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Robustly expansive homoclinic classes

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Abstract. Let $f : M \to M$ be a diffeomorphism defined in a three-dimensional compact boundary-less manifold M. We prove that for an open dense set, C^1-robustly expansive homoclinic classes $H(p)$ for f are hyperbolic. A diffeomorphism f is α-expansive on a compact invariant set K if there is $\alpha > 0$ such that for all $x, y \in K$, if $\operatorname{dist}(f^n(x), f^n(y)) \le \alpha$ for all $n \in \mathbb{Z}$ then $x = y$. By 'robustly' we mean that there is $\alpha > 0$ such that for all nearby diffeomorphisms g, the homoclinic class $H(p_g)$ of the continuation of p is α-expansive.

1. *Introduction*

Let M be a compact connected boundary-less Riemanian three-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and dist : $M \times M \to \mathbb{R}$ a metric on M and $\alpha > 0$. We say that f restricted to K is α-expansive when $\operatorname{dist}(f^n(x), f^n(y)) \le \alpha$ for all $x, y \in K$ and all $n \in \mathbb{Z}$ implies $x = y$. The number $\alpha > 0$ is called a *constant of expansiveness* for f and K, and sometimes we say that f is *expansive* in K if α is fixed. Expansiveness is a property shared by a large class of dynamical systems exhibiting chaotic behavior. Roughly speaking, an expansive dynamical system is one in which two different trajectories can be distinguished by an observer with an instrument capable of distinguishing points at a distance greater than a certain constant $\alpha > 0$ (constant of expansiveness). Examples of expansive systems are for instance diffeomorphisms $f : M \to M$ acting hyperbolically in an f-invariant compact subset Λ of a manifold M. This includes for instance Anosov systems or the non-wandering set of Axiom A diffeomorphisms. Other examples are the pseudo-Anosov homeomorphisms in surfaces of genus greater than 1 [**HT**]. Other examples are given by sub-shifts of finite type defined in $N^{\mathbb{Z}}$ endowed with the Tychonov topology, where N is a finite set of symbols. In this paper we show that for an open dense subset, C^1-robustly expansive homoclinic classes are hyperbolic. By *homoclinic class* we mean the closure of the transversal homoclinic points associated with a hyperbolic periodic orbit p. By *robustly expansive* homoclinic classes we mean that for all nearby diffeomorphisms g, the homoclinic class of the continuation of p is expansive with the same constant of expansiveness as the one for f restricted to $H(p)$. To state our results in a precise way, let us introduce some notation and definitions.

Throughout M is a compact boundary-less three-dimensional manifold, and $\mathrm{Diff}^r(M)$, $r \geq 1$, is the space of C^r diffeomorphisms on M with the C^r topology. If $x \in M$ then $\mathcal{O}_f(x)$ denotes the orbit of x by f, $W^s(x, f)$ and $W^u(x, f)$ denote the stable and unstable manifolds of x respectively, and $H(p, f)$ denotes the homoclinic class of a periodic point p. When no ambiguity is possible we denote these sets by $\mathcal{O}(x)$, $W^s(x)$, $W^u(x)$ and $H(p)$ respectively. Similarly we denote by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ the ϵ-local stable and unstable manifolds of points $x \in M$ respectively. When $\epsilon > 0$ is fixed we omit it and just say local stable manifold etc.

When these local manifolds are one-dimensional the concepts of diameter and length are usually enough for us to describe their properties. But if they are k-dimensional, $k > 1$, we need another concept.

Let us assume that $W^\sigma_\epsilon(x)$, $\sigma = s, u$ is an embedded topological k-disk, $k \geq 1$. Let Γ be the family of all parameterized continuous arcs $\gamma : [0, 1] \to W^\sigma_\epsilon(x)$ joining x with the boundary of $W^\sigma_\epsilon(x)$.

Definition 1.1. We define the size of $W^\sigma_\epsilon(x)$ as $\inf\{\mathrm{diam}(\gamma)/\gamma \in \Gamma\}$.

In the case when $H(p)$ has topological dimension zero (like occurs in Smale's horseshoe, see §2.1) the fact that $H(p)$ is expansive has very poor implications from the hyperbolic point of view. For instance we may have non-hyperbolic periodic points in $H(p)$ (see §2.2). So to obtain good results from expansiveness on $H(p)$, we need to assume that it holds not only for the homoclinic class of a single diffeomorphism f, but that it is shared by diffeomorphisms in a neighborhood of f (see §§2.2 and 2.3). That is, we need the homoclinic class to be robustly expansive. Let us make this notion precise.

Definition 1.2. $H(p)$ is C^r-robustly expansive ($r \geq 1$) iff there exist $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is α-expansive in $H(p_g)$.

To state in a precise way our results we need to introduce the following definitions.

Definition 1.3. We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \quad \text{for all } x \in \Lambda, n \geq 0.$$

Definition 1.4. We say that a compact f-invariant set Λ is hyperbolic if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0, 0 < \lambda < 1$ such that

$$\|Df^n|E(x)\| \leq C\lambda^n \quad \text{for all } x \in \Lambda,\ n \geq 0,$$

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and

$$\|Df^{-n}|F(x)\| \leq C\lambda^n \quad \text{for all } x \in \Lambda,\ n \geq 0.$$

Our main results are the following.

THEOREM A. *Let $f \in \text{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is robustly expansive. Then for an open and dense subset $\mathcal{N}$ in a neighborhood of f in the C^1-topology, $\mathcal{U}(f)$, it holds that if $g \in \mathcal{N}$ then $H(p_g)$ has a codimension-one dominated splitting $E \oplus F$.*

THEOREM B. *Let $f \in \text{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is robustly expansive and has a codimension-one dominated splitting $E \oplus F$. Then for an open and dense subset $\mathcal{N}$ in a neighborhood of f in the C^1-topology, $\mathcal{U}(f)$, it holds that if $g \in \mathcal{N}$ then $H(p_g)$ is hyperbolic.*

From Theorems A and B we have the following theorem.

THEOREM C. *Generically, C^1-robustly expansive homoclinic classes are hyperbolic.*

1.1. *Comments.* Ricardo Mañé studied the case in which for $f : M \to M$ there is a C^1-neighborhood $\mathcal{U}$ of f such that for $g \in \mathcal{U}$, g is expansive (i.e. f is in the C^1-interior of the expansive diffeomorphisms, see [**Ma1**]). He proved that in that case f is quasi-Anosov, that is $Df : TM \to TM$ is expansive (equivalently: if $0 \neq v \in TM$ then $\|Df^n(v)\| \to \infty$ either for $n \to +\infty$ or for $n \to -\infty$). It follows (see [**Ma1**]) that f is Axiom A and verifies the no-cycle condition. In particular all the homoclinic classes are hyperbolic.

In our case we do not assume that f is expansive in the whole manifold, but that there is a homoclinic class $H(p)$ of an f-periodic point p such that f is expansive in $H(p)$ and that the same is valid for g C^1-close to f restricted to $H(p_g)$, the homoclinic class of p_g, the continuation of p.

Without loss of generality we may assume that $H(p)$ is the homoclinic class of an f-hyperbolic fixed point p of index 2, that is, $\dim W^s(p) = 2$. Observe that this hypothesis about the index of $H(p)$ is not a restriction and that the requirement p hyperbolic seems to be necessary to have uniqueness of p_g, the continuation of p.

Moreover we will see in §2.3 that if p is not hyperbolic then the class $H(p)$ cannot be C^1-robustly expansive. This result will follow from applying Lemma 3.1. Although we sketch the proof only in an example it can be extended easily to a general case. We remark that Lemma 3.1 is false in the C^r-topology, $r \geq 2$, as has been proved in [**PS2**]. Thus it is not necessarily true that if $H(p)$ is C^r-robustly expansive with $r \geq 2$ then p (or another periodic point $q \in H(p)$) has to be hyperbolic.

Once we assume that p is hyperbolic it seems natural to assume also that $H(p) \neq \{p\}$, otherwise the dynamics should be trivial. Thus we assume that p is neither a hyperbolic repeller nor a hyperbolic attractor. On the other hand, in these cases $H(p)$ is trivially robustly expansive.

If we consider diffeomorphisms on surfaces then Theorems A and B follow from [**PS1**, Theorem A, Theorem B], taking into account that we cannot have an invariant circle in $H(p)$ if $f|_{H(p)}$ is expansive.

Observe that it is supposed in [**PS1**] that f is C^2 and that all the periodic points are hyperbolic of saddle type. We also have to assume in part of our arguments that f is C^2. But, as C^2-diffeomorphisms are dense in the C^1 topology we finally obtain that such properties hold densely in the C^1 topology and this is enough to achieve our result.

One of the problems we have to face is the following. Let q be a periodic point in $H(p)$. If it were true that *always* for q we have $W^s(q) \cap W^u(p) \neq \emptyset$ and $W^u(q) \cap W^s(p) \neq \emptyset$ then we can show using small C^1-perturbations that all periodic points have to be hyperbolic and all the intersections have to be transverse. But if we do not have this property, when perturbing f it may happen that the point q 'escapes' from the homoclinic class. To avoid this problem we assume certain C^1-generic properties (see Theorems 3.6 and 3.8). On the other hand, it seems natural to try to remove these assumptions.

1.2. *Sketch of the proof.* We now give an idea of the proofs of Theorem A and Theorem B. We first construct a dominated splitting for $H(p)$. For this we start by proving that robustly expansive homoclinic classes do not have Lyapunov stable points and hence there are not sources or sinks on them. Moreover, we cannot have invariant arcs or closed curves in a compact invariant set Λ if f/Λ is expansive. Moreover, if $\Lambda = H(p)$ we cannot have an arc contained in the intersection of the stable manifold of a periodic point $q \in H(p)$ with the unstable manifold of a periodic point $r \in H(p)$ with a heteroclinic connection. Otherwise the use of blenders as in [**BD**] will imply that this curve has to be contained in $H(p)$, which cannot happen by expansiveness (see Lemma 3.10). Thus periodic points homoclinically related have the same index. Moreover, robustly expansive homoclinic classes cannot have tangency between stable and unstable manifolds of their periodic points (see Lemma 3.9). This will imply that the angle between those manifolds is uniformly bounded away from zero (see §4.2). Moreover C^1-robustness of expansiveness will imply that there exists $1 > \delta > 0$ such that if q is a periodic point of period τ then if λ is a contracting eigenvalue of Df_q^τ then $|\lambda| < (1-\delta)^\tau$ and if μ is an expanding eigenvalue of Df_q^τ then $|\mu| > (1+\delta)^\tau$. All these facts together imply the existence of a dominated splitting $E \oplus F$ (with F being one-dimensional (see §4.3)). This will prove Theorem A.

The next step is to prove that this splitting is hyperbolic for a residual subset. First we note that robust expansiveness prevents $H(p)$ from having a periodic point q such that Df_q^τ has a poor rate of contraction in E_q. If we had that Df expands uniformly in F then a result of Mañé [**Ma3**, Theorem II.1, p. 173] would imply hyperbolicity of $E \oplus F$. Since we cannot assume this we prove next that the center-unstable manifolds, which are tangent to F, have good dynamical properties as diameter bounded away from zero and that they are true unstable manifolds when restricted to periodic points (see §4.1). It is here that we need, for technical reasons, to assume that f is C^2 in order to make use of [**PS1**, Proposition 3.1, p. 987] which enables us to control the ω-limit of certain points which rest in a neighborhood of $H(p)$ for all positive iterates. The properties of the local unstable manifolds replace that of uniform expansion in F there, and together with the fact that $H(p)$ has no periodic points where Df contracts weakly in the period in E, are enough to obtain that E is a contracting bundle (see §5.3). Finally, using the result of Mañé mentioned above, we obtain that $E \oplus F$ is hyperbolic. Once we have this, it follows that the same is true for a C^1-neighborhood of f and we can conclude Theorem B.

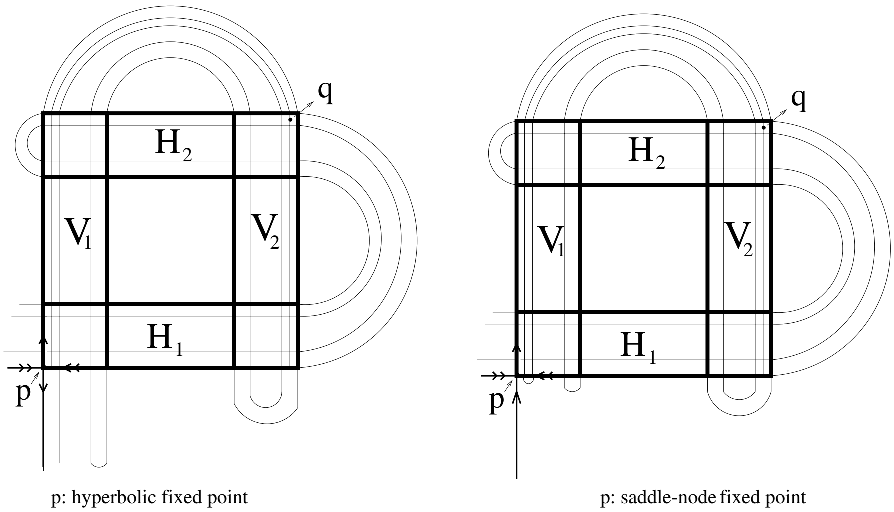

FIGURE 1. Horseshoes, p hyperbolic and saddle-node.

This paper is organized as follows. In §2 we give examples and counterexamples of robustly expansive homoclinic classes. In §3 we recall some results proved elsewhere and that will be used to prove our results and establish some basic properties of robustly expansive homoclinic classes. In §4 we construct a dominated splitting for robustly expansive homoclinic classes proving Theorem A, and in §5 we conclude that this dominated splitting is hyperbolic in a residual subset, proving our Theorem B.

2. *Examples and counterexamples*

2.1. *The horseshoe: C^r robustly expansive for all $r \geq 1$.* The usual Smale's horseshoe is hyperbolic and therefore an example of a C^r-robustly expansive homoclinic class for all $r \geq 1$. It may be seen as the homoclinic class of any of its periodic points which are all hyperbolic. This example exhibits two fixed points, p, q. See Figure 1. Moreover, we may assume that the construction is done in the Oxy plane in $\mathbb{R}^3$. Thus multiplying this two-dimensional horseshoe by a strong linear expansion in the Oz direction we obtain examples of robustly expansive homoclinic classes in $\mathbb{R}^3$ and so in any three-dimensional manifold.

2.2. *Critical saddle-node horseshoe: neither C^2 nor C^1 robustly expansive.* Observe that in Smale's example, the class itself does not require the hyperbolicity of the fixed point p. To see this consider a square C of sides b, t, l, r of length 1, with two horizontal sub-rectangles H_1, H_2 and two vertical sub-rectangles V_1, V_2 as in Figure 1. The length of the horizontal sides of H_1, H_2 is 1 and the vertical sides of H_1 and H_2 have length $1/10$. H_1 has its bottom coinciding with b, the bottom of C, and H_2 has its top coinciding with t, the top of C. V_1, V_2 have horizontal sides of length $1/10$ and vertical sides of length 1. V_1 has its left side coinciding with the left side, l, of C and V_2 has its right side coinciding

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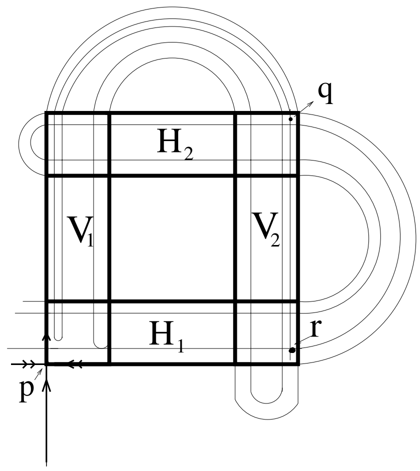

FIGURE 2. Perturbing the critical saddle-node.

with r, the right side of C. Let $f : S^2 \to S^2$ be a diffeomorphism such that $f(H_1) = V_1$, $f(H_2) = V_2$, f contracts linearly horizontal fibers in H_1, but f has eigenvalue 1 at the fixed point p in the leftmost bottom corner intersection of H_1 with V_1. Moreover, it is not difficult to prove that all the other periodic points of this horseshoe are hyperbolic, see [**DRV**, Theorem B]. Furthermore, it is possible to construct this example in such a way that any C^2-perturbation g of $f/H(p)$, preserving the fixed point p, is conjugate to Smale's horseshoe (§2.1). Hence $H(p_g)$ is expansive. As a matter of fact, the constant of expansiveness is equal to the minimum gap between horizontal and vertical rectangles.

But if we perturb in such a way that the fixed point p disappears then we may lose expansiveness (see Figures 1 and 2) due to the appearance of a tangency. Of course, in this case we cannot speak of the class of p. But the class persists: it may be seen, for instance, as the homoclinic class of q, the other fixed point in the example.

Thus a horseshoe with a *critical saddle-node*, that we call *saddle-node horseshoe*, is not C^2-robustly expansive and therefore is not C^1-expansive.

As in the example given above (§2.1) we can transform the saddle-node horseshoe into a three-dimensional one.

2.3. *Non-critical saddle-node horseshoe: C^2-robustly expansive but not C^1-robustly expansive.* Consider a non-critical saddle-node horseshoe $H(p)$ like that of Figure 3, where below p we have another saddle r. It is possible to construct this example in such a way that for any C^2-perturbation such that the fixed point p disappears, all the periodic points of the perturbed system are hyperbolic, and, if μ is an expanding eigenvalue of a periodic point $\hat{q}$ of period τ then $|\mu| > (1 + \delta)^\tau$, for a fixed $\delta > 0$. Indeed, to assure this it is enough to build the example with a Markov partition such that the horizontal sub-rectangles have vertical diameter sufficiently small, and the vertical sub-rectangles have horizontal diameter sufficiently small. Since all the periodic points are hyperbolic for the perturbed system and it has a dominated splitting, using [**PS1**, Theorem 3.1] we obtain

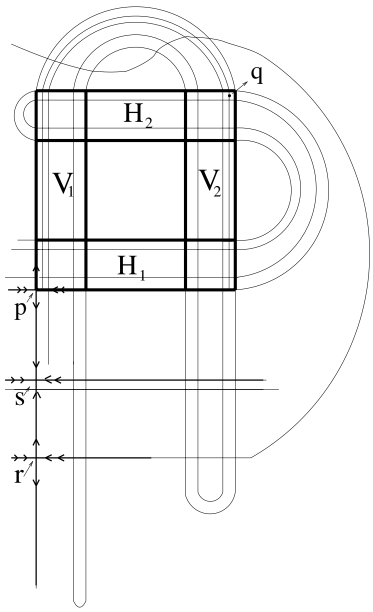

FIGURE 3. Non-critical saddle-node.

that the perturbed system is hyperbolic. Note that, by construction, the perturbed system is conjugated to Smale's horseshoe (§2.1). Hence, the non-critical saddle-node horseshoe gives an example of a C^2-robustly expansive homoclinic class.

Let us see that this example is not C^1-robustly expansive. Using transitions maps as in Proposition 4.4 (see [**BDP**]), we can find a hyperbolic periodic point $\bar{q} \in H(p)$ with a normalized eigenvalue arbitrarily close to 1 accumulated by points in $H(p)$ from both sides of $W^u_\epsilon(\bar{q})$. Thus we can unfold the periodic point $\bar{q}$ into three periodic points $q_1, \bar{q}, q_2$, with $q_1, q_2 \in H(p)$ and with the same period as $\bar{q}$, and such that $\text{dist}(f^n(q_1), f^n(q_2)) \leq \epsilon$ for all $n \in \mathbb{Z}$, contradicting robust expansiveness.

2.4. *Generalized pseudo-Anosov of S^2: a non-expansive homoclinic class.* The following is an example in which the homoclinic class is maximal and almost all periodic points are hyperbolic, but it is not expansive for any $\alpha > 0$. In particular, it is not robustly expansive.

If we consider in S^2 a generalized pseudo-Anosov map like that given in [**BLJ**, Ch. 9], then for all $\epsilon > 0$ the ϵ-local stable and unstable sets of infinitely many periodic points are not locally connected, as has been pointed out in [**Le1**]. As a consequence of this lack of local connectedness the homoclinic class (which for all periodic point is all S^2) is ϵ-expansive for no $\epsilon > 0$ (in fact this is a consequence of the general classification of expansive homeomorphisms on surfaces; see [**Le1, Hi**]).

Let us show that the ϵ-local stable and unstable sets are not locally connected. For this we proceed as follows.

Take in $\mathbb{R}^2$ the action given by the matrix $\left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix}\right)$ and pass it to the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This gives us a very well known Anosov diffeomorphism $a : \mathbb{T}^2 \to \mathbb{T}^2$.

In $\mathbb{T}^2$ (which is an Abelian group) we consider the relation $X \sim -X$. This is the same as to consider this relation in a fundamental domain in $\mathbb{R}^2$. The quotient $\mathbb{T}^2/\sim$ gives the sphere S^2. Moreover the relation $X \sim -X$ is compatible with the Anosov map a, i.e. $a(X) \sim -a(X) = a(-X)$ by linearity, and therefore projects to S^2 as a generalized pseudo-Anosov map $g : S^2 \to S^2$, $g(x) = \Pi(a(\Pi^{-1}(x)))$ that has 1-prongs (that is, points $P \in S^2$ in which the local stable and unstable 'manifolds' are arcs finishing at P (see Figure 4 where O is a point with 1-prongs)). Observe that the projection $\Pi : \mathbb{T}^2 \to S^2$ is a branched covering and that the definition of g does not depend on the pre-image of x by Π^{-1}.

One of the 1-prongs in S^2 is the image O (via the projection map from $\mathbb{T}^2$ to S^2) of $(0, 0) \in \mathbb{T}^2$ which is fixed by g. It is not difficult to see that the unstable manifold of $(0, 0)$ projects to S^2 as an arc ending at O (because $x \sim -x$). The stable and unstable manifolds of the points in $\mathbb{T}^2$ near $(0, 0)$ project to points in S^2 near O like in Figure 4. The intersection of the stable and unstable manifolds of the points $(0, x)$ and $(0, -x)$ consists of four points identified by pairs by the relation $X \sim -X$. If $X \in \mathbb{T}^2$ projects to $x \in S^2$, let us denote by s_x and u_x the projections of the local stable and unstable manifolds respectively of the point X.

One may suppose at first sight that these projections are the local stable and unstable sets of the points in S^2 under the action given by g, but this is not true. To see this, let us choose a 1-prong O and let p be a periodic point very close to O of, say, period k. More precisely, given $\epsilon > 0$ choose $p \neq O$ a periodic point satisfying $\text{dist}(p, O) < \epsilon/4$. Such a point exists since periodic points are dense for the Anosov diffeomorphism a defined on $\mathbb{T}^2$ and projects on S^2 as periodic points for g. Let $\{p, p'\} = s_p \cap u_p$. We take ϵ sufficiently small in order to have $p' \in W^u_{\epsilon/2}(p) \cap W^s_{\epsilon/2}(p)$. Therefore there is a small arc $\sigma \subset u_p$, p' belonging to the interior of σ, such that $\sigma \subset W^u_{3\epsilon/4}(p)$. Then there is a sub-arc $\gamma_1 \subset u_{g^k(p')}$, $g^k(p') \in \gamma_1$ such that $\gamma_1 \subset W^u_\epsilon(p, g^k)$. Iterating by g^k we see that the same is true for $u_{g^{kn}(p')}$ for all integers $n > 0$. That is, for all n, there is an arc $\gamma_n \subset W^u_\epsilon(p, g^k)$. Let us see that the union of these arcs is not locally connected. To see it observe that there is a residual set X of points in the torus $\mathbb{T}^2$ whose orbits under a are dense and therefore these orbits project in S^2 as orbits dense under the action of g. Thus given any point between two arcs γ_n and γ_{n+1} and close to p, there is a point $x \in X$ between γ_n and γ_{n+1}. This point cannot be in $W^u_\epsilon(p, g^k)$ because its orbit is dense. Hence the arc $u_x \not\subset W^u_\epsilon(p, g^k)$ separates γ_n from γ_{n+1} and therefore $W^u_\epsilon(p, g^k)$ is not locally connected.

Arguing similarly we see that the same is true for $W^s_\epsilon(p)$, i.e. there are infinitely many disjoint arcs $\beta_n \subset W^s_\epsilon(p, g^k)$.

We conclude that the orbit of the points $x_{n,m}, x'_{n,m}$ defined by the intersection of the arc $\gamma_n \subset W^s_\epsilon(p, g^k)$ with the arc $\beta_m \subset W^u_\epsilon(p, g^k)$ is ϵ-close to the orbit of p for all the iterates by g (positives and negatives).

As $\epsilon > 0$ is arbitrary, it follows that for no $\epsilon > 0$ we have that g is ϵ-expansive. Moreover, g is transitive and hyperbolic periodic points are dense on $S^2 = \Omega(g)$. In fact all periodic points are hyperbolic except points like those identified with O. But there are only four points like O. As all the stable and unstable manifolds of the Anosov $a : T^2 \to T^2$

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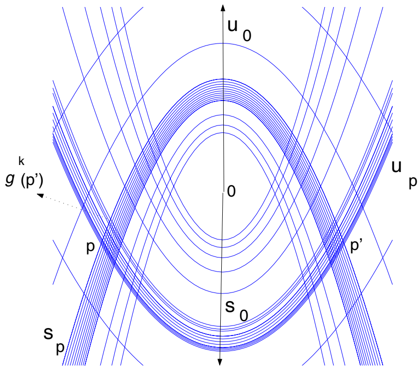

FIGURE 4. Generalized pseudo-Anosov.

are dense in T^2 it follows that in S^2 homoclinic classes $H(p)$ of all the periodic points p coincide and are equal to S^2. Thus the homoclinic class $H(p)$ is maximal for all p periodic (hyperbolic or not).

We point out that there are real analytic models for $g : S^2 \to S^2$ (see [**Ge**] and [**LL**]).

3. *Preliminary results*

We assume in the sequel that p is hyperbolic and that Df_p has one eigenvalue, μ, of modulus greater than 1 and two, λ_1 and λ_2, with moduli less than 1.

We recall that in [**Nh**, Lemma 2.3], it is proved that there exists a residual subset in $H(p)$ of points $x \in H(p)$ such that its forward and backward orbits are dense in $H(p)$. In particular $H(p)$ is always transitive. Moreover hyperbolic periodic points are dense in $H(p)$ [**Sm**].

A fundamental tool to be used is the following result due to J. Franks.

LEMMA 3.1. *Let M be a closed n-manifold, $f : M \to M$ a C^1-diffeomorphism and let $\mathcal{U}(f)$ be any neighborhood of f. Then, there exist $\epsilon > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, \ldots, x_N\}$, a neighborhood U of $\{x_1, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ such that $\|L_1 - Dg_{x_i}\| \leq \epsilon$ for all $1 \leq i \leq N$, then there exists $\hat{g} \in \mathcal{U}(f)$ such that $\hat{g}(x) = g(x)$ if $x \in \{x_1, \ldots, x_N\} \cup (M \backslash U)$ and $Dg_{x_i} = L_i$ for all $1 \leq i \leq N$.*

Proof. See [**Fr**, Lemma 1.1]. The statement given there is slightly different from that above, but the proof of the present statement is contained in [**Fr**]. □

LEMMA 3.2. *Let p be a hyperbolic fixed point for f of index 2 and $H(p)$ its homoclinic class, robustly α-expansive. If $x \in W^s(p) \cap W^u(p)$ then $W^s(p)$ intersects $W^u(p)$ transversally at x.*

Proof. By our assumptions $W^s(p)$ is a Euclidean plane and $W^u(p)$ a line, both immersed in M. If the intersection at x of $W^s(p)$ and $W^u(p)$ is not transversal we should have that $T_x W^u(p) \subset T_x W^s(p)$. Let $D_x \subset W^s_\epsilon(x) \cap W^s(p)$ be a small disk and $N > 0$

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be such that $f^N(D_x) \subset W^s_\epsilon(p)$, and let L_x be a small arc in $W^u_\epsilon(x) \cap W^u(p)$ such that $f^{-N}(L_x) \subset W^u_\epsilon(p)$. A small C^1-perturbation, like those in Lemma 3.1, will produce a diffeomorphism $g \in \mathcal{U}(f)$ in which the corresponding stable and unstable manifolds of p_g will intersect each other in two points x', x'' close enough to x and such that $D_{x',g} \ni x''$, $L_{x',g} \ni x''$. Choosing $0 < \epsilon < \alpha/4$ and g sufficiently close to f in order to have that $g^N(Dx', g) \subset W^s_\epsilon(p_g)$, and $g^{-N}(Lx', g) \subset W^u_\epsilon(p_g)$ and $\text{dist}(g^j(x'), g^j(x'')) < \epsilon$, for all $j : j = -N, \dots, N$, this would imply that $x', x'' \in H(p_g)$ and at the same time $\text{dist}(g^n(x'), g^n(x'')) \leq \alpha$ for all $n \in \mathbb{Z}$, contradicting robust expansiveness. □

A hyperbolic periodic point p is C^1-*far* from homoclinic tangency if every diffeomorphism g C^1-close to f does not have homoclinic tangency associated with p_g, the continuation of p.

COROLLARY 3.3. *Under the hypothesis of Lemma 3.2 we have that p is C^1-far from homoclinic tangency.*

Definition 3.1. We say that x is a Lyapunov stable point in $H(p)$ if for all $\epsilon > 0$ there is $\delta > 0$ such that there is a neighborhood $V(x)$ in $H(p)$ of diameter less than $\delta > 0$ such that for all $n \geq 0$ we have $\text{diam}(f^n(V(x))) < \epsilon$.

LEMMA 3.4. *There are no Lyapunov stable points in $H(p)$ if $H(p)$ is α-expansive.*

Proof. Observe that if $x, y \in H(p)$ satisfy $\text{dist}(f^n(x), f^n(y)) < \epsilon$ for all $n \geq 0$ then $\text{dist}(f^n(x), f^n(y)) \xrightarrow[n\to\infty]{} 0$. For otherwise we may take subsequences $\{f^{n_k}(x)\}, \{f^{n_k}(y)\}$ converging to points z, w respectively such that $\text{dist}(z, w) > 0$ and $\text{dist}(f^n(z), f^n(w)) \leq \epsilon$ for all $n \in \mathbb{Z}$, thus contradicting expansiveness of f in $H(p)$ if $\epsilon < \alpha$.

Now assume that $x \in H(p)$ is Lyapunov stable. Choose $0 < \epsilon < \alpha/4$ and the corresponding $\delta > 0$ and $V(x)$. As $x \in H(p)$ there exists $y \in W^u(p) \cap W^s(p)$ in $V(x)$. Let N be such that $\text{dist}(f^n(y), p) < \epsilon$ if $n \geq N$. It follows that $\text{dist}(f^n(x), p) < 2\epsilon < \alpha$ and therefore by the observation at the beginning of the proof $\text{dist}(f^n(x), p) \xrightarrow[n\to\infty]{} 0$. Thus $x \in W^s(p)$. Hence there is $N' > 0$ such that both $f^{N'}(y)$ and $f^{N'}(x)$ are in $W^s_\epsilon(p)$. As $x, y \in H(p)$ they are accumulated by points in $H(p)$ and therefore there is $y' \in W^u_{\epsilon'}(y) \cap W^s(p)$ different from y and x, as close to y as we wish, $y' \in V(x)$, and with $\epsilon' > 0$ so small that $f^{N'}(y') \in W^u_\epsilon(f^{N'}(y))$. By expansiveness it follows that there is $n > N'$ such that $\text{dist}(f^n(y'), p) > \alpha$. But then $\text{dist}(f^n(x), f^n(y')) > \alpha - \epsilon > \epsilon$, contradicting that $\text{diam}(f^n(V(x))) < \epsilon$ for all $n \geq 0$. □

We denote by *sink* any attracting periodic point and by *source* any repelling periodic point of f.

Remark 3.5. It follows from Lemma 3.4 that there are no sinks in $H(p)$. Similarly, arguing with f^{-1}, we may prove that there are no sources in $H(p)$.

The following result is a straightforward adaptation for diffeomorphisms of the result for flows in [**CMP**].

THEOREM 3.6. *For any $\mathcal{U}$, C^1-open subset of* $\text{Diff}^1(M)$ *there exists a residual set $\mathcal{T}$ in $\mathcal{U}$ such that for $g \in \mathcal{U}$,* $\text{Cl}(W^s(p_g, g)) \cap \text{Cl}(W^u(p_g, g)) = H(p_g)$ *and all the periodic*

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points in $H(p_g)$ are hyperbolic, i.e. g is Kupka–Smale. Moreover, all homoclinic classes are maximal transitive in the sense that if H is a homoclinic class and Λ is a transitive set such that $\Lambda \cap H \neq \emptyset$ then $H \supset \Lambda$.

Remark 3.7. For $g \in \mathcal{T}$, $\mathrm{Cl}(\mathrm{Per}(g))|_{H(p_g)} = H(p_g)$ and the periodic points are hyperbolic.

Given $f \in \mathrm{Diff}^1(M)$ denote by $\mathrm{Per}_h(f)$ the set of hyperbolic points of f. The following theorem is contained in [**Ar**].

THEOREM 3.8. *There exists a residual set in* $\mathrm{Diff}^1(M)$ *such that if* $q \in \mathrm{Per}_h(f) \cap H(p)$ *and* $\dim(W^s(p)) \geq \dim(W^s(q))$ *then there exists a heteroclinic connection between* p *and* q.

Proof. See [**Ar**, Proposition 20]. □

From now on we assume that f is in the residual subset $\mathcal{G}$ of $\mathcal{U}(f)$ in which Theorems 3.6 and 3.8 hold.

LEMMA 3.9. *Let p and $H(p)$ be as in Lemma 3.2 and q a hyperbolic periodic point in $H(p)$. If $2 =$* $\mathrm{index}(p) = \mathrm{index}(q)$ *and $x \in W^s(p) \cap W^u(q)$ then $W^s(p)$ intersects $W^u(q)$ transversally at x. Analogously, if $y \in W^u(p) \cap W^s(q)$ then $W^s(q)$ intersects $W^u(p)$ transversally at y.*

Proof. Arguing by contradiction suppose that $W^u(p)$ is tangent to $W^s(q)$ at a point y. Iterating by f we may assume that the tangency $y \in W^s_\epsilon(q)$. We also may assume that there is a transverse intersection close to y. By the λ-lemma a disk in $W^s(p)$ converges in the C^1-topology to $W^s(q)$. With another small C^1-perturbation we may create a homoclinic tangency between $W^s(p)$ and $W^u(p)$, contradicting Lemma 3.2. The arguments are similar if we suppose that $W^s(p)$ is tangent to $W^u(q)$. □

LEMMA 3.10. *If $f \in \mathcal{G}$ and $H(p)$ is α-expansive then* $\mathrm{index}(q) = \mathrm{index}(p)$ *for all periodic $q \in H(p)$.*

Proof. Assume, by contradiction, that $\mathrm{index}(q) = 1$ for some $q \in \mathrm{Per}(f) \cap H(p)$. By a small C^1 perturbation we may create an intersection between $W^s(p)$ and $W^u_\epsilon(q)$ containing a C^1-circle or a compact arc γ. By Theorem 3.8 we have a heteroclinic connection and using blenders (see [**BD**]) we may assume that $\gamma \subset H(p)$. Let $x \in \gamma$, $x \neq p$ and $x \neq q$. Then there is $N > 0$ such that $f^N(x) \in W^s_\epsilon(p)$ and $f^{-N}(x) \in W^u_\epsilon(\mathcal{O}_f(q))$. By continuity there is $\delta > 0$ such that if $\mathrm{dist}(x, y) < \delta$ then $\mathrm{dist}(f^j(x), f^j(y)) < \epsilon$ for $|j| \leq N$. Hence, taking $y \in \gamma$ such that $\mathrm{dist}(x, y) < \delta$ we contradict α-expansiveness of f if $0 < \epsilon < \alpha/2$. □

COROLLARY 3.11. *Let $f \in \mathcal{G}$. If $q \in H(p)$ is a periodic point then there is no homoclinic tangency associated with q.*

Proof. Similar to that of Lemma 3.9. □

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LEMMA 3.12. *Let q be a hyperbolic periodic point in $H(p)$, $H(p)$ α-expansive, such that $W^s(p)$ cuts $W^u(q)$ and $W^s(q)$ cuts $W^u(p)$. If the intersections are transversal then there exists $\mathcal{U}(q, f) \subset \mathcal{U}(f)$ such that $q_g \in H(p_g)$ for all $g \in \mathcal{U}(q, f)$.*

Proof. Follows from transversality. □

4. *Proof of Theorem A*

In this section we will see that generically C^1-robustly expansive homoclinic classes $H(p)$ have a codimension-one dominated splitting $E \oplus F$, $\dim(F) = 1$. To prove this we will use that the stable and unstable subspaces of the periodic points in $H(p)$ have constant dimension and that the angle they form is bounded away from zero. Moreover, we will see that the eigenvalues of these periodic points are far from the unit circle. These results will allow us to use results of [**We**] that give the desired dominated splitting.

For technical reasons we split the proofs of the results about the eigenvalues into two cases: one in which there exists a periodic point in $H(p)$ with complex eigenvalues, and another in which for all q periodic in $H(p)$ its eigenvalues are real. In the first case we will include as 'complex' the case of real eigenvalues λ_1, λ_2 of a periodic point $q \in H(p)$ such that $|\lambda_1 - \lambda_2| \approx 0$ and also the case of a single eigenvalue with geometric multiplicity 1 and algebraic multiplicity 2 ($D_q f$ not diagonalizable). The reason for treating these eigenvalues as complex is that by a slight C^1-perturbation of the diffeomorphism we will obtain a diffeomorphism g with q a periodic point with (true) complex eigenvalues.

First we reject the possibility of having a periodic point $q \in H(p)$ such that Df_q has more than one contracting eigenvalue of modulus close to 1.

LEMMA 4.1. *With p and $H(p)$ as in Lemma 3.2 and $f \in \mathcal{G}$, for any periodic point q in $H(p)$ of index 2 we have that at most one of its contracting eigenvalues, $|\lambda_1| \le |\lambda_2| < 1$, is close to 1 in modulus. In particular there are not complex eigenvalues of modulus close to 1.*

Proof. Let $q \in H(p)$ be a periodic point of period τ and index 2 and assume first that Df_q has one eigenvalue of modulus greater than 1 and two complex eigenvalues of modulus close to 1. Since $q \in H(p)$, there is a sequence $\{x_n\} \subset W^s(p) \cap W^u(p)$ with $x_n \to q$ as $n \to \infty$. By the generic assumptions we have, there exists a point z_n such that $W^u_\epsilon(x_n) \cap W^s_\epsilon(q) = \{z_n\}$ and, as $W^s_\epsilon(q)$ is accumulated by $W^s(p)$, by the λ-lemma it follows that $z_n \in H(p)$. By forward iteration by f^τ, τ being the period of q, the points z_n accumulate in q. C^1-perturbing f we may assume that the eigenvalues of Dg^τ_q, where g is the perturbation of f, are $\rho e^{\pm i\theta\pi}$ with θ irrational, $\rho < 1$ but close to 1. As in Hopf's bifurcation we let $\rho \to 1$, obtaining an irrational rotation in a small circle $\mathcal{C}$ around q, contained in $W^s_\epsilon(q, f)$ such that the orbit of z_n by g accumulates in $\mathcal{C}$, which therefore is contained in $H(p_g)$. This contradicts expansiveness of $g|_{H(p_g)}$. The same is valid for the case in which both eigenvalues are real, of modulus close to 1 and of the same sign; after an arbitrarily small C^1-perturbation are reduced to the case in which we have complex eigenvalues close to 1. The case in which we have contracting eigenvalues one close to 1 and the other close to -1 can be treated analogously. □

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Let q be a hyperbolic periodic point homoclinically related to p and of period k, such that $Df_q^k/E^s(q)$ has two real eigenvalues λ_1, λ_2, with $1 > |\lambda_1| > (1-(1/n))^k$, $|\lambda_2| < (1-\delta)^k$, $0 < \delta < 1$ fixed. There exist two invariant lines ss and ws in $W_\epsilon^s(q)$ tangent to the directions given by the eigenvectors corresponding to λ_1 and λ_2 respectively. ss separates $W_\epsilon^s(q)$ into two regions B_1 and B_2. Let $\{z_i\}$, $\{z_i'\}$ be sequences of points in $W^u(p) \cap B_1$, and in $W^u(p) \cap B_2$ respectively.

LEMMA 4.2. *There exists g near f with q a periodic point of g of period k and such that g has two more periodic points in ws, q_1, q_2 arbitrarily close to one another, with q_1, q_2 separated by q in the line ws, and with the same period as that of q or with period double the period of q. Moreover the perturbation can be done such that $z_i \in W^u(q', g) \cap B_i$, $i = 1, 2$, and such that $g^{2kn}(z_1) \to q_1$ and $g^{2kn}(z_2) \to q_2$ when $n \to +\infty$.*

Proof. Let $\nu > 0$ be such that $|\lambda_1| = (1-\nu)^k$, and let v_i be the eigenvector associated with λ_i, $i = 1, 2$ and u the eigenvector associated with the eigenvalue μ of modulus greater than 1. Let $Q_q = \oplus_{i=0}^{k-1} TM_{f^i(q)}$ and let us consider a basis of $TM_{f^i(q)}$ formed by $\{Df_q^i(v_1), Df_q^i(v_2), Df_q^i(u)\}$. The union of these bases forms a basis of Q_q. Let $s \in [1-\nu, 1]$ and define $\varphi_i(s) : TM_{f^i(q)} \to TM_{f^i(q)}$, $i = 0, \dots, k-1$ by the matrix

$$\begin{pmatrix} s(1-\nu)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, for each s, $\varphi(s) = (\varphi_0(s), \dots, \varphi_{k-1}(s))$ is a linear map defined on Q. Exploiting that $TM_{f^k(q)} = TM_q$ the composition $F_i(s) = df_{f^i(q)} \circ \varphi_i(s)$ gives a linear map $G(s) = (F_0(s), \dots, F_{k-1}(s))$ defined on Q_q.

It follows that $G^k = G^k(1)$ has k eigenvalues of modulus 1 with eigenvectors $v_1, \dots, Df_q^{k-1}(v_1) \in Q_q$. Moreover, the eigenvalues of G^k are real and hence the eigenvalue associated to v_1 is ± 1. If it is 1 then $G^k(v_1) = v_1$, otherwise $G^{2k}(v_1) = v_1$. Clearly for all $\epsilon_0 > 0$ there exists $\nu > 0$ such that $\|G(s) - Df_q\| < \epsilon_0/10$ if $\nu > 0$ is sufficiently small. By Lemma 3.1, there exists $g(s) : M \to M$ such that $\|f - g(s)\| < \epsilon_0$ and $Dg(s)/Q_q = G(s)/Q_q$. In particular $Dg(1)_{|E_q^s} = G_{|E_q^s}$.

Let us recall how we proved Lemma 3.1. First it is assumed that M is embedded in $\mathbb{R}^N$ for certain N and that the Riemannian metric on M comes from that embedding and the usual metric of $\mathbb{R}^N$. This allows us to assume that $\exp_x(v) \in M$ and $v \in TM_x$ are both in $\mathbb{R}^N$. Then we choose $\delta > 0$ to guarantee that the exponential map $\exp : TM \to M$ when restricted to $\hat{B}_i = \{v \in TM_{f^i(q)}/\|v\| \leq \delta\}$ is near the identity and we define

$$\hat{f} : \hat{B}_i \to TM_{f^{i+1}(q)} \text{ by } \hat{f}(v) = \exp^{-1}(f(\exp(v))).$$

After that we choose a C^∞-function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x \in \mathbb{R}^+$: $0 \leq \sigma(x) \leq 1$, $\sigma(x) = 0$ if $x \geq \delta$, $\sigma(x) = 1$ if $x \leq \delta/4$, and $0 \leq |\sigma'(x)| \leq 2/\delta$ for all $x \in \mathbb{R}^+$. Choosing $\rho : TM \to \mathbb{R}^+$ by $\rho(v) = \sigma(\|v\|)$ and defining the auxiliary function $\hat{g}(v) = \rho(v)G(v) + (1-\rho(v))\hat{f}(v)$ we finally obtain that putting $g(x) = \exp(\hat{g}(\exp^{-1}(x)))$ if $x \in B_i = \exp(\hat{B}_i)$ and $g(x) = f(x)$ elsewhere, $g(x)$ is C^1-near f and $Dg/Q_q = G/Q_q$.

Let $\Delta = \max_{0 \leq i \leq 2k} \|G^i\|$ and $I = \{tv_1/0 \leq |t| \leq (\delta\Delta/4)\}$. It follows by construction of g that for all $u \in L = \exp_q(I)$, u is a periodic point of g of period equal to k.

Observe that L is normally hyperbolic for g. The plane $W^{cs}(q_g)$ defined locally by the strong stable foliation given by the direction defined by v_2 and L is accumulated by the stable manifold of p_g which is transversal to $W^u(q_g)$. Moreover, the unstable manifold of q_g cuts the stable manifold of p_g and so, by the λ-lemma, the unstable manifold of q_g accumulates C^1 in the unstable manifold of p_g. Hence there is a horseshoe Z associated with $W^u(q_g)$ and $W^{cs}(q_g)$. If $x \in Z$ then $x \in H(p_g)$. Forward iterations of the orbit of x will converge to a point $q' \in L$, an end-point of L. If we have points of $W^u(p)$ in both B_1 and B_2 we can obtain points converging to both end-points q_1, q_2. As $W^s(p)$ contains a disk converging in the C^1-topology to $W^{cs}(q_g)$, we have that q_1 and q_2 are in $H(p_g)$, concluding the proof of the lemma. □

4.1. *Estimates of the eigenvalues.* Let $A = Df_p^{\tau_p}$, $B = Df_q^{\tau_q}$ where p and q are hyperbolic periodic points homoclinically related of periods τ_p and τ_q respectively. In [**BDP**] it is proved that there are matrices $T_1 : T_pM \to T_qM$ and $T_2 : T_qM \to T_pM$, called *transition matrices*, that give sense to the composition of A and B. Moreover the corresponding contracting and expanding sub-spaces of T_pM and T_qM are preserved. Using these transition matrices we may compose a finite number of times

$$A^{m_1} \circ T_2 \circ B^{n_1} \circ T_1 \circ A^{m_2} \circ T_2 \circ B^{n_2} \cdots ,$$

obtaining a matrix C which represents the differential map of a periodic point $r \in H(p)$ that spends m_1 iterates near the orbit of p, then n_1 iterates near $\mathcal{O}(q)$, then m_2 iterates near $\mathcal{O}(p)$ and so on. Taking m_j great enough we may assume that there is an iterate of r as close as we wish to p. Moreover the period l of r is about $m_1 + n_1 + m_2 + n_2 + \cdots$. Furthermore, the eigenvalues of r are determined by those of p and q. More precisely, when m_j and n_j grow without bounds the influence of T_1 and T_2 becomes negligible and the differential map Df_r^l is given approximately by

$$A^{m_1} \circ B^{n_1} \circ A^{m_2} \circ B^{n_2} \circ \cdots ,$$

see [**BDP**].

We shall use these transition matrices in the proof of Propositions 4.3 and 4.4 below.

Let us assume $f \in \mathcal{G}$ and that for any q hyperbolic periodic point in $H(p)$ it holds that its eigenvalues are real. As usual we assume that index$(p) = 2$. Observe that we are saying that for a C^1-neighborhood $\mathcal{N}$ of f the result applies to all eigenvalues of all periodic points of g that are in $H(p_g)$.

PROPOSITION 4.3. *There exists $0 < \delta < 1$ such that for any hyperbolic periodic point q in $H(p)$ of period τ it holds that the contracting eigenvalues λ_1, λ_2 of Df_q, $0 < |\lambda_1| < |\lambda_2| < 1$ satisfy $|\lambda_2|^\tau < (1-\delta)^\tau$, where τ is the period of q.*

Proof. First observe that if we have a sequence q_m of hyperbolic periodic points such that both contracting eigenvalues λ_1, λ_2 verify that $(1 - 1/m)^{\tau_m} < |\lambda_j(q_m)| < 1$, $j = 1, 2$ we should have that the period of q_m goes to infinity with m or we will obtain as a limit point a non-hyperbolic periodic point. In the case in which the period goes to infinity we will increase the compressing properties of one of the eigenvalues, say $\lambda_1(q_m)$, perturbing along the orbit of q_m multiplying in the direction of $Df_{f^j(q)}v_1$, where v_1 is the eigenvector

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corresponding to λ_1, by $(1-1/m)$. If $1/m$ is small enough we can apply Lemma 3.1, obtaining a diffeomorphism g for which Dg_{q_m} has an eigenvalue $\lambda_1(q_m, g)$ such that $|\lambda_1(q_m, g)| < (1-1/m)^{\tau_m}$. We then may apply the arguments below.

Taking into account the preceding paragraph we assume that there exists $\{q_m\}$ a sequence of hyperbolic periodic points in $H(p)$ such that

$$(1-1/m)^{\tau_m} < |\lambda_2(q_m)|^{\tau_m} < 1$$

where τ_m is the period of q_m. From that sequence let us pick q a hyperbolic periodic point in $H(p)$ of period τ, such that $1 > |\lambda_2(q)|^\tau = |\lambda_2|^\tau = (1-\nu)^\tau > (1-1/m)^\tau$.

Assume that q has positive eigenvalues and let B be the matrix of Df_q^τ which, in a convenient basis, can be written as

$$B = \begin{pmatrix} (1-\nu)^\tau & 0 & 0 \\ 0 & \lambda_1^\tau & 0 \\ 0 & 0 & \mu^\tau \end{pmatrix}.$$

Let v_1 be the eigenvector corresponding to λ_1^τ, the stronger contracting eigenvalues and let v_2 be the eigenvector corresponding to $(1-\nu)^\tau$. In $W^s_\epsilon(q)$ is defined a local foliation by arcs transverse to v_2 and of the direction of v_1. At q there are defined lines ss and ws tangent to v_1 and v_2 respectively. The arc ss separates $W^s_\epsilon(q)$ into two regions (half-disks) B_1 and B_2.

If there are points of $H(p)$ in both regions B_1 and B_2 different from q then we will have points z_1 and z_2 in different separatrices of the line that under forward iterates will converge to q. In this case by Lemma 4.2 there is g near f with q a periodic point of g and such that g has two more periodic points in ws, q_1, q_2, arbitrarily close one to another, with q_1, q_2 separated by q, and with the same period, that of q or with period double the period of q, and such that $g^{\tau n}(z_1) \to q_1$ and $g^{\tau n}(z_2) \to q_2$ when $n \to +\infty$. Observe that by hypothesis $W^s(q)$ intersects transversely $W^u(p)$ and $W^s(p)$ intersects transversely $W^u(q)$. By the λ-lemma there exists a horseshoe C associated with q and hence we have that $W^s(q) \cap W^u(q)$ is contained in $H(p)$, in fact $H(q) = H(p)$. There is a periodic point R associated with the horseshoe C close to a point $S \in W^s_\epsilon(q) \cap W^u(p)$. By forward iterations by f we obtain points $f^{k\tau}(S)$ closer to q than S. Let R_1 be a hyperbolic periodic point close to $f^{k\tau}(S)$. Generically, the point $S \notin ss$ but $S \in B_1 \cup B_2$. If $S \in ss$ we may slightly perturb f to obtain $S \notin ss$. Hence the same is true for R and R_1, that is, $R,\ R_1 \notin ss$.

Using transition matrices we may compose a finite number of times Df_q with Df_R and Df_{R_1} in order to obtain a hyperbolic periodic point $T \in H(p)$ arbitrarily close to $f^{k\tau}(S)$, of period τ_T, and with matrix $Df_T^{\tau_T}$ near

$$D = \begin{pmatrix} (1-\nu)^{\tau_T} & 0 & 0 \\ 0 & \lambda_1^{\tau_T} & 0 \\ 0 & 0 & \mu^{\tau_T} \end{pmatrix}.$$

We analyze the relative positions between q and T. As the value of the normalized eigenvalues are about the same for q and T, the size of their local stable and unstable manifolds are also the same. We can split (and do) the local stable manifold of T with the strong local stable manifold $ss(T)$ in two regions $B_1(T)$ and $B_2(T)$ where there are segments of $ws(T)$.

We may assume that $f^{k\tau}(S)$ is very close to q and therefore T and q are close, their stable directions are almost parallel, and the same is true with respect to the unstable directions. Thus $W^u_\epsilon(q)$ intersects $W^s_\epsilon(T)$, say at y, and $W^u_\epsilon(T)$ intersects $W^s_\epsilon(q)$ at y'. Also the unstable manifold $W^u(p)$ accumulates in these points. Close to the point $R \in W^s_\epsilon(q) \cap W^u(p)$ there is a point $R' \in W^s_\epsilon(T) \cap W^u(p)$ located in $B_1(T)$ if R is in B_1 and in $B_2(T)$ if R is in B_2 (we assume that locally we can orient ws and $ws(T)$ in the same way). Forward iterations of R' and y' go to the orbit of T and the same is true for y and R with respect to q. Thus T is close to q but the relative positions of y and y' are different. That is, if $y \in B_1$ then $y' \in B_2(S)$. It follows that either q or T is accumulated from both sides of $W^s_\epsilon(\cdot)\backslash ss(\cdot)$. Here $\cdot$ means either q or T. By Lemma 4.2 we may perturb to obtain new periodic points q_1 and q_2 in $ws(\cdot)$ as close to one another as we wish, both of the same period, contradicting expansiveness, and we finish the proof of Proposition 4.3. □

Let us assume now that the hyperbolic fixed point p has either complex eigenvalues or two real eigenvalues λ_1, λ_2 of modulus less than 1 and such that $\lambda_1 - \lambda_2 \approx 0$ or that Df_p is not diagonalizable (recall that we are assuming that $\operatorname{index}(p) = 2$).

PROPOSITION 4.4. *There exists* $0 < \delta < 1$ *such that for every hyperbolic periodic point* $\hat{q}$ *in* $H(p)$ *it holds that if* λ *is a contracting eigenvalue of* $Df_{\hat{q}}$ *then* $|\lambda| < (1-\delta)^\tau$, *where* τ *is the period of* $\hat{q}$.

Proof. First, if Df_p has real contracting eigenvalues λ_1, λ_2, such that $\lambda_1 - \lambda_2 \approx 0$ then under a small C^1-perturbation we obtain true complex contracting eigenvalues. The same is true if Df_p is not diagonalizable. So, without loss of generality, we assume that Df_p has contracting eigenvalues $\lambda_1, \lambda_2 \notin \mathbb{R}$. Arguing by contradiction let us assume that there exists a sequence q_N of hyperbolic periodic points in $H(p)$ such that they have contracting eigenvalues λ_N verifying $(1-(1/N))^{\tau_N} < |\lambda_N| < 1$. Both contracting eigenvalues satisfy this condition and so if $\{\tau_N\}$ has a bounded infinite subsequence, say by T, then we should have a periodic point of period less than or equal to T with two eigenvalues of modulus 1, contradicting Lemma 4.1. Otherwise the sequence τ_N goes to infinity and only one of the contracting eigenvalues is weak and therefore both are real. Assume that the eigenvalues at p are $\rho e^{i\phi}$ and $\rho e^{-i\phi}$ where ϕ defines an irrational rotation (i.e. $\phi \neq (m/n)\pi$ with $m, n \in \mathbb{Z}$) and $0 < \rho < 1$. Let $A_s = Df_{p|_{E^s_p}}$ and $B_s = Df_{\hat{q}}|_{E^s_{\hat{q}}}$. We complete these matrices with the corresponding eigenvalues greater than 1 in modulus in the corresponding unstable subspaces in order to have them acting in the whole tangent space and we call them A and B respectively. In a convenient basis we have

$$A = \begin{pmatrix} \rho\cos\phi & \rho\sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1-\nu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$$

where, assuming that the eigenvalues are positive, $0 < |\nu| < 1/n$.

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Now we use transitions matrices to obtain a periodic point q with differential map $C = A^m \circ T_2 \circ B^n$. Choose $n = m^2$ and let m be sufficiently big that the influence of T_2 is negligible. In this case, the matrix C is near $A^m \circ B^n$. Take m such that $m\phi \approx 0 \bmod (2\pi)$. Recall that ϕ is irrational with respect to π. Then A^m is close to the matrix given by

$$\begin{pmatrix} \rho^m & 0 & 0 \\ 0 & \rho^m & 0 \\ 0 & 0 & \mu_1^m \end{pmatrix}$$

and B^n is given by

$$B^n = \begin{pmatrix} (1-\nu)^n & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \mu_2^n \end{pmatrix}.$$

The composition of the two matrices above yields C with contracting eigenvalues $c_1 = \rho^m(1-\nu)^n$ and $c_2 = \rho^m \lambda^n$. The period of q is $m+n$ and hence

$$c_1^{1/(m+n)} = \rho^{m/(m+n)}(1-\nu)^{n/(m+n)}.$$

As $m \to \infty$ we obtain that $c_1^{1/(m+n)} \to (1-\nu)$, taking into account that $n = m^2$. Thus q behaves like $\hat{q}$ and we may assume that the eigenvalues of Df_q are λ, $1-\nu$ and μ_1, with $0 < \nu < 1/n$. But q has the advantage that it has iterates as close as we wish to p (just take m big), and therefore we will have that $W^u(p)$ intersects $W^s(q)$ and $W^s(p)$ intersects $W^u(q)$. Thus they are homoclinically related. By assumption Df_p has contracting complex eigenvalues, a homoclinic point in $W^s_\epsilon(p)$ will describe a spiral in $W^s_\epsilon(p)$ which projects via the corresponding segments of the unstable manifold $W^u(p)$ (almost parallel to $W^u_\epsilon(p)$ by the λ-lemma) into $W^s_\epsilon(q)$. This implies that $W^u(p) \cap W^s_\epsilon(q) \subset H(p)$ accumulates in both separatrices u^+ and u^- of $W^s_\epsilon(q)$ corresponding to the weak eigenvalue $(1-\nu)$. That is, there are points $z_1 \in H(p) \cap u^+$ and $z_2 \in H(p) \cap u^-$, z_1, z_2 different from q.

Analogously to what we have done in Proposition 4.3, we may perturb f to obtain g with q a periodic point of g and such that g has two more periodic points in u, q_1, q_2, arbitrarily near to one another, with q_1, q_2 separated by q, and with the same period as q or with period double the period of q, and such that $g^{\tau n}(z_1) \to q_1$ and $g^{\tau n}(z_2) \to q_2$ when $n \to +\infty$, see Lemma 4.2.

The existence of q_1, q_2 as above contradicts robust expansiveness, concluding the proof of Proposition 4.4. □

Remark 4.5. It follows from the proof of Proposition 4.4 that it does not matter if the point with complex eigenvalues is another point p' different from p as long as $H(p) = H(p')$.

PROPOSITION 4.6. *Let $f \in \mathcal{G}$ and p and $H(p)$ be as in Lemma 3.2. Let μ be the (real) expanding eigenvalue of a periodic point $q \in H(p)$ of period k. Then there exists $\delta > 0$ such that $|\mu|^k > (1+\delta)^k$.*

Proof. Similar to those of Propositions 4.3 and 4.4. □

Remark 4.7. Propositions 4.3, 4.4 and 4.6 imply in particular that we cannot have periodic points in $H(p)$ homoclinically related to p and which are not hyperbolic whenever $H(p)$ is C^1-robustly expansive.

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4.2. *Estimates of the angles.* We define as in [**We**] wandering rates for linear contractions and expansions.

Definition 4.1. Let V be a finite-dimensional normed linear space and $A : V \to V$ a linear contraction or expansion. Given $v \neq 0$, a ball $B(v, r)$ is A-wandering if $A^n(B(v, r)) \cap B(v, r) = \emptyset$ for all integer $n \neq 0$. Define the A-wandering rate of v with respect to A as

$$\omega(v, A) = \sup\{\lambda > 0 \mid B(v, \lambda\|v\|) \text{ is } A\text{-wandering}\}$$

and the wandering rate $\omega(A)$ of A to be

$$\omega(A) = \inf\{\omega(v, A) \mid v \neq 0\}.$$

The following lemma is in [**We**]. It does not require q to be hyperbolic.

LEMMA 4.8. *Let f be a diffeomorphism and $\epsilon > 0$ be a real number. For any periodic point q of f of period τ, if there is a Df^τ-contracting subspace V^s and a Df^τ-expanding subspace V^u of T_qM such that*

$$\frac{\angle(V^s, V^u)}{\min\{\omega(Df^\tau|_{V^s}), \omega(Df^\tau|_{V^u})\}} < \epsilon$$

then there is g, C^1 ϵ-close to f, that differs from f only in an arbitrarily small ball near q but disjoint from its orbit such that g exhibits homoclinic tangency associated with $q_g = q$.

Proof. See [**We**, Lemma 4.2]. □

PROPOSITION 4.9. *For $f \in \mathcal{G}$ there exist m_0 and $\gamma > 0$ such that for all $q \in H(p)$ of period τ greater than m_0 it holds that $\angle(E^s_q, E^u_q) > \gamma$.*

Proof. It suffices to prove that for any one-dimensional subspace $L^s \subset E^s_q = E^s$, if the period of q is greater than m_0 then $\angle(L^s, L^u) > \gamma$ for certain $\gamma > 0$, where $E^u_q = L^u$. We can assume that $Df^k_q|_{E^s}$ has real eigenvalues. Otherwise, using Lemmas 3.1 and 3.12, we obtain a nearby diffeomorphism g such that q is a periodic point of g with period τ and such that for certain iterate k multiple of τ, $Df^k_q|_{E^s}$ has real eigenvalues. The proof is similar to that of [**We**, Lemma 5.1] or [**PS1**, Lemma 2.2.2].

Let $\epsilon > 0$ be so small that any C^1 ϵ-perturbation of f is in the neighborhood $\mathcal{U}(f)$, and $C_0 = \sup\{\|Dg_x\|; g \in \mathcal{U}(f), x \in M\}$ and let $\epsilon'_0 = \epsilon_0/C_0$ where ϵ_0 is given by Lemma 3.1. We take $\gamma < \epsilon/3$ and $m_0 > 12/\epsilon\epsilon'_0$. Assume without loss of generality that $\omega(Df^\tau|_{L^u}) < \omega(Df^\tau|_{L^s})$. Suppose that the proposition does not hold and $\beta = \angle(L^s, L^u) < \gamma$. Fix $\delta = \beta\epsilon'_0/4$. Let $T_j : Df^j(T_qM) \to Df^j(T_qM)$ be a linear map such that $T_j|_{L^s} = (1-\delta)\,\mathrm{id}$ and $T_j|_{L^u} = (1+\delta)\,\mathrm{id}$, where id stands for the identity map. In the rest of the space T_j acts as the identity. Hence T_j compresses L^s and stretches L^u. The norm of $T_j - \mathrm{id}$ is (see [**Ma2**]) less than

$$\begin{aligned}\|T_j - \mathrm{id}\| &\le \frac{1+\beta}{\beta}(\|(T_j - \mathrm{id})|_{L^s}\| + \|(T_j - \mathrm{id})|_{L^u}\|)\\ &= \frac{1+\beta}{\beta}2\delta = \frac{1+\beta}{\beta}2\beta\epsilon'_0/4 \le \frac{1+\beta}{2}\epsilon'_0 < \epsilon'_0.\end{aligned}$$

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Hence we may apply Lemma 3.1, and assume that $T_j \circ Df_{f^j(q)} = Dg_{g^j(q)}$ (i.e. the orbit of q remains unchanged). Observe that we have not changed the stable and unstable subspaces of q and so $q \in H(p)$. Moreover, if $|\sigma| > 1$ is the eigenvalue of $Df_q^\tau|_{L^u}$ then the eigenvalue of $Dg_q^\tau|_{L^s}$ is $\sigma_1 = (1+\delta)^\tau\sigma$.

Since L^u is one-dimensional the wandering rate of Dg_q^τ restricted to L^u is $\omega_u = (\sigma_1 - 1)/(\sigma_1 + 1)$. Moreover, we still have $\omega(Dg^\tau|_{L^u}) < \omega(Dg^\tau|_{L^s})$ since we have compressed the stable sub-bundle by $(1-\delta)$ while we have stretched L^u by $(1+\delta)$. If λ is the eigenvalue of Df_q^τ restricted to L^s then the condition on the wandering rates is equivalent to $|\lambda\sigma| < 1$ and this still holds if λ is multiplied by $(1-\delta)$ and σ is multiplied by $(1+\delta)$. If $\sigma_1 \geq 2$ then $(\sigma_1 - 1)/(\sigma_1 + 1) \geq \frac{1}{3}$. Hence, in this case,

$$\frac{\beta}{\min\{\omega(Dg^\tau|_{L^s}), \omega(Dg^\tau|_{L^u})\}} = \frac{\beta}{\omega_u} = \frac{\beta(\sigma_1+1)}{\sigma_1 - 1} < \frac{\gamma(\sigma_1+1)}{\sigma_1 - 1} < \epsilon.$$

On the other hand, if $\sigma_1 < 2$ then

$$\frac{\sigma_1 - 1}{\sigma_1 + 1}\epsilon > \frac{\sigma_1 - 1}{3}\epsilon = \frac{(1+\delta)^\tau\sigma - 1}{3}\epsilon \geq m_0\delta\sigma\frac{\epsilon}{3} \geq \gamma \geq \beta.$$

Hence, by Lemma 4.8 we can create a homoclinic tangency between $W^s(q_g)$ and $W^u(q_g)$, contradicting expansiveness since $q \in H(p_g)$. □

COROLLARY 4.10. *Let $f \in \mathcal{G}$. There exists $\gamma > 0$ such that for any periodic point q in $H(p)$ it holds that $\angle(E_q^s, E_q^u) > \gamma$.*

Proof. By Proposition 4.9 this is true for q periodic of period greater than m_0. On the other hand, due to the expansive properties of $H(p)$ there are only finitely many periodic points of period less than or equal to m_0 on $H(p)$. The result follows. □

4.3. *Constructing a dominated splitting.*

LEMMA 4.11. *Let V be an inner product space with* $\dim(V) \geq 2$. *For any $\epsilon > 0$ there is $N_0 \geq 2$ such that if a contracting linear map $A : V \to V$ has norm $\|A\| \geq N_0$ then there is a linear map $T : V \to V$ with $\|T - \mathrm{id}\| < \epsilon$ such that $T \circ A$ has its largest eigenvalue in absolute value greater than or equal to 1. In fact it may be taken to be exactly of modulus 1.*

Proof. See [**We**, Lemma 3.3]. □

LEMMA 4.12. *Assume that $f \in \mathcal{G}$, p and $H(p)$ as in Lemma 3.2. Then there are a C^1-neighborhood $\mathcal{V}(f) = \mathcal{V}$ of f, a number $K \geq 2$ and a number $0 < \lambda < 1$ such that for any hyperbolic periodic point $q \in H(p_g)$ of any period τ and any $g \in \mathcal{V}$ we have, for all $n \geq 0$,*

$$\|Dg^{-\tau n}|_{E^u(q,g)}\| \leq K\lambda^{\tau n}, \tag{1}$$

$$\|Dg^{\tau n}|_{E^s(q,g)}\| \leq K\lambda^{\tau n}. \tag{2}$$

Proof. Observe that $\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K$ is trivial since $E^u(q,g)$ is one-dimensional. Moreover, Proposition 4.6 proves

$$\|Dg^{-\tau}|_{E^u(q,g)}\| \leq K\lambda^\tau.$$

And so we have proved (1).

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To prove (2) we proceed as follows. By Proposition 4.9 and Corollary 4.10 we have that $\angle(E^u_q, E^s_q) > \gamma > 0$. Let $\epsilon > 0$ be so small that any C^1 ϵ-perturbation of f is in the neighborhood $\mathcal{U}(f)$. Recall that any $g \in \mathcal{U}(f)$ is α-expansive in $H(p_g)$.

Let $C_0 = \sup\{\|Dg_x\|; g \in \mathcal{U}(f), x \in M\}$ and $\epsilon'_0 = \epsilon_0/C_0$ where ϵ_0 is given by Lemma 3.1. This gives a neighborhood $\mathcal{V}(f)$ of f which we may assume is contained in $\mathcal{U}(f)$. Let N_0 be that given in Lemma 4.11 to achieve that the perturbation T giving the eigenvalue equal to 1 in modulus satisfies $\|T - \mathrm{id}\| < \epsilon'_0$. Take $K \geq \max\{2, N_0\}$ and let $\delta' = \delta/C_0$ where $\delta > 0$ is that given by Lemma 3.1, with respect to $\mathcal{V}(f)$. Let $\delta_1 > 0$ be so small that $((1+\gamma)/\gamma)(2\delta_1) \leq \delta'$. Finally let $\lambda = 1/(1+\delta_1)$.

We claim that $\|Dg^\tau|_{E_s(q,g)}\| \leq K$. Arguing by contradiction assume that there is g and $q \in H(p_g)$ is hyperbolic periodic point for g of period τ such that $\|Dg^\tau|_{E_s(q,g)}\| > K$. By Lemma 4.11 there exists $T : E^s(q,g) \to E^s(q,g)$ such that $\|T - \mathrm{id}\| < \epsilon'_0$ and such that $T \circ Df_{|E^s(q,g)}$ has its largest eigenvalue in absolute value equal to 1. We extend T to all of T_qM as the identity in $E^u(q,g)$. As the angle between $E^s(q,g)$ and $E^u(q,g)$ is bounded away from zero, we can assume $\|T - \mathrm{id}\| < \epsilon'_0$. By Lemma 3.1 we can think that $T \circ Dg^\tau_q$ is $Dg^\tau_{1_q}$. There is an isotopy $G(t), t \in [0,1]$, close to the identity, connecting $g = G(0)$ to $g_1 = G(1)$. For values close to $t = 1$, $G(t)$ has a contracting eigenvalue of modulus close to 1 associated with a hyperbolic periodic point q_t, contradicting Propositions 4.3 and 4.4. Thus $\|Dg^\tau|_{E^s(q,g)}\| \leq K$.

Next we prove that $\|Dg^\tau|_{E^s(q,g)}\| \leq K\lambda^\tau$. Given q periodic in $H(p)$ of period τ define $T_j : Dg^j(T_qM) \to Dg^j(T_qM)$ by $T_j|_{Dg^j(E^s(q,g))} = (1+\delta_1)\mathrm{id}$. Thus T_j stretches the E^s direction. As $\angle(E^u, E^s) > \gamma$ for all periodic point q, we have that

$$\|T_j - \mathrm{id}\| \leq \frac{1+\gamma}{\gamma}(\delta_1) \leq \delta'.$$

Thus we may assume, using again Lemma 3.1, that $T_j \circ Dg_{g^j(q)} = Dg1_{g_1^j(q)}$. We claim that $E^s(q, g_1)$ is contracted by Dg_1^τ. Otherwise we may choose $t \in [0,1]$ to get $T_j(t)$ that expands E^s by a factor $(1+t\delta)$, obtaining for suitable t an eigenvalue of modulus arbitrarily close to 1 associated with an eigenvector in E^s, and this contradicts Propositions 4.4 and 4.3, proving the claim. Observe that since we do not alter the E^u direction and $\angle(E^u, E^s) > \gamma$ we also get that $E^u(q, g_1)$ is expanded by Dg_1^τ. Hence, applying (1) to g_1 we obtain

$$\|Dg_1^\tau|_{E^s(q,g_1)}\| = (1+\delta_1)^\tau \|Dg^\tau|_{E^s(q,g)}\| \leq K.$$

From this inequality we obtain

$$\|Dg^\tau|_{E^s(q,g)}\| \leq \frac{1}{(1+\delta_1)^\tau} K = \lambda^\tau K,$$

concluding the proof of (2). The proof of Lemma 4.12 is complete. □

THEOREM 4.13. *Let us assume that there are numbers $\gamma > 0$, $K \geq 2$ and $0 < \lambda < 1$ such that $\angle(E^s, E^u) \geq \gamma$ for any periodic point $q \in H(p_g)$ of $g \in \mathcal{V}(f)$. Assume furthermore that*

$$\|Df_q^\tau|_{E^s}\| \leq K \quad \text{and} \quad \|Df_q^{-\tau}|_{E^s}\| \leq K$$

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for any periodic point q for any period τ and that either

$$\|Df_q^{\tau}|_{E^s}\| \leq K\lambda^{\tau}$$

or

$$\|Df_q^{-\tau}|_{E^u}\| \leq K\lambda^{\tau}.$$

Then there exists a dominated splitting $E \oplus F$ in the homoclinic class $H(p)$ with $\dim(E) = 2$ *and* $\dim(F) = 1$. *If $x = q$ is a periodic point then $E(q) = E^s(q)$ and $F(q) = E^u(q)$.*

Proof. It follows from Lemma 4.12 as in [**We**, Proposition 3.7] or [**PS1**, Lemma 2.0.1]. □

The proof of Theorem A is complete.

5. *Proof of Theorem B*

In this section we prove that generically $H(p)$ is hyperbolic. We shall follow the steps of [**Ma3**]. In that article, Mañé used γ-strings (see Definition 5.1 below) to prove that, if in a dominated splitting $E \oplus F$, F is a uniform expanding bundle, then either E is a uniform contracting bundle or there is a periodic point q such that Df_q contracts in the period with a rate arbitrarily close to 1. In our case we cannot assume *a priori* that F is an expanding bundle. So, we replace the former property by the fact that points in a neighborhood of $H(p)$ have local unstable manifolds with both separatrices of diameter bounded away from zero.

5.1. *Hyperbolic times.*

Definition 5.1. We say that a pair of points $(x, f^n(x))$ contained in $H(p)$, $n > 0$, is a γ-string, $0 < \gamma < 1$, if

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| \leq \gamma^n.$$

We say that $(x, f^n(x))$ is a uniform γ-string if $(x, f^k(x))$ is a γ-string for all $0 \leq k \leq n$.

LEMMA 5.1. (Pliss's lemma) *Let $0 < \gamma_1 < \gamma_2 < 1$ and $(x, f^n(x))$ be a γ_1-string. There exist a positive integer $N = N(\gamma_1, \gamma_2, f)$, $c = c(\gamma_1, \gamma_2, f) > 0$ such that if $n \geq N$ then there exist numbers*

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_l \leq n$$

such that $(f^{n_r}(x), f^n(x))$ are uniform γ_2-strings for all $r = 1, 2, \ldots, l$, with $l \geq cn$. That is,

$$\prod_{i=n_r}^{j} \|Df/E(f^i(x))\| \leq \gamma_2^{j-n_r} \quad r = 1, 2, \ldots, l; n_r \leq j \leq n.$$

Proof. The proof of this lemma can be found in [**Pl1, Pl2**]. See also [**Ma3**, Lemma II.3]. □

Remark 5.2. We call the elements of the sequence $n_1, n_2, \ldots$ *hyperbolic times* for x.

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COROLLARY 5.3. *Given* $0 < \gamma_1 < \gamma_2 < 1$ *and* $x \in H(p)$ *such that for some* m

$$\prod_{i=0}^{n} \|Df/E(f^i(x))\| \leq \gamma_1^n \quad \text{for all } n \geq m$$

there exists a sequence $0 \leq n_1 < n_2 < \dots$ *such that*

$$\prod_{i=n_r}^{j} \|Df/E(f^i(x))\| \leq \gamma_2^{j-n_r} \quad \text{for all } j \geq n_r,\ r = 1, 2, \dots.$$

Definition 5.2. We say that $(x, f^n(x))$ is an (N, γ)-obstruction, $0 < \gamma < 1$, $0 < N \leq n$, if $(f^m(x), f^n(x))$ is not a γ-string for all $n - N \leq m \leq n$.

LEMMA 5.4. *Let us take* $0 < \gamma_0 < \gamma_3 < 1$ *and* $0 < \gamma_2 < \gamma_3$ *and let* $(x, f^n(x))$ *be a* γ_0*-string. Moreover let* $0 \leq n_1 < n_2 < \dots < n_k \leq n$ *be the set of integers such that* $(f^{n_r}(x), f^n(x))$ *is a uniform* γ_3*-string and let* $N = N(\gamma_2, \gamma_3, f)$ *be the positive integer given by Lemma 5.1. Then, for all* $1 \leq i < k$ *either* $n_{i+1} - n_i \leq N$ *or* $(f^{n_i}(x), f^{n_{i+1}}(x))$ *is an* (N, γ_2)*-obstruction. Moreover, either* $n_1 \leq N$ *or* $(x, f^{n_1}(x))$ *is an* (N, γ_2)*-obstruction.*

Proof. See [**Ma3**, Lemma II.4]. □

5.2. *Unstable manifolds have uniform size.* Let $f : M \to M$ be a C^2-diffeomorphism and assume that there is a dominated splitting $E \oplus F$ in $H(p)$. Moreover assume that f is Kupka–Smale. This splitting can be uniformly extended to a neighborhood $V(H(p))$ of $H(p)$ (see [**PS1**, p. 985]). We point out that this extension is not necessarily invariant. Nevertheless points whose f-orbits stay in $V(H(p))$, belonging or not to $H(p)$, have a dominated splitting.

For points x in $V(H(p))$ we may define $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(x)$ ϵ-center-stable manifolds and ϵ-center-unstable manifolds respectively such that for $y \in W^{cs}_\epsilon(x)$ we have $T_y(W^{cs}_\epsilon(x)) = E(y)$ and for $y \in W^{cu}_\epsilon(x)$ we have $T_y(W^{cu}_\epsilon(x)) = F(y)$. The local center-stable manifold is a disk and the local center-unstable manifold is an arc. These manifolds vary continuously in $H(p)$.

In [**PS1**, Lemmas 3.3.1 and 3.3.2] there is proved the following result relating to center-unstable manifolds.

LEMMA 5.5. *For all* $\epsilon \leq \delta_0$ *there exists* $\gamma = \gamma(\epsilon)$ *such that for any* x *with orbit in a neighborhood of* $H(p)$ *and for all* $n \geq 0$*:*

(1) $f^{-n}(W^{cu}_\gamma(x)) \subset W^{cu}_\epsilon(f^{-n}(x))$;

(2) *for every* $\gamma \leq \gamma(\delta_0)$*, either*

  (a) $\ell(f^{-n}(W^{cu}_\gamma(x))) \to 0$ *when* $n \to \infty$;

  (b) *or* $x \in W^{cu}_\gamma(p')$ *for some* $p' \in \mathrm{Per}(f_{/H(p)})$ *such that* $p' \in W^{cu}_\gamma(x)$ *and there exists another periodic point* $q \in \mathrm{Cl}(W^u_\gamma(p'))$ *which is a sink or a non-hyperbolic periodic point.*

Remark 5.6. As a matter of fact Lemma 5.5 was proved for diffeomorphisms on surfaces [**PS1**] but it can be extended to the codimension-one case in the same way.

COROLLARY 5.7. *If* $x \in W^s(p) \cap W^u(p)$ *then* $W^{cu}_\epsilon(x) = W^u_\epsilon(x)$.

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Proof. By expansiveness if $x \in W^s(p) \cap W^u(p)$ then $W^s(p)$ intersects $W^u(p)$ transversely and $W^{cu}_\epsilon(x) \subset W^u(p)$. Therefore $W^{cu}_\epsilon(x)$ is a true unstable manifold and the result follows. □

Let $W^{u,+}_\epsilon(p')$ and $W^{u,-}_\epsilon(p')$ denote the two connected components of $W^u_\epsilon(p') \setminus \{p'\}$ with the point p' added. We call them the *local unstable separatrices* of p'.

LEMMA 5.8. *There exist $\eta > 0$ and a neighborhood $V'(H(p)) \supset H(p)$, $V'(H(p)) \subset V(H(p))$ such that* $\operatorname{diam}(W^{u,\sigma}_\epsilon(p')) > \eta$ *for any periodic point p' such that $\mathcal{O}(p') \subset V(H(p))$, $\sigma = +, -$.*

Proof. Assume that the result is not true. Then there exists a sequence p_n accumulating in $H(p)$ such that at least one of its separatrices (say $W^{u,+}_\epsilon(p_n)$) has diameter going to zero with $n \to \infty$ and hence Lemma 5.5 together with the fact that f is Kupka–Smale gives us that there is a sink $q_n \in \operatorname{Cl}(W^{u,+}_\epsilon(p_n))$.

Using again that f is Kupka–Smale we obtain that the period of p_n goes to infinity with n.

Since $\operatorname{diam}(W^{u,+}_\epsilon(p_n)) \xrightarrow[n\to\infty]{} 0$, $\{q_n\}$ also accumulates in $H(p)$ and the same is valid for the orbits of both q_n and p_n.

Using [**PS1**, Lemma 3.2.2 and Corollary 3.3] we may assume that $W^s_\epsilon(p_n)$ has size (see Definition 1.1) greater than some fixed $\delta_2 > 0$. Moreover, the same is true for all points in $\operatorname{Cl}(W^{u,+}_\epsilon(p_n))$.

Let z be an accumulation point of $\{q_n\}$. The point z belongs to $H(p)$, and as $q_n \in \operatorname{Cl}(W^{u,+}_\epsilon(p_n))$ we conclude that the local stable manifold of z, tangent to E, has size bounded away from zero. Moreover, the separatrix $W^{cu,+}_\epsilon(z)$ from the side in which z is accumulated by q_n, is in fact part of the local unstable manifold of z. Otherwise z would be in the center-unstable manifold of a saddle-periodic point p' with a sink $q' \in \operatorname{Cl}(W^{u,+}_\epsilon(z))$, see Lemma 5.5. But if $z = p'$ then by forward iteration points like q_n will fall in the basin of attraction of q', contradicting that q_n itself is periodic. Moreover z cannot be between p' and q' in $W^{cu}_\epsilon(p')$. Otherwise $q' \in H(p)$, which is absurd. Thus $W^{cu,+}_\epsilon(z)$ is part of the local unstable manifold of z.

Let q_{n_1} be a sink close to z. Its strong local stable manifold, $W^{ss}_\epsilon(q_{n_1})$, (tangent to E) cuts the local unstable manifold of z. And the local center-unstable manifold of q_n cuts the local stable manifold of z. As q_{n_1} is a sink, there should exist a saddle-type periodic point p'_{n_1} in the center-unstable manifold of q_{n_1} such that $p'_{n_1} \in H(p)$. For, $W^{cu}_\epsilon(q_{n_1}) \cap W^s_\epsilon(z) \neq \emptyset$ and $W^{ss}_\epsilon(q_{n_1}) \cap W^u_\epsilon(z) \neq \emptyset$. But $z \in H(p)$ implies that it is accumulated by points $x \in W^s(p) \cap W^u(p)$. If x is such a point near to z then its local stable manifold $W^s_\epsilon(x) \subset W^s(p)$ has size bounded away from zero. Therefore $W^{cu}_\epsilon(q_{n_1}) \cap W^s(p) \neq \emptyset$ and $W^u(p) \cap W^{ss}_\epsilon(q_{n_1}) \neq \emptyset$. Backward iteration by multiples of the period of q_{n_1} gives that there is a periodic point of saddle type in its center-unstable manifold. Indeed, otherwise $q_{n_1} \in H(p)$, which is absurd.

The same argument shows that there is a saddle-type periodic point $p'_{n_1} \in W^{cu}_\epsilon(q_{n_1}) \cap H(p)$. As before, the stable manifold of p'_{n_1} has size bounded away from zero.

Let us repeat similar arguments with a sink q_{n_2} between the local stable manifold of z and that of p'_{n_1}. It may be seen that one of the separatrices of the local center-unstable

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manifold of q_{n_2} cuts the stable manifold of z while the other one cuts the stable manifold of p'_{n_1}. Therefore we obtain two different periodic points $p'_{n_2}, p''_{n_2} \in W^{cu}_\epsilon(q_{n_2}) \cap H(p)$. It follows that for all $k \in \mathbb{Z}$, $\mathrm{dist}(f^k(p'_{n_2}), f^k(p''_{n_2})) \leq 2\epsilon < \alpha$, contradicting the expansive properties of f in $H(p)$. □

COROLLARY 5.9. *There exists $\eta > 0$ such that for every point $x \in H(p)$ it holds that* $\mathrm{diam}(W^{u,\sigma}_\epsilon(x)) \geq \eta$, $\sigma = +, -$.

Proof. Hyperbolic periodic points are dense in $H(p)$ and they are in any neighborhood of $H(p)$. □

5.3. *Hyperbolic splitting.* Our next target is to prove that generically $E \oplus F$ is hyperbolic. For this, we shall prove first that E is contracting, and this is done following Mañé's proof of [**Ma3**, Theorem I.4].

LEMMA 5.10. *Let $f : M \to M$ be a C^1 diffeomorphism such that there exists a horseshoe associated with some hyperbolic periodic point p. Then h, the topological entropy of $f_{|H(p)}$, is positive.*

Proof. As $H(p)$ contains a hyperbolic periodic point with a transverse homoclinic intersection the result follows from [**Sm**]. □

From Lemma 5.10 and the variational equation for the metric entropy there exists an ergodic measure μ in $H(p)$ such that $h_\mu > 0$. Using Ruelle's inequality

$$\sum_{\chi_j > 0} \chi_j \geq h_\mu(f)$$

where $\chi_1, \ldots, \chi_k$ are the Lyapunov exponents associated with $(f_{|H(p)}, \mu)$ we conclude that there exists a positive Lyapunov exponent μ-ae and arguing with f^{-1} instead of f a negative Lyapunov exponent μ-ae. Moreover, we have the following.

LEMMA 5.11. *There exist $c > 0$, a positive integer m, and a dense subset D of $H(p)$ such that for x in this subset we have*

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^m / E(f^{-jm}(x))\| \leq -c.$$

Proof. We have that $\mathrm{Per}(f) \cap H(p)$ is dense in $H(p)$. By Lemma 4.12 we have for any periodic point $q \in H(p) \cap \mathrm{Per}(f)$:

$$\|Df^{\tau n}|_{E^s(q)}\| \leq K\lambda^{\tau n} \quad \text{for all } n \geq 0. \tag{3}$$

Here τ is the period of q and $K > 0$, $0 < \lambda < 1$ are independent of the particular periodic point in $H(p)$. Moreover by [**Ma2**, Lemma II.5] there exists $m > 0$, $C > 0$, and μ, $0 < \lambda \leq \mu < 1$ such that

$$\prod_{j=1}^{[\tau/m]} \|Df^m_{f^{jm}(q)}|_{E^s}\| \leq C\mu^{[\tau/m]}, \tag{4}$$

whenever (3) holds. Here $[\tau/m]$ is the greatest integer less than or equal to τ/m.

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Combining (3) and (4) and taking logarithms gives

$$\frac{1}{[\tau/m]}\sum_{j=1}^{[\tau/m]}\log\|Df^m/E(f^{jm}(q))\| \leq \frac{\log C}{[\tau/m]} + \log\mu. \tag{5}$$

Due to expansiveness, periodic points of period $\leq m$ are finite in number. Thus taking $\tau > m$ we have denseness of periodic points of period $> m$ and for them the result follows from inequality (5) since $0 < \mu < 1$. □

The lemma above ensures that there exists a dense subset of points x in $H(p)$ where $\prod_{j=1}^{n}\|Df^m/E(f^{-jm}(x))\|$ converges to 0 exponentially fast. Since if $f : H(p) \to H(p)$ is expansive then the same is true for f^m, we may assume (and do) that $m = 1$ in Lemma 5.11.

Let γ_0 be such that $0 < e^{-c} < \gamma_0 < 1$ where $c > 0$ is given by Lemma 5.11. It follows that there exists D dense in $H(p)$ such that if $x \in D$ there are infinitely many values of n satisfying

$$\prod_{j=1}^{n}\|Df/E(f^j(x))\| < \gamma_0^n.$$

Take $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1.$$

Let $N_0 = N(\gamma_3, \gamma_4)$ be given by Lemma 5.1.

LEMMA 5.12. *If $E_{|H(p)}$ is not a contracting bundle then for all $\epsilon > 0$ there exist a compact invariant set $\Lambda(\epsilon) \subset H(p)$ and $N = N(\epsilon)$ such that every $x \in \Lambda(\epsilon)$ has the following property: there exist x_0 arbitrarily near x, $n_0 \geq 0$ and $y \in \Lambda(\epsilon)$ such that* $\operatorname{dist}(f^{n_0}(x_0), y) < \epsilon$, *$(y, f^n(y))$ is an (N_0, γ_2)-obstruction for all $n \geq N = N(\epsilon)$ and if $n_0 > 0$ then $(x_0, f^{n_0}(x_0))$ is a uniform γ_4-string. Moreover, $\Lambda(\epsilon)$ is the closure of its interior.*

Proof. See [**Ma3**, Lemmas II.6 and II.7]. □

Fix $0 < \gamma < \gamma_0$ and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be as in Lemma 5.12. Choose $k_0 \in (0, 1)$ such that $\gamma < k_0^2\gamma_1$ and $k_0^{-1}\gamma_4 < 1$, i.e. $1 > k_0 > \max\{\gamma_4, \sqrt{\gamma/\gamma_1}\}$.

PROPOSITION 5.13. *If E is not contracting, for all $\epsilon > 0$ there exist sequences $\{x_i\} \subset \Lambda(\epsilon/4)$, ($\Lambda(\epsilon)$ as in Lemma 5.12) and $n_i > 0$ such that:*

(a) $\operatorname{dist}(f^{n_i}(x_i), x_{i+1}) < \epsilon$;

(b) *$(x_i, f^{n_i}(x_i))$ is a uniform γ_4-string but if i is even $(x_i, f^{n_i}(x_i))$ is not a γ_1-string;*

(c) *if $K = \min\{\|Df/E(x)\|; x \in H(p)\}$ then $\gamma_1^{n_i}K^{n_{i-1}} \geq (k_0\gamma_1)^{n_i+n_{i+1}}$ for all even i;*

(d) *there exist odd numbers k, l; $k > l$ such that* $\operatorname{dist}(x_l, x_k) < \epsilon/2$.

Proof. See [**Ma3**, p. 179] for the proof of (a), (b), (c). Item (d) follows from compactness. □

COROLLARY 5.14. *For all $\epsilon > 0$ there exist a sequence $x_1, x_2, \ldots, x_k$ and uniform γ_4-strings $(x_i, f^{n_i}(x_i))$ which are not γ_1-strings, such that* $\operatorname{dist}(x_i, x_{i+1}) < \epsilon$ *and* $\operatorname{dist}(f^{n_k}(x_k), x_1) < \epsilon$ *for all $i = 1, 2, \ldots, k-1$.*

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LEMMA 5.15. *Let us assume that* $(x, f^n(x))$ *is a uniform* γ*-string,* $0 < \gamma < 1$, $n > 0$. *Then there exist positive* ϵ *and* η *such that if for some constant* $c > 0$, $\operatorname{diam}(f^n(W_c^{cu,\sigma}(x))) \geq 2\eta$ *for* $n > n_0$ *but* $\operatorname{diam}(f^j(W_c^{cu,\sigma}(x))) < 2\eta$; $j = 0, \ldots, n-1$ *then* $\operatorname{diam}(f^n(W_c^{cu,\sigma}(y))) \geq \eta$*, for all* $y \in W_\epsilon^{cs}(x)$, $\sigma = +, -$.

(Roughly speaking, if Df contracts in $E(f^j(x))$; $j = 1, \ldots, n$ and in addition $f^j(W_c^{cu,\sigma}(x))$ grows in diameter in n iterates, points in $W_\epsilon^{cs}(x)$ have to have the same property.)

Proof. Let $1 > \gamma_1 > \gamma$. There exists $\epsilon > 0$ such that if $(z, f^n(z))$ is a uniform γ_1 string then $\operatorname{dist}(f^j(z), f^j(y)) \leq \gamma_2^j \epsilon$ for some $1 > \gamma_2 > \gamma_1$ and for all $y \in W_\epsilon^{cs}(z)$. Let ϵ, η be such that if $\operatorname{dist}(z, w) \leq \epsilon + 2\eta$ then

$$1 - c < \frac{\|Df|_{E(w)}\|}{\|Df|_{E(z)}\|} < 1 + c \tag{6}$$

and

$$1 - c < \frac{\|Df|_{F(w)}\|}{\|Df|_{F(z)}\|} < 1 + c. \tag{7}$$

We choose $c > 0$ below and further assume that $\epsilon < \eta/2$. Arguing by contradiction assume that for some $y \in W_\epsilon^{cs}(x)$ and for all $j = 0, 1, \ldots, n$ we have that $\operatorname{diam}(f^j(W_c^{cu,\sigma}(y))) < 2\eta$ and $\operatorname{diam}(f^n(W_c^{cu,\sigma}(y))) < \eta$ then $\operatorname{dist}(f^j(x), f^j(y')) < 2\eta + \epsilon$ for all $y' \in W_c^{cu,\sigma}(y)$. Thus there exists $c > 0$ such that if (6) and (7) hold then $(y', f^n(y'))$ is a uniform γ_1-string and therefore $\operatorname{dist}(f^j(z'), f^j(y')) \leq \gamma_2^j \epsilon$ for all $z' \in W_\epsilon^{cs}(y')$. Moreover by the domination property there is $\delta > 0$ such that for all $x' \in W_c^{cu,\sigma}(x)$ there is $y' \in W_c^{cu,\sigma}(y)$ such that $x' \in W_\epsilon^{cs}(y')$ if $\operatorname{dist}(x, y) < \delta$. Consequently

$$\begin{aligned}\operatorname{dist}(f^n(x), f^n(x')) &\leq \operatorname{dist}(f^n(x), f^n(y)) + \operatorname{dist}(f^n(y), f^n(y')) + \operatorname{dist}(f^n(y'), f^n(x'))\\ &< \epsilon\gamma_2^n + \eta + \epsilon\gamma_2^n < 2\eta,\end{aligned}$$

contradicting the hypothesis.

On the other hand, if there is $0 < k < n$ such that $\operatorname{dist}(f^k(y), f^k(y')) > 2\eta$ and $\operatorname{dist}(f^n(y), f^n(y')) < \eta$, $n \geq n_0$, then let us assume that k is the maximum $j \in [0, n]$ with that property and let us find y'' in the center-unstable arc $[y, y']$ such that $\operatorname{dist}(f^k(y), f^k(y'')) = 2\eta$. Then by the arguments above, interchanging the roles of y and x, we conclude that $\operatorname{dist}(f^k(x), f^k(x')) \geq \eta$. Let $\hat{y} \in W_\eta^{cu,\sigma}(f^k(y)) \cap W_\epsilon^{cs}(f^k(x'))$. Then $\hat{y} \in [f^k(y), f^k(y'')]$ and we may argue with $\hat{y}$ as we have done with y' in the first part of the proof. Thus $\operatorname{diam}(f^n(W_c^{cu,\sigma}(y))) \geq \eta$, finishing the proof. □

The following proposition is similar to a lemma proved in [**Li**] (see also [**Ma3**]). The proof there uses that $Df|F$ is uniformly expanding. We, instead, use that the diameter of the unstable manifolds are uniformly bounded away from zero (Lemma 5.8).

PROPOSITION 5.16. *Let* f *be a* C^2 *diffeomorphism. Given* $\delta > 0$, $0 < \gamma < 1$ *and* $(x_i, f^{n_i}(x_i))$ *a sequence of uniform* γ*-strings in* $H(p)$, $i = 1, \ldots k$*, then there exist* $\mu = \mu(\gamma, \delta) > 0$ *and* $N_0 > 0$ *such that if* $\operatorname{dist}(x_i, x_{i+1}) < \mu$ *and* $\operatorname{dist}(f^{n_k}(x_k), x_1) < \mu$ *for all* $i = 1, 2, \ldots, k-1$ *and* $n_1 + n_2 + \cdots + n_k \geq N_0$ *then there exists a periodic*

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point q of f with period $N = n_1 + n_2 + \cdots + n_k$ such that $\operatorname{dist}(f^n(q), f^n(x_1)) < \delta$ *for all* $0 \le n \le n_1$ *and setting* $N_i = n_1 + \cdots + n_i$ $\operatorname{dist}(f^{N_i+n}(q), f^n(x_{i+1})) < \delta$ *for all* $0 \le n \le n_{i+1}$, $1 \le i \le k-1$.

Proof. Let us first assume $k = 1$. Let $x \in H(p)$ and $W^{cs}_\epsilon(x)$ be the local invariant manifold given by the dominated splitting which is an embedded disk transverse to F. It is proved in [**PS2**] that for any $\epsilon > 0$ there is $r = r(\epsilon) > 0$ such that the size of $W^{cs}_\epsilon(x)$ (see Definition 1.1), $\operatorname{size}(W^{cs}_\epsilon(x)) \ge r$. Moreover, if $x = x_1$ then the fact that $Df_{|E}$ contracts n_1 iterates implies that $W^{cs}_\epsilon(x_1)$ behaves as a stable manifold for those iterates. That is, there is γ_1, $\gamma < \gamma_1 < 1$, such that for $j = 0, 1, \ldots, n_1$, if $y \in W^{cs}_\epsilon(x_1)$ then $\operatorname{dist}(f^j(x_1), f^j(y)) < \gamma_1^j \epsilon$. As usual we choose $\epsilon > 0$ such that $\epsilon < \alpha/2$, where $\alpha > 0$ is a constant of expansiveness. On the other hand, by Corollary 5.9, it holds that there is $\eta > 0$ such that $\operatorname{diam}(W^{u,\sigma}_\epsilon(x_1)) \ge 2\eta$ where $\sigma = +, -$ indicates any one of the separatrices of $W^u_\epsilon(x_1)$. Let us consider local center-unstable manifolds of the points $x \in D_1 \subset W^{cs}_\epsilon(x_1)$ where D_1 is a 2-disk centered in x_1. These center-unstable manifolds are coherent with the local unstable manifolds of points of $H(p) \cap W^{cs}_\epsilon(x_1)$. By expansiveness of f in $H(p)$ for all $c_0 > 0$ there exists $N_0 > 0$ such that $\operatorname{diam}(f^n(W^u_c(y))) > 2\eta$ for all $y \in H(p)$, $c \ge c_0$ and $n \ge N_0$. Hence, by Lemma 5.15 we have that $\operatorname{diam}(f^{n_1}(W^{cu,\sigma}_c(x))) \ge \eta$ because $\operatorname{diam}(f^{n_1}(W^{u,\sigma}_c(x_1))) \ge 2\eta$.

Let $\mathcal{B}_1$ be a cylinder centered in x_1 given by $\mathcal{B}_1 = \bigcup_{x \in D_1} U_x$ where $U_x \subset W^{cu}_\epsilon(x)$ has diameter η and is centered in x. Let $\mathcal{C}_1 \subset \mathcal{B}_1$ be defined by

$$\mathcal{C}_1 = \cup_{x \in D_1}(W^{cu}_c(x)).$$

Take $\mu > 0$ small enough such that when $\operatorname{dist}(f^{n_1}(x_1), x_1) < \mu$ then $f^{n_1}(W^{cs}_\epsilon(x_1))$ is contained in the interior of $\mathcal{B}_1$ and moreover $W^{cu}_\epsilon(x)$ cuts $W^{cs}_\epsilon(y)$ whenever $\operatorname{dist}(x, y) < \mu$ with $x, y \in V(H(p))$. Hence $f^{n_1}(\mathcal{C}_1)$ intersects $\mathcal{C}_1$ and any point in the boundary of $\mathcal{C}_1$ is not fixed by f^{n_1}. Therefore, by a standard argument of index theory, see [**Do**], there exists a fixed point q of f^{n_1} in $\mathcal{C}_1 \cap f^{n_1}(\mathcal{C}_1)$. That is, q is a periodic point of f. Observe that by Lemma 5.8 there is z in $W^u_\epsilon(q) \cap f^{n_1}(D_1)$. By backward iterations the distance between $f^{-j}(z)$ and $f^{-j}(q)$ is bounded by ϵ while $\operatorname{dist}(f^{-j}(z), f^{n_1-j}(x_1)) < \operatorname{diam}(\gamma_1^{n_1-j} r)$ where $\gamma_1^{n_1-j} r = \operatorname{diam}(f^{n_1-j}(D_1))$. Therefore $\operatorname{dist}(f^j(q), f^j(x_1)) \le \epsilon + r$ for all $j = -1, -2, \ldots, -n_1$. Choosing $r < \epsilon < \delta/2$ we conclude that the orbit of q δ-shadows the orbit of x_1 for all $j = 1, 2, \ldots n_1$, proving Proposition 5.16 when $k = 1$.

For $k = 2$ we proceed as follows. Take a small disk $D_2 \subset W^{cs}_\epsilon(x_2)$ and $\mathcal{B}_1$ as in the previous case and set $\mathcal{B}_2 = \bigcup_{x \in D_2}(W^{cu}(x))$ and $\mathcal{C}_2 = \bigcup_{x \in D_2} W^{cu}_\beta(x)$, where β is such that $\operatorname{diam}(f^{n_2}(W^{cu}_\beta(x))) > \eta$. Find $\mu > 0$ such that $\operatorname{dist}(f^{n_1}(x_1), x_2) < \mu$ and $\operatorname{dist}(f^{n_2}(x_2), x_1) < \mu$ imply $f^{n_1}(D_1) \subset \operatorname{int}(\mathcal{B}_2)$ and $f^{n_2}(D_2) \subset \operatorname{int}(\mathcal{B}_1)$, where $\operatorname{int}(A)$ stands for the interior of A. Then $f^{n_2}(f^{n_1}(\mathcal{C}_1) \cap \mathcal{C}_2)$ is a small cylinder that cuts $\mathcal{C}_1$ and no point of the boundary of $\mathcal{C}_1 \cap f^{-n_1}(\mathcal{C}_2)$ is fixed by $f^{n_1+n_2}|_{(\mathcal{C}_1 \cap F^{-n_1}(\mathcal{C}_2))}$. Hence, arguing as before, there is a fixed point q of $f^{n_1+n_2}$ and reasoning as in the case $k = 1$ we conclude the proof for $k = 2$.

The general case follows by induction. □

LEMMA 5.17. *Let $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1$ be as in Lemma 5.12. Moreover let $0 < \gamma < \gamma_0$ and take $0 < k_0 < 1$ such that $\gamma < k_0^2 \gamma_1$ and $k_0^{-1}\gamma_4 < 1$. Then the periodic*

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point q given by Proposition 5.16 satisfies

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^N.$$

Proof. The proof is given in detail in [**Ma3**, §II, pp. 179–181] (the reader should be aware that in Mañé's article our γ_3 is denoted $\bar{\gamma}_2$ and γ_4 is denoted by γ_3). □

Remark 5.18. The number $0 < \gamma < 1$ is arbitrary since the choice of $\gamma_0 \in (e^{-c}, 1)$ is arbitrary. On the other hand, in the construction of [**Ma3**, §II], it is not pointed out that there exists $0 < \gamma_5 = k_0^{-1}\gamma_4 < 1$ such that

$$\prod_{n=1}^{N} \|Df/E(f^n(q))\| < \gamma_5^N.$$

But the proof of this fact is given there.

PROPOSITION 5.19. *We may construct the periodic point q of Proposition 5.16 such that it belongs to $H(p)$.*

Proof. By Lemma 5.17 if $\delta > 0$ is small enough we ensure that $Df_q/E(q)$ contracts at a rate similar to that of $Df_{x_j}/E(x_j)$, $j = 1, 2, \ldots, k$ if $\mathrm{dist}(f^j(q), f^j(x_j)) < \delta$ for all $j = 0, \ldots, N$, in the local center-stable manifold of q. More precisely we have

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\|.$$

And for all $j \in [1, N]$:

$$\prod_{n=1}^{j} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^j.$$

As q is periodic of period N this implies that its center-stable manifold is a stable manifold, of size about the same of that of the center-stable manifold of x_j. Hence the local unstable manifold of a periodic point $q' \in H(p)$ close to x_1 intersects the local stable manifold of q. Therefore $W^u(p)$ accumulates in $W^u(q)$. If we have that for some $j = 1, \ldots, k$, $i = 0, \ldots, n_j$, $W_\epsilon^{cs}(f^i(x_j))$ is a true stable manifold then we are done. For in that case $W^s(p)$ and $W^u(p)$ would accumulate in $W^s(q)$ and in $W^u(q)$ respectively and we obtain that $q \in H(p)$, concluding the proof of Proposition 5.19.

As we cannot make the above assumption, we have to proceed in a different way. Choose $V_l(H(p))$, $l \geq 1$, a sequence of admissible neighborhoods of $H(p)$ such that

$$\mathrm{Cl}(V_{l+1}(H(p))) \subset V_l(H(p)) \text{ and } \bigcap_{l\geq 1} V_l(H(p)) = H(p).$$

For any $V_l(H(p))$ we may find q_l such that q_l shadows the pseudo-orbit given by a sequence $(x_j, f^{n_j}(x_j))$, $j = 1, \ldots, k_l$ of uniform γ_4-strings that are not γ-strings as in Proposition 5.13. Take an accumulation point x of $\{q_l\}$. If there is a subsequence $\{q_{l_h}\}$ of $\{q_l\}$ such that the periods of q_{l_h} are uniformly bounded then $x \in H(p) \cap \mathrm{Per}(f)$ and x is the desired periodic point. Otherwise the periods of q_l are unbounded and $x \in H(p)$ is a uniform γ_4-string for all $n \geq 0$. Therefore by arguments similar to those used in [**PS1**, Corollary 3.3] we obtain a stable manifold for x that the unstable manifold of q_l will cut for $l \geq l_0$. Thus q_l is homoclinically related to p, completing the proof. □

Proof of Theorem B. By Propositions 5.16, 5.19, and Lemma 5.17, if E is not a contracting sub-bundle there exists $q \in H(p) \cap \mathrm{Per}(f)$ such that Df_q contracts in E rather weakly in the period, contradicting Lemma 4.12. Therefore E is a uniform contracting bundle. This in turn implies, using the arguments of [**Ma3**, §II], that F is a uniform expanding bundle. Thus $E \oplus F$ is a hyperbolic splitting. This implies that the same is true for g C^1-close to f. As the set of $g \in \mathrm{Diff}^2(M)$ are dense in the C^1-neighborhood in which the homoclinic class is robustly expansive and taking into account that Kupka–Smale diffeomorphisms are residual in C^r-topology for all $r \geq 1$, the result follows. □

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