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Statistical stability and linear response for random hyperbolic dynamics

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Abstract. We consider families of random products of close-by Anosov diffeomorphisms, and show that statistical stability and linear response hold for the associated families of equivariant and stationary measures. Our analysis relies on the study of the top Oseledets space of a parametrized transfer operator cocycle, as well as ad-hoc abstract perturbation statements. As an application, we show that, when the quenched central limit theorem (CLT) holds, under the conditions that ensure linear response for our cocycle, the variance in the CLT depends differentiably on the parameter.

Key words: linear response, random dynamical systems, uniform hyperbolicity
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1. *Introduction*

The aim of this paper is to study stability for the families of equivariant and stationary measures associated with a random product of (uniformly) hyperbolic diffeomorphisms. Those stability properties are related to the following question: in the context of non-autonomous dynamics, how do the statistical properties change when one perturbs the dynamics?

More precisely, we consider here a family of random hyperbolic diffeomorphisms, $T_{\omega,\varepsilon}$, acting on some Riemannian manifold M and indexed by $\omega \in \Omega$ and $\varepsilon \in I$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is some probability space, and $0 \in I \subset \mathbb{R}$ is some interval. Endowing the probability space with an invertible map $\sigma : \Omega \circlearrowleft$ that is measure-preserving and ergodic, we may form the *random products over* σ, defined by

$$T^n_{\omega,\varepsilon} := T_{\sigma^n\omega,\varepsilon} \circ \cdots \circ T_{\omega,\varepsilon}. \tag{1}$$

Assuming that this random product admits a *physical equivariant measure*, that is, a measure h_ω^ε satisfying the equivariance condition

$$T_{\omega,\varepsilon}^* h_\omega^\varepsilon = h_{\sigma\omega}^\varepsilon, \tag{2}$$

and such that $\mathbb{P}$-almost surely, the ergodic basin of h_ω^ε has positive Riemannian volume (meaning that for $\mathbb{P}$-almost every $\omega \in \Omega$, the set $\{x \in M, (1/n) \sum_{k=0}^{n-1} \delta_{T_{\omega,\varepsilon}^k x} \longrightarrow h_\omega^\varepsilon$ weakly$\}$ has positive Riemannian volume), we ask the following questions: is the map $\varepsilon \in I \mapsto h_\omega^\varepsilon$ continuous at $\varepsilon = 0$ in some suitable sense? Is it differentiable? If so, can one derive an explicit formula for its derivative?

The first question is the *statistical stability* problem, the last two are called the *linear response* problem.

Linear response has received extensive attention, in various context: in the deterministic case (which corresponds, in our setting, to the case where Ω is reduced to a singleton and one considers a smooth family of maps $(T_\varepsilon)_{\varepsilon\in I}$), expanding maps of the circle [6] or in higher dimension [7, 29], piecewise expanding maps of the interval [5, 9] or more general unimodal maps [10], and intermittent maps [3, 11, 27] have been studied. In the setting of hyperbolic dynamics, the problem of linear response was first considered by Ruelle [28] for uniformly hyperbolic maps. A different approach, the so-called weak spectral perturbation (or Gouëzel–Keller–Liverani) theory, was devised by [26] (see also [7]). Finally, we mention the paper [18], where linear response is established for a wide class of partially hyperbolic systems.

The random case may be divided into two different subcases, the annealed case and the quenched case, the latter of which is the focus of this paper. The annealed case may be studied by methods very similar to the deterministic case, namely weak spectral perturbation for the associated family of transfer operators, and often enjoy a convenient 'regularization property' (see, e.g., [23] or [24]). We also mention [2], where the authors deal with annealed perturbation of uniformly and non-uniformly expanding maps. For annealed perturbation of Anosov diffeomorphisms, very general results were obtained in [26].

The study of the quenched case is more recent, and the literature on the subject is sparse. Indeed, in this situation one cannot use the tools devised in the deterministic or annealed case, as the dynamically relevant objects shift from the spectral data of individual transfer operators to the Lyapunov–Oseledets spectra associated with a cocycle of such transfer operators. For the statistical stability problem in this context, we refer to [4, 8, 12, 21]. Recently, the interesting preprint [14] develops an analogue of the Gouëzel–Keller–Liverani theory to study regularity of the exceptional Oseledets spectrum, for quasi-compact cocycles having a dominated splitting, but only up to Lipschitz regularity. This machinery could, in principle, be applied to our setting, to obtain a result similar to our Theorem 8. We observe that, although our result is less general because it only concerns the top Oseledets space, it has the nice property of giving an explicit modulus of continuity, and to have an elementary proof.

For the response problem, a very general study is presented in [30], in the case of a random products of uniformly expanding maps, with a finite or countable number of branches, and in any finite dimension. The idea is to express the equivariant family of measures of the random product as the fixed point of a family of cone-contracting maps that exhibits suitable regularity properties, and to deduce the wanted smoothness of the equivariant measures by some implicit-function-like argument.

We emphasize that the results we present here rely on methods that are quite different from those in the previously discussed paper, as they do not rely on Birkhoff cone contraction techniques. We also remark that, in contrast to the expanding case, the use of the Gouëzel–Liverani scale of anisotropic spaces (or, for that matter, any of the available scale of anisotropic spaces) limits us to products of nearby (in the C^{r+1} topology) diffeomorphisms.

A few months after the present paper was made available as a preprint, the Gouëzel–Keller–Liverani theory for cocycles [**14**] was further generalized in [**15**], to cover the case of quenched linear, as well as higher-order, response. In particular, [**15**, Theorem 3.6] generalizes our Theorem 12 to higher-order Taylor expansions, as remarked in [**15**, Remark 3.8]. The main idea behind this generalization, namely lifting the cocycle to the so-called Mather operator (to which the deterministic Gouëzel–Keller–Liverani theory is then applied), is somehow present in our approach (see, e.g., the proof of Proposition 7), although the latter is independent of weak spectral perturbation theory. We finally remark that, although it should be possible in principle, [**15**] does not state any linear response formula.

Before going any further, we would like to point out a subtle issue, that is peculiar to the quenched case, and related to the 'suitable sense' for which the question of statistical stability and linear response may be answered. In the deterministic case, this means finding a suitable topology into which the invariant measure will live (e.g. $C^r(\mathbb{S}^1, \mathbb{R})$, $r > 1$ for the absolutely continuous invariant probability measure of an expanding map of the circle, or as a distribution of order one for smooth deformations of unimodal maps, see [**10**]). In the quenched case, one also has to take care of the random parameter $\omega \in \Omega$. There are several natural possibilities: the *almost sure* sense (i.e. one studies the almost sure regularity of $\varepsilon \in I \mapsto h_\omega^\varepsilon \in \mathcal{B}$, with $\mathcal{B}$ a suitable Banach space in the range), the *essentially bounded* sense (where one studies the regularity of $\varepsilon \in I \mapsto h_\omega^\varepsilon \in L^\infty(\Omega, \mathcal{B})$), and the L^1 sense (where the map of interest is $\varepsilon \in I \mapsto h_\omega^\varepsilon \in L^1(\Omega, \mathcal{B})$). It is easy to see that the L^∞ sense is the strongest one. Furthermore, given the relation between the equivariant measures and the stationary measure, the L^1 sense implies asking the questions of stability and response for the stationary measure of the skew product. However, an ambiguity arises when one considers the 'almost sure' sense: indeed, it may be that the set of random parameters for which certain estimates on the equivariant measure h_ω^ε holds (let us denote it by Ω_ε) depends on ε. In this situation, it is not clear whether a statement such as '$h_\omega^\varepsilon \to h_\omega^0$ when $\varepsilon \to 0$, $\mathbb{P}$-almost surely' has any probabilistic meaning, because it would hold on $\bigcap_{\varepsilon \in I} \Omega_\varepsilon$, which may be non-measurable set (as the intersection is taken over an uncountable set). For this reason, we refrain from considering the 'almost sure' sense for the regularity results we present, and instead focus on the L^∞-sense.

The paper is organized as follows. In §2, after recalling useful properties of the Gouëzel–Liverani anisotropic Banach spaces, we present and discuss our setup (Hypothesis 1), we state our main result (Theorem 1) as well as a *quenched linear response formula* (3), reminiscent of [**28**, **30**], and give explicit examples of systems to which

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this setting apply (§2.3). In §3, we present abstract theorems on quenched statistical stability (Theorem 8) and quenched linear response (Theorem 12), applicable in particular to the equivariant measure associated with a (sufficiently) smooth family of Anosov diffeomorphisms cocycles. In §4, we give the proof of the main theorem (Theorem 1). In §5, we give various applications of the previous results: first, we remark in Theorem 15 that Theorem 12 easily implies a response for the stationary measure of the skew product associated with the cocycle, and that this can be used to establish linear response for a class of deterministic, partially hyperbolic systems. In §5.2, we prove Theorem 17 which gives the differentiability with respect to the parameter of the variance in the quenched central limit theorem (satisfied by the Birkhoff sum of random observable satisfying certain conditions).

Finally, in §6, we discuss applications of our approach to other type of random hyperbolic systems: random compositions of uniformly expanding maps, or random two-dimensional piecewise hyperbolic maps.

2. *Main theorem*

2.1. *A class of anisotropic Banach spaces introduced by Gouëzel and Liverani.*

The purpose of this subsection is to briefly summarize the main results from [26]. More precisely, we recall the properties of the so-called scale of anisotropic Banach spaces, on which the transfer operator associated to a transitive Anosov diffeomorphism has a spectral gap. The discussion we present here is relevant when building examples under which the abstract results of the present paper are applicable.

Let M denote a C^∞ compact and connected Riemannian manifold. Furthermore, let T be a transitive Anosov diffeomorphism on M of class C^{r+1} for $r > 1$. We denote the transfer operator associated with T by $\mathcal{L}_T$. We recall that the action of $\mathcal{L}_T$ on smooth functions $h \in C^r(M, \mathbb{R})$ is given by

$$\mathcal{L}_T h = (h|\det(DT)|^{-1}) \circ T^{-1}.$$

Let us now briefly summarize the main results from [26]. Take $p \in \mathbb{N}$, $p \leq r$ and $q > 0$ such that $p + q < r$. It is proved in [26] that there exist Banach spaces $\mathcal{B}^{p,q} = (\mathcal{B}^{p,q}, \|\cdot\|_{p,q})$ and $\mathcal{B}^{p-1,q+1} = (\mathcal{B}^{p-1,q+1}, \|\cdot\|_{p-1,q+1})$ with the following properties.

- By construction, $C^r(M, \mathbb{R})$ is dense in $\mathcal{B}^{i,j}$ for $(i, j) = \{(p, q), (p-1, q+1)\}$.
- By [26, Lemma 2.1], $\mathcal{B}^{p,q}$ can be embedded in $\mathcal{B}^{p-1,q+1}$ and the unit ball of $\mathcal{B}^{p,q}$ is relatively compact in $\mathcal{B}^{p-1,q+1}$.
- By [26, Proposition 4.1], elements of $\mathcal{B}^{p,q}$ are distributions of order at most q.
- By [26, Lemma 3.2], multiplication by a C^{k+q} function, $1 \leq k \leq p$, induces a bounded operator on $\mathcal{B}^{p,q}$. Moreover, the action of a C^r vector field induces a bounded operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{p-1,q+1}$.
- Here $\mathcal{L}_T$ acts as a bounded operator on $\mathcal{B}^{i,j}$ for $(i, j) = \{(p, q), (p-1, q+1)\}$. Moreover, for each $h \in \mathcal{B}^{i,j}$ and $\varphi \in C^j(M, \mathbb{R})$, we have that

  $$(\mathcal{L}_T h)(\varphi) = h(\varphi \circ T),$$

  where we denote the action of a distribution h on a test function φ by $h(\varphi)$.

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- By [26, Lemma 2.2], there exist $A > 0$ and $a \in (0, 1)$ such that

$$\|\mathcal{L}_T^n h\|_{p-1,q+1} \le A\|h\|_{p-1,q+1} \quad \text{for } n \in \mathbb{N} \text{ and } h \in \mathcal{B}^{p-1,q+1}$$

  and

$$\|\mathcal{L}_T^n h\|_{p,q} \le Aa^n\|h\|_{p,q} + A\|h\|_{p-1,q+1} \quad \text{for } n \in \mathbb{N} \text{ and } h \in \mathcal{B}^{p,q}.$$

- By [26, Theorem 2.3], $\mathcal{L}_T$ is a quasi-compact operator on $\mathcal{B}^{p,q}$ with spectral radius 1. Moreover, 1 is the only eigenvalue of $\mathcal{L}_T$ on the unit circle. Finally, 1 is a simple eigenvalue of $\mathcal{L}_T$ and the corresponding eigenspace is spanned by the unique S.R.B. measure for T.

2.2. *Regularity assumptions.* In this section, we state precisely our regularity assumptions and our main theorem. We start by fixing, once and for all, the system of C^∞ coordinates chart to be $(\psi_i)_{i=1,\ldots,N}$, where $\psi_i : (-r_i, r_i)^d \to M$, and such that the $X_i = \psi_i((-r_i/2, r_i/2)^d)$ cover M are given by the anisotropic norm construction (see [26]). We also let δ be the Lebesgue number of the previous cover. Recall the following fact: if T and S are C^{r+1} maps from M to itself, such that $\sup_{x\in M} d_M(Tx, Sx) \le \delta/2$, then one has: for any $i \in \{1, \ldots, N\}$,

$$\mathcal{J}_S(i) := \{j \in \{1, \ldots, N\},\ S(X_i) \cap X_j \ne \emptyset\} = \mathcal{J}_T(i),$$

and one may write

$$d_{C^{r+1}}(T, S) = \sum_{i=1}^{N} \sum_{j\in\mathcal{J}(i)} \|T_{ij} - S_{ij}\|_{C^{r+1}},$$

where $\mathcal{J}(i) = \mathcal{J}_S(i) = \mathcal{J}_T(i)$ and $T_{ij} = \psi_j^{-1} \circ T \circ \psi_i : (-r_i, r_i)^d \to (-r_j, r_j)^d$ is a map between open sets in $\mathbb{R}^d$.

For an interval $0 \in I \subset \mathbb{R}$, we consider a C^s mapping $\mathcal{T} : I \to C^{r+1}(M, M)$, such that $T_0 := \mathcal{T}(0)(\cdot)$ is a C^{r+1}, transitive Anosov diffeomorphism. Up to shrinking I, we may and do assume that for all $\varepsilon \in I$, $T_\varepsilon := \mathcal{T}(\varepsilon)(\cdot)$ is a C^{r+1} Anosov diffeomorphism, and that $\sup_{\varepsilon\in I} d_{C^{r+1}}(T_\varepsilon, T_0) \le \delta/4$. In particular, for any $i \in \{1, \ldots, N\}$, the set

$$\mathcal{J}_\varepsilon(i) := \{j \in \{1, \ldots, N\},\ T_\varepsilon(X_i) \cap X_j \ne \emptyset\}$$

is independent of ε. We informally refer to this property by saying that 'the maps T_ε may be read in the same charts'.

Consider now a $\Delta > 0$, and set $V := B_{C^s(I,C^{r+1}(M,M))}(\mathcal{T}, \Delta)$, that is, we consider a small ball, in $C^s(I, C^{r+1}(M, M))$ topology, centered at $\mathcal{T}$. Up to shrinking Δ, we may assume that for any $\mathcal{S} \in V$, any $\varepsilon \in I$, $S_\varepsilon := \mathcal{S}(\varepsilon)(\cdot)$ is an Anosov diffeomorphism, and that $\sup_{\varepsilon\in I} d_{C^{r+1}}(T_\varepsilon, S_\varepsilon) \le \delta/4$. In particular, for any $i \in \{1, \ldots, N\}$, the sets

$$\mathcal{J}_{\mathcal{S}}(i) := \{j \in \{1, \ldots, N\},\ S_\varepsilon(X_i) \cap X_j \ne \emptyset\}$$

are independent of ε and $\mathcal{S}$ both (i.e. they only depend on V).

We may now describe the type of perturbed cocycle we will consider in the following:

Hypothesis 1. Let $r > 4$, $s > 1$, and $0 \in I \subset \mathbb{R}$ an interval; let $\mathcal{T}$ and $V \subset C^s(I, C^{r+1}(M, M))$ be as described previously. Furthermore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\sigma \colon \Omega \to \Omega$ an invertible, ergodic $\mathbb{P}$-preserving transformation and consider a measurable mapping

$$\mathbf{T} \colon \Omega \to V$$

Set $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon)(\cdot)$, $\omega \in \Omega$ and $\varepsilon \in I$.

Let us make a few comments on this assumption, based on the previous discussion.

- We choose the neighborhood V sufficiently small so that for any $\omega \in \Omega$, any $\varepsilon \in I$, the collection of $T_{\omega,\varepsilon}$ can all be read in the same coordinate charts and share the same set of admissible leaves. In particular, one may study their transfer operators on the same anisotropic Banach spaces.
- Our assumption is tailored so that for each fixed $\omega \in \Omega$, $\varepsilon \mapsto T_{\omega,\varepsilon}$ is a smooth curve of Anosov diffeomorphisms, all close-by to a fixed one (namely, $T_{\omega,0}$).

We are now in position to formulate our main result.

THEOREM 1. *Let $(T_{\omega,\varepsilon})_{\omega\in\Omega,\varepsilon\in I}$ be a parametrized cocycle of Anosov diffeomorphisms, satisfying Hypothesis 1. Then, by shrinking I if necessary, there exists a triplet of Banach spaces*

$$\mathcal{B}_{ss} \subset \mathcal{B}_s \subset \mathcal{B}_w,$$

and for each $\varepsilon \in I$ a unique family $(h_\omega^\varepsilon)_{\omega\in\Omega} \subset \mathcal{B}_{ss}$ with the following properties:

- *$\omega \mapsto h_\omega^\varepsilon$ is measurable for each $\varepsilon \in I$;*
- *h_ω^ε is a probability measure for $\varepsilon \in I$ and $\omega \in \Omega$;*
- *$\mathcal{L}_{\omega,\varepsilon} h_\omega^\varepsilon = h_{\sigma\omega}^\varepsilon$ for $\varepsilon \in I$ and $\omega \in \Omega$, where $\mathcal{L}_{\omega,\varepsilon}$ denotes the transfer operator of $T_{\omega,\varepsilon}$;*
- *the map $I \ni \varepsilon \mapsto h_\omega^\varepsilon \in L^\infty(\Omega, \mathcal{B}_w)$ is differentiable at 0, and for $\phi \in C^r(M)$, we have that*

$$\partial_\varepsilon \bigg[\int_M \phi \, dh_\omega^\varepsilon \bigg] \bigg|_{\varepsilon=0} = \sum_{n=0}^{\infty} \int_M \partial_\varepsilon [\phi \circ T^n_{\sigma^{-n}\omega} \circ T_{\sigma^{-n-1}\omega,\varepsilon}] \bigg|_{\varepsilon=0} dh_{\sigma^{-n-1}\omega}, \tag{3}$$

where $h_\omega := h_\omega^0$, $\omega \in \Omega$.

2.3. *Examples.* Here we give explicit examples of systems satisfying Hypothesis 1. In all instances, $r > 4$ and $s > 1$.

Example 2. Let $q \in \mathbb{N}$, $\Omega = \{1, \ldots, q\}^{\mathbb{Z}}$, endowed with a Bernoulli measure. Consider a family $(T_1, \ldots, T_q)$ of (close-enough) C^{r+1} Anosov diffeomorphisms of the d-dimensional torus $\mathbb{T}^d$, where $p : \mathbb{T}^d \to \mathbb{T}^d$ is a C^{r+1} mapping and $0 \in I \subset \mathbb{R}$ is an interval. We set

$$\mathbf{T}(\omega)(\varepsilon, x) := T_i(x) + \varepsilon p(x), \quad \text{if } \omega_0 = i, \tag{4}$$

where $x \in \mathbb{T}^d$, $\varepsilon \in I$, and $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$.

Example 3. Let $q \in \mathbb{N}$, $\Omega = \{1, \ldots, q\}^{\mathbb{Z}}$, endowed with a Bernoulli measure. Consider a C^{r+1} Anosov diffeomorphism T of $\mathbb{T}^d$. Moreover, consider $p_1, \ldots, p_q$ C^{r+1} mappings of $\mathbb{T}^d$ and $0 \in I \subset \mathbb{R}$ an interval, Then we define, for $\varepsilon \in I$, $x \in \mathbb{T}^d$, and $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$, the random map

$$\mathbf{T}(\omega)(\varepsilon, x) = T(x) + \varepsilon p_i(x) \quad \text{if } \omega_0 = i. \tag{5}$$

In both Examples 2 and 3, for each $\omega \in \Omega$, $\mathbf{T}(\omega) \in C^s(I, C^{r+1}(M, M))$. Furthermore, because for each $i \in \{1, \ldots, q\}$, the set $\{\mathbf{T}(\omega) = T_i + \varepsilon p\}$ (respectively, $\{\mathbf{T}(\omega) = T + \varepsilon p_i\}$) is the 1-cylinder $\{\omega_0 = i\}$, one easily checks that the map is measurable.

Example 4. We now consider the following setting: for $\delta > 0$, $\omega \in B_{\mathbb{R}^d}(0, \delta)$ (that is, randomly chosen with respect to Lebesgue measure) and $\varepsilon_0 > 0$, we consider a C^s-smooth curve of Anosov diffeomorphisms

$$I := (-\varepsilon_0, \varepsilon_0) \ni \varepsilon \to T_\varepsilon \in C^{r+1}(\mathbb{T}^d, \mathbb{T}^d).$$

Finally, set

$$\mathbf{T}(\omega)(\varepsilon, x) := T_\varepsilon(x) + \omega, \qquad x \in \mathbb{T}^d.$$

In this last instance, one easily checks that the map $\Omega \ni \omega \mapsto \mathbf{T}(\omega) \in C^s(I, C^{r+1}(M, M))$ is continuous and, thus, measurable.

3. *Some abstract results*

3.1. *Quenched statistical stability for random systems.*

In this section, we formulate an abstract result regarding the statistical stability of certain random dynamical systems that applies, in particular, to random hyperbolic dynamics.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider an invertible transformation $\sigma : \Omega \to \Omega$ which preserves $\mathbb{P}$. Furthermore, let $\mathbb{P}$ be ergodic.

Moreover, let $\mathcal{B}_w = (\mathcal{B}_w, \|\cdot\|_w)$ and $\mathcal{B}_s = (\mathcal{B}_s, \|\cdot\|_s)$ be two Banach spaces such that $\mathcal{B}_s$ is embedded in $\mathcal{B}_w$ and that $\|\cdot\|_w \le \|\cdot\|_s$ on $\mathcal{B}_s$. Suppose that for each $\omega \in \Omega$, $\mathcal{L}_\omega$ is a bounded operator both on $\mathcal{B}_w$ and $\mathcal{B}_s$. In addition, assume that $\omega \to \mathcal{L}_\omega$ is strongly measurable on $\mathcal{B}_s$, that is, that the map $\omega \mapsto \mathcal{L}_\omega h$ is measurable for each $h \in \mathcal{B}_s$. For $\omega \in \Omega$ and $n \in \mathbb{N}$, set

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

We consider a fixed, non-zero $\psi \in \mathcal{B}_s'$ that admits a bounded extension to $\mathcal{B}_w$ that we still denote by ψ, and assume that there exist $D, \lambda > 0$ such that

$$\|\mathcal{L}_\omega^n h\|_s \le D e^{-\lambda n} \|h\|_s, \tag{6}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$, $n \in \mathbb{N}$, and $h \in \mathcal{B}_s^0$, where

$$\mathcal{B}_s^0 = \{h \in \mathcal{B}_s : \psi(h) = 0\}. \tag{7}$$

Obviously, $\mathcal{B}_s^0$ depends on the choice of ψ. However, this dependence has no bearing on our results (see Remark 5), so we do not make it explicit in the notation itself.

Consider now an interval $I \subset \mathbb{R}$ around $0 \in \mathbb{R}$ and suppose that for $\varepsilon \in I$, we have a family $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ of bounded linear operators on spaces $\mathcal{B}_s$ and $\mathcal{B}_w$. Moreover, assume that $\omega \mapsto \mathcal{L}_{\omega,\varepsilon}$ is strongly measurable on $\mathcal{B}_s$ for each $\varepsilon \in I$. Analogously to $\mathcal{L}_\omega^n$, for $\omega \in \Omega$, $\varepsilon \in I$, and $n \in \mathbb{N}$, we define

$$\mathcal{L}_{\omega,\varepsilon}^n := \mathcal{L}_{\sigma^{n-1}\omega,\varepsilon} \circ \cdots \circ \mathcal{L}_{\sigma\omega,\varepsilon} \circ \mathcal{L}_{\omega,\varepsilon}.$$

We set $\mathcal{L}_{\omega,0} = \mathcal{L}_\omega$ and we suppose that there exist $C > 0$, $\lambda_1 \in (0, 1)$, and a measurable $\Omega' \subset \Omega$ satisfying $\mathbb{P}(\Omega') = 1$ such that for each $\varepsilon \in I$:

- for each $\varepsilon \in I$, $\omega \in \Omega'$, $n \in \mathbb{N}$, and $h \in \mathcal{B}_s$,
$$\|\mathcal{L}_{\omega,\varepsilon}^n h\|_s \le C\lambda_1^n \|h\|_s + C\|h\|_w; \tag{8}$$
- for each $\varepsilon \in I$, $\omega \in \Omega'$, and $h \in \mathcal{B}_s$,
$$\|(\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega)h\|_w \le C|\varepsilon| \cdot \|h\|_s; \tag{9}$$
- for each $\varepsilon \in I$, $\omega \in \Omega'$, and $n \in \mathbb{N}$,
$$\|\mathcal{L}_{\omega,\varepsilon}^n\|_w \le C; \tag{10}$$
- for each $\varepsilon \in I$, $\omega \in \Omega'$, we have that
$$\psi(\mathcal{L}_{\omega,\varepsilon} h) = \psi(h) \quad \text{for each } h \in \mathcal{B}_s. \tag{11}$$

We can assume without any loss of generality that Ω' is contained in a full measure set on which (6) holds.

Remark 5.

- Observe that we can assume that Ω' is σ-invariant because we can replace Ω' with $\Omega'' = \bigcap_{k\in\mathbb{Z}} \sigma^k(\Omega')$ which is clearly σ-invariant and also satisfies $\mathbb{P}(\Omega'') = 1$. Therefore, from now on we assume that Ω' is σ-invariant.
- We note that we can deal with the more general situation when Ω' is allowed to depend on ε. However, because the current framework is sufficient for applications we have in mind and for the case of simplicity, we do not explicitly deal with this case.
- The fact that almost every $\mathcal{L}_{\omega,\varepsilon}$ shares a left eigenvector is the reason why the dependence on ψ of the space $\mathcal{B}_s^0$ has no consequence for us. In our examples, ψ will be $\psi(h) := h(1)$ for a finite-order distribution h (and where 1 denotes the constant test function).

We first show that the above assumptions imply that all the perturbed cocycles $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ also satisfy the condition of the type (6) whenever $|\varepsilon|$ is sufficiently small. More precisely, we have the following auxiliary result.

PROPOSITION 6. *There exist ε_0, $D' > 0$ and $\lambda' > 0$ such that*

$$\|\mathcal{L}_{\omega,\varepsilon}^n h\|_s \le D' e^{-\lambda' n} \|h\|_s, \tag{12}$$

for $\varepsilon \in I$ satisfying $|\varepsilon| \le \varepsilon_0$, $\omega \in \Omega'$, $n \in \mathbb{N}$, and $h \in \mathcal{B}_s^0$.

Proof. Let $\varepsilon_0 > 0$ be such that

$$\frac{C^4}{1-\lambda_1}\varepsilon_0 < 1/2, \tag{13}$$

and take an arbitrary $\varepsilon \in I$ satisfying $|\varepsilon| \le \varepsilon_0$.

As

$$\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_\omega = \sum_{k=1}^n \mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon}(\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega})\mathcal{L}^{k-1}_\omega,$$

it follows from (8), (9), and (10) that

$$\begin{aligned}
\|(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_\omega)h\|_w &\le \sum_{k=1}^n \|\mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon}(\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega})\mathcal{L}^{k-1}_\omega h\|_w \\
&\le C\sum_{k=1}^n \|(\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega})\mathcal{L}^{k-1}_\omega h\|_w \\
&\le C^2|\varepsilon|\sum_{k=1}^n \|\mathcal{L}^{k-1}_\omega h\|_s \\
&\le C^2|\varepsilon|\sum_{k=1}^n (C\lambda_1^{k-1}\|h\|_s + C\|h\|_w) \\
&\le C^3|\varepsilon|\bigg(\frac{1}{1-\lambda_1}\|h\|_s + n\|h\|_w\bigg),
\end{aligned}$$

and, thus,

$$\|(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_\omega)h\|_w \le C^3|\varepsilon|\bigg(\frac{1}{1-\lambda_1}\|h\|_s + n\|h\|_w\bigg), \tag{14}$$

for $n \in \mathbb{N}$, $\omega \in \Omega'$, and $h \in \mathcal{B}_s$. Thus, (6), (8), and (14) imply that

$$\begin{aligned}
\|\mathcal{L}^{n+m}_{\omega,\varepsilon}h\|_s &= \|\mathcal{L}^n_{\sigma^m\omega,\varepsilon}\mathcal{L}^m_{\omega,\varepsilon}h\|_s \\
&\le C\lambda_1^n\|\mathcal{L}^m_{\omega,\varepsilon}h\|_s + C\|\mathcal{L}^m_{\omega,\varepsilon}h\|_w \\
&\le C\lambda_1^n(C\lambda_1^m\|h\|_s + C\|h\|_w) + C(\|\mathcal{L}^m_\omega h\|_w + \|(\mathcal{L}^m_{\omega,\varepsilon} - \mathcal{L}^m_\omega)h\|_w) \\
&\le C^2\lambda_1^{n+m}\|h\|_s + C^2\lambda_1^n\|h\|_s + CDe^{-\lambda m}\|h\|_s + C^4|\varepsilon|\bigg(\frac{1}{1-\lambda_1} + m\bigg)\|h\|_s,
\end{aligned}$$

for $n, m \in \mathbb{N}$, $\omega \in \Omega'$, and $h \in \mathcal{B}^0_s$. Hence (recall also (13)), we can find (by decreasing ε_0 if necessary) $a \in (0, 1)$ and $N_0 \in \mathbb{N}$ (independent of ε and ω) such that

$$\|\mathcal{L}^{N_0}_{\omega,\varepsilon}h\|_s \le a\|h\|_s, \tag{15}$$

for $\omega \in \Omega'$ and $h \in \mathcal{B}^0_s$.

On the other hand, it follows readily from (8) that

$$\|\mathcal{L}^n_{\omega,\varepsilon}\|_s \le 2C \quad \text{for } n \in \mathbb{N} \text{ and } \omega \in \Omega'. \tag{16}$$

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Take now an arbitrary $n \in \mathbb{N}$ and write it as $n = mN_0 + k$ for $m, k \in \mathbb{N} \cup \{0\}$, $0 \le k < N_0$. It follows from (15) and (16) that

$$\begin{aligned}\|\mathcal{L}^n_{\omega,\varepsilon} h\|_s = \|\mathcal{L}^{mN_0+k}_{\omega,\varepsilon} h\|_s &\le 2Ca^m \|h\|_s \\ &= 2Ce^{-m \log a^{-1}} \|h\|_s \\ &= 2Ce^{(k/N_0) \log a^{-1}} e^{-(n/N_0) \log a^{-1}} \|h\|_s \\ &\le 2Ce^{\log a^{-1}} e^{-(n/N_0) \log a^{-1}} \|h\|_s,\end{aligned}$$

for $\omega \in \Omega'$, $n \in \mathbb{N}$, $h \in \mathcal{B}^0_s$. We conclude that (12) holds with

$$\lambda' = \log a^{-1}/N_0 > 0 \quad \text{and} \quad D' = 2Ce^{\log a^{-1}} > 0,$$

which are independent on ε. The proof of the proposition is completed. □

We are now in position to establish the existence of a random fixed point for the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega \in \Omega}$ whenever $|\varepsilon| \le \varepsilon_0$.

PROPOSITION 7. *For each $\varepsilon \in I$ satisfying $|\varepsilon| \le \varepsilon_0$, there exists a unique family $(h^\varepsilon_\omega)_{\omega \in \Omega'} \subset \mathcal{B}_s$ such that:*

- *$\omega \mapsto h^\varepsilon_\omega$ is measurable and bounded, that is*
$$\sup_{\omega \in \Omega'} \|h^\varepsilon_\omega\|_s < \infty; \tag{17}$$
- *for $\omega \in \Omega'$,*
$$\psi(h^\varepsilon_\omega) = 1; \tag{18}$$
- *for $\omega \in \Omega'$,*
$$\mathcal{L}_{\omega,\varepsilon} h^\varepsilon_\omega = h^\varepsilon_{\sigma\omega}. \tag{19}$$

Proof. Let $\mathcal{Y}$ denote the set of all measurable functions $v \colon \Omega' \to \mathcal{B}_s$ such that

$$\|v\|_\infty = \sup_{\omega \in \Omega'} \|v(\omega)\|_s < \infty.$$

Then, $(\mathcal{Y}, \|\cdot\|_\infty)$ is a Banach space. Set

$$\mathcal{Z} := \{v \in \mathcal{Y} : \psi(v(\omega)) = 1 \text{ for } \omega \in \Omega'\}.$$

Observe that $\mathcal{Z}$ is nonempty. Indeed, because ψ is non-zero, there exists $g \in \mathcal{B}_s$ such that $\psi(g) = 1$. Set $v_0 \colon \Omega' \to \mathcal{B}_s$ by $v_0(\omega) = g$ for $\omega \in \Omega'$. Then, $v_0 \in \mathcal{Z}$. We claim that $\mathcal{Z}$ is a closed subset of $\mathcal{Y}$. Indeed, let $(v_n)_n$ be a sequence in $\mathcal{Z}$ that converges to some $v \in \mathcal{Y}$. Then, we have that

$$|\psi(v_n(\omega)) - \psi(v(\omega))| \le \|\psi\|_s \cdot \|v_n(\omega) - v(\omega)\|_s \le \|\psi\|_s \cdot \|v_n - v\|_\infty,$$

for $n \in \mathbb{N}$ and $\omega \in \Omega'$, where $\|\psi\|_s$ denotes the norm of $\psi \in \mathcal{B}'_s$. Hence, $\psi(v(\omega)) = 1$ for $\omega \in \Omega'$ and, thus, $v \in \mathcal{Z}$.

For $|\varepsilon| \le \varepsilon_0$, we define a linear operator $\mathbb{L}^\varepsilon \colon \mathcal{Y} \to \mathcal{Y}$ by

$$(\mathbb{L}^\varepsilon v)(\omega) = \mathcal{L}_{\sigma^{-1}\omega,\varepsilon} v(\sigma^{-1}\omega), \quad \omega \in \Omega'.$$

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It follows from (16) (together with our assumption that $\omega \mapsto \mathcal{L}_{\omega,\varepsilon}$ is strongly measurable on $\mathcal{B}_s$ for each ε) that $\mathbb{L}^\varepsilon$ is a well-defined and bounded operator. Moreover, $\mathbb{L}^\varepsilon \mathcal{Z} \subset \mathcal{Z}$. Indeed, for each $v \in \mathcal{Z}$ we have (using (11)) that

$$\psi((\mathbb{L}^\varepsilon v)(\omega)) = \psi(\mathcal{L}_{\sigma^{-1}\omega,\varepsilon} v(\sigma^{-1}\omega)) = \psi(v(\sigma^{-1}\omega)) = 1,$$

for $\omega \in \Omega'$. Thus, $\mathbb{L}^\varepsilon v \in \mathcal{Z}$.

Let us now choose $N \in \mathbb{N}$ such that $D' e^{-\lambda' N} < 1$. It follows from (12) that

$$\begin{aligned} \|(\mathbb{L}^\varepsilon)^N v_1 - (\mathbb{L}^\varepsilon)^N v_2\|_\infty &= \sup_{\omega \in \Omega'} \|\mathcal{L}^N_{\sigma^{-N}\omega,\varepsilon}(v_1(\sigma^{-N}\omega) - v_2(\sigma^{-N}\omega))\|_s \\ &\le D' e^{-\lambda' N} \sup_{\omega \in \Omega'} \|v_1(\sigma^{-N}\omega) - v_2(\sigma^{-N}\omega)\|_s \\ &\le D' e^{-\lambda' N} \|v_1 - v_2\|_\infty, \end{aligned}$$

for $|\varepsilon| \le \varepsilon_0$ and $v_1, v_2 \in \mathcal{Z}$. Hence, $(\mathbb{L}^\varepsilon)^N$ is a contraction on $\mathcal{Z}$ and therefore, $\mathbb{L}^\varepsilon$ has a unique fixed point $v^\varepsilon \in \mathcal{Z}$. Thus, the family $(h^\varepsilon_\omega)_{\omega \in \Omega'}$ defined by $h^\varepsilon_\omega := v^\varepsilon(\omega)$ satisfies (17), (18), and (19).

In order to establish the uniqueness, it is sufficient to note that each family $(h^\varepsilon_\omega)_{\omega \in \Omega'}$ satisfying (17), (18), and (19) gives rise to a fixed point of $\mathbb{L}^\varepsilon$ in $\mathcal{Z}$, which is unique. The proof of the proposition is complete. □

Set

$$h_\omega := h^0_\omega \quad \omega \in \Omega'.$$

The following is our statistical stability result.

THEOREM 8. *Let $\varepsilon \in I$, $|\varepsilon| \le \varepsilon_0$. Then*

$$\sup_{\omega \in \Omega'} \|h^\varepsilon_\omega - h_\omega\|_w \le C|\varepsilon||\log(|\varepsilon|)|, \tag{20}$$

where $C > 0$ is independent on ε.

Before we establish Theorem 8, we need the following auxiliary result. Let h^ε denote the family $(h^\varepsilon_\omega)_{\omega \in \Omega}$ given by Proposition 7.

LEMMA 9. *We have that*

$$\sup_{|\varepsilon| \le \varepsilon_0} \sup_{\omega \in \Omega'} \|h^\varepsilon_\omega\|_s < \infty. \tag{21}$$

Proof. We use the same notation as in the proof of Proposition 7. Take an arbitrary $u \in \mathcal{Z}$. It follows from Banach's contraction principle that

$$h^\varepsilon = \lim_{k \to \infty} (\mathbb{L}^\varepsilon)^{kN} u,$$

for $|\varepsilon| \le \varepsilon_0$. Fix now any ε such that $|\varepsilon| \le \varepsilon_0$. There exists $k_0 \in \mathbb{N}$ such that

$$\|h^\varepsilon - (\mathbb{L}^\varepsilon)^{k_0 N} u\|_\infty < 1.$$

Hence, using (8) we have that

$$\|h^\varepsilon\|_\infty \le 1 + \|(\mathbb{L}^\varepsilon)^{k_0 N} u\|_\infty \le 2C\|u\|_\infty + 1,$$

which readily implies the conclusion of the lemma. □

We are now in a position to prove Theorem 8.

Proof of Theorem 8. Take an arbitrary $\varepsilon \in I$ such that $|\varepsilon| \le \varepsilon_0$. Observe that

$$\begin{aligned}\|h_\omega^\varepsilon - h_\omega\|_w &= \|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega} - \mathcal{L}^n_{\sigma^{-n}\omega} h_{\sigma^{-n}\omega}\|_w \\ &\le \|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega} - \mathcal{L}^n_{\sigma^{-n}\omega} h^\varepsilon_{\sigma^{-n}\omega}\|_w + \|\mathcal{L}^n_{\sigma^{-n}\omega}(h^\varepsilon_{\sigma^{-n}\omega} - h_{\sigma^{-n}\omega})\|_w, \end{aligned} \tag{22}$$

for each $n \in \mathbb{N}$ and $\omega \in \Omega'$. It follows from (6) and (21) that there exists $\tilde{D} > 0$ such that

$$\|\mathcal{L}^n_{\sigma^{-n}\omega}(h^\varepsilon_{\sigma^{-n}\omega} - h_{\sigma^{-n}\omega})\|_w \le \|\mathcal{L}^n_{\sigma^{-n}\omega}(h^\varepsilon_{\sigma^{-n}\omega} - h_{\sigma^{-n}\omega})\|_s \le \tilde{D} e^{-\lambda n}, \tag{23}$$

for $n \in \mathbb{N}$ and $\omega \in \Omega'$.

On the other hand, it follows from (8), (9), and (10) that

$$\begin{aligned}&\|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega} - \mathcal{L}^n_{\sigma^{-n}\omega} h^\varepsilon_{\sigma^{-n}\omega}\|_w \\ &\quad\le \sum_{j=1}^n \|\mathcal{L}^{n-j}_{\sigma^{-n+j}\omega}(\mathcal{L}_{\sigma^{-n+j-1}\omega} - \mathcal{L}_{\sigma^{-n+j-1}\omega,\varepsilon})\mathcal{L}^{j-1}_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega}\|_w \\ &\quad\le C\sum_{j=1}^n \|(\mathcal{L}_{\sigma^{-n+j-1}\omega} - \mathcal{L}_{\sigma^{-n+j-1}\omega,\varepsilon})\mathcal{L}^{j-1}_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega}\|_w \\ &\quad\le C^2|\varepsilon|\sum_{j=1}^n \|\mathcal{L}^{j-1}_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega}\|_s \\ &\quad\le 2nC^3|\varepsilon| \cdot \|h^\varepsilon_{\sigma^{-n}\omega}\|_s.\end{aligned}$$

Hence, by (21) we have that

$$\|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} h^\varepsilon_{\sigma^{-n}\omega} - \mathcal{L}^n_{\sigma^{-n}\omega} h^\varepsilon_{\sigma^{-n}\omega}\|_w \le 2nC^3|\varepsilon| \sup_{|\varepsilon|\le\varepsilon_0}\sup_{\omega\in\Omega'}\|h^\varepsilon_\omega\|_s, \tag{24}$$

for $\omega \in \Omega'$ and $n \in \mathbb{N}$. We conclude from (22), (23), and (24) that

$$\sup_{\omega\in\Omega'}\|h^\varepsilon_\omega - h_\omega\|_w \le 2nC^3|\varepsilon| \sup_{|\varepsilon|\le\varepsilon_0}\sup_{\omega\in\Omega'}\|h^\varepsilon_\omega\|_s + \tilde{D}e^{-\lambda n},$$

for $n \in \mathbb{N}$. Taking $n = \lfloor|\log(|\varepsilon|)|/\lambda\rfloor$, we conclude that (20) holds. □

3.2. *Quenched linear response for random dynamics.* Observe that Theorem 8 gives the continuity (in the appropriate sense) of the map $\varepsilon \mapsto (h^\varepsilon_\omega)_{\omega\in\Omega}$ in $\varepsilon = 0$. We are now concerned with formulating sufficient conditions under which the same map is differentiable in $\varepsilon = 0$.

In addition to requiring the existence of spaces $\mathcal{B}_w$ and $\mathcal{B}_s$ as in §3.1, we also require the existence of a third space $\mathcal{B}_{ss} = (\mathcal{B}_{ss}, \|\cdot\|_{ss})$ that can be embedded in $\mathcal{B}_s$ and such that $\|\cdot\|_s \le \|\cdot\|_{ss}$ on $\mathcal{B}_{ss}$. As in §3.1, we assume that ψ is a non-zero functional on $\mathcal{B}_s$, and we shall also assume that it admits a bounded extension to $\mathcal{B}_w$. We still denote its restriction (respectively, extension) to $\mathcal{B}_{ss}$ (respectively, $\mathcal{B}_w$) by ψ. Furthermore, we let $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega,\varepsilon\in I}$ be a family such that each $\mathcal{L}_{\omega,\varepsilon}$ is a bounded operator on each of those three spaces. In addition, suppose that $\omega\mapsto\mathcal{L}_{\omega,\varepsilon}$ is strongly measurable on both $\mathcal{B}_s$ and $\mathcal{B}_{ss}$ for each $\varepsilon\in I$.

In addition to (6), we also require that

$$\|\mathcal{L}_\omega^n h\|_{ss}\le De^{-\lambda n}\|h\|_{ss}, \tag{25}$$

for $\mathbb{P}$-almost every $\omega\in\Omega$, $n\in\mathbb{N}$, and $h\in\mathcal{B}_{ss}^0$, where

$$\mathcal{B}_{ss}^0=\{h\in\mathcal{B}_{ss}:\psi(h)=0\}.$$

We define $\mathcal{B}_s^0$ and $\mathcal{B}_w^0$ in a similar manner. In particular, $\mathcal{B}_s^0$ is the same as in (7).

In addition, we also assume that there exist $C>0$, $\lambda_1\in(0,1)$, and a measurable $\Omega'\subset\Omega$ with the property that $\mathbb{P}(\Omega')=1$ and:

- for each $\varepsilon\in I$, $\omega\in\Omega'$, $n\in\mathbb{N}$, and $h\in\mathcal{B}_s$, (8) holds;
- for each $\varepsilon\in I$, $\omega\in\Omega'$, and $h\in\mathcal{B}_s$, (9) holds;
- for each $\varepsilon\in I$, $\omega\in\Omega'$, and $n\in\mathbb{N}$, (10) holds;
- for each $\varepsilon\in I$, $\omega\in\Omega'$, $n\in\mathbb{N}$, and $h\in\mathcal{B}_{ss}$,

$$\|\mathcal{L}_{\omega,\varepsilon}^n h\|_{ss}\le C\lambda_1^n\|h\|_{ss}+C\|h\|_s; \tag{26}$$

- for each $\varepsilon\in I$, $\omega\in\Omega'$, and $h\in\mathcal{B}_{ss}$,

$$\|(\mathcal{L}_{\omega,\varepsilon}-\mathcal{L}_\omega)h\|_s\le C|\varepsilon|\|h\|_{ss}; \tag{27}$$

- for each $\varepsilon\in I$ and $\omega\in\Omega'$, we have that for $h\in\mathcal{B}_s$ (and, thus, also for $h\in\mathcal{B}_{ss}$)

$$\psi(\mathcal{L}_{\omega,\varepsilon}h)=\psi(h). \tag{28}$$

As before, we can assume that Ω' is contained in a full-measure set on which (6) and (25) hold and that Ω' is σ-invariant.

The following is a direct consequence of Proposition 6 (applied for the pairs $(\mathcal{B}_s,\mathcal{B}_{ss})$ and $(\mathcal{B}_w,\mathcal{B}_s)$).

LEMMA 10. *There exist ε_0, $D'>0$ and $\lambda'>0$ such that for $\varepsilon\in I$ satisfying $|\varepsilon|\le\varepsilon_0$, $\omega\in\Omega'$, and $n\in\mathbb{N}$, we have that*

$$\|\mathcal{L}_{\omega,\varepsilon}^n h\|_{ss}\le D'e^{-\lambda' n}\|h\|_{ss}\quad \textit{for } h\in\mathcal{B}_{ss}^0, \tag{29}$$

and

$$\|\mathcal{L}_{\omega,\varepsilon}^n h\|_{s}\le D'e^{-\lambda' n}\|h\|_{s}\quad \textit{for } h\in\mathcal{B}_{s}^0. \tag{30}$$

By applying Proposition 7 for $\mathcal{B}_{ss}$ instead of $\mathcal{B}_s$, we deduce the following result.

PROPOSITION 11. *For each ε satisfying $|\varepsilon|\le\varepsilon_0$, there exists a unique family $(h_\omega^\varepsilon)_{\omega\in\Omega'}\subset\mathcal{B}_{ss}$ such that:*

- *$\omega\mapsto h_\omega^\varepsilon$ is measurable and bounded, that is*

$$\sup_{\omega\in\Omega'}\|h_\omega^\varepsilon\|_{ss}<\infty; \tag{31}$$

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- *for* $\omega \in \Omega'$,

$$\psi(h_\omega^\varepsilon) = 1; \tag{32}$$

- *for* $\omega \in \Omega'$,

$$\mathcal{L}_{\omega,\varepsilon} h_\omega^\varepsilon = h_{\sigma\omega}^\varepsilon. \tag{33}$$

Let us now introduce some additional assumptions. We suppose that for $\omega \in \Omega'$, there exists a bounded linear operator $\hat{\mathcal{L}}_\omega \colon \mathcal{B}_{ss} \to \mathcal{B}_s$, admitting a bounded extension (which will also be denoted by $\hat{\mathcal{L}}_\omega$) from $\mathcal{B}_s$ to $\mathcal{B}_w$, and such that

$$\begin{cases} \sup_{\omega\in\Omega'} \|\hat{\mathcal{L}}_\omega\|_{\mathcal{B}_{ss}\to\mathcal{B}_s} < \infty, \\ \sup_{\omega\in\Omega'} \|\hat{\mathcal{L}}_\omega\|_{\mathcal{B}_s\to\mathcal{B}_w} < \infty, \end{cases} \tag{34}$$

and we suppose that there is a function $\alpha : I \to \mathbb{R}_+$, $\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 0$ such that for $\omega \in \Omega'$,

$$\left\| \frac{1}{\varepsilon}(\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega)h - \hat{\mathcal{L}}_\omega h \right\|_w \le \alpha(\varepsilon)\|h\|_{ss} \quad \text{for } h \in \mathcal{B}_{ss} \text{ and } \varepsilon \in I \setminus \{0\}. \tag{35}$$

We emphasize that the inequality (35) only holds in $\mathcal{B}_w$-topology. Obviously, $\hat{\mathcal{L}}_\omega \mathcal{B}_{ss}^0 \subset \mathcal{B}_s^0$, for $\omega \in \Omega'$, but it also follows from (35) and boundedness of ψ on $\mathcal{B}_w$ that $\hat{\mathcal{L}}_\omega : \mathcal{B}_{ss} \to \mathcal{B}_s^0$.

Finally, we assume that for $\omega \in \Omega'$ and every $n \in \mathbb{N}$,

$$\|\mathcal{L}_\omega^n h\|_w \le D' e^{-\lambda' n}\|h\|_w \quad \text{for } h \in \mathcal{B}_w^0. \tag{36}$$

We continue to denote h_ω^0 simply by h_ω. For $\omega \in \Omega'$, set

$$\hat{h}_\omega := \sum_{j=0}^{\infty} \mathcal{L}_{\sigma^{-j}\omega}^j \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} h_{\sigma^{-(j+1)}\omega}. \tag{37}$$

It follows from (6), (31), (34), and the previous discussion that $\hat{h}_\omega \in \mathcal{B}_s^0$ for $\omega \in \Omega'$. In addition,

$$\sup_{\omega\in\Omega'} \|\hat{h}_\omega\|_s < \infty. \tag{38}$$

The following is our linear response result.

THEOREM 12. *We have that*

$$\lim_{\varepsilon\to 0} \sup_{\omega\in\Omega'} \left\| \frac{1}{\varepsilon}(h_\omega^\varepsilon - h_\omega) - \hat{h}_\omega \right\|_w = 0. \tag{39}$$

Proof. Let us begin by introducing some auxiliary notation. Set

$$\tilde{h}_\omega^\varepsilon := h_\omega^\varepsilon - h_\omega \quad \text{and} \quad \tilde{\mathcal{L}}_{\omega,\varepsilon} := \mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega.$$

It follows easily from (33) that

$$\tilde{h}_\omega^\varepsilon - \mathcal{L}_{\sigma^{-1}\omega} \tilde{h}_{\sigma^{-1}\omega}^\varepsilon = \tilde{\mathcal{L}}_{\sigma^{-1}\omega,\varepsilon} h_{\sigma^{-1}\omega}^\varepsilon,$$

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and, thus,

$$\tilde{h}^{\varepsilon}_{\omega} = \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \tilde{\mathcal{L}}_{\sigma^{-(j+1)}\omega,\varepsilon} h^{\varepsilon}_{\sigma^{-(j+1)}\omega}, \tag{40}$$

for $\omega \in \Omega'$. By (37) and (40), we have that

$$\begin{aligned}\left\| \frac{1}{\varepsilon}\tilde{h}^{\varepsilon}_{\omega} - \hat{h}_{\omega} \right\|_{w} &= \left\| \frac{1}{\varepsilon} \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \tilde{\mathcal{L}}_{\sigma^{-(j+1)}\omega,\varepsilon} h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - \hat{h}_{\omega} \right\|_{w} \\ &\le \left\| \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \left(\frac{1}{\varepsilon} \tilde{\mathcal{L}}_{\sigma^{-(j+1)}\omega,\varepsilon} - \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} \right) h^{\varepsilon}_{\sigma^{-(j+1)}\omega} \right\|_{w} \\ &\quad + \left\| \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} \left(h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - h_{\sigma^{-(j+1)}\omega} \right) \right\|_{w}.\end{aligned} \tag{41}$$

By applying Lemma 9, we have

$$\sup_{|\varepsilon| \le \varepsilon_0} \sup_{\omega \in \Omega'} \|h^{\varepsilon}_{\omega}\|_{ss} < \infty.$$

This, together with (35) and (36) implies that

$$\begin{aligned}&\left\| \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \left(\frac{1}{\varepsilon} \tilde{\mathcal{L}}_{\sigma^{-(j+1)}\omega,\varepsilon} - \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} \right) h^{\varepsilon}_{\sigma^{-(j+1)}\omega} \right\|_{w} \\ &\quad \le \sum_{j=0}^{\infty} D' e^{-\lambda' j} \left\| \left(\frac{1}{\varepsilon} \tilde{\mathcal{L}}_{\sigma^{-(j+1)}\omega,\varepsilon} - \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} \right) h^{\varepsilon}_{\sigma^{-(j+1)}\omega} \right\|_{w} \\ &\quad \le \tilde{D}\alpha(\varepsilon) \sup_{|\varepsilon| \le \varepsilon_0} \sup_{\omega \in \Omega'} \|h^{\varepsilon}_{\omega}\|_{ss},\end{aligned} \tag{42}$$

for $\omega \in \Omega'$, where $\tilde{D} > 0$ does not depend on ω and ε. On the other hand, we have by (34) and (36) that

$$\begin{aligned}&\left\| \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} (h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - h_{\sigma^{-(j+1)}\omega}) \right\|_{w} \\ &\quad \le \sum_{j=0}^{\infty} D' e^{-\lambda' j} \|\hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} (h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - h_{\sigma^{-(j+1)}\omega})\|_{w} \\ &\quad \le \sup_{\omega \in \Omega'} \|\hat{\mathcal{L}}_{\omega}\|_{\mathcal{B}_s \to \mathcal{B}_w} \sum_{j=0}^{\infty} D' e^{-\lambda' j} \|h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - h_{\sigma^{-(j+1)}\omega}\|_{s}.\end{aligned}$$

Now, our assumptions ensure that we may apply Theorem 8 for the pair $(\mathcal{B}_s, \mathcal{B}_{ss})$. Hence, we obtain

$$\left\| \sum_{j=0}^{\infty} \mathcal{L}^{j}_{\sigma^{-j}\omega} \hat{\mathcal{L}}_{\sigma^{-(j+1)}\omega} (h^{\varepsilon}_{\sigma^{-(j+1)}\omega} - h_{\sigma^{-(j+1)}\omega}) \right\|_{w} \le C' |\varepsilon| |\log |\varepsilon|| \tag{43}$$

for $\omega \in \Omega'$, where $C' > 0$ is independent on ω and ε. It follows readily from (41), (42), and (43) that (39) holds, which completes the proof of the theorem. □

Remark 13. The purpose of this remark is to interpret Theorem 8 (as well as Theorem 12) in the context of the multiplicative ergodic theory. In order to do so, we first need to introduce two additional assumptions. Namely, we require that:

- $\mathcal{B}_s$ is separable;
- the inclusion $\mathcal{B}_s \hookrightarrow \mathcal{B}_w$ is compact.

We denote the *largest Lyapunov exponent* of the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$, for $\varepsilon \in I$, by $\Lambda(\varepsilon) \in \mathbb{R} \cup \{-\infty\}$. We stress that the existence of $\Lambda(\varepsilon)$ is a direct consequence of (8) (applied to $n = 1$) and the subadditive ergodic theorem. Moreover, we recall that

$$\Lambda(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}^n_{\omega,\varepsilon}\|_s \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

By using (8) together with Proposition 7, it is easy to show (see the proof of [19, Lemma 3.5]) that $\Lambda(\varepsilon) = 0$, for $\varepsilon \in I$ with $|\varepsilon| \le \varepsilon_0$. Moreover, for each such ε, the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ is quasi-compact (in the sense of [25, Definition 2.7]). Hence, it follows from the multiplicative ergodic theorem (see [25, Theorem A]) that for each $\varepsilon \in I$ with $|\varepsilon| \le \varepsilon_0$, there exists:

- $1 \le l = l(\varepsilon) \le \infty$ and a sequence of *exceptional Lyapunov exponents*

  $$0 = \Lambda(\varepsilon) = \lambda_1(\varepsilon) > \lambda_2(\varepsilon) > \cdots > \lambda_l(\varepsilon) > \kappa(\varepsilon)$$

  or in the case $l = \infty$,

  $$0 = \Lambda(\varepsilon) = \lambda_1(\varepsilon) > \lambda_2(\varepsilon) > \cdots \quad \text{with } \lim_{n\to\infty} \lambda_n(\varepsilon) = \kappa(\varepsilon);$$

- a unique measurable Oseledets splitting

  $$\mathcal{B}_s = \left(\bigoplus_{j=1}^{l} Y_j^\varepsilon(\omega) \right) \oplus V^\varepsilon(\omega),$$

  where each component of the splitting is equivariant under $\mathcal{L}_{\omega,\varepsilon}$, that is, $\mathcal{L}_{\omega,\varepsilon}(Y_j^\varepsilon(\omega)) = Y_j^\varepsilon(\sigma\omega)$ and $\mathcal{L}_{\omega,\varepsilon}(V^\varepsilon(\omega)) \subset V^\varepsilon(\sigma\omega)$; the subspaces $Y_j^\varepsilon(\omega)$ are finite-dimensional and for each $y \in Y_j^\varepsilon(\omega) \setminus \{0\}$,

  $$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n_{\omega,\varepsilon} y\| = \lambda_j(\varepsilon);$$

  moreover, for $y \in V(\omega)$, $\lim_{n\to\infty}(1/n) \log \|\mathcal{L}^n_{\omega,\varepsilon} y\| \le \kappa(\varepsilon)$.

It follows easily from Proposition 6 (see the proof of [19, Proposition 3.6]) that $Y_1^\varepsilon(\omega)$ is one-dimensional and is spanned by h_ω^ε, for each $\varepsilon \in I$ such that $|\varepsilon| \le \varepsilon_0$.

Hence, Theorem 8 can be interpreted as a regularity result for the top Oseledets space of $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$. Namely, it shows that it is continuous in appropriate sense in $\varepsilon = 0$. Taking into account that Lyapunov exponents and corresponding Oseledets subspaces represent non-autonomous versions of the classical notions of an eigenvalue and the corresponding eigenspace, we conclude that Theorem 8 is a natural extension of statistical stability results concerned with deterministic systems. In a similar manner, Theorem 12 can be viewed as a non-autonomous generalization linear reponse result.

4. *Proof of the main theorem*

In this section, we prove Theorem 1 by showing that the assumptions of our abstract Theorems 8 and 12 are satisfied.

We place ourselves in the context of §2.2: we fix a small enough interval $0 \in I \subset \mathbb{R}$, and we consider a C^s mapping $\mathcal{T}: I \to C^{r+1}(M, M)$, such that $T_0 := \mathcal{T}(0)(\cdot)$ is a C^{r+1}, transitive Anosov diffeomorphism.

We now let $\Delta > 0$ and consider $V := B_{C^s(I,C^{r+1}(M,M))}(\mathcal{T}, \Delta)$. One has the following lemma.

LEMMA 14. *There exists $C > 0$, depending only on $\mathcal{T}$ and Δ, such that for any $\mathcal{S} \in V$, any $\varepsilon \in I$,*

$$d_{C^{r+1}}(S_\varepsilon, S_0) \leq C|\varepsilon|. \tag{44}$$

Proof. From the discussion in §2.2, it follows that for any $\mathcal{S} \in V$,

$$d_{C^{r+1}}(S_\varepsilon, S_0) = \sum_{i=1}^{N} \sum_{j \in \mathcal{J}(i)} \|S_{ij}(\varepsilon, \cdot) - S_{ij}(0, \cdot)\|_{C^{r+1}},$$

where we use the notation $S_{ij}(\varepsilon, \cdot) = \psi_j^{-1} \circ S_\varepsilon \circ \psi_i$ for $j \in \mathcal{J}(i)$. From the mean value theorem, one obtains $S_{ij}(\varepsilon, \cdot) - S_{ij}(0, \cdot) = \int_0^\varepsilon \partial_\varepsilon S_{ij}(\eta, \cdot)\, d\eta$ and, hence,

$$\begin{aligned}\|S_{ij}(\varepsilon, \cdot) - S_{ij}(0, \cdot)\|_{C^{r+1}} &\leq \int_0^\varepsilon \|\partial_\varepsilon S_{ij}(\eta, \cdot)\|_{C^{r+1}} d\eta \\ &\leq C(\mathcal{T}, \Delta)|\varepsilon|\end{aligned}$$

from which the conclusion follows. □

We consider the following triplet of Banach spaces:

$$\mathcal{B}_{ss} = \mathcal{B}^{3,1}(T_0, M) \hookrightarrow \mathcal{B}_s = \mathcal{B}^{2,2}(T_0, M) \hookrightarrow \mathcal{B}_w = \mathcal{B}^{1,3}(T_0, M). \tag{45}$$

We consider a measurable map $\mathbf{T} : \Omega \to V$, and we write $T_{\omega,\varepsilon} = \mathbf{T}(\omega)(\varepsilon)(\cdot)$. Finally, we let ψ be defined by $\psi(h) = h(1)$, which is a bounded functional on all three spaces in (45).

Proof of Theorem 1.

(1) By Lemma 14 we have, for $\varepsilon > 0$, that $d_{C^{r+1}}(T_\omega, T_{\omega,\varepsilon}) \leq C|\varepsilon|$, with C independent of ε and ω. Hence, [**26**, Lemma 7.1] implies that (9) and (27) hold.

(2) As T is transitive, the deterministic transfer operator associated with T has a spectral gap on all three spaces $\mathcal{B}_{ss}$, $\mathcal{B}_s$ and $\mathcal{B}_w$. (Observe that $\mathcal{B}_w$ is compactly embedded into $\mathcal{B}^{0,4}$.) Consequently, it follows from [**13**, Proposition 2.10] that by shrinking δ is necessary, we have that (6), (25) and (36) hold.

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(3) The uniform Lasota–Yorke inequalities (8), (10), and (26) may be established arguing as in [19, §3.2] or [26, §7].

(4) By arguments analogous to those in [19, §3.1], one has that the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ is strongly measurable on $\mathcal{B}_s$ and $\mathcal{B}_{ss}$.

The previous arguments are enough to apply Proposition 7 and Theorem 8 to our situation, giving us an equivariant family $(h_\omega^\varepsilon)_{\omega\in\Omega} \subset \mathcal{B}_{ss}$, that satisfies our statistical stability estimate (20) with respect to the norm $\|\cdot\|_{2,2}$. We note that (see [19, Proposition 3.3]) for $\varepsilon \in I$, h_ω^ε is actually a positive probability measure on M for $\mathbb{P}$-almost every $\omega \in \Omega$.

What is now left to do is to establish the existence and required properties of the 'derivative operator'. Following the lines of [26, §9], we systematically abuse notation and ignore coordinates charts.

Denote by $g_\omega(\varepsilon, \cdot) := 1/|\det(DT_{\omega,\varepsilon})|$ the weight of the transfer operator $\mathcal{L}_{\omega,\varepsilon}$. Under our assumptions, when viewed in coordinates, the maps $\varepsilon \mapsto g_\omega(\varepsilon,\cdot) \in C^r(M,\mathbb{R}^*)$ and $\varepsilon \mapsto T_{\omega,\varepsilon}(\cdot)^{-1}$ are of class C^s, $s > 1$. In particular, we may, for $\phi \in C^r(M,\mathbb{R})$, differentiate $\mathcal{L}_{\omega,\varepsilon}\phi$ with respect to ε and obtain

$$\partial_\varepsilon[\mathcal{L}_{\omega,\varepsilon}\phi] = \mathcal{L}_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi + v_{\omega,\varepsilon}\phi), \tag{46}$$

$$\begin{aligned}\partial_\varepsilon^2[\mathcal{L}_{\omega,\varepsilon}\phi] = \mathcal{L}_{\omega,\varepsilon}(J_{\omega,\varepsilon}^2\phi + J_{\omega,\varepsilon}(v_{\omega,\varepsilon}\phi) + v_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi)\\ + v_{\omega,\varepsilon}(v_{\omega,\varepsilon}\phi) + [\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi + \partial_\varepsilon[v_{\omega,\varepsilon}\phi]),\end{aligned} \tag{47}$$

where

$$v_{\omega,\varepsilon}\phi := -D\phi(\cdot)\cdot[DT_{\omega,\varepsilon}(\cdot)]^{-1}\cdot\partial_\varepsilon T_\omega(\varepsilon,\cdot), \tag{48}$$

$$J_{\omega,\varepsilon} := \frac{\partial_\varepsilon g_\omega(\varepsilon,\cdot) + v_{\omega,\varepsilon}g_\omega(\varepsilon,\cdot)}{g_\omega(\varepsilon,\cdot)}. \tag{49}$$

Note that both of these expressions are, together with their first s-derivatives with respect to ε, in $C^{r-1}(M,\mathbb{R})$. We also denote by $v_{\omega,\varepsilon}$ the C^r vector field associated with the operator $v_{\omega,\varepsilon}$. As noted in §2.1, multiplication by $J_{\omega,\varepsilon}$ and the action of $v_{\omega,\varepsilon}$ induce the bounded operator from $\mathcal{B}^{i,j}$ to itself (respectively, $\mathcal{B}^{i,j}$ to $\mathcal{B}^{i-1,j+1}$), where $i + j < r$, and the same goes for their derivatives with respect to ε.

Furthermore, by our Assumption 1, $J_{\omega,\varepsilon}$ and $v_{\omega,\varepsilon}$, as well as their derivatives with respect to ε, are bounded uniformly in ω and ε, that is,

$$\max\Big(\sup_{\omega\in\Omega}\sup_{\varepsilon\in I}\|J_{\omega,\varepsilon}\|_{C^{r-1}}, \sup_{\omega\in\Omega}\sup_{\varepsilon\in I}\|\partial_\varepsilon J_{\omega,\varepsilon}\|_{C^{r-1}}\Big) < \infty,$$

$$\max\Big(\sup_{\omega\in\Omega}\sup_{\varepsilon\in I}\|v_{\omega,\varepsilon}\|_{C^{r}}, \sup_{\omega\in\Omega}\sup_{\varepsilon\in I}\|\partial_\varepsilon v_{\omega,\varepsilon}\|_{C^{r}}\Big) < \infty.$$

For $\phi \in C^r(M,\mathbb{R})$, set

$$\hat{\mathcal{L}}_\omega\phi := \partial_\varepsilon[\mathcal{L}_{\omega,\varepsilon}\phi]|_{\varepsilon=0} = \mathcal{L}_\omega(J_{\omega,0}\phi + v_{\omega,0}\phi). \tag{50}$$

By our previous discussion, we conclude that (34) holds.

On the other hand, using Taylor's formula we conclude that for $|\varepsilon|$ small enough,

$$\mathcal{L}_{\omega,\varepsilon}\phi - \mathcal{L}_\omega\phi - \varepsilon\hat{\mathcal{L}}_\omega\phi = \int_0^\varepsilon\int_0^\eta \partial_\varepsilon^2[\mathcal{L}_{\omega,\varepsilon}\phi]|_{\varepsilon=\xi}\, d\xi\, d\eta.$$

By (48) and the following discussion,

$$\|\partial_\varepsilon^2[\mathcal{L}_{\omega,\varepsilon}\phi]|_{\varepsilon=\xi}\|_w \le C\|\phi\|_{ss},$$

where $C > 0$ independent of both ω and ε. Hence, (35) is satisfied, and we may apply Theorem 12, which gives us that the map $\varepsilon \in I \mapsto h_\omega^\varepsilon \in L^\infty(\Omega, \mathcal{B}_w)$ is differentiable at $\varepsilon = 0$. Moreover,

$$\hat{h}_\omega := [\partial_\varepsilon h_\omega^\varepsilon]|_{\varepsilon=0} = \sum_{n=0}^{\infty} \mathcal{L}^{(n)}_{\sigma^{-n}\omega}\hat{\mathcal{L}}_{\sigma^{-n-1}\omega}h_{\sigma^{-n-1}\omega}. \tag{51}$$

To obtain (3), we note that, by the density of smooth functions in $\mathcal{B}^{i,j}$ and (35), $\hat{\mathcal{L}}_\omega$, as a bounded operator from $\mathcal{B}^{i,j}$ to $\mathcal{B}^{i-1,j+1}$, admits the representation (in fact, this formula defines a bounded operator from $\mathcal{D}'_j$ to $\mathcal{D}'_{j+1}$, but we will not need it)

$$(\hat{\mathcal{L}}_\omega f)(\phi) := f(\partial_\varepsilon[\phi\circ T_{\omega,\varepsilon}]|_{\varepsilon=0}),$$

for any $f \in \mathcal{B}^{i,j}$ and $\phi \in C^r(M,\mathbb{R})$. Then, for $\phi \in C^r(M,\mathbb{R})$ we have that

$$\begin{aligned}
\partial_\varepsilon\bigg[\int_M \phi\, dh_\omega^\varepsilon\bigg]\bigg|_{\varepsilon=0} &= \partial_\varepsilon[h_\omega^\varepsilon(\phi)]|_{\varepsilon=0}\\
&= \hat{h}_\omega(\phi)\\
&= \sum_{n=0}^{\infty} \mathcal{L}^{(n)}_{\sigma^{-n}\omega}\hat{\mathcal{L}}_{\sigma^{-n-1}\omega}h_{\sigma^{-n-1}\omega}(\phi)\\
&= \sum_{n=0}^{\infty} \hat{\mathcal{L}}_{\sigma^{-n-1}\omega}h_{\sigma^{-n-1}\omega}(\phi\circ T^n_{\sigma^{-n}\omega})\\
&= \sum_{n=0}^{\infty} h_{\sigma^{-n-1}\omega}(\partial_\varepsilon[\phi\circ T^n_{\sigma^{-n}\omega}\circ T_{\sigma^{-n-1}\omega,\varepsilon}]|_{\varepsilon=0}),
\end{aligned}$$

which gives (3). This completes the proof of Theorem 1. □

5. *Applications*

In this section, we present two applications of our main result. Let us assume that the assumptions in Hypothesis 1 hold. We consider the triplet of spaces given by (45). Furthermore, for $\varepsilon \in I$ sufficiently close to zero, let $(h_\omega^\varepsilon)_{\omega\in\Omega} \subset \mathcal{B}_{ss}$ be as in §4. By shrinking I if necessary, we can assume that h_ω^ε exists for $\varepsilon \in I$ and $\omega \in \Omega$. Moreover, recall that h_ω^ε is a probability measure on M for $\omega \in \Omega$ (see §4). As previously, we write h_ω instead of h_ω^0.

5.1. *Annealed linear response for hyperbolic dynamics.*

As a first application, we establish a form of an annealed linear response.

For $F \in L^\infty(\Omega, C^r(M))$ and $\varepsilon \in I$, we set

$$R(\varepsilon, F) = \int_\Omega\int_M F(\omega, x)\, dh_\omega^\varepsilon(x)\, d\mathbb{P}(\omega). \tag{52}$$

The following is our annealed linear response result.

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THEOREM 15. *The map $R : I \times L^\infty(\Omega, C^r(M)) \to \mathbb{R}$ is differentiable at every* $(0, F)$*, $F \in L^\infty(\Omega, C^r(M))$. Furthermore, one has*

$$\partial_\varepsilon[R(\varepsilon, F)]|_{\varepsilon=0} = \sum_{n=0}^{\infty} \int_\Omega \int_M \partial_\varepsilon[F_\omega \circ T^n_{\sigma^{-n}\omega} \circ T_{\sigma^{-n-1}\omega,\varepsilon}]|_{\varepsilon=0} \, dh_{\sigma^{-n-1}\omega} \, d\mathbb{P}(\omega). \quad (53)$$

Remark 16. The previous result can be interpreted as *linear response for the stationary measure of the skew product*

$$S_\varepsilon(\omega, x) := (\sigma\omega, T_{\omega,\varepsilon} x),$$

acting on $\Omega \times M$. Indeed, the stationary measure μ_ε of this skew-product classically admits the disintegration along fibers

$$\mu_\varepsilon(A \times B) = \int_A h^\varepsilon_\omega(B) \, d\mathbb{P}(\omega),$$

for measurable $A \subset \Omega$, $B \subset M$. In particular, this justifies the 'annealed' terminology, because in the independent and identically distributed case, the measure defined on M by $\tilde{\mu}_\varepsilon(\cdot) = \mu_\varepsilon(\Omega \times \cdot)$ corresponds to the invariant measure of the Markov chain associated with our cocycle.

We also point out that one may use this interpretation to establish a linear response for a class of *deterministic partially hyperbolic skew products*: let us set $\Omega = \mathbb{S}^1$, $\mathbb{P} =$ Lebesgue, and $\sigma(\omega) = \omega + \alpha \mod 1$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then, consider a family $(T_{\omega,\varepsilon})_{\omega\in\mathbb{S}^1,\varepsilon\in I}$ of Anosov diffeomorphisms of $\mathbb{T}^2$, for example,

$$T_{\omega,\varepsilon}(x_1, x_2) := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \omega \\ \omega \end{pmatrix} + \varepsilon \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \end{pmatrix}.$$

This system clearly satisfies our Hypothesis 1 (note that it belongs to the class of Examples 4), and the skew product S_ε acting on $\mathbb{S}^1 \times \mathbb{T}^2 \simeq \mathbb{T}^3$ is clearly a partially hyperbolic system (with central direction tangent to the first coordinate), exhibiting linear response by Theorem 15 and the previous discussion.

Proof of Theorem 15. Fix an arbitrary $F_0 \in L^\infty(\Omega, C^r(M))$. We claim that the derivative of R in $(0, F_0)$ is given by

$$DR(0, F_0)(\varepsilon, H) = \varepsilon \int_\Omega \hat{h}_\omega(F_0(\omega)) \, d\mathbb{P}(\omega) + \int_\Omega h_\omega(H(\omega)) \, d\mathbb{P}(\omega), \quad (54)$$

for $(\varepsilon, H) \in \mathbb{R} \times L^\infty(\Omega, C^r(M))$, where $\hat{h}_\omega$ is given by (51). Indeed, observe that

$$R(\varepsilon, F_0 + H) - R(0, F_0) - \varepsilon \int_\Omega \hat{h}_\omega(F_0(\omega)) \, d\mathbb{P}(\omega) - \int_\Omega h_\omega(H(\omega)) \, d\mathbb{P}(\omega)$$
$$= \int_\Omega (h^\varepsilon_\omega - h_\omega - \varepsilon\hat{h}_\omega)(F_0(\omega)) \, d\mathbb{P}(\omega) + \int_\Omega (h^\varepsilon_\omega - h_\omega)(H(\omega)) \, d\mathbb{P}(\omega).$$

Furthermore, the continuous embedding $\mathcal{B}^{p,q} \hookrightarrow \mathcal{D}'_q$ entails that there is $C > 0$ (independent on both ω and ε) such that

$$\begin{aligned}\bigg|\frac{1}{\varepsilon}\int_\Omega (h^\varepsilon_\omega - h_\omega - \varepsilon\hat{h}_\omega)(F_0(\omega))\, d\mathbb{P}(\omega)\bigg| \\ \leq C\|F_0\|_{L^\infty(\Omega, C^r(M))} \cdot \sup_{\omega\in\Omega}\bigg\|\frac{1}{\varepsilon}(h^\varepsilon_\omega - h_\omega) - \hat{h}_\omega\bigg\|_w,\end{aligned}$$

and, thus, Theorem 12 implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega (h^\varepsilon_\omega - h_\omega - \varepsilon\hat{h}_\omega)(F_0(\omega))\, d\mathbb{P}(\omega) = 0.$$

In addition,

$$\bigg|\int_\Omega (h^\varepsilon_\omega - h_\omega)(H(\omega))\, d\mathbb{P}(\omega)\bigg| \leq C\|H\|_{L^\infty(\Omega, C^r(M))} \cdot \sup_{\omega\in\Omega}\|h^\varepsilon_\omega - h_\omega\|_w,$$

and, consequently, by applying Theorem 8 (for the pair $(\mathcal{B}_s, \mathcal{B}_w)$), we obtain that

$$\lim_{(\varepsilon,H)\to(0,0)}\frac{1}{\|H\|_{L^\infty(\Omega, C^0(M))}}\bigg|\int_\Omega (h^\varepsilon_\omega - h_\omega)(H(\omega))\, d\mathbb{P}(\omega)\bigg| = 0.$$

Thus, (54) holds and the proof of the theorem is completed. In order to establish (53), one can argue as in the proof of formula (3). □

5.2. *Regularity of the variance in the central limit theorem for random hyperbolic dynamics.* In this section, we provide an application of Theorem 12 to the problem of the regularity of the variance (under suitable perturbations) in the quenched version of the central limit theorem for random hyperbolic dynamics.

Let F be as in the previous subsection. For $\omega \in \Omega$ and $\varepsilon \in I$, set

$$f_{\omega,\varepsilon} := F_\omega - h^\varepsilon_\omega(F_\omega) = F_\omega - \int_M F_\omega\, dh^\varepsilon_\omega.$$

Set

$$\begin{aligned}\Sigma^2_\varepsilon := &\int_\Omega\int_M f^2_{\omega,\varepsilon}(x)\, dh^\varepsilon_\omega(x)\, d\mathbb{P}(\omega) \\ &+ 2\sum_{n=1}^\infty \int_\Omega\int_M f_{\omega,\varepsilon}(x) f_{\sigma^n\omega,\varepsilon}(T^n_{\omega,\varepsilon}x)\, dh^\varepsilon_\omega(x)\, d\mathbb{P}(\omega).\end{aligned} \tag{55}$$

Observe that $\Sigma^2_\varepsilon \geq 0$ and that Σ^2_ε does not depend on ω. It is proved in [**19**, Theorem B] that if $\Sigma^2_\varepsilon > 0$, the process $(f_{\omega,\varepsilon} \circ T^n_{\omega,\varepsilon})$ satisfies $\mathbb{P}$-almost surely a quenched central limit theorem. More precisely, for every bounded and continuous $\phi : \mathbb{R} \to \mathbb{R}$ and $\mathbb{P}$-almost every $\omega \in \Omega$, we have that

$$\lim_{n\to\infty}\int \phi\bigg(\frac{S_n(f_{\omega,\varepsilon})}{\sqrt{n}}\bigg)\, dh^\varepsilon_\omega = \int \phi\, d\mathcal{N}(0, \Sigma^2_\varepsilon),$$

where

$$S_n(f_{\omega,\varepsilon}) := \sum_{k=0}^{n-1} f_{\sigma^k\omega,\varepsilon} \circ T^k_{\omega,\varepsilon},$$

and $\mathcal{N}(0, \Sigma_\varepsilon^2)$ denotes the normal distribution with parameters 0 and Σ_ε. Our goal is to establish the following result.

THEOREM 17. *Under the above assumptions, the map $\varepsilon \mapsto \Sigma_\varepsilon^2$ is differentiable at $\varepsilon = 0$.*

We start the proof by making a few remarks related to the map $\varepsilon \mapsto (f_{\omega,\varepsilon})_{\omega\in\Omega} \in C^r(M)$.

- For each ε, $\omega \mapsto f_{\omega,\varepsilon}$ is an element of $L^\infty(\Omega, C^r(M))$. Moreover, by Lemma 9 we have that

$$\sup_{|\varepsilon|\le\varepsilon_0} \operatorname{esssup}_{\omega\in\Omega} \|f_{\omega,\varepsilon}\|_{C^r} \le \Big(1 + \sup_{|\varepsilon|\le\varepsilon_0} \operatorname{esssup}_{\omega\in\Omega} \|h^\varepsilon_\omega\|_{ss}\Big) \operatorname{esssup}_{\omega\in\Omega} \|F_\omega\|_{C^r}. \tag{56}$$

- It is differentiable at $\varepsilon = 0$. Indeed, we have

$$\frac{1}{\varepsilon}(f_{\omega,\varepsilon} - f_\omega) = \frac{1}{\varepsilon}(h_\omega - h^\varepsilon_\omega)(F_\omega),$$

  which yields

$$\operatorname{esssup}_{\omega\in\Omega} \left| \frac{1}{\varepsilon}(f_{\omega,\varepsilon} - f_\omega) + \hat{h}_\omega(F_\omega) \right| \to 0, \tag{57}$$

  as $\varepsilon \to 0$, via Theorem 12. Here, we write f_ω instead of $f_{\omega,0}$.

The above observations together with Theorem 12 easily imply the following lemma.

LEMMA 18. *The map*

$$\varepsilon \mapsto \int_\Omega \int_M f^2_{\omega,\varepsilon}(x)\, dh^\varepsilon_\omega(x)\, d\mathbb{P}(\omega)$$

is differentiable at $\varepsilon = 0$.

Proof. For ε sufficiently close to zero, let $H(\varepsilon) \in L^\infty(\Omega, C^r(M))$ be defined by

$$H(\varepsilon)(\omega) = f^2_{\omega,\varepsilon}, \quad \omega \in \Omega.$$

Then, the discussion preceding the statement of the lemma implies that the map H is differentiable at $\varepsilon = 0$. Now the conclusion of the lemma follows from Theorem 15 and the simple observation that

$$\int_\Omega \int_M f^2_{\omega,\varepsilon}(x)\, dh^\varepsilon_\omega(x)\, d\mathbb{P}(\omega) = R(\varepsilon, H(\varepsilon)),$$

with R given by (52). □

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We recall that (see §2.1) that for $h \in \mathcal{B}^{p,q}$ and $f \in C^q(M)$, we can define $f \cdot h \in \mathcal{B}^{p,q}$ whose action as a distribution is given by

$$(f \cdot h)(\phi) = h(f\phi), \quad \text{for } \phi \in C^q(M).$$

Moreover, there exists $C > 0$ (depending only on M) such that

$$\|f \cdot h\|_{p,q} \leq C\|h\|_{p,q} \cdot \|f\|_{C^q}.$$

The above inequality is frequently used in what follows and, thus, we do not explicitly refer to it. Moreover, in what follows, $C > 0$ denotes a constant which is independent on all parameters (ω, n, etc.) involved.

Observe that

$$(f_{\omega,\varepsilon} \cdot h_\omega^\varepsilon)(f_{\sigma^n\omega,\varepsilon} \circ T_{\omega,\varepsilon}^n) = \mathcal{L}_{\omega,\varepsilon}^n(f_{\omega,\varepsilon} \cdot h_\omega^\varepsilon)(f_{\sigma^n\omega,\varepsilon}).$$

In addition, $(f_{\omega,\varepsilon} \cdot h_\omega^\varepsilon)(1) = h_\omega^\varepsilon(f_{\omega,\varepsilon}) = 0$. We now write

$$\frac{1}{\varepsilon}(\mathcal{L}_{\omega,\varepsilon}^n(f_{\omega,\varepsilon} \cdot h_\omega^\varepsilon)(f_{\sigma^n\omega,\varepsilon}) - \mathcal{L}_\omega^n(f_\omega \cdot h_\omega)(f_{\sigma^n\omega})) = (I)_{n,\omega,\varepsilon} + (II)_{n,\omega,\varepsilon} + (III)_{n,\omega,\varepsilon}, \tag{58}$$

where

$$(I)_{n,\omega,\varepsilon} := \mathcal{L}_\omega^n(f_\omega \cdot h_\omega)\bigg(\frac{1}{\varepsilon}(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega})\bigg),$$
$$(II)_{n,\omega,\varepsilon} := \frac{1}{\varepsilon}(\mathcal{L}_{\omega,\varepsilon}^n - \mathcal{L}_\omega^n)(f_\omega \cdot h_\omega)(f_{\sigma^n\omega,\varepsilon}),$$
$$(III)_{n,\omega,\varepsilon} := \mathcal{L}_{\omega,\varepsilon}^n\bigg(\frac{f_{\omega,\varepsilon} \cdot h_\omega^\varepsilon - f_\omega \cdot h_\omega}{\varepsilon}\bigg)(f_{\sigma^n\omega,\varepsilon}).$$

LEMMA 19. *For each $n \in \mathbb{N}$,*

$$\lim_{\varepsilon \to 0} \operatorname{esssup}_{\omega\in\Omega}|(I)_{n,\omega,\varepsilon} - \hat{h}_{\sigma^n\omega}(F_{\sigma^n\omega})\mathcal{L}_\omega^n(f_\omega \cdot h_\omega)(1)| = 0.$$

In addition, for ε sufficiently close to zero, we have that

$$\operatorname{esssup}_{\omega\in\Omega}|(I)_{n,\omega,\varepsilon}| \leq Ce^{-\lambda n}.$$

Proof. The first assertion follows directly from (25), (56), and (57). In addition, observe that for ε sufficiently close to zero,

$$\operatorname{esssup}_{\omega\in\Omega}|(I)_{n,\omega,\varepsilon} - \hat{h}_{\sigma^n\omega}(F_{\sigma^n\omega})\mathcal{L}_\omega^n(f_\omega \cdot h_\omega)(1)| \leq Ce^{-\lambda n}.$$

On the other hand, (25), (38), and (57) imply that

$$\operatorname{esssup}_{\omega\in\Omega}|\hat{h}_{\sigma^n\omega}(F_{\sigma^n\omega})\mathcal{L}_\omega^n(f_\omega \cdot h_\omega)(1)| \leq Ce^{-\lambda n}.$$

The above two estimates readily give the second assertion of the lemma. $\square$

LEMMA 20. *For each $n \in \mathbb{N}$,*

$$\lim_{\varepsilon \to 0} \operatorname{esssup}_{\omega\in\Omega} |(II)_{n,\omega,\varepsilon} - \hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega)(f_{\sigma^n\omega})| = 0, \tag{59}$$

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where

$$\hat{\mathcal{L}}_{n,\omega} = \sum_{k=1}^{n} \mathcal{L}^{n-k}_{\sigma^k\omega} \hat{\mathcal{L}}_{\sigma^{k-1}\omega} \mathcal{L}^{k-1}_{\omega}.$$

Furthermore, for ε *sufficiently close to zero, we have that*

$$\operatorname{esssup}_{\omega\in\Omega} |(II)_{n,\omega,\varepsilon}| \le Cne^{-\lambda' n}.$$

Proof. In order to prove (59), we first claim that

$$\left\| \frac{1}{\varepsilon}(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega})(f_\omega \cdot h_\omega) - \hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega) \right\|_w \le \tilde{\alpha}(\varepsilon), \tag{60}$$

with $\tilde{\alpha}(\varepsilon) \to 0$ when $\varepsilon \to 0$. Observe that

$$\frac{1}{\varepsilon}(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega}) = \sum_{k=1}^{n} \mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon} \frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} \mathcal{L}^{k-1}_{\omega},$$

and, therefore,

$$\begin{aligned}
\frac{1}{\varepsilon}(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega}) - \hat{\mathcal{L}}_{n,\omega} &= \sum_{k=1}^{n} \left[\mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon} \frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} - \mathcal{L}^{n-k}_{\sigma^k\omega} \hat{\mathcal{L}}_{\sigma^{k-1}\omega} \right] \mathcal{L}^{k-1}_{\omega} \\
&= \sum_{k=1}^{n} \Bigg[(\mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon} - \mathcal{L}^{n-k}_{\sigma^k\omega}) \frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} \\
&\quad + \mathcal{L}^{n-k}_{\sigma^k\omega} \left(\frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} - \hat{\mathcal{L}}_{\sigma^{k-1}\omega} \right) \Bigg] \mathcal{L}^{k-1}_{\omega}.
\end{aligned}$$

By the arguments in the proof of Proposition 6, (25), (27), and (56), we have that

$$\begin{aligned}
&\left\| (\mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon} - \mathcal{L}^{n-k}_{\sigma^k\omega}) \frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} \mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega) \right\|_w \\
&\quad \le C|\varepsilon|(n-k) \left\| \frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} \mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega) \right\|_s \\
&\quad \le C|\varepsilon|(n-k) e^{-\lambda(k-1)} \operatorname{esssup}_{\omega\in\Omega} \|f_\omega \cdot h_\omega\|_{ss} \\
&\quad \le C|\varepsilon|(n-k) e^{-\lambda k}.
\end{aligned} \tag{61}$$

Similarly, using (25), (31), (35), (36), and (56), we obtain that

$$\begin{aligned}
&\left\| \mathcal{L}^{n-k}_{\sigma^k\omega} \left(\frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} - \hat{\mathcal{L}}_{\sigma^{k-1}\omega} \right) \mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega) \right\|_w \\
&\quad \le Ce^{-\lambda'(n-k)} \left\| \left(\frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} - \hat{\mathcal{L}}_{\sigma^{k-1}\omega} \right) \mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega) \right\|_w \\
&\quad \le Ce^{-\lambda'(n-k)} \alpha(\varepsilon) \|\mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega)\|_{ss} \\
&\quad \le Ce^{-\lambda' n} \alpha(\varepsilon) \operatorname{esssup}_{\omega\in\Omega} \|f_\omega \cdot h_\omega\|_{ss} \\
&\quad \le C\alpha(\varepsilon) e^{-\lambda' n}.
\end{aligned} \tag{62}$$

Then, (61) and (62) imply (60).

Furthermore, (6), (25), (31), (34), and (56) imply that

$$
\begin{aligned}
\|\hat{\mathcal{L}}_{\omega,n}(f_\omega \cdot h_\omega)\|_w &\le \sum_{k=1}^{n} \|\mathcal{L}^{n-k}_{\sigma^k\omega}\hat{\mathcal{L}}_{\sigma^{k-1}\omega}\mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega)\|_s \\
&\le C\sum_{k=1}^{n} e^{-\lambda(n-k)} \operatorname{esssup}_{\omega\in\Omega}(\|\hat{\mathcal{L}}_{\sigma^{k-1}\omega}\|_{B_{ss}\to B_s} \cdot \|\mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega)\|_{ss}) \\
&\le C\sum_{k=1}^{n} e^{-\lambda(n-k)}e^{-\lambda(k-1)} \operatorname{esssup}_{\omega\in\Omega} \|f_\omega \cdot h_\omega\|_{ss} \\
&\le Cne^{-\lambda n}.
\end{aligned} \tag{63}
$$

Using Theorem 8, (56), (60), and (63), we have that

$$
\begin{aligned}
&\operatorname{esssup}_{\omega\in\Omega} |(II)_{n,\omega,\varepsilon} - \hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega)(f_{\sigma^n\omega})| \\
&\quad\le \operatorname{esssup}_{\omega\in\Omega} \left|\frac{1}{\varepsilon}(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega})(f_\omega \cdot h_\omega)(f_{\sigma^n\omega,\varepsilon}) - \hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega)(f_{\sigma^n\omega,\varepsilon})\right| \\
&\qquad + \operatorname{esssup}_{\omega\in\Omega} |\hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega)(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega})| \\
&\quad\le \tilde{\alpha}(\varepsilon) \operatorname{esssup}_{\omega\in\Omega}\|f_{\sigma^n\omega,\varepsilon}\|_{C^r} + Cne^{-\lambda n} \operatorname{esssup}_{\omega\in\Omega} |(h^\varepsilon_\omega - h_\omega)(F_\omega)| \\
&\quad\le C\tilde{\alpha}(\varepsilon) + Cne^{-\lambda n}|\varepsilon||\log(|\varepsilon|)|,
\end{aligned}
$$

which implies the first assertion of the lemma.

On the other hand, using (36) (which also persists under small perturbations), (31), (34), and (56), we have that for ε sufficiently small,

$$
\operatorname{esssup}_{\omega\in\Omega} \left\|(\mathcal{L}^{n-k}_{\sigma^k\omega,\varepsilon} - \mathcal{L}^{n-k}_{\sigma^k\omega})\frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon}\mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega)\right\|_w \le Ce^{-\lambda' n}. \tag{64}
$$

Moreover, from (62) it follows that for ε sufficiently small,

$$
\operatorname{esssup}_{\omega\in\Omega} \left\|\mathcal{L}^{n-k}_{\sigma^k\omega}\left(\frac{\mathcal{L}_{\sigma^{k-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{k-1}\omega}}{\varepsilon} - \hat{\mathcal{L}}_{\sigma^{k-1}\omega}\right)\mathcal{L}^{k-1}_{\omega}(f_\omega \cdot h_\omega)\right\|_w \le Ce^{-\lambda' n}. \tag{65}
$$

By (64) and (65), we have that for sufficiently small ε,

$$
\operatorname{esssup}_{\omega\in\Omega} \left\|\frac{1}{\varepsilon}(\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega})(f_\omega \cdot h_\omega) - \hat{\mathcal{L}}_{n,\omega}(f_\omega \cdot h_\omega)\right\|_w \le Cne^{-\lambda' n}.
$$

The above estimate together with (63) easily implies that the second assertion of the lemma also holds. □

By using similar arguments, one can establish the following lemma.

LEMMA 21. *For each $n \in \mathbb{N}$,*

$$
\lim_{\varepsilon\to 0} \operatorname{esssup}_{\omega\in\Omega}|(III)_{n,\omega,\varepsilon} - \mathcal{L}^n_{\omega}(\hat{h}_\omega(F_\omega)h_\omega + f_\omega \cdot \hat{h}_\omega)(f_{\sigma^n\omega})| = 0.
$$

Moreover, for ε sufficiently small, we have that

$$
\operatorname{esssup}_{\omega\in\Omega}|(III)_{n,\omega,\varepsilon}| \le Ce^{-\lambda' n}.
$$

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The conclusion of Theorem 17 follows from previous lemmas and the dominated convergence theorem.

Remark 22. In [**20**] the authors have extended the results from [**19**] to the case of vector-valued observables. In particular, the quenched version of the central limit theorem for vector-valued observables was established. In this setting, the variance is a symmetric matrix which is, in general, positive semi-definite (for the central limit theorem to hold it needs to be positive-definite). One can easily establish the version of Theorem 17 in this setting, essentially by repeating the arguments in the proof of Theorem 17 for each matrix component.

6. *Application to other types of random systems*

In this paper, we focused our efforts on studying (quenched) statistical stability and linear response for random compositions of Anosov diffeomorphisms. Nevertheless, our approach, or a slight modification thereof, is applicable to other types of random hyperbolic systems.

6.1. *Random uniformly expanding dynamics.*

In this subsection, let us describe the application of Theorems 8 and 12 to a simple class of fiberwise perturbations of random compositions of uniformly expanding maps of the unit circle $\mathbb{S}^1$. The setting is close to [**24**, §6]: consider a family $(D_\varepsilon)_{\varepsilon\in I}$ of diffeomorphisms of $\mathbb{S}^1$ (where $0 \in I \subset \mathbb{R}$ is an interval), satisfying

$$D_\varepsilon = \mathrm{Id} + \varepsilon S,$$

where $S : \mathbb{S}^1 \to \mathbb{R}$ is a C^4 map. Letting $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, endowed with an invertible, measure-preserving and ergodic map $\sigma : \Omega \circlearrowleft$. We consider a measurable map $\omega \in \Omega \mapsto T_\omega \in C^4(\mathbb{S}^1, \mathbb{S}^1)$ such that:

(1) there exists $\lambda > 1$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, $\inf_{x\in\mathbb{S}^1} |T'_\omega(x)| \geq \lambda$;

(2) $\operatorname{esssup}_{\omega\in\Omega} \|T_\omega\|_{C^4} \leq \Delta$ for some small $\Delta > 0$.

We then set

$$T_{\omega,\varepsilon} := D_\varepsilon \circ T_\omega \quad \text{for } \varepsilon \in I \text{ and } \omega \in \Omega,$$

and we review the assumptions of Theorems 8 and 12 for the spaces $\mathcal{B}_{ss} = W^{3,1}(\mathbb{S}^1)$, $\mathcal{B}_s = W^{2,1}(\mathbb{S}^1)$, and $\mathcal{B}_w = W^{1,1}(\mathbb{S}^1)$.

- Equations (8) and (10) are established in [**14**, §5].
- Equation (9) follows from [**24**, Proposition 35].
- By applying [**13**, Proposition 2.10] (provided that Δ is sufficiently small), we conclude that (6) holds on $(\mathcal{B}_{ss}, \mathcal{B}_s, \mathcal{B}_w)$.
- To define the derivative operator $\hat{\mathcal{L}}_\omega$, we start by remarking that because $\mathcal{L}_{\omega,\varepsilon} = \mathcal{L}_{D_\varepsilon}\mathcal{L}_\omega$, one has (see [**24**, Eq. (51)]) that

$$\hat{\mathcal{L}}_\omega = \left[\frac{d\mathcal{L}_{D_\varepsilon}}{d\varepsilon}\right]\Big|_{\varepsilon=0} \mathcal{L}_\omega = -(\mathcal{L}_\omega(\cdot)S)'.$$

It is easy to see that $\hat{\mathcal{L}}_\omega$ defines a bounded operator from $\mathcal{B}_{ss}$ to $\mathcal{B}_s$ (respectively, from $\mathcal{B}_s$ to $\mathcal{B}_w$) and satisfies (34).

As for condition (35), we have for $\phi \in \mathcal{B}_s$

$$\begin{aligned}\|\varepsilon^{-1}(\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega)\phi - \hat{\mathcal{L}}_\omega\phi\|_w &\leq \|\varepsilon^{-1}(\mathcal{L}_{D_\varepsilon} - \mathrm{Id}) + (\cdot S)'\|_{\mathcal{B}_s\to\mathcal{B}_w} \operatorname{esssup}_{\omega\in\Omega} \|\mathcal{L}_\omega\phi\|_s \\ &\leq C\alpha(\varepsilon)\|\phi\|_s\end{aligned}$$

by using (8), with $\alpha(\varepsilon) = \|\varepsilon^{-1}(\mathcal{L}_{D_\varepsilon} - \mathrm{Id}) + (\cdot S)'\|_{\mathcal{B}_s\to\mathcal{B}_w}$, which goes to zero as $\varepsilon \to 0$ by [**24**, Proposition 36].

6.2. *Random piecewise hyperbolic dynamics.* Let us discuss the application of Theorem 8 to random compositions of close-by piecewise hyperbolic maps, defined on a two-dimensional compact Riemann manifold X, as described in [**19**, §10] and [**16**, §2]. It is noteworthy that one *cannot* directly apply Theorem 8, because, as noted in [**19**, §10.2.1], the transfer operator map $\omega \mapsto \mathcal{L}_\omega$ is not strongly measurable. Still, the conclusion of Theorem 8 holds; let us explain why.

In [**16**, §2.4], the set Γ_A of maps T satisfying the assumptions of [**16**, §2], with second derivative $|D^2T| < A$ is introduced, as well as the distance γ between two such maps.

Let us fix a (small enough) $\varepsilon_0 > 0$, a $T \in \Gamma_A$ and let $X_{\varepsilon_0} := \{S \in \Gamma_A : \gamma(T, S) < \varepsilon_0\}$. We let $\mathcal{B}_s$ and $\mathcal{B}_w$ be the Banach spaces defined in [**16**, §2.2] (where $\mathcal{B}_s$ is denoted $\mathcal{B}$). In particular, we recall that elements of $\mathcal{B}_s$ are distributions of order at most one. Letting $I := [-\varepsilon_0/2, \varepsilon_0/2]$, we set, for a fixed $L > 0$:

$$B_{\varepsilon_0,L} := \{\mathcal{T} : I \to X_{\varepsilon_0},\ \gamma(\mathcal{T}(\varepsilon), \mathcal{T}(\varepsilon')) \leq L|\varepsilon - \varepsilon'|, \quad \text{for all } \varepsilon, \varepsilon' \in I\}.$$

This can be viewed as a ball of Lipschitz (with respect to the distance γ) curves from I to X_{ε_0}. We now consider a measurable, countably valued mapping $\mathbf{T} : \Omega \to B_{\varepsilon_0,L}$. As before, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space endowed with an invertible, measure-preserving, and ergodic map σ and we use the notation $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon, \cdot)$.

We claim that for any $\varepsilon \in I$, there exists a measurable family $(h_\omega^\varepsilon)_{\omega\in\Omega} \subset \mathcal{B}_s$ such that $\mathcal{L}_{\omega,\varepsilon} h_\omega^\varepsilon = h_{\sigma\omega}^\varepsilon$ for $\mathbb{P}$-almost every $\omega \in \Omega$ and

$$\operatorname{esssup}_{\omega\in\Omega} \|h_\omega^\varepsilon - h_\omega\|_w \leq C\varepsilon^\beta |\log(|\varepsilon|)|,$$

for some $C > 0$, $0 < \beta < 1$, independent on ω and ε, with $h_\omega := h_\omega^0$.

Let us review the assumptions for Theorem 8 in this context.

- Equation (6) holds by [**19**, Eq. (70)], where $\psi \in \mathcal{B}_s'$ is given by $\psi(h) = h(1)$, $h \in \mathcal{B}_s$.
- Up to shrinking ε_0, we have (8) and (10) by [**19**, Eq. (71)].
- Up to replacing ε by ε^β, (9) follows from the definition of $B_{\varepsilon_0,L}$ and [**16**, Lemma 6.1].
- As usual, (11) holds as $\mathcal{L}_{\omega,\varepsilon}$ is a transfer operator associated with $T_{\omega,\varepsilon}$.

In particular, Proposition 6 (uniform in ε and ω exponential decay of correlations) holds in the present setting. We cannot use here the fixed-point construction of Proposition 7, because we do not know whether the cocycle of transfer operators $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ is strongly measurable. However, we can use (8), (10), and that, for each $\varepsilon \in I$, $T_{\omega,\varepsilon}$ is countably valued to apply the version of the MET for the so-called $\mathbb{P}$-continuous cocycles (see

[22, Theorem 17]): this gives us, as in Remark 13, that for each $\varepsilon \in I$ there exists:

- $1 \leq l = l(\varepsilon) \leq \infty$ and a sequence of *exceptional Lyapunov exponents*

$$0 = \Lambda(\varepsilon) = \lambda_1(\varepsilon) > \lambda_2(\varepsilon) > \cdots > \lambda_l(\varepsilon) > \kappa(\varepsilon)$$

  or in the case $l = \infty$,

$$0 = \Lambda(\varepsilon) = \lambda_1(\varepsilon) > \lambda_2(\varepsilon) > \cdots \quad \text{with } \lim_{n\to\infty} \lambda_n(\varepsilon) = \kappa(\varepsilon);$$

- a full-measure set Ω_ε such that for each $\omega \in \Omega_\varepsilon$, there is a unique measurable Oseledets splitting

$$\mathcal{B}_s = \left(\bigoplus_{j=1}^{l} Y_j^\varepsilon(\omega) \right) \oplus V^\varepsilon(\omega),$$

  where each component of the splitting is equivariant under $\mathcal{L}_{\omega,\varepsilon}$, that is, $\mathcal{L}_{\omega,\varepsilon}(Y_j^\varepsilon(\omega)) = Y_j^\varepsilon(\sigma\omega)$ and $\mathcal{L}_{\omega,\varepsilon}(V^\varepsilon(\omega)) \subset V^\varepsilon(\sigma\omega)$. The subspaces $Y_j^\varepsilon(\omega)$ are finite-dimensional and for each $y \in Y_j^\varepsilon(\omega) \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_{\omega,\varepsilon}^n y\|_s = \lambda_j(\varepsilon).$$

  Moreover, for $y \in V(\omega)$, $\lim_{n\to\infty}(1/n) \log \|\mathcal{L}_{\omega,\varepsilon}^n y\|_s \leq \kappa(\varepsilon)$.

It follows easily from Proposition 6 (see the proof of [19, Proposition 3.6]) that $Y_1^\varepsilon(\omega)$ is one-dimensional: for each $\varepsilon \in I$, we may, thus, consider a generator h_ω^ε, normalized by $\psi(h_\omega^\varepsilon) = 1$, which satisfies $\mathcal{L}_{\omega,\varepsilon} h_\omega^\varepsilon = h_{\sigma\omega}^\varepsilon$. We now claim that

$$\sup_{\varepsilon\in I} \operatorname{esssup}_{\omega\in\Omega} \|h_\omega^\varepsilon\|_s < +\infty. \tag{66}$$

In order to establish (66), we start by observing that using (12) we have that

$$\|h_\omega^\varepsilon - \mathcal{L}_{\sigma^{-n}\omega,\varepsilon}^n 1\|_s = \|\mathcal{L}_{\sigma^{-n}\omega,\varepsilon}^n (h_{\sigma^{-n}\omega}^\varepsilon - 1)\|_s \leq D' e^{-\lambda' n} \|h_{\sigma^{-n}\omega}^\varepsilon - 1\|_s, \tag{67}$$

for $n \in \mathbb{N}$, $\omega \in \Omega$ and $\varepsilon \in I$. Furthermore, because $\lambda_1(\varepsilon) = 0$, we have that the random variable $\omega \mapsto \|h_\omega^\varepsilon\|_s$ is tempered (we recall that a random variable $K\colon \Omega \to (0, +\infty)$ is tempered if $\lim_{n\to\pm\infty}(1/n) \log K(\sigma^n\omega) = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$) for each $\varepsilon \in I$. Hence, by [1, Proposition 4.3.3] for each $\varepsilon \in I$, there exists a random variable $K_\varepsilon\colon \Omega \to (0, +\infty)$ such that

$$\|h_\omega^\varepsilon - 1\|_s \leq K_\varepsilon(\omega) \quad \text{and} \quad K_\varepsilon(\sigma^n\omega) \leq e^{\lambda'|n|/2} K_\varepsilon(\omega), \tag{68}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ and $n \in \mathbb{Z}$. By (67) and (68), we obtain that

$$\|h_\omega^\varepsilon - \mathcal{L}_{\sigma^{-n}\omega,\varepsilon}^n 1\|_s \leq D' K_\varepsilon(\omega) e^{-(\lambda' n/2)} \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega \text{ and } n \in \mathbb{N},$$

which implies that for $\varepsilon \in I$,

$$h_\omega^\varepsilon = \lim_{n\to\infty} \mathcal{L}_{\sigma^{-n}\omega,\varepsilon}^n 1 \quad \text{in } \mathcal{B}_s, \text{ for } \mathbb{P}\text{-almost every } \omega \in \Omega. \tag{69}$$

Clearly, (66) follows readily from (8) and (69). From there, we can reproduce the proof of Theorem 8, to obtain the announced result.

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Remark 23. It is natural to ask whether Theorem 12 can be applied in the piecewise hyperbolic setting described above. First, we note that there is no natural candidate for a $\mathcal{B}_{ss}$ space. Indeed, as noted in the introduction of [**16**] (and in contrast with the situation in [**26**]), considering a (piecewise) C^r or a (piecewise) C^s, $r > s > 2$, system yields the same couple ($\mathcal{B}_w$ and $\mathcal{B}_s$) of Banach spaces. In other words, the degree of the smoothness of maps does not influence the construction of the anisotropic spaces, which makes unclear whether this line of reasoning can produce a space $\mathcal{B}_{ss}$ satisfying our requirements. In fact, to the best of the authors' knowledge there are currently no results dealing with the linear response for classes of piecewise hyperbolic dynamics described above even in the deterministic setting (that is, when we take Ω to be a singleton).

Second, the case of deterministic, one-dimensional piecewise expanding maps [**5**, **9**] suggests that, in general, the linear response does not hold in a piecewise smooth setting.

Finally, we note that for random compositions of billiard maps such as described, e.g., in [**17**] do not fall under the setup of Theorem 8, as they do not satisfy Lasota–Yorke inequalities of the type (8) and (10) (the $\|\cdot\|_w$ carries a factor η^n for some $\eta \geq 1$).

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