{i \in I} \Psi_i \mathbf{A}_i \chi_i \right)^* \tag{43}$$

where $(\sum_{i \in I} \Psi_i \mathbf{A}_i \chi_i)^*$ is the formal adjoint of $\sum_{i \in I} \Psi_i \mathbf{A}_i \chi_i$ for the Hilbert space structure on $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$ defined above. The operator $\mathbf{A}_m^{(k)}$ is elliptic in the sense of

[15, Definition 8] since its principal symbol reads $A_m(x; \xi) \otimes \mathbf{Id}$ where \mathbf{Id} means the identity map $\Lambda^k(T^*M_x) \otimes \mathcal{E}_x \mapsto \Lambda^k(T^*M_x) \otimes \mathcal{E}_x$ and $\mathbf{A}_m^{(k)}$ is formally self-adjoint. Hence by [15, Lemma 11], it has a self-adjoint extension on the Hilbert space $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$. Without loss of generality, we may assume $\mathbf{A}_m^{(k)}$ to be invertible in $\Omega^k(M, \mathcal{E})$ since by [15, Lemma 12], there is a smoothing, self-adjoint operator \widehat{r} such that $\mathbf{A}_m^{(k)} + \widehat{r}$ is self-adjoint, elliptic and invertible in $\Omega^k(M, \mathcal{E})$. We define an anisotropic Sobolev space of currents by setting

$$\mathcal{H}_k^m(M, \mathcal{E}) = (\mathbf{A}_m^{(k)})^{-1} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E}).$$

Mimicking the proofs of [15], we can deduce some properties of these spaces of currents. First of all, they are endowed with a Hilbert structure inherited from the L^2 -structure on M . The topological dual $\mathcal{H}_k^m(M, \mathcal{E})'$ of $\mathcal{H}_k^m(M, \mathcal{E})$ can be identified with the space

$$\mathcal{H}_k^m(M, \mathcal{E})' \simeq \mathbf{A}_m^{(k)} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$$

by the pairing

$$\psi_1 \in \mathbf{A}_m^{(k)} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E}), \psi_2 \in \mathcal{H}_k^m(M, \mathcal{E}) \mapsto \int_M \langle \psi_1 \wedge \star_k \psi_2 \rangle_{\mathcal{E}}$$

and $\mathcal{H}_k^m(M, \mathcal{E})$ is in fact reflexive. We also note that the space $\mathcal{H}_k^m(M, \mathcal{E})$ can be identified with $\mathcal{H}_0^m(M, \mathbb{R}) \otimes_{\mathcal{C}^\infty(M)} \Omega^k(M, \mathcal{E})$, and one has

$$\Omega^k(M, \mathcal{E}) \subset \mathcal{H}_k^m(M, \mathcal{E}) \subset \mathcal{D}'^k(M, \mathcal{E}),$$

where the injections are continuous. Using the Hodge star, we finally find that

$$\mathcal{H}_k^m(M, \mathcal{E})' \simeq \star_k \mathbf{A}_m^{(k)} L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E}) = \mathbf{A}_m^{(n-k)} L^2(M, \Lambda^{n-k}(T^*M) \otimes \mathcal{E})$$

which is also equipped with a natural Hilbert space structure. The Hilbert space

$$\mathbf{A}_m^{(n-k)} L^2(M, \Lambda^{n-k}(T^*M) \otimes \mathcal{E})$$

will also be denoted by $\mathcal{H}_{n-k}^{-m}(M, \mathcal{E})$ by some little abuse of notation.

9.3. Pollicott–Ruelle resonances and their resonant states

Now that we have defined our Sobolev in a similar fashion as for the Anosov setting of [16], we can follow almost verbatim the argument of Faure and Sjöstrand in [16] in order to show the existence of a discrete dynamical spectrum on these spaces. Indeed, this part of their arguments only made use of the dynamical properties of m combined with microlocal tools and analytic Fredholm theory. More precisely, the main result on the spectral properties of $-\mathcal{L}_{V, \nabla}^{(k)}$ acting on these anisotropic spaces is the following Proposition:

Proposition 9.3 (Discrete spectrum). *The operator $-\mathcal{L}_{V,\nabla}^{(k)}$ defines a maximal closed unbounded operator on $\mathcal{H}_k^m(M, \mathcal{E})$,*

$$-\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

with domain given by $\mathcal{D}(-\mathcal{L}_{V,\nabla}^{(k)}) := \{\psi \in \mathcal{H}_k^m(M, \mathcal{E}) : -\mathcal{L}_{V,\nabla}^{(k)}\psi \in \mathcal{H}_k^m(M, \mathcal{E})\}$. It coincides with the closure of $-\mathcal{L}_{V,\nabla}^{(k)} : \Omega^k(M, \mathcal{E}) \rightarrow \Omega^k(M, \mathcal{E})$ in the graph norm for operators. Moreover, there exists a constant C_0 in \mathbb{R} (that depends on the choice of the order function $m(x; \xi)$) such that $-\mathcal{L}_{V,\nabla}^{(k)}$ has empty spectrum for $\operatorname{Re}(z) > C_0$. Finally, the operator

$$-\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}),$$

has a discrete spectrum with finite multiplicity in the domain

$$\operatorname{Re}(z) > -C_m + C_{\mathcal{E}},$$

where $C_{\mathcal{E}} > 0$ depends only on the choice of the metric $g, \langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $C_m > 0$ is the constant from Lemma 8.1.

The second part on the discrete spectrum is obtained by showing that the operator $(-\mathcal{L}_{V,\nabla}^{(k)} - z)$ is a Fredholm operator of index 0 depending analytically on z in the corresponding half-plane [25, 46]. In the case of Anosov flows, the proof of this result was given by Faure–Sjöstrand in [16, §3] for $k = 0$ while the extension to the case of currents was done by Dyatlov and Zworski in [12, §3]. Note that the proofs in both references are of slightly different nature but they both crucially rely on the properties of the escape function used to define the anisotropic space $\mathcal{H}_k^m(M, \mathcal{E})$. The proof of this Proposition was given in great details in [16, Theorem 1.4] in the case $k = 0$ and $\mathcal{E} = M \times \mathbb{C}$. As was already mentioned, the extension to the case where $0 \leq k \leq n$ and where \mathcal{E} is an arbitrary vector bundle can be adapted almost verbatim except that we have to deal with pseudodifferential operators with symbols in $\operatorname{Hom}(\Lambda^k(T^*M) \otimes \mathcal{E})$. As was already observed in [8, 12], the main point to adapt to the vector bundle framework is that the (pseudodifferential) operators under consideration have a *scalar symbol*. In fact, given any local basis $(e_j)_{j=1, \dots, J_k}$ of $\Lambda^k(T^*M) \otimes \mathcal{E}$ and any family $(u_j)_{j=1, \dots, J_k}$ of smooth functions $\mathcal{C}^\infty(M)$, one has

$$\mathcal{L}_{V,\nabla}^{(k)} \left(\sum_{j=1}^{J_k} u_j e_j \right) = \sum_{j=1}^{J_k} \mathcal{L}_V(u_j) e_j + \sum_{j=1}^{J_k} \mathcal{L}_{V,\nabla}^{(k)}(e_j) u_j,$$

where each term in the second part of the sum on the right-hand side is of order 0 as a differential operator acting on the u_j . In other words, the principal symbol of $\mathcal{L}_{V,\nabla}^{(k)}$ is $\xi(V(x)) \mathbf{Id}_{\Lambda^k(T^*M)_x \otimes \mathcal{E}_x}$. This scalar form allows to adapt the proofs of [16] to this vector bundle framework – see [16] for a detailed proof.

For the sake of completeness, we refer to §9.5 where we give a brief picture of the strategy developed by Faure and Sjöstrand to prove Proposition 9.3 starting from Lemma 8.1. Note that, in this reference, the authors use the convention $-i\mathcal{L}_V$ instead of $-\mathcal{L}_V$.

Remark 9.4. We also note that they implicitly show [16, Lemma 3.3] that, for every z in \mathbb{C} satisfying $\text{Im}z > C_0$, one has

$$\left\| \left(\mathcal{L}_{V,\nabla}^{(k)} + z \right)^{-1} \right\|_{\mathcal{H}_k^m(M,\mathcal{E}) \rightarrow \mathcal{H}_k^m(M,\mathcal{E})} \leq \frac{1}{\text{Re}(z) - C_0}. \tag{44}$$

In particular, combining Proposition 9.3 to the Hille–Yosida theorem [14, Corollary 3.6, p. 76], one knows that

$$(\varphi^{-t})^* : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}), \tag{45}$$

generates a strongly continuous semigroup which is defined for every $t \geq 0$ and whose norm is bounded by e^{tC_0} .

We now list some properties of this spectrum:

- As in [16, Theorem 1.5], we can show that the eigenvalues (counted with their algebraic multiplicity) and the eigenspaces of $-\mathcal{L}_{V,\nabla}^{(k)} : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E})$ are in fact independent of the choice of the escape function. For every $0 \leq k \leq n$, we call the eigenvalues the *Pollicott–Ruelle resonances of index k* and we denote by $\mathcal{R}_k(V, \nabla)$ this set.
- By duality, the same spectral properties hold for the dual operator

$$(-\mathcal{L}_{V,\nabla}^{(k)})^* = -\mathcal{L}_{-V,\nabla}^{(n-k)} : \mathcal{H}_{n-k}^{-m}(M, \mathcal{E}) \rightarrow \mathcal{H}_{n-k}^{-m}(M, \mathcal{E}). \tag{46}$$

- Given any z_0 in $\mathcal{R}_k(V, \nabla)$, the corresponding spectral projector $\pi_{z_0}^{(k)}$ is given by [25, Appendix]:

$$\pi_{z_0}^{(k)} := \frac{1}{2i\pi} \int_{\gamma_{z_0}} (z + \mathcal{L}_{V,\nabla}^{(k)})^{-1} dz : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}), \tag{47}$$

where γ_{z_0} is a small contour around z_0 which only contains the eigenvalue z_0 in its interior.

- Given any z_0 in \mathbb{C} with $\text{Re}(z_0) > -C_m + C_{\mathcal{E}}$, there exists $m_k(z_0) \geq 1$ such that, in a small neighborhood of z_0 , one has

$$(z + \mathcal{L}_{V,\nabla}^{(k)})^{-1} = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{(\mathcal{L}_{V,\nabla}^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}}{(z - z_0)^l} + R_{z_0,k}(z) : \mathcal{H}_k^m(M, \mathcal{E}) \rightarrow \mathcal{H}_k^m(M, \mathcal{E}), \tag{48}$$

with $R_{z_0,k}(z)$ a holomorphic function near z_0 and with $\pi_{z_0}^{(k)} = 0$ whenever $z_0 \notin \mathcal{R}_k(V, \nabla)$.

9.4. Proof of Theorems 2.1 and 2.2

We will now briefly deduce the proofs of our main theorems in the case where $\mathcal{E} = M \times \mathbb{C}$. We will in fact prove something slightly stronger as we will verify that these statements

hold for any $\psi_1 \in \mathcal{H}_k^m(M)$ and any $\psi_2 \in \mathcal{H}_{n-k}^{-m}(M)$. Let ψ_1 be an element in $\mathcal{H}_k^m(M)$. Thanks to Remark 9.4, we know that

$$\int_0^{+\infty} e^{-tz} \varphi^{-t*}(\psi_1) dt = (\mathcal{L}_V^{(k)} + z)^{-1}(\psi_1)$$

holds in $\mathcal{H}_k^m(M)$ for $\text{Re}(z) > C_0$. From Proposition 9.3, we have a meromorphic extension of the right-hand side on $\text{Re}(z) > -C_m + C$ whose poles are included in the set of Pollicott–Ruelle resonances $\mathcal{R}_k(V)$. Moreover, from (48), we also know that the following holds true in $\mathcal{H}_k^m(M)$:

$$(z + \mathcal{L}_V^{(k)})^{-1}(\psi_1) = \sum_{l=1}^{m_k(z_0)} (-1)^{l-1} \frac{(\mathcal{L}_V^{(k)} + z_0)^{l-1} \pi_{z_0}^{(k)}(\psi_1)}{(z - z_0)^l} + R_{z_0,k}(z)(\psi_1)$$

in a neighborhood of any z_0 in $\mathcal{R}_k(V)$ which satisfies $\text{Re}(z_0) > -C_m + C$. Finally, as all the results hold in $\mathcal{H}_k^m(M)$, we can always pair these equalities with some ψ_2 in $\mathcal{H}_{n-k}^{-m}(M)$.

9.5. A few words on Faure–Sjöstrand’s construction

As was already explained, once we are given an escape function satisfying the properties of Lemma 8.1, we are in a position to apply the strategy of reference [16]. This lemma was indeed the only lemma from this reference that used the dynamical properties of the flow. After that, the authors proceeded to a detailed analytic work based on microlocal techniques and Fredholm theory. For the sake of completeness, let us outline the strategy of their proof and we refer to this reference for more details. Their first observation is that studying the spectrum on the above anisotropic Sobolev spaces is equivalent to studying the conjugated operator

$$\mathbf{A}_m^{(k)} \circ \left(-\mathcal{L}_{V,\nabla}^{(k)}\right) \circ (\mathbf{A}_m^{(k)})^{-1}$$

on the more standard space $L^2(M, \Lambda^k(T^*M) \otimes \mathcal{E})$. This is a pseudodifferential operator in $\Psi^1(M, \Lambda^k(T^*M) \otimes \mathcal{E})$, and we can write

$$\mathbf{A}_m^{(k)} \circ \left(-\mathcal{L}_{V,\nabla}^{(k)}\right) \circ (\mathbf{A}_m^{(k)})^{-1} \approx \text{Op} \left((-i\xi(V(x)) + X_H G_m(x; \xi)) \mathbf{Id}_{\Lambda^k(T^*M)_x \otimes \mathcal{E}_x} \right),$$

where \approx should be understood as an equality up to an error belonging to $\mathcal{O}(\Psi^0) + \mathcal{O}_m(\Psi^{-1+0})$. In order to explain the proof of [16], we will focus ourselves on the ‘principal term’ $(-i\xi(V(x)) + X_H G_m(x; \xi)) \mathbf{Id}_{\Lambda^k(T^*M)_x \otimes \mathcal{E}_x}$ even if some attention has to be paid to these remainder terms. The next step is that Faure and Sjöstrand verify for which z in \mathbb{C} , the operator

$$\mathbf{A}_m^{(k)} \circ \left(-\mathcal{L}_{V,\nabla}^{(k)}\right) \circ (\mathbf{A}_m^{(k)})^{-1} - z \tag{49}$$

is invertible in L^2 . There are three regions that we may distinguish using the convention of Lemma 8.1. First, when $(x; \xi) \notin N_0$ with $\|\xi\|_x \geq R$, we can use the fact that $X_H G_m(x; \xi) \leq -c \min(|u|, s)$ in order to invert the symbol $X_H G_m$ in this region of phase space, at least for some z verifying $\text{Re}(z) \geq C_{\mathcal{E}} - c \min(|u|, s)$. On the other hand, if $(x; \xi) \in N_0$ and $\|\xi\|_x \geq R$, then $X_H G_m$ is not anymore uniformly strictly negative. Yet, we may subtract to the operator defined by (49) a (self-adjoint)

pseudodifferential operator $\hat{\chi}_0$ in $\Psi^0(M, \Lambda^k(T^*M) \otimes \mathcal{E})$ whose principal symbol is proportional to $c \min(|u|, s) \mathbf{Id}_{\Lambda^k(T^*M) \otimes \mathcal{E}}$ near N_0 and identically vanishes away from it. It remains to deal with the compact part of phase space $\|\xi\|_x \leq R$. Here, we can subtract a compact operator $\hat{\chi}_1$ in order to make the real part invertible in this region too. In the end, the operator

$$P(z) := \mathbf{A}_m^{(k)} \circ \left(-\mathcal{L}_{V, \nabla}^{(k)}\right) \circ \left(\mathbf{A}_m^{(k)}\right)^{-1} - \hat{\chi}_1 - \hat{\chi}_0 - z$$

is invertible in L^2 for $\text{Re}(z) \geq C_{\mathcal{E}} - c \min(|u|, s)$. We shall denote its inverse by $r(z)$. This rough picture can be made rigorous and we refer to [16, pp. 340–345] for a detailed proof. Then Faure and Sjöstrand conclude using arguments from analytic Fredholm theory. Precisely, they write that for $\text{Re}(z) \geq C_{\mathcal{E}} - c \min(|u|, s)$,

$$\mathbf{A}_m^{(k)} \circ \left(-\mathcal{L}_{V, \nabla}^{(k)}\right) \circ \left(\mathbf{A}_m^{(k)}\right)^{-1} - z = (\text{Id} + (\hat{\chi}_0 + \hat{\chi}_1)r(z))P(z).$$

Using the ellipticity of $\xi(V(x))$ near N_0 , they show that $(\text{Id} + (\hat{\chi}_0 + \hat{\chi}_1)r(z))$ is a Fredholm operator of index 0 [16, Lemma 3.4] as $P(z)$ is. Hence, on $\text{Re}(z) \geq C_{\mathcal{E}} - c \min(|u|, s)$, we have a holomorphic family of Fredholm operators of index 0 which are invertible for $\text{Re}(z)$ large enough. Then, the conclusion follows from classical theorems of analytic Fredholm theory – see e.g., [46, Theorem D.4].

Acknowledgements. We warmly thank Frédéric Faure for many explanations on his works with Johannes Sjöstrand and Masato Tsujii. We also acknowledge useful discussions related to this article and its companion articles [9, 10] with Yannick Bonthonneau, Livio Flaminio, Colin Guillarmou, Benoit Merlet, Frédéric Naud and Patrick Popescu Pampu. The presentation of the article highly benefited from a detailed and careful report of an anonymous referee that we kindly acknowledge. The second author is partially supported by the Agence Nationale de la Recherche through the Labex CEMPI (ANR-11-LABX-0007-01) and the ANR project GERASIC (ANR-13-BS01-0007-01).

Appendix A. Hyperbolic critical elements

In the definition of Morse–Smale flows, we implicitly assumed some results on hyperbolic fixed points and hyperbolic closed orbits that we will briefly review in this appendix. For more details, we invite the reader to look at the classical textbook of Palis and de Melo [33, Chapters 2, 3] – see also [26] for general results on (partially) hyperbolic invariant subsets.

A.1. Limit sets

We start with some terminology from the theory of dynamical systems. We say that a point x in M is wandering if there exist some open neighborhood U of x and some $t_0 > 0$ such that

$$U \cap \left(\bigcup_{|t| > t_0} \varphi^t(U)\right) = \emptyset.$$

The *nonwandering set* of the flow is given by the points which are not wandering. The set of nonwandering points is denoted by $\text{NW}(\varphi^t)$. Given any $x \in M$, we define

$$\alpha(x) := \bigcap_{T \leq 0} \overline{\{\varphi^t(x) : t \leq T\}},$$

and

$$\omega(x) := \bigcap_{T \geq 0} \overline{\{\varphi^t(x) : t \geq T\}}.$$

We note that for every x in M , $\alpha(x)$ and $\omega(x)$ are contained in $\text{NW}(\varphi^t)$. For any invariant closed subset Λ of M , we define the *unstable and stable manifolds* of Λ :

$$W^u(\Lambda) := \{x \in M : \alpha(x) \subset \Lambda\},$$

and

$$W^s(\Lambda) := \{x \in M : \omega(x) \subset \Lambda\}.$$

A.2. Hyperbolic fixed points

We say that a point x_0 in M is a *hyperbolic fixed point* of $(\varphi^t)_{t \in \mathbb{R}}$ if $V(x_0) = 0$ and $d_{x_0} V : T_{x_0} M \rightarrow T_{x_0} M$ has no eigenvalue on the imaginary axis. Equivalently, it means that for every $t \neq 0$, x_0 is a fixed point of the smooth diffeomorphism φ^t and that $d_{x_0} \varphi^t : T_{x_0} M \rightarrow T_{x_0} M$ has no eigenvalue of modulus one.

Consider now a hyperbolic fixed point x_0 and some small enough $\delta > 0$. We can define the local unstable and stable manifolds as follows:

$$W_\delta^u(x_0) := \{x \in B(x_0, \delta) : \forall t \leq 0, \varphi^t(x) \in B(x_0, \delta)\},$$

and

$$W_\delta^s(x_0) := \{x \in B(x_0, \delta) : \forall t \geq 0, \varphi^t(x) \in B(x_0, \delta)\}.$$

Then, one has [33, Chapter 2, Proposition 6.1 and Theorem 6.2]

$$W^u(x_0) = \bigcup_{t \geq 0} \varphi^t(W_\delta^u(x_0)), \quad \text{and} \quad W^s(x_0) = \bigcup_{t \geq 0} \varphi^{-t}(W_\delta^s(x_0)).$$

Moreover, $W_\delta^u(x_0)$ (resp. $W_\delta^s(x_0)$) is a smooth embedded disk whose dimension is that of the unstable (resp. stable) space of $d_{x_0} V$ while $W^u(x_0)$ (resp. $W^s(x_0)$) is a smooth injectively immersed manifold in M whose tangent space at x_0 is the unstable (resp. stable) space of $d_{x_0} V : T_{x_0} M \rightarrow T_{x_0} M$.

A.3. Hyperbolic closed orbits

We say that a point x_0 in M is a *hyperbolic periodic point* if $V(x_0) \neq 0$, there exists $T_0 > 0$ such that $\varphi^{T_0}(x_0) = x_0$ and $d_{x_0} \varphi^{T_0} : T_{x_0} M \rightarrow T_{x_0} M$ has 1 as a simple eigenvalue and no other eigenvalue of modulus 1. Equivalently, we will say that $\{\varphi^t(x_0) : 0 \leq t \leq T_0\}$ is a hyperbolic closed orbit.

This can also be defined in terms of Poincaré sections which allows to make the connection with the case of hyperbolic fixed points. Let Σ be a smooth hypersurface

containing x_0 which is transversal to the vector field V . We denote by $P_\Sigma : O \subset \Sigma \rightarrow \Sigma$ the corresponding Poincaré map. Then, the point x_0 is said to be a hyperbolic periodic point of the flow if x_0 is a hyperbolic point for the Poincaré map. Note that this definition does not depend on the choice of Poincaré section Σ . Fix now a neighborhood \tilde{O} of the closed orbit Λ generated by the point x_0 . We can define the (local) unstable and stable manifolds:

$$W^u_{\tilde{O}}(x_0) := \{x \in \tilde{O} : \forall t \leq 0, \varphi^t(x) \in \tilde{O}\},$$

and

$$W^s_{\tilde{O}}(x_0) := \{x \in \tilde{O} : \forall t \geq 0, \varphi^t(x) \in \tilde{O}\}.$$

As in the case of fixed points, the following holds:

$$W^u(\Lambda) = \bigcup_{t \geq 0} \varphi^t(W^u_{\tilde{O}}(x_0)), \quad \text{and} \quad W^s(\Lambda) = \bigcup_{t \leq 0} \varphi^t(W^s_{\tilde{O}}(x_0)).$$

Moreover, $W^u_{\tilde{O}}(x_0)$ and $W^s_{\tilde{O}}(x_0)$ are smooth submanifolds of M which are transverse [33, Chapter 3, Proposition 1.5] and $W^u_{\tilde{O}}(x_0) \cap W^s_{\tilde{O}}(x_0) = \Lambda$. In fact, for a given Poincaré section Σ , one can define $W^u_\delta(x_0)$ and $W^s_\delta(x_0)$ for the induced Poincaré map P_Σ and $W^u_{\tilde{O}}(x_0)$ (resp. $W^s_{\tilde{O}}(x_0)$) is an open neighborhood of Λ inside $\bigcup_{t \in (0, 2T_0)} \varphi^{-t}(W^u_\delta(x_0))$ (resp. $\bigcup_{t \in (0, 2T_0)} \varphi^t(W^s_\delta(x_0))$). Also, $W^u(\Lambda)$ and $W^s(\Lambda)$ are smooth immersed submanifolds of M [33, Chapter 3, Corollary 1.6].

Finally, $W^u(\Lambda)$ and $W^s(\Lambda)$ are invariantly fibered by smooth submanifolds $(W^{uu}(x_0))_{x_0 \in \Lambda}$ (resp. $(W^{ss}(x_0))_{x_0 \in \Lambda}$) tangent to the unstable (resp. stable) space at $\varphi^{T_0}(x_0)$ [26, Theorem 4.1]. Points of these submanifolds are characterized as follows, for every x_0 in Λ ,

$$W^{uu}(x_0) := \left\{ x \in M : \lim_{n \rightarrow +\infty} \varphi^{-nT_0}(x) = x_0 \right\},$$

and

$$W^{ss}(x_0) := \left\{ x \in M : \lim_{n \rightarrow +\infty} \varphi^{nT_0}(x) = x_0 \right\}.$$

A.4. λ -lemma

In the previous paragraphs, we saw that understanding the dynamics near a critical element is related to understanding the dynamics of a diffeomorphism f near a hyperbolic point $\Lambda := \{x_0\}$. This reduction can be done either by considering the time one map of the flow in the case of fixed points, or by looking at the Poincaré map associated with a certain transversal to the orbit. In this paragraph, we would like to give some quantitative features of the dynamics for a diffeomorphism $f : M \rightarrow M$ near a hyperbolic point x_0 . In particular, we would like to recall the λ -lemma [32, 38] (sometimes called the inclination lemma). We follow closely the presentation of [33, Chapter 2] – see also [5, Chapter 5] in the case of more general hyperbolic subsets. Before that, recall that two smooth submanifolds S and S' of M are ϵ - \mathcal{C}^1 close if there exists a \mathcal{C}^1 diffeomorphism $h : S \rightarrow S'$ such that $i' \circ h$ is ϵ -close to i in the \mathcal{C}^1 topology.⁷

⁷Here $i : S \rightarrow M$ and $i' : S' \rightarrow M$ denote the inclusion maps.

We fix some local coordinates $(x, y) \in \mathbb{R}^{n'} = E_s \oplus E_u$ around the hyperbolic point x_0 (with $n' = n$ or $n - 1$). The local stable (resp. unstable) manifold is then the graph of a smooth function $\kappa_s : B_s(0, r_1) \rightarrow E_u$ (resp. $\kappa_u : B_u(0, r_1) \rightarrow E_s$) where $B_s(0, r_1)$ (resp. $B_u(0, r_1)$) is the ball of radius r_1 centered at 0 inside E_s (resp. E_u). Moreover, we have that $\kappa_u(0) = \kappa_s(0) = 0$ and $d_0\kappa_s = d_0\kappa_u = 0$. We then introduce the following change of coordinates:

$$\kappa : B_s(0, r_1) \oplus B_u(0, r_1) \rightarrow E_s \oplus E_u, \quad (x, y) \mapsto (x - \kappa_u(y), y - \kappa_s(x)).$$

From our construction, one can verify that for $r_1 > 0$ small enough, κ induces a diffeomorphism near the origin. Hence, one can assume without loss of generality that we are working in a local chart where the local stable (resp. unstable) manifold is represented by the stable (resp. unstable) linear space. The λ -lemma can then be formulated as follows [33, Chapter 2, Lemma 7.1] (see also [5, Theorem 5.7.2]):

Theorem A.1 (λ -lemma). *We use the above conventions. Let $O = B_s(x_0, r) \times B_u(x_0, r)$, let x be an element in $W^s(x_0)$ and let D^u be a small disk of dimension $\dim(E^u(x_0))$ which is transversal to $W^s(x_0)$ at x .*

If we denote by D_N^u the connected component of $f^N(D^u) \cap O$ containing $f^N(x)$, then for every $\epsilon > 0$, there exists N_0 such that, for every $N \geq N_0$, D_N^u is ϵ -close to $W^u(x_0)$.

A.5. Sternberg–Chen’s theorem

In this paragraph, we collect a few results on the linearization of vector fields near hyperbolic critical elements in order to illustrate that our assumption of being \mathcal{C}^1 linearizable is in some sense generic.

A.5.1. The case of a fixed point. Recall Sternberg–Chen’s theorem on the linearization of vector fields near hyperbolic critical points [7] (see also [31, Theorem 9, p. 50]):

Theorem A.2 (Sternberg–Chen). *Let $V(x) = \sum_j a_j(x)\partial_{x_j}$ be a smooth vector field defined in a neighborhood of 0 in \mathbb{R}^n . Suppose that $V(0) = 0$ and that 0 is a hyperbolic fixed point. Denote by (μ_j) the eigenvalues of $A := (\partial_{x_k} a_j(0))_{k,j}$. Suppose that the eigenvalues satisfy the nonresonant assumption,*

$$\forall \alpha_1, \dots, \alpha_n \in \mathbb{N} \quad \text{s.t.} \quad \alpha_1 + \dots + \alpha_n \geq 2, \quad \forall 1 \leq j \leq n, \quad \mu_j \neq \sum_{i=1}^d \alpha_i \mu_i.$$

Then, there exists a smooth diffeomorphism h which is defined in a neighborhood of 0 such that $h(0) = 0$ and such that

$$h^*(V)(x) = D_x h(L(x))$$

where $L(x) = Ax \cdot \partial_x$.

Here \mathbb{N} denotes the set of nonnegative integers. The classical Grobman–Hartman theorem ensures the existence of a conjugating homeomorphism. The crucial observation for us is that the conjugating map is smooth provided some nonresonance assumption is made.

Remark A.3. We stated here a version of the theorem which gives conditions to have a smooth diffeomorphism. Yet, if we only search a C^k conjugating diffeomorphism (with $k \geq 1$), we can restrict our assumptions by imposing only a *finite* number of conditions on the eigenvalues – see [31, Theorem 10, p. 52] for the precise statement.

A.5.2. The case of a closed orbit. Sternberg–Chen’s theorem can be generalized in the case of a closed hyperbolic orbit [44]. Consider a smooth vector field $V(x, \theta)$ defined on $B_{n-1}(0, r) \times (\mathbb{R}/\mathcal{P}_\Lambda\mathbb{Z})$ where $B_{n-1}(0, r)$ is a small ball of radius $r > 0$ centered at 0 in \mathbb{R}^{n-1} . We make the assumption that

$$V(x, \theta) = (1 + g(x, \theta))\partial_\theta + (A(\theta)x + f(x, \theta))\cdot\partial_x, \tag{50}$$

with $f(x, \theta) = \mathcal{O}(\|x\|^2)$, $A(\theta)$ smooth and $g(x, \theta) = \mathcal{O}(\|x\|)$.

Simplifying coordinates near the periodic orbit. Let us first explain how the vector field can be put under the form (50). Recall that the tubular neighborhood theorem states that some neighborhood of $\Lambda \subset M$ is C^∞ diffeomorphic to some neighborhood of the zero section of the normal bundle $N(\Lambda \subset M)$ induced by some Riemannian metric on M . But Λ is a circle (with global coordinates θ in $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$); hence $N(\Lambda \subset M)$ is an oriented real vector bundle over \mathbb{S}^1 . Hence, $N(\Lambda)$ is trivial and diffeomorphic to a cartesian product $\mathbb{S}^1 \times \mathbb{R}^{d-1}$. Equivalently, triviality and the tubular neighborhood theorem guarantee that we have some germ of coordinate system (x, θ) near Λ where Λ is defined by the global equation $\{x = 0\}$. In these coordinates, the vector field V associated with φ^t can be written as

$$V(x, \theta) = \tilde{g}(x, \theta)\partial_\theta + \tilde{f}(x, \theta)\partial_x, \tag{51}$$

where (x, θ) belongs to $\mathcal{O}_\Lambda \times \mathbb{S}^1$ for some small neighborhood $\mathcal{O}_\Lambda \subset \mathbb{R}^{n-1}$ of 0. As Λ is a closed orbit, we can suppose that $\tilde{f}(0, \theta) = 0$ for all $\theta \in \mathbb{S}^1$ and $\tilde{g}(0, \theta) > 0$ for all $\theta \in \mathbb{S}^1$. Before explaining the analogue of the Sternberg–Chen’s theorem for closed hyperbolic orbit, let us first reparametrize the θ variable. Set $\tilde{\theta}(\theta) = \int_0^\theta \tilde{g}^{-1}(0, s) ds$ and $\mathcal{P}_\Lambda := \int_0^{2\pi} \tilde{g}^{-1}(0, s) ds$. In these new coordinates, one has

$$d\tilde{\theta} = d \int_0^\theta \tilde{g}^{-1}(0, s) ds = \tilde{g}^{-1}(0, \theta) d\theta \implies d\tilde{\theta}(\tilde{g}(0, \theta)\partial_\theta) = 1.$$

The fact that this is a diffeomorphism relies on the fact that $\tilde{g}(0, s) > 0$ for all $s \in [0, 2\pi]$. Therefore up to doing this reparametrization, we may assume that

$$V(x, \theta) = \tilde{f}(x, \theta)\partial_x + \tilde{g}(x, \theta)\partial_\theta \tag{52}$$

with $\theta \in \mathbb{R}/(\mathcal{P}_\Lambda\mathbb{Z})$,

$$\tilde{f}(x, \theta) = A(\theta)x + f(x, \theta), \quad \text{and} \quad \tilde{g}(x, \theta) = 1 + g(x, \theta),$$

Moreover, $f(x, \theta) = \mathcal{O}(\|x\|^2)$ and $g(x, \theta) = \mathcal{O}(\|x\|)$ uniformly in a small neighborhood of $\{x = 0\}$.

Following [44] and using the conventions of §4.3, we introduce the following nonresonance conditions:

- $(\mu_j(\Lambda))_{j=1,\dots,n-1}$ are *nonresonant in space* if, for every $\alpha \in \mathbb{N}^{n-1}$ with $\sum_{k=1}^{n-1} \alpha_k \geq 2$ and for every $1 \leq j \leq n-1$,

$$\mu_j(\Lambda) - \sum_{k=1}^{n-1} \alpha_k \mu_k(\Lambda) \notin \frac{2i\pi}{\mathcal{P}_\Lambda} \mathbb{Z};$$

- $(\mu_j(\Lambda))_{j=1,\dots,n-1}$ are *nonresonant in time* if, for every $\alpha \in \mathbb{N}^{n-1}$ with $\sum_{k=1}^{n-1} \alpha_k \geq 1$,

$$\sum_{k=1}^{n-1} \alpha_k \mu_k(\Lambda) \notin \frac{2i\pi}{\mathcal{P}_\Lambda} \mathbb{Z}.$$

We can now state the analogue of Sternberg–Chen’s theorem in the case of a hyperbolic closed orbit [44, Theorem 3]:

Theorem A.4. *Let $V(x, \theta)$ be a smooth vector field of the form (50). Suppose that $\{0\} \times \mathbb{R}/(\mathcal{P}_\Lambda \mathbb{Z})$ is a hyperbolic closed orbit for the flow generated by V and that $(\mu_j(\Lambda))_{j=1,\dots,n-1}$ are both nonresonant in time and space. Then, there exists a smooth diffeomorphism h which is defined in a neighborhood of $\{0\} \times (\mathbb{R}/\mathcal{P}_\Lambda \mathbb{Z})$ such that*

$$Vh^*(V)(x, \theta) = D_x h(L(x, \theta)),$$

with

$$L(x, \theta) = \partial_\theta + A(\theta)x \cdot \partial_x.$$

Remark A.5. Fix $k \geq 1$. As in [31], the proof of [44] could be adapted to ensure that under a *finite* number of nonresonant conditions in space and times, h can be chosen to be \mathcal{C}^k . However, we are not aware of a place in the literature where this condition is explicitly written.

Appendix B. Proof of the dynamical statements from § 3

In this appendix, for the sake of completeness, we briefly review the instructive proofs of some dynamical results due to Smale [38].

B.1. Proof of Lemma 3.5

Let x be an element in $W^u(\Lambda_i) \cap W^s(\Lambda_j)$ which does not belong to Λ_i (otherwise $i = j$ and the conclusion is trivial). Then, the flow line $t \mapsto \varphi^t(x)$ is contained in the intersection $W^u(\Lambda_i) \cap W^s(\Lambda_j)$ which should have dimension at least 1. Also, one knows

$$\begin{aligned} \dim(T_x W^u(\Lambda_i)) + \dim(T_x W^s(\Lambda_j)) &= \dim(T_x W^u(\Lambda_i) + T_x W^s(\Lambda_j)) \\ &\quad + \dim(T_x W^u(\Lambda_i) \cap T_x W^s(\Lambda_j)). \end{aligned}$$

From the transversality assumption $T_x W^u(\Lambda_i) + T_x W^s(\Lambda_j) = T_x M$ which, combined to the previous observation, implies

$$\dim(W^u(\Lambda_i)) + \dim(W^s(\Lambda_j)) \geq n + 1.$$

We now distinguish two cases. On the one hand, if Λ_j is a critical point, then $\dim(W^s(\Lambda_j)) = n - \dim(W^u(\Lambda_j))$ (by transversality at Λ_j). Hence, $\dim(W^u(\Lambda_i)) \geq 1 + \dim(W^u(\Lambda_j))$ as expected. On the other hand, if Λ_j is a closed orbit, one has, by transversality at Λ_j , $\dim(W^s(\Lambda_j)) = n - \dim(W^u(\Lambda_j)) + 1$ from which one can conclude that $\dim(W^u(\Lambda_i)) \geq \dim(W^u(\Lambda_j))$.

B.2. Proof of Lemma 3.6

From Lemma 3.5, we note that, if $W^u(\Lambda_j) \cap W^s(\Lambda_j) \neq \Lambda_j$, then Λ_j is necessarily a closed orbit. In fact, it means that there exists x that belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$ but not to Λ_j . From the case of equality in Lemma 3.5, it follows that Λ_j is a closed orbit.

Suppose now that there exists such a point x_0 , i.e., x_0 belongs to $W^u(\Lambda_j) \cap W^s(\Lambda_j)$ but not to Λ_j . Note that x_0 does not belong to Λ_i for every $i \neq j$ – see Lemma 3.3. Let U be a small open set containing x_0 . We would like to prove that, for every $t_0 > 0$, there exists $t \geq t_0$ such that $\varphi^t(U) \cap U \neq \emptyset$, which would contradict the fact that $x_0 \notin NW(\varphi^t)$.

For that purpose, fix Σ a small Poincaré section associated with the closed orbit Λ_j and centered at the point $y_0 \in \Lambda_j$ – see Appendix A. We also set D^u to be a small open disk containing x_0 inside $W^u(\Lambda_j) \cap U$ and D^s to be a small open disk containing x_0 inside $W^s(\Lambda_j) \cap U$. Fix now $t_0 > 0$ and let us show the expected contradiction. As x_0 belongs to $W^s(\Lambda_j)$, we know that there exists $t \geq t_0$ such that $\varphi^t(x_0)$ belongs to Σ . We denote by $D^u(t, \Sigma)$ the connected component of $\varphi^t(D^u) \cap \Sigma$ containing $\varphi^t(x_0)$. From the λ -lemma A.1, we know that, for t large enough, $D^u(t, \Sigma)$ is ϵ C^1 -close to the unstable manifold of the Poincaré map near y_0 (for some small ϵ). Similarly, we can work in negative times and construct $D^s(-t', \Sigma)$ which is ϵ C^1 -close to the stable manifold of the Poincaré map near y_0 . As $\dim W^u(\Lambda_j) + \dim W^s(\Lambda_j) = n + 1$, $D^u(t, \Sigma)$ (resp. $D^s(-t', \Sigma)$) have the same dimension as the unstable (resp. stable) manifolds of the induced Poincaré map on Σ . Therefore, there exists a point y_1 which lies in the intersection of $D^u(t, \Sigma)$ and $D^s(-t', \Sigma)$. Then, we set $y_2 = \varphi^{-t}(y_1) = \varphi^{-(t+t')}(y_3)$ which belongs to $U \cap \varphi^{-(t+t')}(U)$. This gives the contradiction.

B.3. Proof of Theorem 3.7

The proof of Theorem 3.7 was given by Smale in [38]. The proof of this classical result contains important ideas that enlighten our general strategy. Hence, it seems useful to recall Smale’s argument. The proof starts with the following Lemma [38, Lemma 3.3]:

Lemma B.1. *Suppose $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$. Then, $W^u(\Lambda_j) \subset \overline{W^u(\Lambda_i)}$.*

This lemma means the following. If $\varphi^t(x)$ is repelled by Λ_i and attracted by Λ_j , then the unstable manifold $W^u(\Lambda_j)$ of the set Λ_j at the arrival is contained in the closure of the bigger stratum $W^u(\Lambda_i)$.

Proof. Let x_0 be a point in $W^u(\Lambda_j)$ and let $\epsilon > 0$. We aim at constructing a point $y \in W^u(\Lambda_i)$ which is at a distance $\leq \epsilon$ of x_0 .

We start with the case where Λ_j is a hyperbolic fixed point. We first note that there exists $T_0 > 0$ such that $\varphi^{-T_0}(x_0)$ belongs to an ϵ -neighborhood O of Λ_j inside $W^u(\Lambda_j)$. We also fix a small neighborhood O' of Λ_j inside M (containing x_0). In order to construct y ,

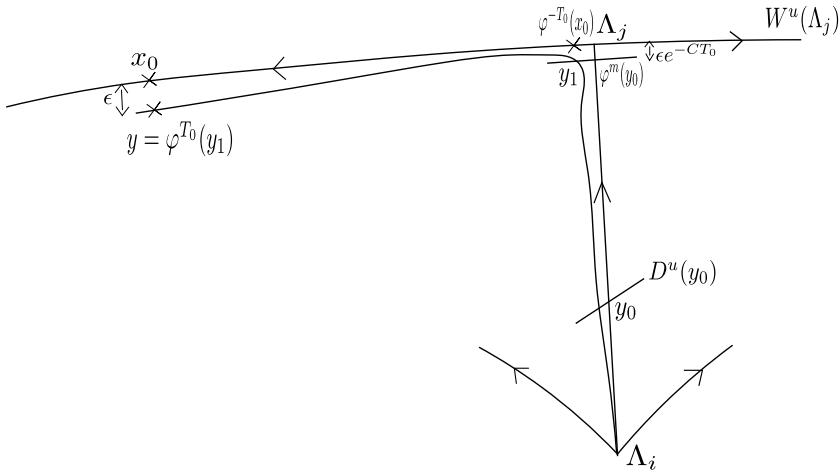


Figure 3. Proof of Lemma B.1.

we will make use of a point lying inside $W^u(\Lambda_i) \cap W^s(\Lambda_j)$. Such a point y_0 exists from our assumption. Let $D^u(y_0)$ be a small disk contained in $W^u(\Lambda_i)$ which is of dimension $\dim(W^u(\Lambda_j))$, which contains y_0 in its interior and which is transversal to $W^s(\Lambda_j)$. The existence of such a disk is provided by the dimension bound $\dim(W^u(\Lambda_j)) \leq \dim(W^u(\Lambda_i))$ from Lemma 3.5. Recall that, for a fixed point, $\dim(W^u(\Lambda_j)) + \dim(W^s(\Lambda_j)) = n$.

Using the map $f = \varphi^1$ in the λ -lemma, we can find m large enough such that the connected component of $\varphi^m(D^u(y_0)) \cap O'$ containing $\varphi^m(y_0)$ is ϵe^{-CT_0} C^1 -close to $W^u(\Lambda_j)$ near Λ_j (for some $C > 0$ larger than the maximal expansion rate of φ^t). In particular, we can find a point $y_1 \in W^u(\Lambda_i)$ which is at a distance ϵe^{-CT_0} of $\varphi^{-T_0}(x_0)$ – see Figure 3. Recall that we said that the maximal expansion rate of the flow φ^t is less than C . Hence, we have constructed a point $y = \varphi^{T_0}(y_1) \in W^u(\Lambda_i)$ which is ϵ -close to x_0 as expected.

In the case where Λ_j is a closed orbit, we shall fix Σ to be a Poincaré section transversal to Λ_j . For every $T_0 > 0$ large enough, there will exist $T > T_0$ such that $\varphi^{-T}(x_0)$ belongs to Σ . As in the proof of Lemma 3.6, we have to use the λ -lemma for the induced Poincaré map and we deduce the results following the same lines as for a fixed point. \square

We continue with the following lemma [38, Lemma 3.5]:

Lemma B.2. *Suppose $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_2}) \neq \emptyset$ and $W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_3}) \neq \emptyset$. Then, one has*

$$W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_3}) \neq \emptyset.$$

Moreover, if $i_1 = i_3$, then $i_1 = i_2 = i_3$. In particular, from Lemma 3.6, all these intersections are reduced to Λ_{i_1} .

Proof. The argument looks very much like the proof of the no-cycle Lemma 3.6. Let x_2 be an element in $W^u(\Lambda_{i_2}) \cap W^s(\Lambda_{i_3})$. Let \tilde{D}^s be a small disk inside $W^s(\Lambda_{i_3})$ which is of dimension $\dim(W^s(\Lambda_{i_2}))$ and which contains x_2 . Again, this is possible according to

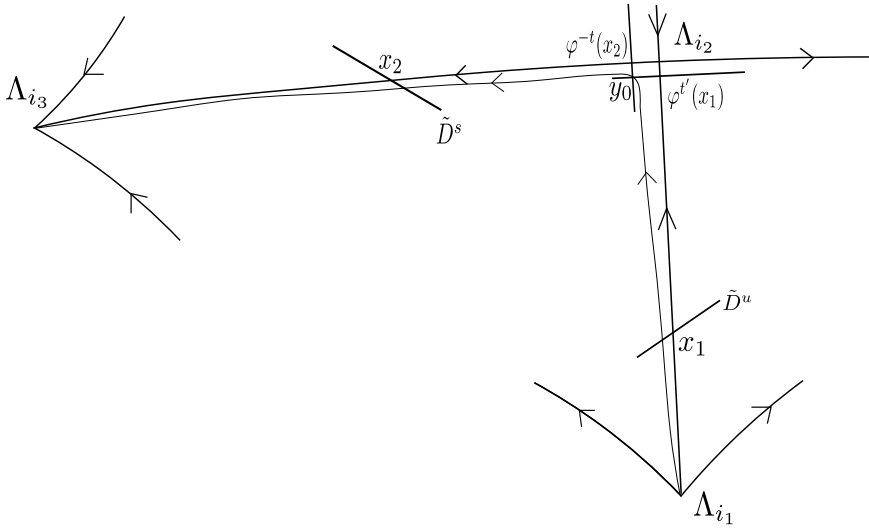


Figure 4. Proof of Lemma B.2.

Lemma 3.5. By applying φ^{-t} with $t > 0$ large enough, we have two options which follow from the λ -lemma:

- Λ_{i_2} is a hyperbolic fixed point – see Figure 4. If we set $D^s(-t)$ to be connected component of $\varphi^{-t}(\tilde{D}^s) \cap \mathcal{O}$ (where \mathcal{O} is a neighborhood of Λ_{i_2}) containing $\varphi^{-t}(x_2)$ in its interior, then $D^s(-t)$ is $\epsilon \mathcal{C}^1$ -close to the local stable manifold of Λ_{i_2} . Note that any element in $D^s(-t)$ belongs to $W^s(\Lambda_{i_3})$ and that this small piece of disk has dimension $\dim(W^s(\Lambda_{i_2}))$.
- Λ_{i_2} is a hyperbolic closed orbit. As in the proof of Lemma 3.6, we fix a Poincaré section Σ and we define a small disk $D^s(-t, \Sigma)$ inside $W^s(\Lambda_{i_3}) \cap \Sigma$ which is ϵ close to the stable manifold of the induced Poincaré map on Σ . Again, any point in $D^s(-t, \Sigma)$ is contained in the stable manifold of Λ_{i_3} . Note that this small disk has dimension $\dim(W^s(\Lambda_{i_2})) - 1$.

Then, we fix x_1 be an element in $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_2})$. Working with positive times, we can similarly define either $D^u(t')$ (which is of dimension $\dim(W^u(\Lambda_{i_2}))$) or $D^u(t', \Sigma)$ (which is of dimension $\dim(W^u(\Lambda_{i_2})) - 1$) inside $W^u(\Lambda_{i_1})$. Using the relations on the dimension as in the proof of Lemma 3.6, we know that these small pieces of disks have at least one point of intersection y_0 . Hence, the point y_0 belongs to $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_3})$ which concludes the proof of the first part.

Now, if $i_1 = i_3$, then we have constructed a point y_0 in a neighborhood of Λ_{i_2} which is contained in $W^u(\Lambda_{i_1}) \cap W^s(\Lambda_{i_1})$. Then, the no-cycle lemma implies finally $i_2 = i_1$ as expected. □

The combination of Lemmas B.1, B.2 and 6.2 gives the proof of Smale’s Theorem 3.7 as we shall now explain. First, we note that, according to Lemma 3.3, one can find, for

every $x \in M$, a unique $1 \leq j(x) \leq K$ such that $x \in W^u(\Lambda_{j(x)})$. To prove the first part of Smale’s theorem, we can verify that, for every $1 \leq i \leq K$,

$$\overline{W^u(\Lambda_i)} = \bigcup_{x \in \overline{W^u(\Lambda_i)}} W^u(\Lambda_{j(x)}).$$

Note that the first inclusion \subset is obvious by definition while the other one follows from Lemma 6.2. It now remains to show the partial order relation on the unstable manifolds inside M . We start with the transitivity property. Suppose that $\overline{W^u(\Lambda_{i_1})} \leq W^u(\Lambda_{i_2})$ and that $W^u(\Lambda_{i_2}) \leq W^u(\Lambda_{i_3})$. This exactly means that $W^u(\Lambda_{i_1}) \subset \overline{W^u(\Lambda_{i_2})}$ and $W^u(\Lambda_{i_2}) \subset \overline{W^u(\Lambda_{i_3})}$. Applying Lemma 6.2 to (i_1, i_2) and to (i_2, i_3) , we find a sequence $i_3 = j_1, \dots, j_m = i_1$ such that, for every $1 \leq p \leq m - 1$, $W^u(\Lambda_{j_p}) \cap W^s(\Lambda_{j_{p+1}}) \neq \emptyset$. Then, a combination of Lemmas B.1 and B.2 shows that $W^u(\Lambda_{i_1}) \subset \overline{W^u(\Lambda_{i_3})}$ which implies that $W^u(\Lambda_{i_1}) \leq W^u(\Lambda_{i_3})$. For the reflexivity, we suppose that both $W^u(\Lambda_i) \leq W^u(\Lambda_j)$ and $W^u(\Lambda_j) \leq W^u(\Lambda_i)$ hold true for some couple (i, j) with $i \neq j$. Then, applying Lemmas 6.2 and B.2 in this order, and as $i \neq j$, it provides the existence of a point x accumulating to Λ_i in the past and to Λ_j in the future. Similarly, we can find a point y such that the roles of i and j are reversed. Proceeding as in the proof of Lemma B.2, we can find a point z outside Λ_i which accumulates to Λ_i in the past and in the future. This contradicts Lemma 3.6 and concludes the proof of the reflexivity.

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