Can. J. Math. Vol. 47 (2), 1995 pp. 436–448

ADDITION FORMULA FOR BIG q-LEGENDRE POLYNOMIALS FROM THE QUANTUM SU(2) GROUP

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ABSTRACT. From Koornwinder's interpretation of big q-Legendre polynomials as spherical elements on the quantum SU(2) group an addition formula is derived for the big q-Legendre polynomial. The formula involves Al-Salam-Carlitz polynomials, little q-Jacobi polynomials and dual q-Krawtchouk polynomials. For the little q-ultraspherical polynomials a product formula in terms of a big q-Legendre polynomial follows by q-integration. The addition and product formula for the Legendre polynomials are obtained when q tends to 1.

1. **Introduction.** Quantum groups provide a powerful approach to special functions of basic hypergeometric type, *cf.* the survey papers by Koornwinder [12] and by Noumi [16], where the reader will also find (more) references to the literature on quantum groups and basic hypergeometric functions. In this paper we show how the quantum group theoretic interpretation of basic Jacobi polynomials leads to an addition formula for the big q-Legendre polynomials involving little q-Jacobi polynomials, dual q-Krawtchouk polynomials and Al-Salam-Carlitz polynomials.

There are now several addition formulas available for basic analogues of the Legendre polynomial. The addition formula for the continuous q-Legendre polynomial is proved analytically by Rahman and Verma [20], and a quantum SU(2) group theoretic proof of this addition formula is given by Koelink [10]. However, the quantum group theoretic proof more or less uses knowledge concerning the structure of the addition formula for the continuous q-Legendre polynomials. On the other hand, Koornwinder's [13] addition formula for the little q-Legendre polynomials follows naturally from the interpretation of the little q-Jacobi polynomials on the quantum SU(2) group and this formula would have been hard to guess without this interpretation. Rahman [19], knowing what to prove, has given an analytic proof of the addition formula for the little q-Legendre polynomials. As a follow-up to Koornwinder's [14] paper, in which he establishes an interpretation of a two-parameter family of Askey-Wilson polynomials as zonal spherical elements on the quantum SU(2) group, abstract addition formulas, *i.e.* involving non-commuting variables, have been given by Noumi and Mimachi [17] (see also [18]) and by Koelink [10].

Supported by a NATO-Science Fellowship of the Netherlands Organization for Scientific Research (NWO).
Received by the editors September 7, 1994.
AMS subject classification: 33D80, 33D45, 33C45, 42C05.
Key words and phrases: addition formula, product formula, big q-Legendre polynomials, little q-Jacobi polynomials, Al-Salam-Carlitz polynomials, quantum groups, SU(2), q-Charlier polynomials, Legendre polynomials.

As a result of this approach there is a (degenerate) addition formula for the two-parameter family of Askey-Wilson polynomials, *cf.* [17], [10].

The group theoretic proof of the addition formula for the Legendre polynomials starts with the spin l ($l \in \mathbb{Z}$) representation t^l of the group SU(2). The matrix elements $t^l_{n,m}$ are known in terms of Jacobi polynomials and the matrix element $t^l_{0,0}$ is expressible in terms of the Legendre polynomial. Moreover, $t^l_{0,0}$ is the zonal spherical function with respect to the one-parameter subgroup $K = S\big(U(1) \times U(1)\big)$ of SU(2), *i.e.* $t^l_{0,0}(gk) = t^l_{0,0}(kg) = t^l_{0,0}(g)$ for all $g \in \mathrm{SU}(2)$ and for all $k \in K$. Using the homomorphism property we get

$$t^l_{0,0}(gh) = \sum_k t^l_{0,k}(g)t^l_{k,0}(h), \quad \forall g, h \in \mathrm{SU}(2), \tag{1.1}$$

which yields the addition formula for the Legendre polynomials. We can also view (1.1) as an expression for the unique (up to a scalar) function $\mathrm{SU}(2) \ni g \longmapsto t^l_{0,0}(gh)$ in the span of the matrix elements $t^l_{n,m}$, which is left K-invariant and right hKh^{-1}-invariant. It is this view of (1.1) we adopt in this paper.

This view of (1.1) implies that we are not using the comultiplication in the quantum group theoretic derivation of the addition formula, in contrast with the quantum group theoretic proofs of addition formulas mentioned. We start with a formula relating the unique (up to a scalar) zonal spherical element, which is left and right invariant with respect to different quantum "subgroups", to the matrix elements of the standard irreducible unitary representations of the quantum SU(2) group. This formula is proved by Koornwinder in his paper [14] on zonal spherical elements on the quantum SU(2) group. In [14] Koornwinder interpreted a two-parameter family of Askey-Wilson polynomials as zonal spherical elements on the quantum SU(2) group. For a suitable choice of the parameters a quantum group theoretic interpretation of the big q-Legendre polynomials is obtained, which is a quantum group analogue of (1.1).

This identity involves non-commuting variables, so we use a representation to obtain an identity for operators acting on a Hilbert space. By letting these operators act on suitable vectors of the Hilbert space and taking inner products we obtain in a natural way an addition formula for the big q-Legendre polynomial. The addition formula involves Al-Salam-Carlitz polynomials, little q-Jacobi polynomials and dual q-Krawtchouk polynomials. The big q-Legendre polynomial corresponds to the term $t^l_{0,0}(gh)$ on the left hand side of (1.1) and the little q-Jacobi polynomials, respectively the dual q-Krawtchouk polynomials, correspond to $t^l_{0,k}(g)$, respectively $t^l_{k,0}(h)$, in (1.1). The Al-Salam-Carlitz polynomials stem from the non-commutativity.

The dual q-Krawtchouk polynomial tends to the Krawtchouk polynomial as $q \uparrow 1$ and the Krawtchouk polynomial can be rewritten as a Jacobi polynomial, *cf.* Koornwinder [11, Section 2], Nikiforov and Uvarov [15, Sections 12 and 22]. On the level of basic hypergeometric series we can rewrite the dual q-Krawtchouk polynomial as a rational function resembling a Jacobi polynomial of argument $z/(1+z)$, *cf.* [9, p. 429]. From the addition formula we obtain an expression for the product of a little q-ultraspherical polynomial times a dual q-Krawtchouk polynomial as a q-integral transformation of the big q-Legendre polynomials. We show that a special case of this addition formula is related to a special case of the addition formula for little q-Legendre polynomials, *cf.* [13].

Although our initial relation is a special case of the initial relation for Koornwinder's second addition formula for q-ultraspherical polynomials, which he announced in [14, Remark 5.4], the addition formula for the big q-Legendre polynomial proved here is not a special case of that second addition formula. This is due to the fact that we use an infinite dimensional $*$-representation on our initial relation, whereas Koornwinder uses a one-dimensional $*$-representation to obtain the q-Legendre case of his addition formula for q-ultraspherical polynomials.

It should be noted that there is an abstract addition formula for the big q-Legendre polynomial as a special case of the general abstract addition formula mentioned before, *cf.* [10], [17]. It is (at present) unknown whether it is possible to derive an addition formula for the big q-Legendre polynomials from the abstract addition formula. It might give an extension of the result presented in this paper.

This paper is organised as follows. In Sections 2 and 3 we recall the necessary information on basic hypergeometric orthogonal polynomials and on the quantum SU(2) group. The main result is proved in Section 4. Finally, in Section 5 the limit $q \uparrow 1$ is considered. This limit transition can be handled with the devices developed by Van Assche and Koornwinder [22] to prove that the addition and product formula for the little q-Legendre polynomials tend to the familiar addition and product formula for the Legendre polynomial.

2. **Preliminaries on basic hypergeometric orthogonal polynomials.** The notation for q-shifted factorials and basic hypergeometric series is taken from the book [7] by Gasper and Rahman. We will assume $q \in (0, 1)$.

The big q-Jacobi polynomials were introduced by Andrews and Askey [3, Section 3] and are defined by

$$P_n(x; a, b, c, d; q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, qax/c \\ qa, -qad/c \end{matrix}; q, q\right). \tag{2.1}$$

The polynomial $P_n(x; 1, 1, c, d; q)$ is the big q-Legendre polynomial.

The monic big q-Jacobi polynomials $\hat{P}_n$ with $a = 0$, $b = 0$, can be obtained as a limit case of (2.1). First calculate the coefficient of x^n in (2.1) and next apply [7, (3.2.3)] before taking $a \longrightarrow 0$, $b \longrightarrow 0$. We find

$$\begin{aligned} \hat{P}_n(x; 0, 0, c, d; q) &= d^n q^{\frac{1}{2}n(n-1)} {}_2\varphi_1\left(\begin{matrix} q^{-n}, c/x \\ 0 \end{matrix}; q, -\frac{qx}{d}\right), \\ \hat{P}_n(x; 0, 0, c, d; q) &= (-c)^n q^{\frac{1}{2}n(n-1)} {}_2\varphi_1\left(\begin{matrix} q^{-n}, -d/x \\ 0 \end{matrix}; q, \frac{qx}{c}\right). \end{aligned} \tag{2.2}$$

These polynomials satisfy the three-term recurrence relation

$$\begin{aligned} x\hat{P}_n(x; 0, 0, c, d; q) = \hat{P}_{n+1}(x; 0, 0, c, d; q) &+ q^n(c - d)\hat{P}_n(x; 0, 0, c, d; q) \\ &+ q^{n-1}cd(1 - q^n)\hat{P}_{n-1}(x; 0, 0, c, d; q). \end{aligned} \tag{2.3}$$

Comparison of (2.3) with the three-term recurrence relation for the Al-Salam-Carlitz polynomials, *cf.* [1, Section 4], [6, Chapter VI, Section 10], shows that these monic big q-Jacobi polynomials are Al-Salam-Carlitz polynomials with dilated argument, $\hat{P}_n(x;0,0,c,d;q) = c^n U_n^{(-d/c)}(x/c;q)$. The orthogonality relations for the $\hat{P}_n(\cdot;0,0,c,d;q)$ can be phrased as

(2.4)
$$\int_{-d}^{c} \big(\hat{P}_n\hat{P}_m\big)(x;0,0,c,d;q)(qx/c,-qx/d;q)_\infty \, d_qx \\ = \delta_{n,m} q^{\frac{1}{2}n(n-1)}(cd)^n(q;q)_n(1-q)c(q,-d/c,-qc/d;q)_\infty.$$

Here the q-integral is defined by, *cf.* [7, Section 1.11],

$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx, \quad \int_0^a f(x)\,d_qx = a(1-q)\sum_{k=0}^{\infty} f(aq^k)q^k.$$

We will also need the little q-Jacobi polynomials $p_n(x;a,b;q)$, *cf.* Andrews and Askey [2, Section 3], [3, Section 3]. The little q-Jacobi polynomials are big q-Jacobi polynomials with $c=1$ and $d=0$ and normalised such that the value at 0 is 1. Explicitly,

$$p_n(x;a,b;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{n+1}ab \\ qa \end{matrix}; q, qx\right). \tag{2.5}$$

The last set of orthogonal polynomials needed is the set of dual q-Krawtchouk polynomials, *cf.* [21, Section 4], which is a special case of the q-Racah polynomials, *cf.* [5, Section 4].

$$R_n(q^{-x} - s^{-1}q^{x-N}; s, N; q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-x}, -s^{-1}q^{x-N} \\ q^{-N}, 0 \end{matrix}; q, q\right) \tag{2.6}$$

for $n \in \{0,\dots,N\}$.

3. **Results on the quantum** SU(2) **group.** Let $q \in (0,1)$ be a fixed number. The unital $*$-algebra $\mathcal{A}_q$ is generated by the elements α and γ subject to the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \\ \alpha^*\alpha + \gamma\gamma^* = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1. \tag{3.1}$$

For $q \uparrow 1$ the algebra can be identified with the algebra of polynomials on the group SU(2). The algebra $\mathcal{A}_q$ is actually a Hopf $*$-algebra. See [12], [16] for references to the literature.

The irreducible unitary corepresentations of the Hopf $*$-algebra $\mathcal{A}_q$ have been classified. For each dimension $2l+1$, $l \in \frac{1}{2}\mathbb{Z}_+$, there is precisely one such corepresentation, which we denote by $t^l = \big(t^l_{n,m}\big)$, $n,m \in \{-l,-l+1,\dots,l\}$. The matrix coefficients

$t^l_{n,m} \in \mathcal{A}_q$ are explicitly known in terms of little q-Jacobi polynomials. For our purposes it suffices to have

$$(3.2)\qquad \begin{aligned} t^l_{0,m} &= d^l_m(\alpha^*)^m p_{l-m}(\gamma\gamma^*;q^{2m},q^{2m};q^2)(-q\gamma^*)^m \\ t^l_{0,-m} &= d^l_m\gamma^m p_{l-m}(\gamma\gamma^*;q^{2m},q^{2m};q^2)\alpha^m \end{aligned}$$

with

$$d^l_m = \frac{q^{-m(l-m)}}{(q^2;q^2)_m}\sqrt{\frac{(q^2;q^2)_{l+m}}{(q^2;q^2)_{l-m}}}$$

for $l \in \mathbb{Z}_+$, $m = 0,\ldots,l$. See [12], [16] for this result as well as for references to the literature.

Next we recall a special case of Koornwinder's result [14, Theorem 5.2] on general spherical elements on the quantum SU(2) group. The case we consider is the case $\tau \longrightarrow \infty$ of [14, Theorem 5.2]. Explicitly, the following identity in $\mathcal{A}_q$ is valid;

$$(3.3)\qquad \sum_{m=-l}^{l} q^{-m/2}c^{l,\sigma}_m t^l_{0,m} = C_l(\sigma)P_l(\rho_{\sigma,\infty};1,1,q^{2\sigma},1;q^2).$$

where $\sigma \in \mathbb{R}$,

$$c^{l,\sigma}_m = c^{l,\sigma}_{-m} = \frac{i^m q^{-(l+\sigma)m+\frac{1}{2}m^2}}{\sqrt{(q^2;q^2)_{l+m}(q^2;q^2)_{l-m}}}R_{l-m}(q^{-2l}-q^{-2l-2\sigma};q^{2\sigma},2l;q^2),$$

$$C_l(\sigma) = (-1)^l q^{-l^2-l}\frac{(-q^{2-2\sigma};q^2)_l}{(q^{2l+2};q^2)_l}$$

are constants and

$$\rho_{\sigma,\infty} = \lim_{\tau\longrightarrow\infty} 2q^{\sigma+\tau-1}\rho_{\sigma,\tau} = iq^\sigma(\alpha^*\gamma^*-\gamma\alpha)-(1-q^{2\sigma})\gamma^*\gamma \in \mathcal{A}_q.$$

Here $\rho_{\sigma,\tau}$ is defined in [14, (4.8)]. Equation (3.3) can be proved by redoing Koornwinder's [14] analysis with X_τ replaced by X_∞ or by taking the limit $\tau \longrightarrow \infty$ in his result [14, Theorem 5.2]. In the latter case we use the limit transition of the Askey-Wilson polynomials to the big q-Legendre polynomials as described in [14, Theorem 6.2], $c^{l,\sigma}_l = i^l q^{-\frac{1}{2}l^2-l\sigma}(q^2;q^2)_{2l}^{-1/2}$, and the limit

$$\lim_{\tau\longrightarrow\infty} q^{2\tau l}c^{l,\tau}_m = \frac{(-1)^l\delta_{m,0}}{(q^{2l+2};q^2)_l}.$$

This follows for $m \geq 0$ from [7, (3.2.3) with $e = 0$, (1.5.3)] and by the symmetry $c^{l,\tau}_{-m} = c^{l,\tau}_m$ for all m.

A $*$-representation π of the commutation relations (3.1) is acting on $\ell^2(\mathbb{Z}_+)$ equipped with an orthonormal basis $\{e_n\}_{\{n\in\mathbb{Z}_+\}}$, and the explicit action of the generators is given by

$$(3.4)\qquad \pi(\alpha)e_n = \sqrt{1-q^{2n}}e_{n-1},\qquad \pi(\gamma)e_n = q^n e_n.$$

The irreducible $*$-representations of $\mathcal{A}_q$ have been classified, *cf.* [12] and the references therein. The infinite dimensional $*$-representations are parametrised by the unit circle; $\pi_\theta(\alpha) = \pi(\alpha)$ and $\pi_\theta(\gamma) = e^{i\theta}\pi(\gamma)$ for $\theta \in [0,2\pi)$.

4. **Addition formula for big q-Legendre polynomials.** In this section we prove an addition formula for the big q-Legendre polynomials. We start by representing the relation (3.3) in $\mathcal{A}_q$ as an identity for operators in the Hilbert space $\ell^2(\mathbb{Z}_+)$. Letting these operators act on suitable vectors and taking inner products yields the addition formula. This addition formula involves Al-Salam-Carlitz polynomials, little q-Jacobi polynomials and dual q-Krawtchouk polynomials. From the addition formula we find a q-integral representation for the product of a little q-Jacobi polynomial and a dual q-Krawtchouk polynomial.

Consider the action of the infinite dimensional $*$-representation π in $\ell^2(\mathbb{Z}_+)$ on $\rho_{\sigma,\infty}$. The operator $\pi(\rho_{\sigma,\infty})$ is a bounded self-adjoint operator and the action on a basis vector e_n of the standard orthonormal basis is given by

$$\pi(\rho_{\sigma,\infty})e_n = -iq^{\sigma+n-1}\sqrt{1-q^{2n}}e_{n-1} - q^{2n}(1-q^{2\sigma})e_n + iq^{\sigma+n}\sqrt{1-q^{2n+2}}e_{n+1}.$$

Consequently, $\sum_{n=0}^{\infty} p_n e_n$ is an eigenvector of $\pi(\rho_{\sigma,\infty})$ for the eigenvalue λ if and only if

$$(4.1)\qquad \lambda p_n = -iq^{\sigma+n}\sqrt{1-q^{2n+2}}p_{n+1} - q^{2n}(1-q^{2\sigma})p_n + iq^{\sigma+n-1}\sqrt{1-q^{2n}}p_{n-1} \quad \forall n.$$

Since $p_{-1}=0$ and $p_0=1$, we view (4.1) as a three-term recurrence for polynomials in λ. In order to determine the polynomials from (4.1) we calculate the leading coefficient $lc(p_n) = i^n q^{-\sigma n} q^{-\frac{1}{2}n(n-1)}(q^2;q^2)_n^{-\frac{1}{2}}$ and determine the three-term recurrence relation for the monic polynomials $\hat{p}_n$;

$$(4.2)\qquad \lambda\hat{p}_n(\lambda) = \hat{p}_{n+1}(\lambda) - q^{2n}(1-q^{2\sigma})\hat{p}_n(\lambda) + (1-q^{2n})q^{2\sigma+2n-2}\hat{p}_{n-1}(\lambda).$$

Comparison of (4.2) with the three-term recurrence relation (2.3) for the big q-Jacobi polynomials with $a=0$ and $b=0$ leads to

$$(4.3)\qquad p_n(\lambda) = i^n q^{-\sigma n} q^{-\frac{1}{2}n(n-1)}(q^2;q^2)_n^{-\frac{1}{2}}\hat{P}_n(\lambda;0,0,q^{2\sigma},1;q^2).$$

Denote the corresponding vector by $v_\lambda = \sum_{n=0}^{\infty} p_n(\lambda)e_n$.

PROPOSITION 4.1. *For $\lambda=-q^{2x}$, $x\in\mathbb{Z}_+$, and $\lambda=q^{2\sigma+2x}$, $x\in\mathbb{Z}_+$, the vectors v_λ constitute an orthogonal basis of $\ell^2(\mathbb{Z}_+)$.*

PROOF. From the asymptotic formula, *cf.* [8, (1.17)], as $n\longrightarrow\infty$

$$\hat{P}_n(\lambda;0,0,c,d;q) \sim \lambda^n(c/\lambda,-d/\lambda;q)_\infty$$

for $\lambda\neq 0$, $\lambda\neq cq^x$ and $\lambda\neq -dq^x$, $x\in\mathbb{Z}_+$, it follows that $v_\lambda\notin\ell^2(\mathbb{Z}_+)$ for $\lambda\neq -q^{2x}$ and $\lambda\neq q^{2\sigma+2x}$, $x\in\mathbb{Z}_+$.

In the remaining cases we use the straightforward estimate

$$(4.4)\qquad \left|{}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix}; q, z\right)\right| \leq q^{-xn}(-q^{-x};q)_x(q,-|z|;q)_\infty,$$

for fixed $x\in\mathbb{Z}_+$, in combination with the series representation (2.2) for the monic big q-Jacobi polynomials $\hat{P}_n(\cdot;0,0,c,d;q)$ to see that we obtain eigenvectors $v_\lambda\in\ell^2(\mathbb{Z}_+)$ for $\pi(\rho_{\sigma,\infty})$ for the eigenvalues $\lambda=-q^{2x}$, $x\in\mathbb{Z}_+$, and $\lambda=q^{2\sigma+2x}$, $x\in\mathbb{Z}_+$.

The orthogonality follows, since the vectors are eigenvectors of a self-adjoint operator for different eigenvalues. It remains to prove the completeness of the set of eigenvectors in $\ell^2(\mathbb{Z}_+)$. To do this we first calculate the length of the eigenvectors in $\ell^2(\mathbb{Z}_+)$. Consider λ of the form $q^{2\sigma+2x}$, $x \in \mathbb{Z}_+$, then we have proved the orthogonality relations

$$(4.5)\qquad \begin{aligned} h_x\delta_{x,y} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}q^{-2\sigma n}}{(q^2;q^2)_n} {}_2\varphi_1\left(\begin{matrix} q^{-2x}, q^{-2n} \\ 0 \end{matrix}; q^2, -q^{2+2\sigma+2x}\right) \\ \times {}_2\varphi_1\left(\begin{matrix} q^{-2y}, q^{-2n} \\ 0 \end{matrix}; q^2, -q^{2+2\sigma+2y}\right), \end{aligned}$$

for $x, y \in \mathbb{Z}_+$, $h_x > 0$. We view the ${}_2\varphi_1$-series as a polynomial of degree x in the variable q^{-2n}. It has leading coefficient $(-1)^x q^{2x(x+\sigma)}$. Since (4.5) holds, we have

$$(4.6)\qquad h_x = (-1)^x q^{2x(x+\sigma)} \sum_{n=0}^{\infty} \frac{q^{n(n-1)}q^{-2\sigma n}}{(q^2;q^2)_n} {}_2\varphi_1\left(\begin{matrix} q^{-2x}, q^{-2n} \\ 0 \end{matrix}; q^2, -q^{2+2\sigma+2x}\right) q^{-2nx}.$$

In (4.6) we replace the ${}_2\varphi_1$-series by its terminating series representation

$$\sum_{k=0}^{x} \frac{(q^{-2x};q^2)_k (q^{-2n};q^2)_k}{(q^2;q^2)_k} (-1)^k q^{2k(1+\sigma+x)}$$

and we interchange the summations, which is justified by the estimate (4.4). The inner sum over n starts at $n = k$ and after a shift in the summation parameter the inner sum can be evaluated using ${}_0\varphi_0(-;-;q,z) = (z;q)_\infty$, *cf.* [7, (1.3.16)]. The remaining sum over k can be summed using the q-binomial theorem ${}_1\varphi_0(q^{-p};-;q,z) = (q^{-p}z;q)_p$, *cf.* [7, (1.3.14)]. The result is

$$(4.7)\qquad h_x = q^{-2x}(q^2;q^2)_x(-q^{2\sigma+2};q^2)_x(-q^{-2\sigma};q^2)_\infty.$$

So $w_x = v_{q^{2\sigma+2x}}/\|v_{q^{2\sigma+2x}}\|$ is an eigenvector of length 1 of the self-adjoint operator $\pi(\rho_{\sigma,\infty})$.

The orthogonality relations for the eigenvectors corresponding to eigenvalues of the form $-q^{2x}$, $x \in \mathbb{Z}_+$, is (4.5) with σ replaced by $-\sigma$. So $u_x = v_{-q^{2x}}/\|v_{-q^{2x}}\|$ is an eigenvector of length 1 of the self-adjoint operator $\pi(\rho_{\sigma,\infty})$.

The set of orthonormal eigenvectors $\{u_x\}_{\{x\in\mathbb{Z}_+\}} \cup \{w_x\}_{\{x\in\mathbb{Z}_+\}}$ forms a complete set of basis vectors for $\ell^2(\mathbb{Z}_+)$ if and only if the dual orthogonality relations

$$(4.8)\qquad \delta_{n,m} = \sum_{x=0}^{\infty} \langle u_x, e_n\rangle \overline{\langle u_x, e_m\rangle} + \sum_{x=0}^{\infty} \langle w_x, e_n\rangle \overline{\langle w_x, e_m\rangle}$$

hold. It is easily seen that (4.8) is equivalent to the orthogonality relations (2.4) for the monic big q-Jacobi polynomials $\hat{P}_n(\cdot;0,0,q^{2\sigma},1;q^2)$. The first sum in (4.8) corresponds to the q-integral over $[-1,0]$ and the second sum corresponds to the q-integral over $[0,q^{2\sigma}]$. ∎

The orthogonality relations $\langle u_x, u_y\rangle = \delta_{x,y}$ (or $\langle w_x, w_y\rangle = \delta_{x,y}$) and $\langle u_x, w_y\rangle = 0$ can be stated in terms of the q-Charlier polynomials, *cf.* [7, exercise 7.13]. (Note that the factor on the right hand side in [7, exercise 7.13] has to be replaced by its reciprocal.)

https://doi.org/10.4153/CJM-1995-024-8 Published online by Cambridge University Press

COROLLARY 4.2. *Define the q-Charlier polynomials by*

$$c_n(x;a;q) = {}_2\varphi_1(q^{-n}, x; 0; q, -q^{n+1}/a), \quad a > 0,$$

then

$$\sum_{x=0}^{\infty} \frac{a^x q^{\frac{1}{2}x(x-1)}}{(q;q)_x} \big(c_n c_m\big)(q^{-x};a;q) = \delta_{n,m} q^{-n} (q;q)_n (-q/a;q)_n (-a;q)_\infty$$

and

$$\sum_{x=0}^{\infty} \frac{(-1)^x q^{\frac{1}{2}x(x-1)}}{(q;q)_x} c_n(q^{-x};a;q) c_m(q^{-x};a^{-1};q) = 0.$$

In order to convert (3.3) into a relation involving commuting variables we apply the infinite dimensional $*$-representation π to it. We let the resulting bounded operator act on a standard basis vector e_p and we take inner products with an eigenvector $v_\lambda \in \ell^2(\mathbb{Z}_+)$, *cf.* Proposition 4.1. Next we use the fact that π is a $*$-representation to get the following identity

$$(4.9) \qquad \sum_{m=-l}^{l} q^{-m/2} c_m^{l,\sigma} \langle \pi\big(t^l_{0,m}\big) e_p, v_\lambda \rangle = C_l(\sigma) P_l(\lambda; 1, 1, q^{2\sigma}, 1; q^2) \langle e_p, v_\lambda \rangle,$$

since $P_l(\lambda; 1, 1, q^{2\sigma}, 1; q^2)$ is a polynomial with real coefficients and $\pi(\rho_{\sigma,\infty})$ is self-adjoint. The operator on the left hand side of (4.9) can be calculated explicitly by (3.2) and (3.4), since the standard basis vector e_p is an eigenvector of $\pi(\gamma)$. Explicitly, for $m \geq 0$,

$$(4.10) \qquad \begin{aligned} \pi\big(t^l_{0,m}\big) e_p &= d^l_m (-1)^m q^{m(p+1)} \sqrt{(q^{2p+2};q^2)_m} p_{l-m}(q^{2p}; q^{2m}, q^{2m}; q^2) e_{p+m}, \\ \pi\big(t^l_{0,-m}\big) e_p &= d^l_m q^{m(p-m)} \sqrt{(q^{2p};q^{-2})_m} p_{l-m}(q^{2(p-m)}; q^{2m}, q^{2m}; q^2) e_{p-m}, \end{aligned}$$

with the convention $e_n = 0$ for $n < 0$. Furthermore, from (4.3) it follows that for all $p \in \mathbb{Z}_+$

$$(4.11) \qquad \langle e_p, v_\lambda \rangle = \frac{i^{-p} q^{-\sigma p} q^{-\frac{1}{2}p(p-1)}}{\sqrt{(q^2;q^2)_p}} \hat{P}_p(\lambda; 0, 0, q^{2\sigma}, 1; q^2).$$

Now we use (4.10) and (4.11) in (4.9) together with the explicit values for $C_l(\sigma)$, $c^{l,\sigma}_m$ and d^l_m, *cf.* (3.2), (3.3). Divide the resulting identity by the factor in front of the monic big q-Jacobi polynomial in (4.11) to obtain

$$(4.12) \qquad \begin{aligned} &(-1)^l q^{-l^2-l} \frac{(-q^{2-2\sigma};q^2)_l}{(q^{2l+2};q^2)_l} P_l(\lambda; 1, 1, q^{2\sigma}, 1; q^2) \hat{P}_l(\lambda; 0, 0, q^{2\sigma}, 1; q^2) \\ &= \frac{1}{(q^2;q^2)_l} R_l(q^{-2l} - q^{-2l-2\sigma}; q^{2\sigma}, 2l; q^2) p_l(q^{2p}; 1, 1; q^2) \hat{P}_p(\lambda; 0, 0, q^{2\sigma}, 1; q^2) \\ &\quad + \sum_{m=1}^{l} (-1)^m \frac{q^{2m(p-l)} (q^{2p};q^{-2})_m}{(q^2;q^2)_{l-m} (q^2;q^2)_m} R_{l-m}(q^{-2l} - q^{-2l-2\sigma}; q^{2\sigma}, 2l; q^2) \\ &\qquad \times p_{l-m}(q^{2(p-m)}; q^{2m}, q^{2m}; q^2) \hat{P}_{p-m}(\lambda; 0, 0, q^{2\sigma}, 1; q^2) \\ &\quad + \sum_{m=1}^{l} (-1)^m \frac{q^{m(m+1)-2m(\sigma+l)}}{(q^2;q^2)_{l-m} (q^2;q^2)_m} R_{l-m}(q^{-2l} - q^{-2l-2\sigma}; q^{2\sigma}, 2l; q^2) \\ &\qquad \times p_{l-m}(q^{2p}; q^{2m}, q^{2m}; q^2) \hat{P}_{p+m}(\lambda; 0, 0, q^{2\sigma}, 1; q^2) \end{aligned}$$

for $\lambda = -q^{2x}, x \in \mathbb{Z}_+$, or $\lambda = q^{2\sigma+2x}, x \in \mathbb{Z}_+$.

We can now state and prove the main theorem of the paper.

THEOREM 4.3 (ADDITION FORMULA FOR THE BIG q-LEGENDRE POLYNOMIAL). *With the notation of (2.1), (2.2), (2.5) and (2.6) we have for $c, d > 0$, $p, l \in \mathbb{Z}_+$, $x \in \mathbb{C}$,*

$$
\begin{aligned}
(-1)^l q^{-\frac{1}{2}l(l+1)} \frac{(-qd/c;q)_l}{(q^{l+1};q)_l} P_l(x;1,1,c,d;q)\hat{P}_p(x;0,0,c,d;q) \\
= (q;q)_l^{-1} R_l\Big(q^{-l} - \frac{d}{c}q^{-l}; \frac{c}{d}, 2l; q\Big) p_l(q^p;1,1;q)\hat{P}_p(x;0,0,c,d;q) \\
+ \sum_{m=1}^{l} (-1)^m \frac{d^m q^{m(p-l)}(q^p;q^{-1})_m}{(q;q)_{l-m}(q;q)_m} R_{l-m}\Big(q^{-l} - \frac{d}{c}q^{-l}; \frac{c}{d}, 2l; q\Big) \\
\times p_{l-m}(q^{p-m};q^m,q^m;q)\hat{P}_{p-m}(x;0,0,c,d;q) \\
+ \sum_{m=1}^{l} (-1)^m \frac{q^{\frac{1}{2}m(m+1)-lm}}{c^m(q;q)_{l-m}(q;q)_m} R_{l-m}\Big(q^{-l} - \frac{d}{c}q^{-l}; \frac{c}{d}, 2l; q\Big) \\
\times p_{l-m}(q^p;q^m,q^m;q)\hat{P}_{p+m}(x;0,0,c,d;q).
\end{aligned}
\tag{4.13}
$$

PROOF. Since (4.12) only involves polynomials, it holds for all values of λ. In (4.12) we replace $q^2, q^{2\sigma}, \lambda$ by $q, c/d, x/d$. Now (4.13) follows from

$$P_n(x/d;a,b,c/d,1;q) = P_n(x;a,b,c,d;q),$$
$$\hat{P}_n(x/d;0,0,c/d,1;q) = d^{-n}\hat{P}_n(x;0,0,c,d;q),$$

which is a consequence of (2.1) and (2.2). ■

REMARKS. 1. The choice of the infinite dimensional $*$-representation does not influence the result. We would obtain the same addition theorem if we had considered the development of a (∞,τ)-spherical element in terms of the standard matrix elements instead of (3.3).

2. If we specialise $c = 1$ and $d = 0$ in (4.13), then we can sum the dual q-Krawtchouk polynomials R_{l-m} by the q-Chu-Vandermonde sum [7, (1.5.3)], from which we see that R_{l-m} equals $(q^{m+1};q)_{l-m}/(q^{l+m+1};q)_{l-m}$. The monic big q-Jacobi polynomial $\hat{P}_p$ with $a = b = d = 0$, $c = 1$ is summable by the q-binomial theorem [7, (1.3.14)], which results in $(-1)^p q^{\frac{1}{2}p(p-1)}(q^{1-p}x;q)_p$. Furthermore, the big q-Legendre polynomial reduces to $(-1)^l q^{\frac{1}{2}l(l+1)} p_l(x;1,1;q)$, so that we obtain the following special case of (4.13);

$$p_l(x;1,1;q)(q^{1-p}x;q)_p = \sum_{m=0}^{l} \frac{q^{m(m-l+p)}(q;q)_{l+m}}{(q;q)_{l-m}(q;q)_m^2} p_{l-m}(q^p;q^m,q^m;q)(q^{1-p-m}x;q)_{p+m}.$$

This corresponds to the case $x \longrightarrow \infty$ of Koornwinder's addition formula for the little q-Legendre polynomials [13, Theorem 4.1 with $q^z = x$].

The following q-integral representation for the product of a dual q-Krawtchouk polynomial and a little q-ultraspherical polynomial is a direct consequence of Theorem 4.3 and the orthogonality relations (2.4). Just multiply (4.13) by $\hat{P}_{p+m}(x;0,0,c,d;q)$ and q-integrate over $[-d, c]$ with respect to the weight function $(qx/c, -qx/d;q)_\infty$.

COROLLARY 4.4. *For $c,d>0$, $p,l\in\mathbb{Z}_+$, $m\in\{0,\ldots,l\}$ we have*

(4.14)
$$R_{l-m}\Big(q^{-l}-\frac{d}{c}q^{-l};\frac{c}{d},2l;q\Big)p_{l-m}(q^p;q^m,q^m;q)$$
$$=C\int_{-d}^{c}P_l(x;1,1,c,d;q)(\hat{P}_p\hat{P}_{p+m})(x;0,0,c,d;q)(qx/c,-qx/d;q)_\infty\,d_qx$$

with
$$C=\frac{(-1)^{l+m}q^{-\frac12 l(l+1)-\frac12 p(p-1)+m(l-p-m)}c^{-p}d^{-p-m}(-qd/c;q)_l(q;q)_{l-m}}{(1-q)c(q^{l+1};q)_l(q^{m+1};q)_p(q,-d/c,-qc/d;q)_\infty}.$$

Multiplying (4.13) by $\hat{P}_{p-m}(x;0,0,c,d,;q)$ and q-integrating over $[-d,c]$ yields the same result (4.14). Specialising $m=0$ in (4.14) shows that the product of the little q-Legendre polynomial and a dual q-Krawtchouk polynomial can be written as a q-integral transform with a positive kernel of the big q-Legendre polynomial.

5. **The limit case $q\uparrow 1$.** In this section we show that the addition formula for the big q-Legendre polynomials (4.13) and the product formula (4.14) tend to the addition and product formula for the Legendre polynomials as $q\uparrow 1$. The general theorems of Van Assche and Koornwinder [22] used to obtain the addition and product formula for the Legendre polynomials form the addition and product formula for the little q-Legendre polynomials, *cf.* [13], are applicable in this case as well. See Askey [4, Lecture 4] for information on addition formulas for classical orthogonal polynomials.

We use the notation $R_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomial normalised by $R_n^{(\alpha,\beta)}(1)=1$. First we note that the little and big q-Jacobi polynomials tend to the Jacobi polynomials of shifted argument as $q\uparrow 1$;

(5.1)
$$\lim_{q\uparrow 1}P_n(x;q^\alpha,q^\beta,c,d;q)=R_n^{(\alpha,\beta)}\Big(\frac{2x+d-c}{c+d}\Big),$$
$$\lim_{q\uparrow 1}p_n(x;q^\alpha,q^\beta;q)=R_n^{(\alpha,\beta)}(1-2x).$$

The dual q-Krawtchouk polynomial can be rewritten as a ${}_2\varphi_2$-series, which tends to a Jacobi polynomial as $q\uparrow 1$. This has also been used in [9, p. 429] to prove that the q-Krawtchouk polynomial tends to Jackson's q-Bessel function. We can also let the dual q-Krawtchouk tend to the Krawtchouk polynomial and use the relation between Krawtchouk polynomials and Jacobi polynomials, *cf.* [11, Section 2], [15, Sections 12, 22]. The result is

(5.2)
$$\lim_{q\uparrow 1}R_{l-m}\Big(q^{-l}-\frac{d}{c}q^{-l};\frac{c}{d},2l;q\Big)=\frac{(m+1)_{l-m}}{(l+m+1)_{l-m}}\Big(1+\frac{d}{c}\Big)^{l-m}R_{l-m}^{(m,m)}\Big(\frac{c-d}{c+d}\Big).$$

In order to apply the theorems of Van Assche and Koornwinder [22] we have to consider the orthonormal big q-Jacobi polynomials with $a=0$, $b=0$. Define

(5.3)
$$p_k(x;q)=\frac{\hat{P}_k(x;0,0,c,d;q)}{q^{\frac14 k(k-1)}(cd)^{k/2}(1-q)^{1/2}c^{1/2}\sqrt{(q;q)_k(q,-d/c,-qc/d;q)_\infty}},$$

then the polynomials $p_k(x;q)$ satisfy the recurrence relation

$$xp_k(x;q) = a_{k+1}(q)p_{k+1}(x;q) + b_k(q)p_k(x;q) + a_k(q)p_{k-1}(x;q)$$

with

$$a_k(q) = q^{\frac{1}{2}(k-1)}\sqrt{cd(1-q^k)}, \quad b_k(q) = q^k(c-d).$$

Fix $r \in (0,1)$ and define $a_{k,n} = a_k(r^{1/n})$ and $b_{k,n} = b_k(r^{1/n})$. The following limits are easily established;

$$\lim_{n\to\infty} a_{n,n} = \sqrt{rcd(1-r)} > 0, \quad \lim_{n\to\infty} b_{n,n} = r(c-d) \in \mathbb{R}$$

and

$$\lim_{n\to\infty} (a_{k,n}^2 - a_{k-1,n}^2) = 0, \quad \lim_{n\to\infty} (b_{k,n} - b_{k-1,n}) = 0$$

uniformly in k. Now [22, Theorem 1] can be applied and it yields

$$\lim_{n\to\infty} \frac{p_{n+1}(x;r^{1/n})}{p_n(x;r^{1/n})} = \rho\Big(\frac{x-r(c-d)}{2\sqrt{rcd(1-r)}}\Big) \tag{5.4}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-d,c]$. Here $\rho(x) = x + \sqrt{x^2-1}$ and the square root is the one for which $|\rho(x)| > 1$ for $x \notin [-1,1]$. Rewriting (5.4) in terms of the big q-Jacobi polynomial and iterating yields

$$\lim_{p\to\infty} \frac{\hat{P}_{p+m}(x;0,0,c,d;r^{1/p})}{\hat{P}_p(x;0,0,c,d;r^{1/p})} = \big(cdr(1-r)\big)^{m/2}\rho^m\Big(\frac{x-r(c-d)}{2\sqrt{rcd(1-r)}}\Big) \tag{5.5}$$

for all $m \in \mathbb{Z}$ and $x \in \mathbb{C}\setminus[-d,c]$.

Now the proof that (4.13) tends to the addition formula for Legendre polynomials can be finished. Replace q by $r^{1/p}$ in (4.13), divide both sides by $(q^{l+1};q)_l^{-1}\hat{P}_p(x;0,0,c,d;r^{1/p})$ and let $p \to \infty$, *i.e.* $q \uparrow 1$, then we can use (5.1), (5.2) and (5.5) to obtain, after a short calculation,

$$\begin{aligned} R_l^{(0,0)}\Big(\frac{2x+d-c}{c+d}\Big) &= (-1)^l R_l^{(0,0)}\Big(\frac{c-d}{c+d}\Big)R_l^{(0,0)}(1-2r) \\ &+ \sum_{m=1}^{l} \frac{(l+1)_l(m+1)_{l-m}}{(l-m)!\,m!(l+m+1)_{l-m}}(-1)^{l+m}(1+\frac{d}{c})^{-m}\Big(\frac{d}{c}r(1-r)\Big)^{m/2} \\ &\times R_{l-m}^{(m,m)}\Big(\frac{c-d}{c+d}\Big)R_{l-m}^{(m,m)}(1-2r)\Big[\rho^m\Big(\frac{x-r(c-d)}{2\sqrt{rcd(1-r)}}\Big) + \rho^{-m}\Big(\frac{x-r(c-d)}{2\sqrt{rcd(1-r)}}\Big)\Big]. \end{aligned} \tag{5.6}$$

The term in square brackets equals $2T_m\Big(\big(x-r(c-d)\big)/2\sqrt{rcd(1-r)}\Big)$, where $T_m(\cos\theta) = \cos m\theta$ is the Chebyschev polynomial of the first kind. In (5.6) we also use $R_n^{(m,m)}(-x) = (-1)^n R_n^{(m,m)}(x)$, then we find, after a short manipulation of the Pochhammer symbols,

$$\begin{aligned} R_l^{(0,0)}\Big(\frac{2x+d-c}{c+d}\Big) &= R_l^{(0,0)}\Big(\frac{c-d}{c+d}\Big)R_l^{(0,0)}(1-2r) \\ &+2\sum_{m=1}^{l} \frac{(l+m)!}{(l-m)!\,(m!)^2}(1+\frac{d}{c})^{-m}\Big(\frac{d}{c}r(1-r)\Big)^{m/2}R_{l-m}^{(m,m)}\Big(\frac{d-c}{c+d}\Big) \\ &\times R_{l-m}^{(m,m)}(1-2r)T_m\Big(\frac{x-r(c-d)}{2\sqrt{rcd(1-r)}}\Big). \end{aligned} \tag{5.7}$$

https://doi.org/10.4153/CJM-1995-024-8 Published online by Cambridge University Press

Since the dependence on x in (5.7) is polynomial, the restriction $x \in \mathbb{C}\backslash[-d,c]$ can be removed. Formula (5.7) is equivalent to the addition formula for the Legendre polynomial, *cf.* [4, Lecture 4],

$$(5.8)\qquad \begin{aligned} R_l^{(0,0)}\big(xy+t\sqrt{(1-x^2)(1-y^2)}\big) &= R_l^{(0,0)}(x)R_l^{(0,0)}(y)\\ &+2\sum_{m=1}^{l}\frac{(l+m)!}{(l-m)!\,(m!)^2}2^{-2m}\big(\sqrt{(1-x^2)(1-y^2)}\big)^m R_{l-m}^{(m,m)}(x)R_{l-m}^{(m,m)}(y)T_m(t)\end{aligned}$$

by identifying $(d-c)/(c+d)$, $1-2r$, $\big(x-r(c-d)\big)/2\sqrt{rcd(1-r)}$ with x, y and t.

The limit case of the product formula (4.14) can also be handled with the methods developed by Van Assche and Koornwinder [22]. Note that

$$A=\lim_{n\to\infty}a_{n+k,n}=\sqrt{rcd(1-r)},\quad B=\lim_{n\to\infty}b_{n+k,n}=r(c-d)$$

for all $k\in\mathbb{Z}$. Now [22, Theorem 2] can be applied to yield

$$(5.9)\qquad \lim_{p\to\infty}\int_{-d}^{c}f(z)p_p(z;r^{1/p})p_{p+m}(z;r^{1/p})\,d\mu_p(z)=\frac{1}{\pi}\int_{B-2A}^{B+2A}\frac{f(z)T_m\big((z-B)/2A\big)}{\sqrt{4A^2-(z-B)^2}}\,dz$$

for all continuous functions f on $[-d,c]$. Here

$$\int_{-d}^{c}f(z)\,d\mu_p(z)=\int_{-d}^{c}f(z)(qz/c,-qz/d;q)_\infty\,d_qz$$

with q on the right hand side replaced by $r^{1/p}$, and the $p_p(z;q)$ are the orthonormal big q-Jacobi polynomials with $a=b=0$, *cf.* (5.3).

If we now use (5.9), (5.3), (5.2) and (5.1) to take the limit $q=r^{1/p}\uparrow 1$, *i.e.* $p\longrightarrow\infty$, in (4.14), we obtain

$$\begin{aligned}(-1)^{l+m}\big(dr(1-r)/c\big)^{-m/2}\frac{(l-m)!\,m!}{(l+1)_l\pi}\int_{B-2A}^{B+2A}R_l^{(0,0)}\Big(\frac{2z+d-c}{c+d}\Big)\frac{T_m\big((z-B)/2A\big)}{\sqrt{4A^2-(z-B)^2}}\,dz\\ =\frac{(m+1)_{l-m}}{(l+m+1)_{l-m}}(1+\frac{d}{c})^{-m}R_{l-m}^{(m,m)}\Big(\frac{c-d}{c+d}\Big)R_{l-m}^{(m,m)}(1-2r)\end{aligned}$$

with $A=\sqrt{rcd(1-r)}$, $B=r(c-d)$. By changing the integration variable to $t=(z-B)/2A$, replacing $(d-c)/(c+d)$, $1-2r$ by x, y and using $R_n^{(m,m)}(-x)=(-1)^nR_n^{(m,m)}(x)$ we obtain the product formulas

$$(5.10)\qquad \begin{aligned}R_{l-m}^{(m,m)}(x)R_{l-m}^{(m,m)}(y)&=2^{2m}\frac{(l-m)!\,(m!)^2}{\pi(l+m)!}\big(\sqrt{(1-x^2)(1-y^2)}\big)^{-m}\\ &\times\int_{-1}^{1}R_l^{(0,0)}\big(xy+t\sqrt{(1-x^2)(1-y^2)}\big)\frac{T_m(t)}{\sqrt{1-t^2}}\,dt.\end{aligned}$$

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