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INDETERMINACY AND THE ROLE OF FACTOR SUBSTITUTABILITY

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We discuss the role of the elasticity of substitution in the local determinacy properties of a steady state or a stationary balanced growth path in a general multisector economy with CES technologies. Our main results are the following: We give some sufficient conditions for the occurrence of local indeterminacy in exogenous and endogenous growth models. We show that local indeterminacy takes place even without a capital intensity reversal from the private to the social level if the productive factors are weakly substitutable. Moreover, we show that the conditions for local indeterminacy in exogenous growth models and in endogenous growth models may be qualitatively different.

Keywords: Sector-Specific Externalities, Constant Returns, Factor Substitutability, Indeterminacy

1. INTRODUCTION

The aim of this paper is to discuss the role of the elasticity of substitution or factor substitutability in the local determinacy properties of a steady state or a stationary balanced growth path in a general multisector economy with CES technologies. We prove two new results. First, local indeterminacy occurs without a capital intensity reversal, provided the elasticity of substitution between factors is "weak" (less than 1). Second, allowing the elasticity of substitution between factors to be different from unity provides sufficient flexibility in these models to recognize that conditions for local indeterminacy in exogenous growth models are qualitatively different from those for endogenous growth models, unlike what the literature might suggest.

Under perfect competition, it is now well known since the contributions of Kehoe and Levine (1985), Muller and Woodford (1988), and Kehoe et al. (1990), that equilibria are generically locally determinate in models with a finite number of infinitely lived agents whereas local indeterminacy, that is, the existence of a

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continuum of equilibrium paths, is a rather standard feature of models with an infinite number of finitely lived agents, such as OLG models.[1] On the contrary, under imperfect competition, locally indeterminacy also arises in economies "à la Ramsey" with infinitely lived agents.

There is a large literature on indeterminacy in macroeconomics. Increasing returns, imperfect competition, and money are some of the main transmission mechanisms that provide room for the existence of sunspot fluctuations based on shocks on expectations.[2] Following Woodford (1986) who proves that local indeterminacy is a sufficient condition for the existence of sunspot equilibria, many contributions have focused on coordination problems raised by multiple equilibria[3] and on macroeconomic instability based on the volatility of expectations.[4] Indeterminacy and sunspots indeed provide a possible explanation of business-cycle fluctuations. The most recent contributions, such as those from Benhabib and Nishimura (1998, 1999), show that exogenous-growth multisector models are compatible with realistic market imperfections and can provide a good match to macroeconomic time series.[5] Most of the papers in the literature deal with infinitely lived agents in order to match the frequency of fluctuations observed in the data.[6]

Since Romer (1986) and Lucas (1988), much attention has also been given to stability of equilibria in endogenous growth models.[7] The common feature of most of these contributions is the consideration of Cobb-Douglas technologies and sector-specific and/or intersectoral external effects.[8] Focusing on production functions with unitary elasticity of substitution prevented any discussion of the role of factor substitutability.

Although Cobb-Douglas technologies are widely used in growth theory, recent papers have questioned the empirical relevance of this specification. Duffy and Papageorgiou (2000), for instance, consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that for the entire sample of countries the assumption of unitary elasticity of substitution may be rejected. Moreover, dividing the sample of countries into several subsamples, they find that capital and labor have an elasticity of substitution greater than unity in the richest group of countries whereas the elasticity is less than unity in the poorest group of countries. It therefore seems necessary to question the robustness of indeterminacy with respect to the elasticity of substitution.[9]

Boldrin and Rustichini (1994) and Drugeon et al. (2003) consider two-sector growth models with general neoclassical technologies and intersectoral externalities. Bond et al. (1996) also examine the effects of distortionary factor taxation, which is formally equivalent to sector-specific external effects. These authors provide conditions for local indeterminacy of equilibrium paths. However, no clear condition on the size of the elasticity of substitution is given.

We consider in this paper an $(n+1)$-sector continuous-time economy with CES technologies and sector-specific externalities. Our goal is to characterize the local stability of a steady state or a stationary balanced growth path and to evaluate the influence of the input elasticity of substitution. The model with exogenous growth consists of a pure consumption good and n capital goods that are produced using capital and labor. We assume that the instantaneous utility function is linear. The model with endogenous growth consists of one consumable capital good and n pure capital goods. We assume that each good is produced without fixed factors. As in the standard literature, we assume that the instantaneous utility function is homogeneous with a constant intertemporal elasticity of substitution.

Building on some recent empirical studies of disaggregated U.S. data by Basu and Fernald (1997), we assume that the aggregate technology of each sector has constant social returns, which implies that individual firms exhibit small decreasing returns. This divergence between private and social returns is explained by the existence of mild external effects. It follows that the standard duality between Rybczynski and Stolper-Samuelson effects, which holds in optimal growth models, is broken in our framework.

Based on the destruction of this duality, it has been shown by Benhabib and Nishimura (1998) in a two-sector exogenous growth model with sector-specific externalities that the steady state is locally indeterminate if the consumption good is capital intensive at the private level but labor intensive at the social level.[10] Local indeterminacy then requires a capital intensity reversal.

However, all these results have been established using Cobb-Douglas technologies. We use CES production functions and study the role of the elasticity of substitution in the stability properties of the steady state or the stationary balanced growth path in general multisector models. We show that local indeterminacy takes place even without a capital intensity reversal if factor substitutability is weak, that is, the elasticity of substitution is less than 1. Moreover, we prove that factor substitutability must be sufficiently weak in exogenous growth models for local indeterminacy to take place whereas this is not necessarily true in endogenous growth models.

This paper is organized as follows. The next section sets up the basic model of production. Section 3 examines the conditions for indeterminacy in the exogenous growth model while Section 4 deals with the endogenous growth model. Section 5 contains some concluding comments. Most of the proofs are gathered in the Appendix.

2. BASIC MODEL OF PRODUCTION

Output y_j is produced by inputs $x_{0j}, \ldots, x_{nj}$. There are $n+1$ outputs $y_0, y_1, \ldots, y_n$. The production functions are CES, and the representative firm in each industry faces the following function:

$$y_j = \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho_j} + e_j(\boldsymbol{X}_j) \right)^{-1/\rho_j}, \quad j = 0, \ldots, n, \qquad \textbf{(1)}$$

with $\rho_j > -1$ and $\sigma_j = 1/(1+\rho_j) \geq 0$ the elasticity of substitution. The positive externalities, $e_j(X_j)$, will be equal to $\sum_{i=0}^{n} b_{ij} X_{ij}^{-\rho_j}$, with $b_{ij} \geq 0$ and $\boldsymbol{X}_j = (X_{0j}, X_{1j}, \ldots, X_{nj})$ where X_{ij} denotes the average use of input i in sector j. We assume that these economywide averages are taken as given by the individual firm. At the equilibrium, since all firms of sector j are identical, we have $X_{ij} = x_{ij}$ and we may define the *social production functions* as follows:

$$y_j = \left(\sum_{i=0}^{n} (\beta_{ij} + b_{ij}) x_{ij}^{-\rho_j} \right)^{-1/\rho_j}. \tag{2}$$

We assume that in each sector $\sum_{i=0}^{n} (\beta_{ij} + b_{ij}) = 1$ so that the production functions collapse to Cobb-Douglas in the particular case $\rho_j = 0$. Notice also that the returns to scale are constant at the social level, and decreasing at the private level.[11] Our formulation is, however, compatible with constant returns at the private level if we assume that there exists a factor in fixed supply such as land.[12] In this case, the income of the representative consumer will be increased by the rental of land.

A firm in each industry maximizes its profit, given output price p_j and input prices $w_0, \ldots, w_n$. Its profit is

$$\pi_j = p_j y_j - \sum_{i=0}^{n} w_i x_{ij}.$$

Assumption 1. For any $i, j = 0, \ldots, n$, $\beta_{ij} > 0$.

It is well known that with CES technologies, depending on the value of the elasticity of substitution, the Inada conditions may not be satisfied. It follows that Assumption 1 is necessary but not sufficient to guarantee that a positive amount of every good is produced. Therefore, in the rest of the paper, we restrict our analysis to the case of interior solutions for which every good is produced by a positive amount and every input is used by a positive amount in the production of every good.

To focus the analysis on the elasticity of substitution, we assume that it is identical accross sectors, that is, $\rho_j = \rho$, $j = 0, \ldots, n$. The first-order conditions subject to the private technologies (1) are the following:

$$p_j \beta_{ij} (y_j / x_{ij})^{1+\rho} = w_i, \qquad i, j = 0, \ldots, n. \tag{3}$$

From (3), we have

$$x_{ij} / y_j = (p_j \beta_{ij} / w_i)^{\frac{1}{1+\rho}} \equiv a_{ij}(w_i, p_j), \qquad i, j = 0, \ldots, n. \tag{4}$$

We call a_{ij} the input coefficients from the *private* viewpoint. If the agents take account of externalities as endogenous variables in profit maximization, the first-order conditions subject to the social technologies (2) are

$$p_j (\beta_{ij} + b_{ij})(y_j / x_{ij})^{1+\rho} = w_i, \qquad i, j = 0, \ldots, n,$$

and the input coefficients become

$$\bar{a}_{ij}(w_i, p_j) = (p_j \hat{\beta}_{ij} / w_i)^{1/(1+\rho)}, \qquad i, j = 0, \ldots, n,$$

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with $\hat{\beta}_{ij} = \beta_{ij} + b_{ij}$. We call $\bar{a}_{ij}$ the input coefficients from the *social* viewpoint. However, as we show below, the factor-price frontier, which gives a relationship between input prices and output prices, is not exactly expressed with the input coefficients from the social viewpoint. We define

$$\hat{a}_{ij}(w_i, p_j) \equiv (\hat{\beta}_{ij}/\beta_{ij}) a_{ij}(w_i, p_j),$$

which we will call the *quasi*-input coefficients from the *social* viewpoint, and it is easy to derive that

$$\hat{a}_{ij}(w_i, p_j) = \bar{a}_{ij}(w_i, p_j)(\hat{\beta}_{ij}/\beta_{ij})^{\rho/(1+\rho)}.$$

Notice that $\hat{a}_{ij} = \bar{a}_{ij}$ if $b_{ij} = 0$, i.e., there is no externality coming from input i in sector j, or $\rho = 0$, that is, the production function is Cobb-Douglas.

On the basis of these input coefficients, we now establish various lemmas. We first show that the factor-price frontier is determined by the quasi-input coefficients from the social viewpoint.

LEMMA 1. *Denote* $\boldsymbol{p} = (p_0, \dots, p_n)'$, $\boldsymbol{w} = (w_0, \dots, w_n)'$, *and* $\hat{A}(\boldsymbol{w}, \boldsymbol{p}) = [\hat{a}_{ij}(w_i, p_j)]$. *Then,* $\boldsymbol{p} = \hat{A}(\boldsymbol{w}, \boldsymbol{p})'\boldsymbol{w}$.

Proof. Substituting (4) into the real production function (2) gives

$$y_j = \left[p_j^{\frac{-\rho}{1+\rho}} y_j^{-\rho} \sum_{i=0}^{n} (\beta_{ij} + b_{ij}) \left(\frac{w_i}{\beta_{ij}} \right)^{\frac{\rho}{1+\rho}} \right]^{-1/\rho}.$$

It follows that

$$p_j^{\frac{\rho}{1+\rho}} = \sum_{i=0}^{n} \hat{\beta}_{ij} \left(\frac{w_i}{\beta_{ij}} \right)^{\frac{\rho}{1+\rho}} = \sum_{i=0}^{n} \left[(\hat{\beta}_{ij}/\beta_{ij}) \left(\frac{\beta_{ij}}{w_i} \right)^{\frac{1}{1+\rho}} w_i \right]. \tag{5}$$

Multiplying both sides of this equality by $p_j^{1/(1+\rho)}$ gives

$$p_j = \sum_{i=0}^{n} \left[(\hat{\beta}_{ij}/\beta_{ij}) \left(\frac{p_j \beta_{ij}}{w_i} \right)^{\frac{1}{1+\rho}} w_i \right] = \sum_{i=0}^{n} \hat{a}_{ij} w_i.$$

■

The total stock of factors is a vector $\boldsymbol{x} = (x_0, \dots, x_n)'$ with $\boldsymbol{x}_i = \sum_{j=0}^{n} \boldsymbol{x}_{ij}$. From the full employment conditions, we derive the factor market-clearing equation, which depends on the input coefficients from the private perspective.

LEMMA 2. *Denote* $\boldsymbol{x} = (x_0, \dots, x_n)'$, $\boldsymbol{y} = (y_0, \dots, y_n)'$, *and* $A(\boldsymbol{w}, \boldsymbol{p}) = [a_{ij}(w_i, p_j)]$. *Then,* $A(\boldsymbol{w}, \boldsymbol{p})\boldsymbol{y} = \boldsymbol{x}$.

Proof. By definition, $x_{ij} = a_{ij} y_j$, and thus,

$$\sum_{j=0}^{n} x_{ij} = \sum_{j=0}^{n} a_{ij} y_j = x_i.$$

■

We now examine some comparative statics. Since the function $\hat{A}(\boldsymbol{w}, \boldsymbol{p})$ is homogeneous of degree zero in $\boldsymbol{w}$ and $\boldsymbol{p}$, the envelope theorem implies that the factor-price frontier satisfies Lemma 3.

LEMMA 3. $d\boldsymbol{p} = \hat{A}(\boldsymbol{w}, \boldsymbol{p})' d\boldsymbol{w}$.

Proof. Differentiating equation (5) gives

$$p_j^{\frac{-1}{1+\rho}} dp_j = \sum_{i=0}^{n} \frac{\hat{\beta}_{ij}}{\beta_{ij}} \left(\frac{w_i}{\beta_{ij}} \right)^{\frac{-1}{1+\rho}} dw_i,$$

and therefore,

$$dp_j = \sum_{i=0}^{n} \frac{\hat{\beta}_{ij}}{\beta_{ij}} \left(\frac{p_j \beta_{ij}}{w_i} \right)^{\frac{1}{1+\rho}} dw_i = \sum_{i=0}^{n} \hat{a}_{ij} dw_i.$$

■

The factor market-clearing equation finally satisfies

LEMMA 4.

$$A(\boldsymbol{w}, \boldsymbol{p}) d\boldsymbol{y} + \sum_{j=0}^{n} y_j \left(\frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{w}} d\boldsymbol{w} + \frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{p}} d\boldsymbol{p} \right) = d\boldsymbol{x} \quad \textit{with} \quad \frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{w}} = \left[\frac{\partial a_{ij}}{\partial w_s} \right]_{s=0,\dots,n} \quad \textit{and} \quad \frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{p}} = \left[\frac{\partial a_{ij}}{\partial p_s} \right]_{s=0,\dots,n}.$$

Proof. Starting from $\sum_{j=0}^{n} a_{ij} y_j = x_i$, we have

$$\sum_{j=0}^{n} a_{ij} dy_j + \sum_{j=0}^{n} y_j \left(\sum_{s=0}^{n} \frac{\partial a_{ij}}{\partial w_s} dw_s + \sum_{s=0}^{n} \frac{\partial a_{ij}}{\partial p_s} dp_s \right) = dx_i.$$

■

The total derivative of the system can be summarized by the following equation:

$$H \begin{pmatrix} d\boldsymbol{y} \\ d\boldsymbol{w} \end{pmatrix} = \begin{pmatrix} I & -\sum_{j=0}^{n} y_j \frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{p}} \\ 0 & I \end{pmatrix} \begin{pmatrix} d\boldsymbol{x} \\ d\boldsymbol{p} \end{pmatrix} \quad \text{with} \quad H = \begin{pmatrix} A & \sum_{j=0}^{n} y_j \frac{\partial \boldsymbol{a}^j}{\partial \boldsymbol{w}} \\ 0 & \hat{A}' \end{pmatrix}.$$

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Under Assumption 1, we now define $(n+1)\times(n+1)$ positive matrices

$$B = \left[\beta_{ij}^{1/(1+\rho)}\right] \quad \text{and} \quad \hat{B} = \left[\hat{\beta}_{ij}/\beta_{ij}^{\rho/(1+\rho)}\right].$$

Assumption 2. B and $\hat{B}$ are nonsingular matrices.

Let us also define the two following diagonal matrices:

$$W = \begin{pmatrix} w_0^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n^{\frac{1}{1+\rho}} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} p_0^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n^{\frac{1}{1+\rho}} \end{pmatrix}.$$

From (4), we get $A = W^{-1}BP$, $\hat{A} = W^{-1}\hat{B}P$, and thus under Assumption 2, $A^{-1} = P^{-1}B^{-1}W$, $\hat{A}^{-1} = P^{-1}\hat{B}^{-1}W$. Note also from Lemmas 1–2 and the above diagonal matrices that

$$W^{-1}\boldsymbol{w} = \hat{B}'^{-1}P^{-1}\boldsymbol{p} \quad \Leftrightarrow \quad \begin{pmatrix} w_0^{\frac{\rho}{1+\rho}} \\ \vdots \\ w_n^{\frac{\rho}{1+\rho}} \end{pmatrix} = \hat{B}'^{-1} \begin{pmatrix} p_0^{\frac{\rho}{1+\rho}} \\ \vdots \\ p_n^{\frac{\rho}{1+\rho}} \end{pmatrix} \tag{6}$$

and

$$\boldsymbol{y} = P^{-1}B^{-1}W\boldsymbol{x}. \tag{7}$$

Factor rentals are functions of output prices only, $w_i = w_i(\boldsymbol{p})$, whereas outputs are functions of factor stocks and output prices, $y_i = y_i(\boldsymbol{x},\boldsymbol{p})$, $i = 0,\ldots,n$. Finally, we obtain

$$\begin{pmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial p} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial p} \end{pmatrix} = H^{-1} = \begin{pmatrix} A^{-1} & * \\ 0 & \hat{A}'^{-1} \end{pmatrix}. \tag{8}$$

Without external effects, that is, $b_{ij} = 0$, the matrix $[\partial y/\partial x]$ reflects the Rybczynski theorem whereas the matrix $[\partial w/\partial p]$ reflects the Stolper–Samuelson theorem. From the duality between these two effects well known in trade theory, we get $[\partial y/\partial x] = [\partial w/\partial p]'$. However, in the presence of externalities, the Rybczynski effects depend on the input coefficients from the private perspective whereas the Stolper–Samuelson effects depend on the quasi-input coefficients from the social perspective. The duality between these two effects is thus destroyed. This follows from the fact that with market distortions, true costs are not being minimized. Local indeterminacy of equilibria will be a consequence of this property.

3. EXOGENOUS GROWTH MODEL WITH LINEAR UTILITY

3.1. Model

A representative agent optimizes a linear additively separable utility function with discount rate $\delta \geq 0$. This problem can be described as

$$\begin{aligned}
&\max_{\{x_{ij}(t)\}} \int_0^{+\infty} y_0(t)e^{-\delta t}\,dt \\
&\text{s.t. } y_j(t) = \left(\sum_{i=0}^{n} \beta_{ij}x_{ij}(t)^{-\rho} + e_j(\boldsymbol{X}_j(t))\right)^{-1/\rho} && j = 0,\ldots,n, \\
&\dot{x}_i(t) = y_i(t) - gx_i(t) && i = 1,\ldots,n, \\
&x_i(t) = \sum_{j=0}^{n} x_{ij}(t) && i = 0,\ldots,n, \\
&x_i(0) \text{ given} && i = 1,\ldots,n, \\
&\{e_j(\boldsymbol{X}_j(t))\}_{t\geq 0} \text{ given} && j = 0,\ldots,n,
\end{aligned} \tag{9}$$

where $x_0(t)$, interpreted as labor, is always equal to 1; $y_0(t)$ is the output of the pure consumption good; and $g \geq 0$ is the depreciation rate of the capital stocks.[13] We can write the modified Hamiltonian in current value as

$$\mathcal{H} = \left(\sum_{i=0}^{n} \beta_{i0}x_{i0}^{-\rho}(t) + e_0(\boldsymbol{X}_0(t))\right)^{-1/\rho} + \sum_{i=0}^{n} w_i(t)\left(x_i(t) - \sum_{j=0}^{n} x_{ij}(t)\right)$$
$$+ \sum_{j=1}^{n} p_j(t)\left(\left(\sum_{i=0}^{n} \beta_{ij}x_{ij}^{-\rho}(t) + e_j(\boldsymbol{X}_j(t))\right)^{-1/\rho} - gx_j(t)\right).$$

Here, $p_j(t)$ and $w_i(t)$ are, respectively, co-state variable and Lagrange multipliers, representing utility prices of the capital goods, their rental rates, and the wage rate, with $p_0(t) = 1$. The static first-order conditions are given by

$$w_s = p_j\beta_{sj}(y_j/x_{sj})^{1+\rho} = \beta_{s0}(y_0/x_{s0})^{1+\rho}$$

for $j = 1,\ldots,n$, $s = 0,1,\ldots,n$, and they are equivalent to (3). Let $\boldsymbol{x}^1 = (x_1,\ldots,x_n)'$, $\boldsymbol{y}^1 = (y_1,\ldots,y_n)'$, $\boldsymbol{p}^1 = (p_1,\ldots,p_n)'$ and $\boldsymbol{w}^1 = (w_1,\ldots,w_n)'$. It follows from (6) and (7) that the necessary conditions that describe the solution to problem (9) are given by the equations of motion:

$$\begin{aligned}
\dot{x}_i(t) &= y_i(\boldsymbol{x}^1(t),\boldsymbol{p}^1(t)) - gx_i(t) && i = 1,\ldots,n, \\
\dot{p}_i(t) &= (\delta+g)p_i(t) - w_i(\boldsymbol{p}^1(t)) && i = 1,\ldots,n.
\end{aligned} \tag{10}$$

Assumption 3. There exists a stationary point $(\boldsymbol{x}^{1*},\boldsymbol{p}^{1*})$ of the dynamical system (10) that solves $\dot{x}_i(t) = \dot{p}_i(t) = 0$, $i = 1,\ldots,n$.[14]

Linearizing around $(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})$ gives the $2n \times 2n$ Jacobian matrix

$$J = \begin{pmatrix} \left(\dfrac{\partial y^1(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})}{\partial x^1}\right) - gI & \left(\dfrac{\partial y^1(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})}{\partial p^1}\right) \\ 0 & -\left(\dfrac{\partial w^1(\boldsymbol{p}^{1*})}{\partial p^1}\right) + (\delta + g)I \end{pmatrix}.$$

Since in this model we have one pure consumption good, we need to eliminate from equality (8) the columns and rows that are associated with x_0, y_0, p_0, and w_0. To do so, we introduce the following $n \times n$ matrices:

$$A_1 = \left[a_{ij} - \frac{a_{i0}a_{0j}}{a_{00}}\right] \quad \text{and} \quad \hat{A}_1 = \left[\hat{a}_{ij} - \frac{\hat{a}_{i0}\hat{a}_{0j}}{\hat{a}_{00}}\right]$$

for $i, j = 1, \ldots, n$. The Jacobian matrix is thus as follows:

$$J = \begin{pmatrix} A_1^{-1} - gI & * \\ 0 & (\delta + g)I - \hat{A}_1'^{-1} \end{pmatrix}.$$

In the current economy, there are n capital goods whose initial values are given. Any solution from (10) that converges to the steady state $(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})$ satisfies the transversality condition and constitutes an equilibrium. Therefore, given $\boldsymbol{x}(0)$, if there is more than one set of initial prices $\boldsymbol{p}(0)$ in the stable manifold of $(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})$, the equilibrium path from $\boldsymbol{x}(0)$ will not be unique. In particular, if J has more than n roots with negative real parts, there will be a continuum of converging paths and thus a continuum of equilibria.

DEFINITION 1. *If the locally stable manifold of the steady state $(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})$ has dimension greater than n, then $(\boldsymbol{x}^{1*}, \boldsymbol{p}^{1*})$ is said to be locally indeterminate.*

The roots of J are determined by the roots of $[A_1^{-1} - gI]$ and $[(\delta + g)I - \hat{A}_1'^{-1}]$. A_1 is the matrix of factor intensity differences from the private viewpoint and $\hat{A}_1$ is the matrix of quasi factor intensity differences from the social viewpoint. Using the definitions of input coefficients given in Section 2, we may indeed interpret the elements of A_1 and $\hat{A}_1$ as follows:

DEFINITION 2. *The consumption good is said to be*

(i) *more intensive in capital good i than the capital good j at the* private *level if* $a_{ij}a_{00} - a_{i0}a_{0j} < 0$;
(ii) *more* quasi *intensive in capital good i than the capital good j at the* social *level if* $\hat{a}_{ij}\hat{a}_{00} - \hat{a}_{i0}\hat{a}_{0j} < 0$;
(iii) *more intensive in capital good i than the capital good j at the* social *level if* $\bar{a}_{ij}\bar{a}_{00} - \bar{a}_{i0}\bar{a}_{0j} < 0$.

As in a Cobb-Douglas framework, it is usual to formulate the factor intensity differences in terms of the β_{ij} and $\hat{\beta}_{ij}$ coefficients. A similar convenient formulation

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can be achieved with CES technologies. To do so at the private level, we need to define an $n \times n$ matrix B_1 as follows:

$$B_1 = \left[\beta_{ij}^{\frac{1}{1+\rho}} - \frac{\beta_{i0}^{\frac{1}{1+\rho}} \beta_{0j}^{\frac{1}{1+\rho}}}{\beta_{00}^{\frac{1}{1+\rho}}}\right] \tag{11}$$

for $i, j = 1, \ldots, n$. Considering also

$$W_1 = \begin{pmatrix} w_1^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n^{\frac{1}{1+\rho}} \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} p_1^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n^{\frac{1}{1+\rho}} \end{pmatrix},$$

we easily obtain from (4) $A_1 = W_1^{-1} B_1 P_1$. Similarly, using the quasi input coefficients at the social level and defining an $n \times n$ matrix $\hat{B}_1$ as

$$\hat{B}_1 = \left[\frac{\hat{\beta}_{ij}}{\beta_{ij}^{\frac{\rho}{1+\rho}}} - \frac{\hat{\beta}_{i0}\hat{\beta}_{0j}}{\hat{\beta}_{00}} \frac{\beta_{00}^{\frac{\rho}{1+\rho}}}{\beta_{i0}^{\frac{\rho}{1+\rho}} \beta_{0j}^{\frac{\rho}{1+\rho}}}\right] \tag{12}$$

for $i, j = 1, \ldots, n$, it follows that $\hat{A}_1 = W_1^{-1} \hat{B}_1 P_1$. Under Assumption 2, the matrices B_1 and $\hat{B}_1$ are invertible. By the steady state conditions, $\delta + g = w_i/p_i$, the Jacobian matrix J becomes

$$J = \begin{pmatrix} P_1^{-1}\left(B_1^{-1} - \frac{g}{(\delta+g)^{\frac{1}{1+\rho}}} I\right) W_1 & * \\ 0 & W_1\left((\delta+g)^{\frac{\rho}{1+\rho}} I - \hat{B}_1'^{-1}\right) P_1^{-1} \end{pmatrix}.$$

We may thus relate the input coefficients to the CES parameters.

PROPOSITION 1. *Let Assumptions* 1–3 *hold. At the steady state*,

(i) *the consumption good is more intensive in capital good i than the capital good j from the private perspective if and only if* $\beta_{ij}\beta_{00} - \beta_{i0}\beta_{0j} < 0$;

(ii) *the consumption good is more quasi labor intensive than the capital good j from the social perspective if and only if*

$$\left(\frac{\hat{\beta}_{ij}\hat{\beta}_{00}}{\hat{\beta}_{i0}\hat{\beta}_{0j}}\right) > \left(\frac{\beta_{ij}\beta_{00}}{\beta_{i0}\beta_{0j}}\right)^{\frac{\rho}{1+\rho}};$$

(iii) *the consumption good is more intensive in capital good i* (*labor*) *than the capital good j from the social perspective if and only if* $\hat{\beta}_{ij}\hat{\beta}_{00} - \hat{\beta}_{i0}\hat{\beta}_{0j} < (>)\, 0$.

3.2. Two-Sector Model

In the two-sector model with $n = 1$, the matrices A_1 and $\hat{A}_1$ are scalars. From Definition 2, if $a_{11}a_{00} - a_{10}a_{01} < 0$, the consumption good is capital intensive from the private viewpoint. Moreover, if $\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} < 0$, the consumption good is *quasi* capital intensive from the social viewpoint. The following proposition establishes that local indeterminacy requires a capital intensity reversal from the private input coefficients to the quasi input coefficients.

PROPOSITION 2. *Let* $n = 1$ *and Assumptions* 1–3 *hold. The steady state is locally indeterminate if and only if the consumption good is capital intensive from the private perspective*, *but quasi labor intensive from the social perspective.*

To get indeterminacy in a framework with constant returns to scale at the social level, we need a mechanism that nullifies the duality between the Rybczynski and Stolper–Samuelson effects. As shown in Section 2, the Rybczynski effect is given by the input coefficients from the private perspective whereas the Stolper–Samuelson effect is given by the quasi input coefficients from the social perspective. In the presence of external effects, the duality between these coefficients is broken and local indeterminacy may appear. Starting from an arbitrary equilibrium, consider an increase in the rate of investment induced by an instantaneous increase in the relative price of the investment good. When the investment good is labor intensive at the private level, an increase in the capital stock decreases its output at constant prices through the Rybczynski effect. If, on the contrary, the investment good is quasi capital intensive at the social level, the initial rise in its price causes, through the Stolper–Samuelson effect, an increase in one of the components of its return and requires a price decline to maintain the overall return to capital equal to the discount rate. This offsets the initial rise in the relative price of the investment good so that the transversality condition still holds.

This mechanism is very similar to the one exhibited in the contribution of Benhabib and Nishimura (1998). There is, however, a major difference with the current paper, which is based on the fact that as soon as the factor elasticity of substitution is nonunitary, the quasi input coefficients at the social level depend on the elasticity of substitution whereas the social input coefficients do not. It follows that, depending on the value of the elasticity of substitution, the capital intensity reversal from the private input coefficients to the quasi input coefficients does not necessarily requires a capital intensity reversal from the private to the social level.

Let us now precisely study the role of the factor elasticity of substitution. Consider (ii) of Proposition 1. When $\rho \geq 0$, if $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} > 0$ and $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} \leq 0$, then the consumption-good sector is always quasi labor intensive from the social perspective. When $-1 < \rho < 0$, even if $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0$, the consumption-good sector may be quasi labor intensive from the social perspective when $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} < 0$ and dominates $\beta_{11}\beta_{00} - \beta_{10}\beta_{01}$. We may therefore derive the following result, which shows that when the true social input coefficients

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are considered, local indeterminacy also takes place without a capital intensity reversal.

PROPOSITION 3. *Let* $n = 1$, *Assumptions* 1–3 *hold*, *and the consumption good be capital intensive from the private perspective. Then*,

(i) *if the consumption good is labor intensive from the social perspective*, *there exists* $\rho_1^* \in (-1, 0)$ *such that the steady state is locally indeterminate for any* $\rho > \rho_1^*$ *and saddle-point stable for any* $\rho \in (-1, \rho_1^*)$;

(ii) *if the consumption good is also capital intensive from the social perspective and*

$$1 > \frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}} \geq \frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}},$$

there exists $\rho_2^* > 0$ *such that the steady state is locally indeterminate for any* $\rho > \rho_2^*$ *and saddle-point stable for any* $\rho \in (-1, \rho_2^*)$.

This proposition shows that when the productive factors are sufficiently substitutable, local indeterminacy occurs if the consumption good is capital intensive at the private level. Note that local indeterminacy is still possible even if the consumption good is capital intensive at the social level. Condition (i) only coincides with the result obtained by Benhabib and Nishimura (1988) in the particular case of Cobb-Douglas technologies. Therefore, when CES production functions are considered, a capital intensity reversal is not always necessary for local indeterminacy.

COROLLARY 1. *Let* $n = 1$, *Assumptions* 1–3 *hold*, *the consumption good be capital intensive from the private perspective*, *and*

$$\frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}} \geq \frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}}$$

hold. Then, *there exists* $\rho^* > -1$ *such that for any* $\rho > \rho^*$, *the steady state is locally indeterminate.*

3.3. General Multisector Model

In a general multisector model with n capital goods, the local conditions for stability or instability require strong properties on matrices of factor intensity differences B_1 or $\hat{B}_1$. In this section, we attempt to provide a generalization of the results obtained for the two-sector model and propose new conditions for local indeterminacy. As in the two-sector case, we have to consider the coefficients β_{ij} and $\hat{\beta}_{ij}$ of the CES technologies. We impose restrictions only on the sign of the diagonal terms of B_1 and $\hat{B}_1$ but not on the sign of the off-diagonal coefficients. We introduce the following sets of indices which characterize the sign of the diagonal terms in B_1 and $\hat{B}_1$ when $\rho = 0$:

$$\mathcal{B} = \{j \in \{1, \ldots, n\} \text{ such that } \beta_{jj}\beta_{00} - \beta_{j0}\beta_{0j} < 0\}, \tag{13}$$

$$\widehat{\mathcal{B}} = \{k \in \{1, \ldots, n\} \text{ such that } \hat{\beta}_{kk}\hat{\beta}_{00} - \hat{\beta}_{k0}\hat{\beta}_{0k} < 0\}. \tag{14}$$

When $j \in \mathcal{B}$, the consumption good is more intensive in capital good j than the capital good j itself at the private level. Similarly, when $k \in \hat{\mathcal{B}}$, the consumption good is more intensive in capital good k than the capital good k itself at the social level. Let us denote the number of elements in $\mathcal{B}$ and $\hat{\mathcal{B}}$, respectively, by $\#\mathcal{B}$ and $\#\hat{\mathcal{B}}$. If $\#\mathcal{B} = n$, then $\beta_{ii}\beta_{00} - \beta_{i0}\beta_{0i} < 0$ for any $i = 1, \ldots, n$, and if $\#\hat{\mathcal{B}} = 0$ then $\hat{\beta}_{ii}\hat{\beta}_{00} - \hat{\beta}_{i0}\hat{\beta}_{0i} \geq 0$ for any $i = 1, \ldots, n$. These restrictions are similar to the Metzler and Minkowsky conditions of Benhabib and Nishimura (1999) which imply the existence of $2n$ eigenvalues with negative real parts.[15] Our main objective is to give conditions for the existence of at least $n+1$ eigenvalues with negative real parts from the sign patterns of the diagonal elements of matrices B_1 and $\hat{B}_1$.

To relate the sign of diagonal elements to the sign of the real parts of eigenvalues, we introduce the following dominant diagonal properties of matrices.

DEFINITION 3. *An $n \times n$ matrix $C = [c_{ij}]$ has a dominant diagonal if $|c_{ii}| > \sum_{j \neq i} |c_{ij}|$ for each $i = 1, \ldots, n$ or $|c_{ii}| > \sum_{i \neq j} |c_{ij}|$ for each $j = 1, \ldots, n$.*

This definition is stronger than the quasi-dominant diagonal introduced by McKenzie (1960) since the weighting parameters are here equal to one. We also introduce a strong dominant diagonal that requires both row dominance and column dominance.

DEFINITION 4. *An $n \times n$ matrix $C = [c_{ij}]$ has a strong dominant diagonal if $|c_{ii}| > \sum_{j \neq i} |c_{ij}|$ for each $i = 1, \ldots, n$ and $|c_{ii}| > \sum_{i \neq j} |c_{ij}|$ for each $j = 1, \ldots, n$.*

From now on we explicitly parameterize the matrices B_1 and $\hat{B}_1$ by ρ, namely $B_1(\rho)$ and $\hat{B}_1(\rho)$, in order to simplify the exposition.

Assumption 4. There exists $\bar{\rho} > -1$ such that for any $\rho > \bar{\rho}$, $B_1(\rho)$ has a strong dominant diagonal, and $\hat{B}_1(\rho)$ has only real eigenvalues with dominant diagonal.

Remark 1. In a two-sector model, when we consider constructing an alternative equilibrium with a higher investment rate, we have to decrease the initial level of consumption. If the elasticity of intertemporal substitution in consumption is finite, the desire to smooth consumption over time may overwhelm the technological effects coming from the Rybczynski and Stolper–Samuelson theorems. This is why we assume a linear specification for the utility function. As shown by Benhabib and Nishimura (1998), such an assumption is no longer necessary as soon as a third nonconsumption good is introduced. In this case, indeterminacy may arise from compositional changes in outputs without too much affecting the output of consumption. However, we still assume in the following that the elasticity of intertemporal substitution in consumption is infinite. Such an assumption is necessary to get precise results without resorting to numerical computations. It is easy to notice indeed that if the utility function is nonlinear, the Jacobian matrix J is no longer triangular and the characteristic roots cannot be analyzed. Our strategy is therefore to give conditions on the technological fundamentals to get local indeterminacy when the utility function is linear. As numerically illustrated by Benhabib and Nishimura (1998) in a Cobb-Douglas framework, the argument mentioned above then guarantees that local indeterminacy will persist for finite values of the elasticity of intertemporal substitution in consumption.

THEOREM 1. *Let Assumptions* 1–4 *hold*, $\#\mathcal{B} \geq 1$, *and*

$$\frac{\hat{\beta}_{ii}\hat{\beta}_{00}}{\hat{\beta}_{i0}\hat{\beta}_{0i}} \geq \frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}} \tag{15}$$

for all $i = 1, \ldots, n$. *Then, there exists* $-1 < \rho^* \leq \bar{\rho}$ *such that the steady state is locally indeterminate for any* $\rho > \rho^*$.

Theorem 1 suggests that local indeterminacy cannot arise with high substitutability. We may thus provide conditions for saddle-point stability. We first introduce an alternative restriction to Assumption 4.

Assumption 5. There exists $\bar{\rho} > -1$ such that for any $\rho \in (-1, \bar{\rho}]$, $B_1(\rho)$ has a strong dominant diagonal, and $\hat{B}_1(\rho)$ has only real eigenvalues with dominant diagonal.

PROPOSITION 4. *Let Assumptions* 1–3 *and* 5 *hold with* $\#\mathcal{B} = n$. *Then, there exists* $\hat{\rho} \geq \bar{\rho}$ *such that the steady state is saddle-point stable for any* $\rho \in (-1, \hat{\rho})$.

Condition $\#\mathcal{B} = n$ with dominant diagonal guarantee that the Jacobian matrix J has at least n negative eigenvalues. However, strong factor substitutability leads to the existence of a unique equilibrium path.

3.4. Examples

In Theorem 1, the lower bound ρ^*, above which local indeterminacy is obtained, may be positive or negative depending on the values of $\#\mathcal{B}$ and $\#\hat{\mathcal{B}}$. If $\rho^* < 0$, then the results cover the Cobb-Douglas case. Under $\#\mathcal{B} \geq 1$ and suitable dominant diagonal assumptions, it may be shown that $\rho^* < 0$ if $\#\hat{\mathcal{B}} = 0$ whereas $\rho^* > 0$ if $\#\hat{\mathcal{B}} > 0$. Simple examples may illustrate the possibility of indeterminacy in three different interesting configurations. Following Benhabib and Nishimura (1998), the production parameters are calibrated along the lines of a standard RBC model.[16] We also consider extremely small external effects.

Example 1. We first illustrate the case $\#\mathcal{B} = 1$ and $\#\hat{\mathcal{B}} = 0$. Consider the following matrices of private CES coefficients and external-effects coefficients

$$B(\rho) = \begin{pmatrix} (0.66)^{\frac{1}{1+\rho}} & (0.64)^{\frac{1}{1+\rho}} & (0.61)^{\frac{1}{1+\rho}} \\ (0.24)^{\frac{1}{1+\rho}} & (0.2)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} & (0.1)^{\frac{1}{1+\rho}} & (0.1)^{\frac{1}{1+\rho}} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.06 \end{pmatrix}.$$

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It follows that

$$B_1(\rho) = \begin{pmatrix} (0.2)^{\frac{1}{1+\rho}} - (0.233)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} - (0.222)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} - (0.097)^{\frac{1}{1+\rho}} & (0.1)^{\frac{1}{1+\rho}} - (0.094)^{\frac{1}{1+\rho}} \end{pmatrix},$$

$$\hat{B}_1(\rho) = \begin{pmatrix} \frac{0.26}{(0.2)^{\frac{\rho}{1+\rho}}} - \frac{0.233}{(0.233)^{\frac{\rho}{1+\rho}}} & \frac{0.23}{(0.23)^{\frac{\rho}{1+\rho}}} - \frac{0.222}{(0.222)^{\frac{\rho}{1+\rho}}} \\ \frac{0.1}{(0.1)^{\frac{\rho}{1+\rho}}} - \frac{0.097}{(0.097)^{\frac{\rho}{1+\rho}}} & \frac{0.16}{(0.1)^{\frac{\rho}{1+\rho}}} - \frac{0.094}{(0.094)^{\frac{\rho}{1+\rho}}} \end{pmatrix},$$

and, when $\rho = 0$,

$$B_1(0) = \begin{pmatrix} -0.032 & 0.004 \\ 0.003 & 0.006 \end{pmatrix} \quad \text{and} \quad \hat{B}_1(0) = \begin{pmatrix} 0.027 & 0.004 \\ 0.003 & 0.066 \end{pmatrix}.$$

We have $\#\mathcal{B} = 1$ for row 1 of matrix $B_1(\rho)$, which has a strong dominant diagonal for any $\rho > -1$. Moreover $\#\hat{\mathcal{B}} = 0$ and the matrix $\hat{B}_1(\rho)$ has a dominant diagonal for any $\rho \in (-1, -0.55) \cup (-0.29, +\infty)$. The eigenvalues are positive when $\rho > -0.409$ and have an opposite sign when $\rho \in (-1, -0.409)$. Then, there exists $\rho^* = -0.409$ such that the steady state is locally indeterminate for any $\rho > \rho^*$ and saddle-point stable when $\rho \in (-1, \rho^*)$. The values of $\#\mathcal{B}$ and $\#\hat{\mathcal{B}}$, respectively, give the number of negative diagonal terms of $B_1(0)$ and $\hat{B}_1(0)$. Under dominant diagonal assumptions, these give the number of negative roots of $B_1(0)$ and $\hat{B}_1(0)$. Therefore, if $\#\mathcal{B} = 1$ and $\#\hat{\mathcal{B}} = 0$, the Jacobian matrix when $\rho = 0$ has $n + 1$ roots with negative real parts.

Example 2. We now illustrate the case $\#\mathcal{B} = 1$ and $\#\mathcal{B} - \#\hat{\mathcal{B}} < 1$. Consider a slight modification of the previous matrices B and b:

$$B(\rho) = \begin{pmatrix} (0.66)^{\frac{1}{1+\rho}} & (0.68)^{\frac{1}{1+\rho}} & (0.65)^{\frac{1}{1+\rho}} \\ (0.24)^{\frac{1}{1+\rho}} & (0.2)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} & (0.1) & (0.11)^{\frac{1}{1+\rho}} \end{pmatrix}, \; b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \quad \mathbf{(16)}$$

We easily obtain

$$B_1(\rho) = \begin{pmatrix} (0.2)^{\frac{1}{1+\rho}} - (0.247)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} - (0.236)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} - (0.103)^{\frac{1}{1+\rho}} & (0.11)^{\frac{1}{1+\rho}} - (0.098)^{\frac{1}{1+\rho}} \end{pmatrix},$$

$$\hat{B}_1(\rho) = \begin{pmatrix} \frac{0.22}{(0.2)^{\frac{\rho}{1+\rho}}} - \frac{0.247}{(0.247)^{\frac{\rho}{1+\rho}}} & \frac{0.23}{(0.23)^{\frac{\rho}{1+\rho}}} - \frac{0.236}{(0.236)^{\frac{\rho}{1+\rho}}} \\ \frac{0.1}{(0.1)^{\frac{\rho}{1+\rho}}} - \frac{0.103}{(0.103)^{\frac{\rho}{1+\rho}}} & \frac{0.12}{(0.11)^{\frac{\rho}{1+\rho}}} - \frac{0.098}{(0.098)^{\frac{\rho}{1+\rho}}} \end{pmatrix},$$

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and, when $\rho = 0$,

$$B_1(0) = \begin{pmatrix} -0.047 & -0.006 \\ -0.003 & 0.012 \end{pmatrix}, \qquad \hat{B}_1(0) = \begin{pmatrix} -0.027 & -0.006 \\ -0.003 & 0.022 \end{pmatrix}.$$

We have $\#\mathcal{B} = 1$ for row 1 and $B_1(\rho)$ has a strong dominant diagonal for any $\rho > -1$. Condition (15) is satisfied, $\#\hat{\mathcal{B}} = 1$ and $\hat{B}_1(\rho)$ has a dominant diagonal for any $\rho \in (-1, 0.150) \cup (0.231, +\infty)$. The eigenvalues are positive when $\rho > 0.196$ and have opposite sign when $\rho \in (-1, 0.196)$. Moreover, $\#\mathcal{B} - \#\hat{\mathcal{B}} = 0$ and there exists $\rho^* = 0.196$ such that the steady state is locally indeterminate for any $\rho > \rho^*$ and saddle-point stable when $\rho \in (-1, \rho^*)$. Note that since we use a dominant diagonal property for $B_1(\rho)$ and $\hat{B}_1(\rho)$, local indeterminacy requires at least $n + 1$ negative diagonal terms in $B_1(\rho)$ and $\hat{B}_1(\rho)$. When $\rho = 0$ and under $\#\mathcal{B} \geq 1$, this requires that $\#\hat{\mathcal{B}} \geq \#\mathcal{B} - 1$. Therefore, local indeterminacy cannot occur in the Cobb-Douglas case when $\#\mathcal{B} - \#\hat{\mathcal{B}} < 1$.

Example 3. We finally illustrate the case $\#\mathcal{B} = 1$ and $\#\mathcal{B} - \#\hat{\mathcal{B}} \geq 1$. Consider a slight modification of matrices (16) as follows:

$$B(\rho) = \begin{pmatrix} (0.66)^{\frac{1}{1+\rho}} & (0.68)^{\frac{1}{1+\rho}} & (0.65)^{\frac{1}{1+\rho}} \\ (0.24)^{\frac{1}{1+\rho}} & (0.2)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} & (0.1)^{\frac{1}{1+\rho}} & (0.09)^{\frac{1}{1+\rho}} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.03 \end{pmatrix}.$$

We easily obtain

$$B_1(\rho) = \begin{pmatrix} (0.2)^{\frac{1}{1+\rho}} - (0.247)^{\frac{1}{1+\rho}} & (0.23)^{\frac{1}{1+\rho}} - (0.236)^{\frac{1}{1+\rho}} \\ (0.1)^{\frac{1}{1+\rho}} - (0.103)^{\frac{1}{1+\rho}} & (0.09)^{\frac{1}{1+\rho}} - (0.098)^{\frac{1}{1+\rho}} \end{pmatrix},$$

$$\hat{B}_1(\rho) = \begin{pmatrix} \dfrac{0.22}{(0.2)^{\frac{\rho}{1+\rho}}} - \dfrac{0.247}{(0.247)^{\frac{\rho}{1+\rho}}} & \dfrac{0.23}{(0.23)^{\frac{\rho}{1+\rho}}} - \dfrac{0.236}{(0.236)^{\frac{\rho}{1+\rho}}} \\ \dfrac{0.1}{(0.1)^{\frac{\rho}{1+\rho}}} - \dfrac{0.103}{(0.103)^{\frac{\rho}{1+\rho}}} & \dfrac{0.12}{(0.09)^{\frac{\rho}{1+\rho}}} - \dfrac{0.098}{(0.098)^{\frac{\rho}{1+\rho}}} \end{pmatrix},$$

and, when $\rho = 0$,

$$B_1(0) = \begin{pmatrix} -0.047 & -0.006 \\ -0.003 & -0.008 \end{pmatrix} \quad \text{and} \quad \hat{B}_1(0) = \begin{pmatrix} -0.027 & -0.006 \\ -0.003 & 0.022 \end{pmatrix}.$$

We have $\#\mathcal{B} = 2$ and $B_1(\rho)$ has a strong dominant diagonal for any $\rho > -1$. Condition (15) is satisfied and $\#\hat{\mathcal{B}} = 1$. $\hat{B}_1(\rho)$ has a dominant diagonal for any $\rho \in (-1, -0.856) \cup (-0.596, 0.150) \cup (0.231, +\infty)$ and its eigenvalues are positive when $\rho > 0.195$, have the opposite sign when $\rho \in (-0.714, 0.195)$, and are negative when $\rho \in (-1, -0.714)$. Moreover, we have $\#\mathcal{B} - \#\hat{\mathcal{B}} = 1$ and there exist $\rho^* = 0.195$ and $\underline{\rho}^* = -0.714$ such that the steady state is locally indeterminate

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for any $\rho \in (\underline{\rho}^*, \rho^*) \cup (\rho^*, +\infty)$ and saddle-point stable when $\rho \in (-1, \underline{\rho}^*)$. The Cobb-Douglas case falls into this configuration. Although the sufficient conditions of Theorem 1 imply that the lower bound ρ^* is positive, local indeterminacy of the steady state may still hold for some ρ that is strictly less than ρ^*. When $\#\mathcal{B} \geq 1$, $\#\hat{\mathcal{B}} > 0$, and $\#\mathcal{B} - \#\hat{\mathcal{B}} \geq 1$, there are at least $n+1$ negative diagonal terms in $B_1(0)$ and $\hat{B}_1(0)$. If, in addition, dominant diagonal properties are satisfied at $\rho = 0$, the steady state is locally indeterminate at $\rho = 0$ even though the lower bound given by Theorem 1 is strictly positive.

4. ENDOGENOUS GROWTH MODEL WITH NONLINEAR UTILITY

4.1. Model

We now consider an economy without fixed factors that exhibits unbounded growth. A representative agent optimizes a nonlinear additively separable utility function with discount rate $\delta \geq 0$. This problem can be described as

$$\begin{aligned}
&\max_{c(t),\{x_{ij}(t)\}} \int_0^{+\infty} \frac{[c(t)]^{1-\sigma}}{1-\sigma} e^{-\delta t}\, dt \\
&\text{s.t.}\quad y_j(t) = \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}(t)^{-\rho} + e_j(\boldsymbol{X}_j(t))\right)^{-1/\rho} && j = 0, \ldots, n, \\
&\qquad \dot{x}_0(t) = y_0(t) - g x_0(t) - c(t), \\
&\qquad \dot{x}_i(t) = y_i(t) - g x_i(t) && i = 1, \ldots, n, \\
&\qquad x_i(t) = \sum_{j=0}^{n} x_{ij}(t) && i = 0, \ldots, n, \\
&\qquad x_i(0) \text{ given} && i = 0, \ldots, n, \\
&\qquad \{e_j(\boldsymbol{X}_j(t))\}_{t \geq 0} \text{ given} && j = 0, \ldots, n.
\end{aligned} \tag{17}$$

There is no pure consumption good: the good 0 is both a factor of production and a consumption good. The modified Hamiltonian in current value is

$$\begin{aligned}
\mathcal{H} = \frac{[c(t)]^{1-\sigma}}{1-\sigma} &+ p_0(t)\left(\left(\sum_{i=0}^{n} \beta_{i0} x_{i0}^{-\rho}(t) + e_0(\boldsymbol{X}_0(t))\right)^{-1/\rho} - g x_0(t) - c(t)\right) \\
&+ \sum_{j=1}^{n} p_j(t)\left(\left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho}(t) + e_j(\boldsymbol{X}_j(t))\right)^{-1/\rho} - g x_j(t)\right) \\
&+ \sum_{i=0}^{n} w_i(t)\left(x_i(t) - \sum_{j=0}^{n} x_{ij}(t)\right).
\end{aligned}$$

Here $p_j(t)$ and $w_i(t)$ are, respectively, co-state variables and Lagrange multipliers,

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representing utility prices of the capital goods and their rental rates. The static first-order conditions for this problem are given by

$$c^{-\sigma} = p_0, \tag{18}$$

$$w_s = p_j \beta_{sj} (y_j / x_{sj})^{1+\rho} \tag{19}$$

for $j = 1, \ldots, n, s = 0, 1, \ldots, n$, and equations (19) are equivalent to (3). It follows from (6) and (7) that the necessary conditions that describe the solution to problem (17) are given by the equations of motion:

$$\begin{aligned} \dot{x}_0(t) &= y_0(\boldsymbol{x}(t), \boldsymbol{p}(t)) - g x_0(t) - c(t), & & \\ \dot{x}_i(t) &= y_i(\boldsymbol{x}(t), \boldsymbol{p}(t)) - g x_i(t) & i &= 1, \ldots, n, \\ \dot{p}_i(t) &= (\delta + g) p_i(t) - w_i(\boldsymbol{p}(t)) & i &= 0, \ldots, n. \end{aligned} \tag{20}$$

The production functions being homogeneous of degree one, let the growth rate of c and x_i along the balanced growth path be μ. From equation (18), prices must then decline at the rate $\sigma\mu$. We define discounted variables as

$$\bar{x}_i(t) = e^{-\mu t} x_i(t), \bar{y}_i(t) = e^{-\mu t} y_i(t), \bar{p}_i(t) = e^{\sigma\mu t} p_i(t), \bar{w}_i(t) = e^{\sigma\mu t} w_i(t).$$

Note that $e^{-\mu t} c(t) = e^{-\mu t} [p_0(t)]^{-1/\sigma} = [\bar{p}_0(t)]^{-1/\sigma}$. Since there are no fixed factors, outputs $\boldsymbol{y}$ are homogeneous of degree one in stocks $\boldsymbol{x}$, and homogeneous of degree zero in prices $\boldsymbol{p}$, and the factor prices $\boldsymbol{w}$ are homogeneous of degree one in prices. Then equations (20) can be written as:

$$\begin{aligned} \dot{\bar{x}}_0(t) &= \bar{y}_0(\bar{\boldsymbol{x}}(t), \bar{\boldsymbol{p}}(t)) - (g + \mu)\bar{x}_0(t) - [\bar{p}_0(t)]^{-1/\sigma} & & \\ \dot{\bar{x}}_i(t) &= \bar{y}_i(\bar{\boldsymbol{x}}(t), \bar{\boldsymbol{p}}(t)) - (g + \mu)\bar{x}_i(t) & i &= 1, \ldots, n, \\ \dot{\bar{p}}_i(t) &= (\delta + g + \sigma\mu)\bar{p}_i(t) - w_i(\bar{\boldsymbol{p}}(t)) & i &= 0, \ldots, n. \end{aligned} \tag{21}$$

The stationary balanced growth rate μ corresponds to the stationary point $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$ of the above system.

Assumption 6. There exists a stationary point $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$ of the dynamical system (21) that solves $\dot{\bar{x}}_i(t) = \dot{\bar{p}}_i(t) = 0, i = 0, 1, \ldots, n$.[17]

From the price equations in the dynamical system (21) evaluated at $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$ and Lemma 1, we have $(\delta + g + \sigma\mu)\hat{A}'\boldsymbol{w} = \boldsymbol{w}$, which can be reformulated as

$$[\hat{A}' - (\delta + g + \sigma\mu)^{-1} I]\boldsymbol{w} = 0. \tag{22}$$

Thus, μ is obtained from the Frobenius root of $\hat{A}'$, that is, $(\delta + g + \sigma\mu)^{-1}$, which has $\boldsymbol{w}$ as eigenvector. Linearizing around $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$ gives the $2(n + 1) \times 2(n + 1)$

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Jacobian matrix,

$$J = \begin{pmatrix} \left(\frac{\partial \bar{y}(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)}{\partial \bar{x}}\right) - (g+\mu)I & \left(\frac{\partial \bar{y}(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)}{\partial \bar{p}}\right) + Z \\ 0 & (\delta + g + \sigma\mu)I - \left(\frac{\partial \bar{w}(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)}{\partial \bar{p}}\right) \end{pmatrix},$$

where Z is a matrix of zeros except for the element of the first row and the first column, which is $(1/\sigma)p_0^{-1-1/\sigma}$. From equation (8), J becomes

$$J = \begin{pmatrix} A^{-1} - (g+\mu)I & \left(\frac{\partial \bar{y}(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)}{\partial \bar{p}}\right) + Z \\ 0 & (\delta + g + \sigma\mu)I - \hat{A}'^{-1} \end{pmatrix}. \tag{23}$$

In the current economy, there are $n+1$ capital goods whose initial values are given. Any solution from (21) that converges to the steady state $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$ satisfies the transversality condition and constitutes an equilibrium. Therefore, given $\bar{\boldsymbol{x}}(0)$, if there is more than one set of initial prices $\bar{\boldsymbol{p}}(0)$ in the stable manifold of $(\bar{\boldsymbol{x}}^*, \bar{\boldsymbol{p}}^*)$, the equilibrium path from $\bar{\boldsymbol{x}}(0)$ will not be unique. Notice that the Frobenius root of $\hat{A}'$ implies the existence of one zero root for J. Therefore, if J has more than n roots with negative real parts, there will be a continuum of converging paths and thus a continuum of equilibria. Definition 1 of local indeterminacy still applies.

The roots of J are given by the roots of $[A^{-1} - (g+\mu)I]$ and $[(\delta + g + \sigma\mu)I - \hat{A}'^{-1}]$. As in the exogenous growth formulation, we may formulate the factor intensity differences in terms of the β_{ij} and $\hat{\beta}_{ij}$ coefficients. From Lemmas 3 and 4 we have $d\bar{\boldsymbol{p}} = \hat{A}' d\bar{\boldsymbol{w}}$ and $A d\bar{\boldsymbol{y}} + \sum_{j=0}^{n} \bar{\boldsymbol{y}}_j [(\partial \boldsymbol{a}^j / \partial \bar{\boldsymbol{w}}) d\bar{\boldsymbol{w}} + (\partial \boldsymbol{a}^j / \partial \bar{\boldsymbol{p}}) d\bar{\boldsymbol{p}}] = d\bar{\boldsymbol{x}}$. Consider the following two diagonal matrices:

$$\bar{W} = \begin{pmatrix} (\bar{w}_0)^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\bar{w}_n)^{\frac{1}{1+\rho}} \end{pmatrix} \quad \text{and} \quad \bar{P} = \begin{pmatrix} (\bar{p}_0)^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\bar{p}_n)^{\frac{1}{1+\rho}} \end{pmatrix}.$$

The steady-state conditions give $\delta + g + \sigma\mu = (\bar{w}_i / \bar{p}_i)$. Then, we can rewrite J as follows:

$$J = \begin{pmatrix} \bar{P}^{-1}\left(B^{-1} - \frac{g+\mu}{(\delta + g + \sigma\mu)^{\frac{1}{1+\rho}}} I\right)\bar{W} & \left(\frac{\partial \bar{y}(\bar{x}^*, \bar{p}^*)}{\partial \bar{p}}\right) + Z \\ 0 & \bar{W}((\delta + g + \sigma\mu)^{\frac{\rho}{1+\rho}} I - \hat{B}'^{-1})\bar{P}^{-1} \end{pmatrix}$$

Applying the same proof as the one of Proposition 1, we finally obtain Proposition 5.

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PROPOSITION 5. *Let Assumptions* 1, 2, *and* 6 *hold. At the steady state*,

(i) *the consumable capital good is more intensive in the pure capital good i than the pure capital good j at the private level iff* $\beta_{ij}\beta_{00} - \beta_{i0}\beta_{0j} < 0$;

(ii) *the consumable capital good is more quasi intensive in itself* (*the pure capital good i*) *than the pure capital j at the social level iff*

$$\left(\frac{\hat{\beta}_{ij}\hat{\beta}_{00}}{\hat{\beta}_{i0}\hat{\beta}_{0j}}\right) > (<) \left(\frac{\beta_{ij}\beta_{00}}{\beta_{i0}\beta_{0j}}\right)^{\frac{\rho}{1+\rho}};$$

(iii) *the consumable capital good is more intensive in itself* (*the pure capital good i*) *than the pure capital j at the social level iff* $\hat{\beta}_{ij}\hat{\beta}_{00} - \hat{\beta}_{i0}\hat{\beta}_{0j} > (<) 0$.

4.2. Two-Sector Model

Considering a two-sector model, we first show that, as in the exogenous growth framework, local indeterminacy requires a factor intensity reversal from the private input coefficients to the social quasi input coefficients.

PROPOSITION 6. *Let* $n = 1$ *and Assumptions* 1, 2, *and* 6 *hold. If the consumable capital good is intensive in the pure capital good from the private perspective, but it is quasi intensive in itself from the social perspective*, *then the balanced growth path is locally indeterminate.*

This result is based on the quasi input coefficients from the social viewpoint which do not have real economic meaning. Therefore, we need to use the true social input coefficients. Now, however, we have to consider explicitely the factor elasticity of substitution. Applying the same proof as in the exogenous growth case, we show that local indeterminacy occurs without such a reversal when the elasticity of substitution is less than one.

PROPOSITION 7. *Let* $n = 1$, *Assumptions* 1, 2, *and* 6 *hold*, *and the consumable capital good be intensive in the pure capital good from the private perspective. Then*,

(i) *if the consumable capital good is intensive in itself from the social perspective, there exists* $\rho^* \in (-1, 0)$ *such that the balanced growth path is locally indeterminate for any* $\rho > \rho^*$;

(ii) *if the consumable capital good is also intensive in the pure capital good from the social perspective and*

$$1 > \frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}} > \frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}}, \qquad (24)$$

there exists $\rho^* > 0$ *such that the steady state is locally indeterminate for any* $\rho > \rho^*$.

When the technologies are close enough to a Leontief function, this Corollary provides a new condition for local indeterminacy based on a consumable capital good intensive in the pure capital good at the private and social levels. Moreover, it extends the conclusions of Bond et al. (1996), Benhabib et al. (2000), and Mino (2001) to any economies with technologies between Cobb-Douglas and Leontief functions.

4.3. General Multisector Model

Consider now a general multisector model with $n+1$ goods. As we can see in the Jacobian matrix (23), n roots are determined by the matrix $[A^{-1} - (g+\mu)I]$ without externalities and the other n roots by the matrix $[(\delta + g + \sigma\mu)I - \hat{A}'^{-1}]$ with externalities. If A^{-1} has a root with negative real part, it makes one root with negative real part in the Jacobian matrix. If $\hat{A}'^{-1}$ has a root with positive real part, it makes one root with negative real part in the Jacobian matrix unless the root corresponds to the Frobenius root of $\hat{A}'^{-1}$. Therefore, to provide sufficient conditions in the multisector case, we start with an assumption that implies that B and thus A have at least one root with negative real part. From now on, we explicitely parameterize the matrices B and $\hat{B}$ by ρ, namely $B(\rho)$ and $\hat{B}(\rho)$, in order to simplify the exposition.

Assumption 7. For any $\rho > -1$, $B(\rho)$ has a negative determinant.

From this Assumption, we derive Lemma 5.[18]

LEMMA 5. *Under Assumptions* 1, 2, 6, *and* 7, *there exist at least one row* $i \in \{0, \dots, n\}$ *such that* $\beta_{ii}^{1/(1+\rho)} \leq \sum_{j \neq i} \beta_{ij}^{1/(1+\rho)}$.

Now, define the set of rows i that satisfies the inequality given in Lemma 5:

$$\mathcal{I}(\rho) = \left\{ i \in \{0, \dots, n\} \text{ such that } \beta_{ii}^{1/(1+\rho)} \leq \sum_{j \neq i} \beta_{ij}^{1/(1+\rho)} \right\}. \tag{25}$$

Next, we look at the matrix with externalities and we introduce a dominant diagonal assumption given by Definition 4 in the preceding section for $\hat{B}(0)$ in order to obtain $n+1$ roots with positive real parts.

Assumption 8. $\hat{B}(0)$ has a dominant diagonal.

Moreover, we introduce slight restrictions on some components of matrix B.

Assumption 9. Any row $k = 0, \dots, n$ of B satisfies $\beta_{kk} \neq \beta_{kj}$, $j \neq k$.

This implies that the amount of capital k used in its own industry is different from the amount of capital k used in any other industries.

LEMMA 6. *Let Assumptions* 1–2 *and* 6–9 *hold. Then, there exist* $\underline{\rho} \in [-1, 0)$ *and* $\bar{\rho} > 0$ *such that for any* $\rho \in (\underline{\rho}, \bar{\rho})$, $\hat{B}(\rho)$ *has a dominant diagonal. Moreover.*[19]

(i) *if there is some* $i \in \mathcal{I}(\rho)$ *and at least one* $j \neq i$ *such that* $\beta_{ii} < \beta_{ij}$, *then* $\underline{\rho} > -1$;
(ii) *let* $n \geq 2$. *If for all* $i \in \mathcal{I}(\rho)$ *and all* $j \neq i$, $\beta_{ii} > \beta_{ij}$, *then* $\underline{\rho} = -1$.

We now introduce Assumption 10, based on the open interval $(\underline{\rho}, \bar{\rho})$ given in Lemma 6.

Assumption 10. The matrix $\hat{B}(\rho)$ has only real eigenvalues for any $\rho \in (\underline{\rho}, \bar{\rho})$.

This finally allows us to state the result in Theorem 2.

THEOREM 2. *Let Assumptions* 1–2 *and* 6–10 *hold. Then, the following cases occur*:

(i) *If there is some* $i \in \mathcal{I}(\rho)$ *and at least one* $j \neq i$ *such that* $\beta_{ii} < \beta_{ij}$, *then there exist* $\rho_1^* \in (-1, \underline{\rho}]$ *and* $\rho_2^* \geq \bar{\rho}$ *such that for any* $\rho \in (\rho_1^*, \rho_2^*)$, *the stationary balanced growth path is locally indeterminate.*
(ii) *Let* $n \geq 2$. *If for all* $i \in \mathcal{I}(\rho)$ *and all* $j \neq i$, $\beta_{ii} > \beta_{ij}$, *then there exists* $\rho^* \geq \bar{\rho}$ *such that the stationary balanced growth path is locally indeterminate for any* $\rho \in (-1, \rho^*)$.

This theorem provides a generalization of Proposition 7 (i). In a two-sector model with $n = 1$, if the consumable capital good is intensive in the pure capital good at the private level then B has a negative determinant and Lemma 5 implies that case (i) of Lemma 6 necessarily holds. Moreover, if $\hat{B}(0)$ has a dominant diagonal, then $\hat{\beta}_{00} > \hat{\beta}_{01}$ and $\hat{\beta}_{11} > \hat{\beta}_{10}$, which implies that the consumable capital good is intensive in itself from the social perspective.

When compared with the exogenous growth case, Theorem 2 shows that the role of the factor elasticity of substitution in the occurrence of local indeterminacy is quite different in the endogenous growth model. In fact, $\hat{B}(\rho)$ is defined by input coefficients and not by capital intensity coefficients in the current formulation. This implies that local indeterminacy does not occur with low factor substitutability when external effects are mild. Moreover, Theorem 2 shows with case (ii) that local indeterminacy occurs with an arbitrarily large elasticity of substitution. This is not the case in exogenous growth models.

In a recent contribution that corresponds to case $\rho = 0$ in our formulation, Benhabib et al. (2000) show that if B has n roots with negative real parts and $\hat{B}$ has at least two roots with positive real parts, the Jacobian matrix has one zero root and at least $n + 1$ roots with negative real parts, and then the stationary balanced growth path is locally indeterminate. Notice that their conditions are fundamentally based on the private input coefficients while our results rely on the social input coefficients. Indeed, it clearly appears that B being a nonnegative $(n+1) \times (n+1)$ matrix, the ocurrence of n roots with negative real parts is difficult to obtain. On the contrary, $\hat{B}$ being also a nonnegative matrix, a dominant diagonal property ensures the existence of $n + 1$ roots with positive real parts, and local indeterminacy is obtained under a mild additional condition on B.

5. CONCLUDING COMMENTS

The main objective of this paper has been to discuss the role of factor substitutability on the stability properties of a steady state or a stationary balanced growth path

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in a general multisector economy with CES technologies. We have given some easily tractable sufficient conditions for the occurrence of local indeterminacy in exogenous and endogenous growth models. We have proved this result without a capital intensity reversal from the private to the social level when the elasticity of substitution is less than one. Finally, we have shown that factor substitutability must be sufficiently weak in exogenous growth models for local indeterminacy to take place whereas this is not necessary in endogenous growth models.

We conclude with some comments about two possible further researches. We have assumed throughout the paper that all the sectors have the same elasticity of substitution. It would be interesting to study how our results are modified if some heterogeneity is introduced on this parameter. Moreover, although we have provided indeterminacy examples with mild external effects, we have not discussed the relationship between the elasticity of substitution and the size of externalities.

NOTES

1. See also Diamond (1965) and Galor (1992).
2. See the recent survey by Benhabib and Farmer (1999).
3. See Evans and Guesnerie (2003).
4. See Benhabib and Rustichini (1994), Benhabib (1998).
5. See also Benhabib and Farmer (1996), Benhabib et al. (2002), Nishimura and Venditti (2002).
6. From this point of view, two-period OLG models are not appropriate to deal with short-run business-cycle fluctuations.
7. See Xie (1994), Mitra (1998), Benhabib et al. (2000), and Mino (2001).
8. All these papers are representative-agent two-sector models. However, these models possess features of agents' interactions through external effects. See Becker and Tsyganov (2002) for a model that explicitly introduces a heterogeneity of agents.
9. On the basis of interest-rate elasticity of U.S. demand for money, Benhabib and Farmer (2000) use a CES production function, with labor and real money balance as inputs, to investigate the monetary transmission mechanism.
10. Similar results have been derived in endogenous growth models by Bond et al. (1996), Mino (2001) and Benhabib et al. (2000).
11. Denoting by $f^j(x_{0j}, \dots, x_{nj}, e_j(\boldsymbol{X}_j))$ the technology of sector j, we have for any $\lambda > 1$

$$f^j(\lambda x_{0j}, \dots, \lambda x_{nj}, e_j(\boldsymbol{X}_j)) = \lambda \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho_j} + \lambda^{\rho_j} e_j(\boldsymbol{X}_j) \right)^{-\frac{1}{\rho_j}} < \lambda f^j(x_{0j}, \dots, x_{nj}, e_j(\boldsymbol{X}_j)).$$

12. The technology of sector j may be formulated as follows:

$$y_j = \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho_j} + e_j(\boldsymbol{X}_j) T_j^{-\rho_j} \right)^{-1/\rho_j},$$

with T_j the amount of land used in the production of good j, which is normalized to 1 in our formulation.

13. We assume that the instantaneous utility is linear. This implies that the objective functional coincides with the social production function of the consumption-good sector which is nonlinear and concave.
14. From a general point of view, existence of steady state is proved under conditions (continuity of functions, some boundary properties of the functions on the domain) that are fairly independent from

the local stability properties at the steady state. In our model, following the procedure introduced by Benhabib and Nishimura (1979) and Benhabib and Rustichini (1990) for optimal growth models and Benhabib and Nishimura (1998) for growth models with externalities, the existence of a steady state may be proved by constructing CES production functions from some values of the prices p_j, the rental rates w_j and the coefficients β_{ij}, b_{ij} that satisfy the local indeterminacy conditions. However, to avoid unnecessary complications, we simply assume the existence of the steady state to begin with.

15. A Metzler matrix has negative diagonal elements and positive off-diagonal elements, whereas a Minkowski matrix has positive diagonal elements and negative off-diagonal elements.

16. We use indeed very similar values for the input coefficients. See also Benhabib et al. (1997).

17. When endogenous growth is considered, the same comment as in note 14 applies.

18. The result of Lemma 5 refers to some diagonal and off-diagonal elements in each column. However, similar results hold true for diagonal and off-diagonal elements in each row, too. In the case in which the dominant diagonal property does not hold, similar inequalities follow for each column and each row. To avoid unnecessary complications, we only use inequalities between diagonal elements and off-diagonal elements for each column throughout the rest of this paper.

19. The critical value $\bar{\rho}$ may be $+\infty$ when $n = 1$ under condition (i) in Proposition 7. However, when $n \geq 2$, $\bar{\rho} < +\infty$ if the externalities are mild.

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APPENDIX

A.1. PROOF OF PROPOSITION 1

At the steady state, $(\delta+g)p_i = w_i$, and thus

$$a_{ij}a_{00} - a_{i0}a_{0j} = \left(\frac{w_j}{(\delta+g)^2 w_i}\right)^{\frac{1}{1+\rho}} \left((\beta_{ij}\beta_{00})^{\frac{1}{1+\rho}} - (\beta_{i0}\beta_{0j})^{\frac{1}{1+\rho}}\right),$$

$$\hat{a}_{ij}\hat{a}_{00} - \hat{a}_{i0}\hat{a}_{0j} = \left(\frac{w_j}{(\delta+g)^2 w_i}\right)^{\frac{1}{1+\rho}} \left(\frac{\hat{\beta}_{ij}\hat{\beta}_{00}}{(\beta_{ij}\beta_{00})^{\frac{\rho}{1+\rho}}} - \frac{\hat{\beta}_{i0}\hat{\beta}_{0j}}{(\beta_{i0}\beta_{0j})^{\frac{\rho}{1+\rho}}}\right),$$

$$\bar{a}_{ij}\bar{a}_{00} - \bar{a}_{i0}\bar{a}_{0j} = \left(\frac{w_j}{(\delta+g)^2 w_i}\right)^{\frac{1}{1+\rho}} \left((\hat{\beta}_{ij}\hat{\beta}_{00})^{\frac{1}{1+\rho}} - (\hat{\beta}_{i0}\hat{\beta}_{0j})^{\frac{1}{1+\rho}}\right).$$

■

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A.2. PROOF OF PROPOSITION 2

In the two-sector model, A_1 and $\hat{A}_1$ are scalars. From Lemma 2, $x_1 = a_{10}y_0 + a_{11}y_1$. Moreover, at the steady state, $y_1 = gx_1$, and it follows that

$$a_{10}y_0 + ga_{11}x_1 = x_1 \quad \Leftrightarrow \quad a_{10}y_0 = [1 - ga_{11}]x_1 > 0.$$

Therefore,

$$A_1^{-1} - gI = \frac{a_{00}}{a_{11}a_{00} - a_{10}a_{01}} - g = \frac{a_{00}[1 - ga_{11}] + a_{10}a_{01}g}{a_{11}a_{00} - a_{10}a_{01}} < 0$$

if and only if $a_{11}a_{00} - a_{10}a_{01} < 0$. From Lemma 1, $p_1 = \hat{a}_{01}w_0 + \hat{a}_{11}w_1$. Moreover, at the steady state, $(\delta + g)p_1 = w_1$, and it follows that

$$(\delta + g)\hat{a}_{01}w_0 + (\delta + g)\hat{a}_{11}w_1 = w_1 \quad \Leftrightarrow \quad (\delta + g)\hat{a}_{01}w_0 = [1 - (\delta + g)\hat{a}_{11}]w_1 > 0.$$

Therefore,

$$\begin{aligned}(\delta + g)I - \hat{A}'^{-1} &= (\delta + g) - \frac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01}} \\ &= -\frac{[1 - (\delta + g)\hat{a}_{11}]\hat{a}_{00} + (\delta + g)\hat{a}_{10}\hat{a}_{01}}{\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01}} < 0\end{aligned}$$

if and only if $\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} > 0$. ■

A.3. PROOF OF PROPOSITION 3

Since the consumption good is capital intensive at the private level, we have $\beta_{11}\beta_{00}/\beta_{10}\beta_{01} < 1$. When $\rho \geq 0$, this implies that $(\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)} < 1$ with $\lim_{\rho\to+\infty}(\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)} = (\beta_{11}\beta_{00}/\beta_{10}\beta_{01})$. On the contrary, when $\rho \in (-1, 0)$, this implies that $(\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)} > 1$ with $\lim_{\rho\to-1}(\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)} = +\infty$. Note that $(\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)}$ is decreasing in ρ.

(i) If the consumption good is labor intensive at the social level, then $\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01} > 1$. Therefore, there exists $\rho_1^* \in (-1, 0)$ such that $(\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01}) > (\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)}$ for all $\rho > \rho_1^*$ and $(\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01}) < (\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)}$ for all $\rho \in (-1, \rho_1^*)$. Local indeterminacy when $\rho > \rho_1^*$ and thus saddle-point stability when $\rho \in (-1, \rho_1^*)$ follow from Propositions 1 and 2.

(ii) If the consumption good is labor intensive at the social level, then $\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01} < 1$. Since we have

$$1 > \frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}} \geq \frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}},$$

there exists $\rho_2^* > 0$ such that $(\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01}) > (\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)}$ for any $\rho > \rho_2^*$ and $(\hat{\beta}_{11}\hat{\beta}_{00}/\hat{\beta}_{10}\hat{\beta}_{01}) < (\beta_{11}\beta_{00}/\beta_{10}\beta_{01})^{\rho/(1+\rho)}$ for any $\rho \in (-1, \rho_2^*)$. Local indeterminacy when $\rho > \rho_2^*$ and thus saddle-point stability when $\rho \in (-1, \rho_2^*)$ follow from Propositions 1 and 2. ■

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A.4. PROOF OF THEOREM 1

We need to establish various important lemmas. To prove the first one, we use the following Wielandt (1973) theorem:

THEOREM A.1. *Let $B = CD$, where D is symmetric and $C + C'$ is positive definite. Let b_+, b_0, and b_- denote the number of positive, vanishing, and negative real parts of eigenvalues of B and let d_+, d_0, and d_- denote the number of positive, vanishing, and negative eigenvalues of D. Then $b_+ = d_+$, $b_0 = d_0$, and $b_- = d_-$.*

Consider the following set of indices

$$\mathcal{N} = \{i \in \{1, \ldots, n\} \text{ such that } b_{ii} < 0\}.$$

We can now prove Lemma A.1.

LEMMA A.1. *Let $B = [b_{ij}]$ be an $n \times n$ matrix with strong dominant diagonal. Assume that $\#\mathcal{N} = p < n$. Then, B has p eigenvalues with negative real parts and $n - p$ eigenvalues with positive real parts.*

Proof. We can write B as the following product: $B = CD$ with C a strong dominant diagonal matrix with positive diagonal and D a diagonal matrix with p diagonal elements equal to -1 and $n - p$ diagonal elements equal to 1. Since C has a strong dominant diagonal, $C + C'$ is symmetric with positive strong dominant diagonal. Therefore, all the eigenvalues of $C + C'$ are positive and $C + C'$ is positive definite. Moreover, D satisfies $d_- = p$, $d_0 = 0$, and $d_+ = n - p$. The result follows from the Wielandt theorem. ■

LEMMA A.2. *Let Assumptions 1–3 hold. If the $n \times n$ matrix $\hat{A}_1'$ has n real positive eigenvalues, then $[(\delta + g)I - \hat{A}_1'^{-1}]$ has n negative eigenvalues.*

Proof. From Lemma 1, we have $\boldsymbol{p} = \hat{A}'\boldsymbol{w}$, that is,

$$\begin{pmatrix} p_0 \\ \boldsymbol{p}^1 \end{pmatrix} = \begin{pmatrix} \hat{a}_{00} & \hat{a}'_{.0} \\ \hat{a}'_{0.} & \hat{A}'_1 \end{pmatrix} \begin{pmatrix} w_0 \\ \boldsymbol{w}^1 \end{pmatrix}.$$

Since at the steady state, $(\delta + g)\boldsymbol{p}^1 = \boldsymbol{w}^1$, we have

$$\begin{aligned} & (\delta + g)^{-1}\boldsymbol{w}^1 = \hat{a}'_{0.}w_0 + \hat{A}'_1\boldsymbol{w}^1 \\ \Leftrightarrow \quad & \\ & [I - (\delta + g)\hat{A}'_1]\boldsymbol{w}^1 = (\delta + g)\hat{a}'_{0.}w_0 > 0. \end{aligned} \tag{A.1}$$

It follows that $[I - (\delta + g)\hat{A}'_1]$ has a positive quasi-dominant diagonal and thus n roots with positive real parts [see McKenzie (1960) and Takayama (1997)]. Now, denote $\lambda\{\hat{A}'_1\}$ an eigenvalue of $\hat{A}'_1$ and assume that each $\lambda\{\hat{A}'_1\}$ is real and positive. It follows that $1 - (\delta + g)\lambda\{\hat{A}'_1\} > 0$, which is equivalent to $(\delta + g) - \lambda\{\hat{A}_1'^{-1}\} < 0$. Therefore, $[(\delta + g)I - \hat{A}_1'^{-1}]$ has n negative real roots. ■

As proved by Benhabib et al. (2000), at the steady state, the sign pattern of roots of A_1 is the same as that of B_1, and the sign pattern of roots of $\hat{A}_1$ is the same as that of $\hat{B}_1$. Consider then the set of indices $\mathcal{B}$ defined by equation (13).

LEMMA A.3. *Let Assumptions 1–3 hold. If B_1 has a strong dominant diagonal with $\#\mathcal{B} = p \geq 1$, then $[A_1^{-1} - gI]$ has at least p roots with negative real parts.*

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Proof. B_1 has a strong dominant diagonal with p strictly negative diagonal coefficients and $n-p$ strictly positive diagonal coefficients. It follows from Lemma A.2 that B_1 has $p \geq 1$ eigenvalues with negative real parts and $n-p<n$ eigenvalues with positive real parts. Therefore $A_1^{-1}=P_1^{-1}B_1^{-1}W_1$ has p eigenvalues with negative real parts and $n-p$ eigenvalues with positive real parts, and thus $[A_1^{-1}-gI]$ has at least p roots with negative real parts. ■

We characterize now the roots of the matrix $[(\delta+g)I-\hat{A}_1'^{-1}]$.

LEMMA A.4. *Let Assumptions* 1–4 *hold and*

$$\frac{\hat{\beta}_{ii}\hat{\beta}_{00}}{\hat{\beta}_{i0}\hat{\beta}_{0i}} > \frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}} \tag{A.2}$$

for all $i=1,\ldots,n$. *Then, there exists* $-1<\hat{\rho}\leq\bar{\rho}$ *such that for any* $\rho>\hat{\rho}$, $[(\delta+g)I-\hat{A}_1'^{-1}]$ *has* n *negative roots.*

Proof. The following limits are obtained when ρ goes to -1 or $+\infty$:

$$\lim_{\rho\to+\infty}\left(\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}\right)^{\frac{\rho}{1+\rho}} = \frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}},$$

$$\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}<1 \quad\Rightarrow\quad \lim_{\rho\to-1}\left(\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}\right)^{\frac{\rho}{1+\rho}} = +\infty,$$

$$\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}>1 \quad\Rightarrow\quad \lim_{\rho\to-1}\left(\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}\right)^{\frac{\rho}{1+\rho}} = 0.$$

Under Assumptions 1–4, if the inequalities (A.2) are satisfied, every diagonal element of $\hat{B}_1(\rho)$ is positive when ρ is sufficiently large. If one diagonal coefficient becomes zero at some value $\tilde{\rho}>\bar{\rho}$, the dominant diagonal property of $\hat{B}_1(\rho)$ is lost at $\tilde{\rho}$. Therefore, $\hat{B}_1(\rho)$ has a positive dominant diagonal for any $\rho\geq\bar{\rho}$, and under Assumption 4, there exists $-1<\hat{\rho}\leq\bar{\rho}$ such that $\hat{B}_1(\rho)$ has n positive real roots for any $\rho>\hat{\rho}$. It follows that $\hat{A}_1'=P_1^{-1}\hat{B}_1(\rho)^{-1}W_1$ also has n positive real roots for any $\rho>\hat{\rho}$. Lemma A.3 then implies that $[(\delta+g)I-\hat{A}_1'^{-1}]$ has n negative real roots. ■

Consider the set of indices $\hat{\mathcal{B}}$ defined by (14). Theorem 1 is a direct consequence of Lemmas A.2 to A.5. ■

A.5. PROOF OF PROPOSITION 4

Lemma A.4 with $\#\mathcal{B}=n$ implies that the matrix $[A_1^{-1}-gI]$ has n roots with negative real parts. Moreover, we have for each $i=1,\ldots,n$

$$\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}<1 \quad\Rightarrow\quad \lim_{\rho\to-1}\left(\frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}}\right)^{\frac{\rho}{1+\rho}} = +\infty.$$

It follows that if $\hat{B}_1(\rho)$ has a strong dominant diagonal for $\rho\in(-1,\bar{\rho}]$ with $\bar{\rho}>-1$, then its diagonal is necessarily negative. Assuming now that $\hat{B}_1(\rho)$ has real roots for any $\rho\in(-1,\bar{\rho}]$, it follows that the roots of $\hat{A}'$ are real and negative. Then, the same argument as in the proof of Lemma A.3 implies that $[(\delta+g)I-\hat{A}_1^{\prime -1}]$ has n positive eigenvalues. ■

A.6. PROOF OF PROPOSITION 6

In the two-sector endogenous growth model, A and $\hat{A}$ are 2×2 matrices. The determinant of A^{-1} is the inverse of the determinant of A. If the consumable capital good is intensive in the pure capital good from the private perspective, the determinant of A is negative, so that one root of A^{-1} is negative. Therefore, at least one root of $A^{-1}-(g+\mu)I$ is negative. Since $(\delta+g+\sigma\mu)^{-1}$ is a Frobenius root of $\hat{A}'$, if the consumable capital good is quasi intensive in itself from the private perspective, the matrix $(\delta+g+\sigma\mu)I-\hat{A}^{\prime -1}$ has one zero root and one negative root [see Benhabib et al. (2000)]. Therefore, J has one zero root and at least two negative roots. ■

A.7. PROOF OF LEMMA 5

We prove this result by contradiction. Assume that $\beta_{ii}^{1/(1+\rho)} > \sum_{j\neq i}\beta_{ij}^{1/(1+\rho)}$ for all $i=1,\dots,n$. This is a dominant diagonal property for B. Since $\beta_{ii}>0$ by Assumption 2, all the roots of B have positive real parts and the determinant of B is positive. This is in contradiction to Assumption 7. Therefore, there is some i such that $\beta_{ii}^{1/(1+\rho)} \leq \sum_{j\neq i}\beta_{ij}^{1/(1+\rho)}$. ■

A.8. PROOF OF LEMMA 6

The matrix $\hat{B}(\rho)$ has a dominant diagonal if for any $i=0,\dots,n$

$$\hat{\beta}_{ii}/\beta_{ii}^{\frac{\rho}{1+\rho}} > \sum_{j\neq i}\hat{\beta}_{ij}/\beta_{ij}^{\frac{\rho}{1+\rho}} \quad\Leftrightarrow\quad \left[\sum_{j\neq i}\hat{\beta}_{ij}(\beta_{ij}/\beta_{ii})^{\frac{-\rho}{1+\rho}}\right]^{-1} > 1/\hat{\beta}_{ii}. \tag{A.3}$$

If $\hat{B}(0)$ has a dominant diagonal, then there exist $\underline{\rho}<0<\bar{\rho}$ such that this property still holds for any $\rho\in(\underline{\rho},\bar{\rho})$.

If $B(\rho)$ has a negative determinant, then from Lemma 5 there exists at least one row $i\in\{0,1,\dots,n\}$ such that $\beta_{ii}^{1/(1+\rho)} \leq \sum_{j\neq i}\beta_{ij}^{1/(1+\rho)}$. Consider therefore the set of rows $\mathcal{I}(\rho)$ defined by (25). Two cases need to be considered:

(i) If there is some $i\in\mathcal{I}(\rho)$ and at least one $j\neq i$ such that $\beta_{ii}<\beta_{ij}$, then for $\rho<0$, $(\beta_{ij}/\beta_{ii})^{-\rho/(1+\rho)}>1$ and $\lim_{\rho\to -1}\left(\beta_{ij}/\beta_{ii}\right)^{-\rho/(1+\rho)}=+\infty$. It follows that

$$\lim_{\rho\to -1}\left[\sum_{j\neq i}\hat{\beta}_{ij}(\beta_{ij}/\beta_{ii})^{\frac{-\rho}{1+\rho}}\right]^{-1}=0$$

Therefore equation (A.3) cannot hold when ρ is sufficiently close to -1 and $\underline{\rho}>-1$.

(ii) If for all $i\in\mathcal{I}(\rho)$ and all $j\neq i$, $\beta_{ii}>\beta_{ij}$, then when $\rho<0$, we have $(\beta_{ij}/\beta_{ii})^{-\rho/(1+\rho)}<1$. Moreover, any row $k\notin\mathcal{I}(\rho)$ of $B(\rho)$ will be such that $\beta_{kk}^{1/(1+\rho)} \geq \sum_{j\neq k}\beta_{kj}^{1/(1+\rho)}$. In this case and under Assumption 9, we necessarily have $\beta_{kk}>\beta_{kj}$ for any $j\neq k$. Since $\hat{B}(0)$ has a dominant diagonal, equation (A.3) holds for $\rho=0$, but the left-hand side of (A.3) increases with ρ. It follows that (A.3) holds for all $\rho\in(-1,0]$, and $\hat{B}(\rho)$ has a dominant diagonal within this interval. It follows that $\underline{\rho}=-1$. ■

A.9. PROOF OF THEOREM 2

We need first to establish the following lemma.

LEMMA A.5. *Let Assumptions* 1, 2, *and* 6 *hold. If the* $(n+1)\times(n+1)$ *matrix* $\hat{A}'$ *has* $n+1$ *real positive eigenvalues, then* $[(\delta+g+\sigma\mu)I-\hat{A}'^{-1}]$ *has n negative eigenvalues and one zero eigenvalue.*

Proof. The Frobenius root of $\hat{A}'$ is $\hat{\lambda}=(\delta+g+\sigma\mu)^{-1}$. Denote $\lambda\{X\}$ an eigenvalue λ of a matrix X. Under Assumption 1, $\hat{a}_{ij}>0$ for all $i, j=0,\dots,n$, and the matrix $\hat{A}'$ is indecomposable. It follows that $\hat{\lambda}$ is a simple eigenvalue. Moreover, if $\lambda\{\hat{A}'\}$ is real and positive, we have

$$\begin{aligned}\lambda\{(\delta+g+\sigma\mu)I-\hat{A}'^{-1}\} &= \hat{\lambda}^{-1}-\lambda\{\hat{A}'^{-1}\}=\lambda\{\hat{A}'^{-1}\}(\hat{\lambda}^{-1}\lambda\{\hat{A}'^{-1}\}^{-1}-1)\\ &=\lambda\{\hat{A}'\}^{-1}(\hat{\lambda}^{-1}\lambda\{\hat{A}'\}-1).\end{aligned}$$

Since $\hat{\lambda}>\lambda\{\hat{A}'\}>0$, we have $\lambda\{(\delta+g+\sigma\mu)I-\hat{A}'^{-1}\}<0$. When we consider the Frobenius root of $\hat{A}'$, $\hat{\lambda}$, it follows that $\lambda\{(\delta+g+\sigma\mu)I-\hat{A}'^{-1}\}=0$. ■

Theorem 2 is a consequence of Lemma 5, Lemma 6, and Lemma A.5. As proved by Benhabib et al. (2000), along the balanced growth path, the sign pattern of roots of A is the same as that of B, and the sign pattern of roots of $\hat{A}$ is the same as that of $\hat{B}$. If B has a negative determinant, Lemma 5 implies that $[A^{-1}-(g+\mu)I]$ has at least one real negative eigenvalue. Under Assumption 10, the roots of $\hat{A}$ are real. Case (i) is obtained by continuity from Lemma A.5 and Lemma 6. There exists $\rho^*\geq\bar{\rho}$ such that the stationary balanced growth path is locally indeterminate for any $\rho\in(-1,\rho^*)$. Similarly, case (ii) is obtained by continuity from Lemma A.5 and case (ii) in Lemma 6. There exist $\rho_1^*\leq\underline{\rho}$ and $\rho_2^*\geq\bar{\rho}$ such that for any $\rho\in(\rho_1^*,\rho_2^*)$, the stationary balanced growth path is locally indeterminate. ■