



COMPOSITIO MATHEMATICA

Singular Gauduchon metrics

Chung-Ming Pan

Compositio Math. **158** (2022), 1314–1328.

[doi:10.1112/S0010437X22007618](https://doi.org/10.1112/S0010437X22007618)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865





Singular Gauduchon metrics

Chung-Ming Pan

ABSTRACT

In 1977, Gauduchon proved that on every compact hermitian manifold (X, ω) there exists a conformally equivalent hermitian metric ω_G which satisfies $dd^c \omega_G^{n-1} = 0$. In this note, we extend this result to irreducible compact singular hermitian varieties which admit a smoothing.

Introduction

Let X be an n -dimensional compact complex manifold equipped with a positive-definite smooth $(1, 1)$ -form ω . We also call ω a hermitian metric because such ω corresponds to a hermitian metric. A famous theorem of Gauduchon [Gau77] says that there exists a metric ω_G in the conformal class of ω such that $dd^c \omega_G^{n-1} = 0$ and the metric ω_G is unique up to a positive multiple. These kind of metrics are since then called *Gauduchon metrics*. The conformal factor ρ satisfying $\omega_G^{n-1} = \rho \omega^{n-1}$ is called *Gauduchon factor*.

In complex geometry, finding canonical metrics on complex manifolds is a central problem. Two celebrated examples are Yau's solution of the Calabi conjecture [Yau78] and Uhlenbeck and Yau's characterization of the existence of hermitian–Einstein metrics on stable vector bundles [UY86]. These theorems are established on Kähler manifolds. When the manifold is non-Kähler, the analysis is more difficult because hermitian metrics are no longer closed. In such cases, Gauduchon metrics provide a useful substitute. For instance, Li and Yau [LY87] used Gauduchon metrics to define the slope stability of vector bundles on compact non-Kähler manifolds. As a consequence, they generalized the Uhlenbeck–Yau theorem to non-Kähler setting. For generalized Calabi–Yau type problem in non-Kähler context, Tosatti and Weinkove [TW10] showed that for arbitrary representative $\Psi \in c_1^{BC}(X)$ of the first Bott–Chern class of X , there exists a hermitian metric ω such that $\text{Ric}(\omega) = \Psi$ by solving complex Monge–Ampère equations. In their proof, Gauduchon metrics play an important role to simplify calculations. Furthermore, Székelyhidi, Tosatti and Weinkove [STW17] proved that one can even find a Gauduchon metric with prescribed Chern–Ricci curvature. On the other hand, Angella, Calamai and Spotti [ACS17] studied the Chern–Yamabe problem (i.e. finding constant Chern scalar curvature metrics in the conformal class of a given metric ω). They used Gauduchon metrics to define a conformal invariant called the Gauduchon degree and showed that if a metric ω has non-positive Gauduchon degree then the Chern–Yamabe problem admits a solution. For more applications and results about Gauduchon metrics, the interested reader is referred to [FU13, FWW13, Li21] and the references therein.

Received 17 June 2021, accepted in final form 12 May 2022, published online 17 August 2022.

2020 Mathematics Subject Classification 53C55 (primary), 14D06, 32C15, 53C15 (secondary).

Keywords: Gauduchon metrics, complex spaces, smoothable singularities.

© 2022 The Author(s). The publishing rights in this article are licensed to Foundation Compositio Mathematica under an exclusive licence.

From an algebraic point of view, singularities are ubiquitous as they occur in various contexts, notably in the minimal model program and moduli theory. Ueno [Uen75] found a birational class of three-dimensional complex manifolds which does not admit a smooth minimal model. In moduli theory, it is necessary to deal with singular varieties when compactifying moduli spaces of smooth manifolds. Already in dimension one, the fundamental domain of moduli space of elliptic curves is non-compact and nodal curves lie on its boundary. On the other hand, in non-Kähler geometry, investigating singular varieties admitting a non-Kähler smoothing is an essential issue due to close interactions of string theory and mathematics established over the past 40 years. In the 1980s, a large class of non-Kähler Calabi–Yau threefolds was built via conifold transitions introduced by Clemens [Cle83] and Friedman [Fri86]. Roughly speaking, the process goes as follows: contracting a collection of disjoint $(-1, -1)$ -curves from a Kähler Calabi–Yau threefold X to obtain a singular Calabi–Yau variety X_0 and then smoothing singularities of X_0 , one obtains a family of Calabi–Yau threefolds $(X_t)_{t \neq 0}$ which are generally non-Kähler. Thus, it is important to understand the geometric structure on X_t induced by the original Calabi–Yau manifold X . Experts believe that the Hull–Strominger system [Hul86, Str86] provides a natural candidate. These models attracted a lot of attentions in recent years (cf. [Rei87, Fri91, Tia92, Ros06, FY08, Chu12, FLY12, PPZ18, CPY21] and the references therein).

Given these considerations, it is legitimate to look for canonical metrics or special metrics on singular hermitian varieties. In this note, we focus on Gauduchon metrics. A standard way to give a metric structure on a singular complex space is to restrict an ambient metric in local embeddings (see Definition 1.1 for the precise definition). Then we address the following question.

Question. Suppose that X is an irreducible, reduced, compact complex space equipped with a hermitian metric ω . Can one find a Gauduchon metric ω_G in the conformal class of ω ?

The purpose of this note is to give partial answers in the setup of smoothable singularities. This means that the variety can be approximated by a family of smooth manifolds and the hermitian metric is the restriction of an ambient smooth metric. The precise statements are as follows.

Setup (S). Let \mathcal{X} be an $(n + 1)$ -dimensional, irreducible, reduced, complex analytic space. Suppose that

- (i) $\pi : \mathcal{X} \rightarrow \mathbb{D}$ is a proper, surjective, holomorphic map with connected fibres;
- (ii) π is smooth over the punctured disk \mathbb{D}^* ;
- (iii) the central fibre X_0 is an n -dimensional, irreducible, reduced, compact complex analytic space.

Let ω be a hermitian metric on \mathcal{X} in the sense of Definition 1.1. For each $t \in \mathbb{D}$, we define the hermitian metric ω_t on fibre X_t by restriction (i.e. $\omega_t := \omega|_{X_t}$).

In Setup (S), on each smooth fibre X_t , there exists a Gauduchon factor ρ_t with respect to ω_t by Gauduchon’s theorem. We may normalize these Gauduchon factors such that $\inf_{X_t} \rho_t = 1$. Then we show the existence of a smooth Gauduchon factor on the smooth part of the central fibre.

THEOREM A (Cf. Corollary 2.3 and Theorem 3.1). *In Setup (S), we have the following properties.*

- (i) *There is a uniform constant $C_G \geq 1$ such that the normalized Gauduchon factors ρ_t are bounded between 1 and C_G on each smooth fibre X_t for all $t \in \mathbb{D}_{1/2}^*$.*
- (ii) *There exists a smooth bounded Gauduchon factor ρ of ω_0 on X_0^{reg} .*

Thus, X_0 admits a *bounded Gauduchon metric* (see Definition 1.2) in the conformal class of ω_0 . The idea of proof is to approximate the Gauduchon factor on the singular fibre by the normalized Gauduchon factors on nearby smooth fibres.

Next, we assume that X is a variety endowed with a bounded Gauduchon metric ω_G . We show that the trivial extension of the $(n - 1, n - 1)$ -form ω_G^{n-1} through X^{sing} is a pluriclosed current on X . Moreover, we also prove the analogous uniqueness result of Gauduchon.

THEOREM B. *Suppose that X is an n -dimensional, irreducible, reduced, compact complex space endowed with a bounded Gauduchon metric ω_G . Then we have the following uniqueness and extension properties.*

- (i) *If ω'_G is another bounded Gauduchon metric in the conformal class of ω_G , then ω'_G must be a positive multiple of ω_G ;*
- (ii) *Let T be the positive $(n - 1, n - 1)$ -current obtained as the trivial extension of ω_G^{n-1} . Then T is a pluriclosed current.*

The main strategy is to use ‘good’ cutoff functions. Complement of proper analytic subsets (e.g. $X^{\text{reg}} = X \setminus X^{\text{sing}}$) admit exhaustion functions with small L^2 -gradient. This enables us to show that the trivial extension of ω_G^{n-1} as a current on X satisfies $dd^c T = 0$ in the sense of currents globally on X . The uniqueness property follows similarly.

This note is organized as follows: §1 provides some backgrounds. Section 2 contains sup-estimate of normalized Gauduchon factors in families (the first property in Theorem A). In §3, we show the existence of bounded Gauduchon factors on the central fibre (the second part of Theorem A) and give the proof of Theorem B.

1. Preliminaries

In this section, we recall some notation and conventions which will be used in the sequel. We define the twisted exterior derivative by $d^c = (i/2)(\bar{\partial} - \partial)$ and we then have $dd^c = i\partial\bar{\partial}$. We say that a form is *pluriclosed* if it is dd^c -closed. We denote by:

- (i) $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$ the open disk of radius r ;
- (ii) $\mathbb{D}_r^* := \{z \in \mathbb{C} \mid 0 < |z| < r\}$ the punctured disk of radius r .

When $r = 1$, we simply write $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$.

1.1 Metrics on singular spaces

Let X be a reduced complex analytic space of pure dimension $n \geq 1$. We denote by X^{reg} the complex manifold of regular points of X and $X^{\text{sing}} := X \setminus X^{\text{reg}}$ the singular set of X . Now, we give the definition of hermitian metrics on reduced complex analytic space X .

DEFINITION 1.1. A hermitian metric ω on X is the data of a hermitian metric ω on X^{reg} such that given any local embedding $X \xrightarrow{\text{loc}} \mathbb{C}^N$, ω extends smoothly to a hermitian metric on \mathbb{C}^N .

Remark 1.1. The notion of smooth forms (and hermitian metrics) as above does not depend on the choice of local embeddings (see [Dem85, p. 14]). Hermitian metrics always exist: one can use local embeddings and then glue local data of hermitian metrics by a partition of unity.

Note that in Definition 1.1 a hermitian metric on X is more than just a metric on X^{reg} . Now, we define Gauduchon metrics in the following two different concepts.

DEFINITION 1.2. We say that a hermitian metric ω_G on X^{reg} is:

- (i) *Gauduchon* if it satisfies $\text{dd}^c \omega_G^{n-1} = 0$ on X^{reg} ;
- (ii) *bounded Gauduchon metric* on X if there exist a hermitian metric ω and a positive bounded smooth function ρ defined on X^{reg} such that $\omega_G = \rho^{1/(n-1)} \omega$ and $\text{dd}^c \omega_G^{n-1} = 0$ on X^{reg} .

We define the complex Laplacian and the norm of gradients with respect to ω by

$$\Delta_\omega f := \text{tr}_\omega(\text{dd}^c f) = \frac{n \text{dd}^c f \wedge \omega^{n-1}}{\omega^n}$$

$$|df|_\omega^2 := \text{tr}_\omega(df \wedge d^c f) = \frac{n df \wedge d^c f \wedge \omega^{n-1}}{\omega^n}.$$

1.2 Currents on singular spaces

Recall that smooth forms on X are defined as restriction of smooth forms in local embeddings. We denote by

- (i) $\mathcal{D}_{p,q}(X)$ the space of compactly supported smooth forms of bidegree (p, q) ;
- (ii) $\mathcal{D}_{p,p}(X)_\mathbb{R}$ the space of real smooth (p, p) -forms with compact support.

The notion of currents on X , $\mathcal{D}'_{p,q}(X)$ and $\mathcal{D}'_{p,p}(X)_\mathbb{R}$, is defined by acting on compactly supported smooth forms on X . The operators d , d^c and dd^c are well-defined by duality (see [Dem85] for detail arguments).

1.3 Example

We give an example of a non-Kähler variety satisfying Setup (S) extracted from [LT94]. The manifold $M = (\Gamma_1 \times \Gamma_2) \cap H \subset \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3$ is a simply connected Calabi–Yau threefold with $b_2 = 14$, where

$$\Gamma_1 = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{C}\mathbb{P}^3,$$

$$\Gamma_2 = \{y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0\} \subset \mathbb{C}\mathbb{P}^3,$$

$$H = \{x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0\} \subset \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3.$$

There exists 15 disjoint rational curves ℓ_1, \dots, ℓ_{15} with normal bundles $\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-1)^{\oplus 2}$, ℓ_1, \dots, ℓ_{14} generate $H_2(M, \mathbb{Z})$, and there exists $a_j \neq 0$ for $j \in \{1, \dots, 15\}$ such that $\sum_{j=1}^{15} a_j [\ell_j] = 0$. Then one can contract these curves and obtain a singular space X_0 with 15 ordinary double points. Moreover, there is a smoothing $\pi : \mathcal{X} \rightarrow \mathbb{D}$ of X_0 such that for all $t \neq 0$, X_t is diffeomorphic to a connected sum of $\mathbb{S}^3 \times \mathbb{S}^3$. This implies that X_t does not admit a Kähler metric since $b_2(X_t) = 0$. One can check that X_0 is not Kähler.

1.4 Remarks on the family setting

Under Setup (S), the family of metrics $(\omega_t)_t$ satisfies the following properties: For all $t \in \overline{\mathbb{D}}_{1/2}$, there is a constant $B \geq 0$ independent of t such that

$$-B\omega_t^n \leq \text{dd}_t^c \omega_t^{n-1} \leq B\omega_t^n, \tag{1.1}$$

where d_t^c is the twisted exterior derivative with respect to the complex structure of X_t . Indeed, in a local embedding, we have $-B\omega^n \leq \text{dd}^c \omega^{n-1} \leq B\omega^n$ on \mathcal{X} ; hence (1.1) is just the restriction on each fibre X_t . On the other hand, the volume of (X_t, ω_t) is comparable to the volume of (X_0, ω_0) for all $t \in \overline{\mathbb{D}}_{1/2}$. Namely, we have a uniform constant $C_V \geq 1$ such that

$$C_V^{-1} \leq \text{Vol}_{\omega_t}(X_t) \leq C_V, \quad \forall t \in \overline{\mathbb{D}}_{1/2}. \tag{1.2}$$

The lower bound is obvious. One can prove the upper bound by the continuity of the total mass of currents $(\omega^n \wedge [X_t])_{t \in \overline{\mathbb{D}}_{1/2}}$. The proof goes as follows: the current of integration $[X_t]$ can be written as $\text{dd}^c \log |\pi - t|$ by the Poincaré–Lelong formula. As $|\pi - t|$ converges to $|\pi|$ uniformly when $t \rightarrow 0$, $\log |\pi - t|$ converges to $\log |\pi|$ almost everywhere and, thus, $\log |\pi - t| \rightarrow \log |\pi|$ in L^1 when $t \rightarrow 0$ by Hartogs’ lemma. Therefore, $\omega^n \wedge [X_t] \xrightarrow[t \rightarrow 0]{} \omega^n \wedge [X_0]$ in the sense of currents and this implies $\text{Vol}_{\omega_t}(X_t) \xrightarrow[t \rightarrow 0]{} \text{Vol}_{\omega_0}(X_0)$. Thus, using the compactness of X_0 , we obtain a uniform upper bound C_V of $\text{Vol}_{\omega_t}(X_t)$ for all $t \in \overline{\mathbb{D}}_{1/2}$.

Finally, we give a remark on non-smoothable singularities: The first non-smoothable example was given by Thom and reproduced by Hartshorne [Har74]. They considered a cone in \mathbb{C}^6 over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^2$ into $\mathbb{C}P^5$ and they proved that the cone does not admit a smoothing because of a topological constraint.

2. Gauduchon metrics on smooth fibres

The aim of this section is to prove the uniform boundedness of normalized Gauduchon factors ρ_t with respect to ω_t on smooth fibres X_t in Setup (S). First, we fix a compact hermitian manifold (X, ω) . Suppose that $B \geq 0$ is a constant such that $-B\omega^n \leq \text{dd}^c \omega^{n-1} \leq B\omega^n$. From Gauduchon’s theorem [Gau77], there exists a unique positive smooth function $\rho \in \ker \Delta_\omega^*$ or, equivalently,

$$\text{dd}^c(\rho \omega^{n-1}) = \text{dd}^c \rho \wedge \omega^{n-1} + d\rho \wedge d^c \omega^{n-1} - d^c \rho \wedge d\omega^{n-1} + \rho \text{dd}^c \omega^{n-1} = 0$$

such that $\inf_X \rho = 1$ and $\omega_G = \rho^{1/(n-1)} \omega$ is a Gauduchon metric. Then we prove that the Gauduchon factor is bounded by geometric quantities.

THEOREM 2.1. *Suppose that (X, ω) is an n -dimensional compact hermitian manifold. If $\rho \in \ker \Delta_\omega^*$ and $\inf_X \rho = 1$, then there is a constant C_G depending only on n, B, C_S, C_P and $\text{Vol}_\omega(X)$ such that*

$$\sup_X \rho \leq C_G,$$

where C_S and C_P are Sobolev and Poincaré constants.

The proof of Theorem 2.1 is inspired by the paper of Tosatti and Weinkove [TW10]. We apply Moser’s iteration twice to obtain an upper bound of ρ . On the other hand, under Setup (S), the Sobolev and Poincaré constants of the fibres X_t are uniformly bounded independently of t . The uniform Sobolev constant in family comes from Wirtinger inequality and Michael and Simon’s Sobolev inequality on minimal submanifolds [MS73]. The study of Poincaré constant in families goes back to Yoshikawa [Yos97] and Ruan and Zhang [RZ11]. For convenience, the reader is also referred to [DGG20, Propositions 3.8 and 3.10]. Although they only stated the properties on a family of Kähler spaces, the proof does not rely on Kähler structures.

PROPOSITION 2.2. *Suppose that $\pi : (\mathcal{X}, \omega) \rightarrow \mathbb{D}$ is a family of compact hermitian varieties in Setup (S). For all $t \in K \Subset \mathbb{D}$, there exists uniform Sobolev and Poincaré constants $C_S(K)$ and $C_P(K)$ such that*

$$\forall f \in L^2_1(X_t^{\text{reg}}), \quad \left(\int_{X_t} |f|^{2n/(n-1)} \omega_t^n \right)^{(n-1)/n} \leq C_S \left(\int_{X_t} |df|_{\omega_t}^2 \omega_t^n + \int_{X_t} |f|^2 \omega_t^n \right)$$

and

$$\forall f \in L^2_1(X_t^{\text{reg}}) \quad \text{and} \quad \int_{X_t} f \omega_t^n = 0, \quad \int_{X_t} |f|^2 \omega_t^n \leq C_P \int_{X_t} |df|_{\omega_t}^2 \omega_t^n.$$

Remark 2.1. The irreducible condition is crucial in the proof of uniform Poincaré inequality. Assume that X_0 has two irreducible components X'_0 and X''_0 . Consider a function f on X_0^{reg} and it is defined by

$$\begin{cases} f = 1/\text{Vol}_{\omega_t}(X'_0) & \text{on } (X'_0)^{\text{reg}}, \\ f = -1/\text{Vol}_{\omega_t}(X''_0) & \text{on } (X''_0)^{\text{reg}}, \\ f = 0 & \text{otherwise.} \end{cases}$$

Then it is not hard to see that the right-hand side of Poincaré inequality is zero but the left-hand side is positive. This yields a contradiction. One can also construct a ‘quantitative’ version of that example. Namely, the Poincaré constant $C_{P,t}$ on each smooth fibre X_t blows up when $t \rightarrow 0$ (see, e.g., [Yos97, DGG20]).

Combining Theorem 2.1, Proposition 2.2, (1.1) and (1.2), we obtain the following uniform estimate in the family setting.

COROLLARY 2.3. *Let $\pi : (\mathcal{X}, \omega) \rightarrow \mathbb{D}$ be a family of compact hermitian manifolds as in Setup (S). Then there exists a constant $C_G > 0$ such that*

$$\sup_{X_t} \rho_t \leq C_G, \quad \text{and} \quad \inf_{X_t} \rho_t = 1 \quad \text{for all } t \in \mathbb{D}_{1/2}^*,$$

where ρ_t is the Gauduchon factor with respect to (X_t, ω_t) .

2.1 Proof of Theorem 2.1

In this subsection, we establish two gradient estimates ((2.1) and (2.3)) and then apply Moser’s iteration argument to obtain an upper bound of ρ . In order to check the dependence on each data, we formulate the dependence of given constants. For convenience, we write $V := \text{Vol}_{\omega}(X)$. We start the proof with a useful formula.

LEMMA 2.4. *Suppose that $\rho \in \ker \Delta_{\omega}^*$. Then we have*

$$\int_X F'(\rho) d\rho \wedge d^c \rho \wedge \omega^{n-1} = \int_X G(\rho) dd^c \omega^{n-1},$$

where F and G are two real \mathcal{C}^1 -functions defined on $\mathbb{R}_{>0}$ which satisfy $x F'(x) = G'(x)$.

Proof. As $\rho \in \ker \Delta_{\omega}^*$, we have $dd^c(\rho \omega^{n-1}) = 0$ equivalently. From Stokes’ theorem, it follows that

$$\begin{aligned} 0 &= \int_X F(\rho) dd^c(\rho \omega^{n-1}) = - \int_X F'(\rho) d\rho \wedge d^c(\rho \omega^{n-1}) \\ &= - \int_X F'(\rho) d\rho \wedge d^c \rho \wedge \omega^{n-1} - \int_X F'(\rho) \rho d\rho \wedge d^c \omega^{n-1}. \end{aligned}$$

By the assumption $x F'(x) = G'(x)$, we obtain the desired formula

$$\begin{aligned} \int_X F'(\rho) d\rho \wedge d^c \rho \wedge \omega^{n-1} &= - \int_X \rho F'(\rho) d\rho \wedge d^c \omega^{n-1} \\ &= - \int_X dG(\rho) \wedge d^c \omega^{n-1} = \int_X G(\rho) dd^c \omega^{n-1}. \quad \square \end{aligned}$$

Consider $F'(x) = (np^2/4)x^{-p-2}$ and $G(x) = -(np/4)x^{-p}$ for $p \geq 1$. By Lemma 2.4, we have the following inequality

$$\begin{aligned} \int_X |d(\rho^{-p/2})|_\omega^2 \omega^n &= \int_X nd(\rho^{-p/2}) \wedge d^c(\rho^{-p/2}) \wedge \omega^{n-1} \\ &= \int_X F'(\rho)d\rho \wedge d^c\rho \wedge \omega^{n-1} = \int_X G(\rho)dd^c\omega^{n-1} \\ &= \frac{np}{4} \int_X \rho^{-p} (-dd^c\omega^{n-1}) \\ &\leq p\frac{nB}{4} \int_X \rho^{-p}\omega^n. \end{aligned} \tag{2.1}$$

Put $\beta = n/(n - 1) > 1$. Combining Sobolev inequality and (2.1), we obtain

$$\begin{aligned} \left(\int_X (\rho^{-1})^{p\beta} \omega^n \right)^{1/\beta} &\leq C_S \left(\int_X |d(\rho^{-p/2})|_\omega^2 \omega^n + \int_X |\rho^{-p/2}|^2 \omega^n \right) \\ &\leq C_S \left(p\frac{nB}{4} \int_X (\rho^{-1})^p \omega^n + \int_X (\rho^{-1})^p \omega^n \right) \\ &\leq p \left(\frac{nB}{4} + 1 \right) C_S \int_X \rho^{-p} \omega^n. \end{aligned}$$

For all $p \geq 1$, we have

$$\|\rho^{-1}\|_{L^{p\beta}(X,\omega)} \leq p^{1/p} C_1^{1/p} \|\rho^{-1}\|_{L^p(X,\omega)},$$

where C_1 is a constant depending only on n, B, C_S . Inductively, we obtain

$$\begin{aligned} \|\rho^{-1}\|_{L^{p\beta^k}(X,\omega)} &\leq p^{(1/p)\sum_{j=0}^{k-1} 1/\beta^j} \beta^{(1/p)\sum_{j=0}^{k-1} j/\beta^j} C_1^{(1/p)\sum_{j=0}^{k-1} 1/\beta^j} \|\rho^{-1}\|_{L^p(X,\omega)} \\ &\leq p^{n/p} \beta^{n(n-1)/p} C_1^{n/p} \|\rho^{-1}\|_{L^p(X,\omega)}. \end{aligned}$$

Let $p = 1$ and $k \rightarrow \infty$ and, thus, we obtain

$$1 = \sup_X \rho^{-1} \leq \beta^{n(n-1)} C_1^n \int_X \rho^{-1} \omega^n.$$

Therefore, the L^1 -norm of ρ^{-1} is bounded away from zero by a constant δ which depends only on n, B, C_S :

$$\int_X \rho^{-1} \omega^n \geq \frac{1}{\beta^{n(n-1)} C_1^n} =: 2\delta.$$

Choosing sufficiently small δ , we may assume $A := V/\delta \geq 1$. Then we have

$$2\delta \leq \int_{\{\rho < A\}} \rho^{-1} \omega^n + \int_{\{\rho \geq A\}} \rho^{-1} \omega^n \leq \int_{\{\rho < A\}} \omega^n + \frac{1}{A} \int_X \omega^n.$$

Hence, the volume of $\{\rho < A\}$ is bounded away from zero:

$$\int_{\{\rho < A\}} \omega^n \geq \delta. \tag{2.2}$$

Now, we consider

$$F'(x) = \frac{n(p+1)^2 (\log x)^{p-1}}{4x^2} \quad \text{and} \quad G(x) = \frac{n(p+1)^2 (\log x)^p}{4p} \quad \text{for } p \geq 1.$$

From Lemma 2.4, we find the following estimate

$$\begin{aligned} \int_X |d(\log \rho)^{(p+1)/2}|_\omega^2 \omega^n &= \frac{n(p+1)^2}{4} \int_X \frac{(\log \rho)^{p-1}}{\rho^2} d\rho \wedge d^c \rho \wedge \omega^{n-1} \\ &= \int_X F'(\rho) d\rho \wedge d^c \rho \wedge \omega^{n-1} = \int_X G(\rho) dd^c \omega^{n-1} \\ &= \frac{n(p+1)^2}{4p} \int_X (\log \rho)^p dd^c \omega^{n-1} \\ &\leq pnB \int_X (\log \rho)^p \omega^n. \end{aligned} \tag{2.3}$$

Again, Sobolev inequality, (2.3) and Hölder inequality yield the following inequalities:

$$\begin{aligned} \left(\int_X (\log \rho)^{(p+1)\beta} \omega^n \right)^{1/\beta} &\leq C_S \left(\int_X |d(\log \rho)^{(p+1)/2}|_\omega^2 \omega^n + \int_X (\log \rho)^{p+1} \omega^n \right) \\ &\leq C_S \left(pnB \int_X (\log \rho)^p \omega^n + \int_X (\log \rho)^{p+1} \omega^n \right) \\ &\leq C_S \left(pnBV^{1/(p+1)} \left(\int_X (\log \rho)^{p+1} \omega^n \right)^{p/(p+1)} + \int_X (\log \rho)^{p+1} \omega^n \right) \\ &\leq (p+1)2nBC_S \max\{V, 1\} \max \left\{ \int_X (\log \rho)^{p+1} \omega^n, 1 \right\}. \end{aligned}$$

Write $q = p + 1 \geq 2$. We have

$$\|\log \rho\|_{L^{q\beta}(X,\omega)} \leq q^{1/q} C_2^{1/q} \max \{ \|\log \rho\|_{L^q(X,\omega)}, 1 \},$$

where $C_2 > 0$ is a constant depending only on n, B, C_S, V . Using the similar strategy of Moser’s iteration again, we derive

$$\begin{aligned} \sup_X (\log \rho) &\leq 2^{n/2} \beta^{n(n-1)/2} C_2^{n/2} \max \{ \|\log \rho\|_{L^2(X,\omega)}, 1 \} \\ &\leq 2^{n/2} \beta^{n(n-1)/2} C_2^{n/2} \max \left\{ \left(\sup_X \log \rho \right)^{1/2} \left(\int_X \log \rho \omega^n \right)^{1/2}, 1 \right\} \end{aligned}$$

and, thus,

$$\sup_X (\log \rho) \leq C_3 \max \left\{ \int_X \log \rho \omega^n, 1 \right\}$$

for some constant $C_3 = C_3(n, B, C_S, V)$.

Now, everything comes down to bounding $\int_X \log \rho \omega^n$ from above. Using Poincaré inequality and (2.3) with $p = 1$, we obtain

$$\int_X |\log \rho - \underline{\log \rho}|^2 \omega^n \leq C_P \int_X |d \log \rho|_\omega^2 \omega^n \leq C_4 \int_X \log \rho \omega^n, \tag{2.4}$$

where $\underline{\log \rho} = (1/V) \int_X \log \rho \omega^n$ is the average of $\log \rho$ and $C_4 = C_4(n, B, C_P)$. Then by (2.2), we can infer that

$$\begin{aligned} \delta \int_X \log \rho \omega^n &= V \delta \underline{\log \rho} \leq V \int_{\{\rho < A\}} \frac{\log \rho \omega^n}{\omega^n} \\ &\leq V \int_{\{\rho < A\}} (\log A - \log \rho + \underline{\log \rho}) \omega^n \\ &\leq V \int_X (|\log \rho - \underline{\log \rho}| + \log A) \omega^n. \end{aligned} \tag{2.5}$$

We use (2.4) and (2.5) to obtain

$$\begin{aligned} \int_X \log \rho \omega^n &\leq \frac{V}{\delta} \left(\int_X |\log \rho - \underline{\log \rho}| \omega^n + V \log A \right) \\ &\leq \frac{V}{\delta} \left(V^{1/2} \left(\int_X |\log \rho - \underline{\log \rho}|^2 \omega^n \right)^{1/2} + V \log A \right) \\ &\leq \frac{V}{\delta} \left(V^{1/2} C_4^{1/2} \left(\int_X \log \rho \omega^n \right)^{1/2} + V \log A \right). \end{aligned}$$

Note that if $x^2 \leq ax + b$ for $a, b > 0$, then $x \leq a/2 + (b + a^2/4)^{1/2}$. Eventually, we obtain

$$\int_X \log \rho \omega^n \leq C_5(n, B, C_S, C_P, V)$$

and this completes the proof of Theorem 2.1.

3. Gauduchon current on the singular fibre

In this section, we construct a Gauduchon factor ρ_0 on X_0^{reg} as the limit of the Gauduchon factors on the nearby fibres X_t . In particular, we derive that the limit ρ is bounded and, thus, $\rho^{1/(n-1)}\omega_0$ is a bounded Gauduchon metric on X_0^{reg} . On the other hand, for a fixed irreducible, reduced, compact complex analytic space X , we show that the $(n - 1)$ -power of a bounded Gauduchon metric can be extend trivially to a pluriclosed current on whole X and also prove a uniqueness result.

3.1 Gauduchon metric on the central fibre

THEOREM 3.1. *Suppose that X_0 is the central fibre in Setup (S). There exists a smooth function $1 \leq \rho \leq C_G$ on X_0^{reg} such that $\rho^{1/(n-1)}\omega_0$ is a bounded Gauduchon metric. Here C_G is the constant introduced in Corollary 2.3.*

Proof. We shall apply standard elliptic theory on some relatively compact subsets of X_0^{reg} to get a smooth function ρ . This ρ is the limit of $(\rho_{t_j})_{j \in \mathbb{N}}$ defined on the fibre X_{t_j} for some sequence $t_j \rightarrow 0$ when $j \rightarrow +\infty$.

Recall that $\pi : \mathcal{X} \rightarrow \mathbb{D}$ is submersive on X_0^{reg} . By the tubular neighborhood theorem, there is an open neighborhood \mathcal{M} of X_0^{reg} in \mathcal{X}^{reg} such that the following statements hold.

- (i) For all $U \in X_0^{\text{reg}}$, there exists an open set $\mathcal{M}_U \subset \mathcal{X}^{\text{reg}}$, a constant $\delta_U > 0$, and a diffeomorphism $\psi_U : U \times \mathbb{D}_{\delta_U} \xrightarrow{\sim} \mathcal{M}_U$ such that the diagram

$$\begin{array}{ccc} U \times \mathbb{D}_{\delta_U} & \xrightarrow[\text{diffeo.}]{\psi_U} & \mathcal{M}_U \subset \mathcal{M} \subset \mathcal{X}^{\text{reg}} \\ \downarrow \text{pr}_2 & \swarrow \pi & \\ \mathbb{D}_{\delta_U} & & \end{array}$$

commutes. In particular, for all $t \in \mathbb{D}_{\delta_U}$, $\psi_U(\cdot, t) : U \rightarrow \mathcal{M}_U$ is a diffeomorphism onto its image $M_{U,t} = \psi_U(U, t) = \mathcal{M}_U \cap X_t$.

- (ii) If $U \Subset V \in X_0^{\text{reg}}$, we have $\delta_U \geq \delta_V > 0$ and $\psi_U(x, t) = \psi_V(x, t)$ for all $x \in U$ and $t \in \mathbb{D}_{\delta_V}$.

Now, we denote by $P_t = \Delta_{\omega_t}^*$ and fix $U_1 \Subset U_2 \Subset X_0^{\text{reg}}$ which are connected open subsets. Note that U_i can be identified with $M_{U_i,t} = \psi_{U_2}(U_i, t)$ for $i \in \{1, 2\}$ and for all $t \in \mathbb{D}_{\delta_{U_2}}$ and, hence, P_t can act on smooth functions defined on U_2 . In addition, on U_2 , the Riemannian metric g_0 induced by ω_0 is quasi-isometric to g_t induced by ω_t , and the volume form ω_0^n is comparable with ω_t^n . In other words, we have a uniform constant $C_{U_2} > 0$ such that

$$C_{U_2}^{-1} \langle \cdot, \cdot \rangle_{\omega_0} \leq \langle \cdot, \cdot \rangle_{\omega_t} \leq C_{U_2} \langle \cdot, \cdot \rangle_{\omega_0} \quad \text{and} \quad C_{U_2}^{-1} \omega_0^n \leq \omega_t^n \leq C_{U_2} \omega_0^n. \tag{3.1}$$

By Gårding inequality, we have

$$\|u\|_{L^2(U_1, \omega_0)} \leq C_{U_1, U_2} (\|P_t u\|_{L^2(U_2, \omega_0)} + \|u\|_{L^2(U_2, \omega_0)})$$

for all $u \in C_c^\infty(U_2)$. The constant C_{U_1, U_2} can be chosen independent of t because the coefficients of P_t move smoothly in t . Choose a cutoff function χ such that $\text{supp}(\chi) \subset U_2$ and $\chi \equiv 1$ on U_1 . From Gårding inequality, we obtain

$$\|\rho_t\|_{L^2(U_1, \omega_0)} \leq C_{U_1, U_2} (\|P_t(\chi\rho_t)\|_{L^2(U_2, \omega_0)} + \|\chi\rho_t\|_{L^2(U_2, \omega_0)}). \tag{3.2}$$

In (3.2), the second term $\|\chi\rho_t\|_{L^2(U_2, \omega_0)}$ is uniformly bounded because of Corollary 2.3. Hence, we only need to estimate $\|P_t(\chi\rho_t)\|_{L^2(U_2, \omega_0)}$. Note that

$$\begin{aligned} P_t(\chi\rho_t) &= \frac{n}{\omega_t^n} dd_t^c(\chi\rho_t\omega_t^{n-1}) \\ &= \frac{n}{\omega_t^n} (2d\rho_t \wedge d_t^c\chi \wedge \omega_t^{n-1} + \chi dd_t^c(\rho_t\omega_t^{n-1}) + \rho_t dd^c(\chi\omega_t^{n-1}) - \chi\rho_t dd_t^c\omega_t^{n-1}) \\ &= 2\langle d\rho_t, d\chi \rangle_{\omega_t} + \rho_t P_t(\chi) - \rho_t \chi \frac{dd_t^c\omega_t^n}{\omega_t^n}. \end{aligned} \tag{3.3}$$

Obviously, $\rho_t P_t(\chi)$ and $\rho_t \chi (dd_t^c\omega_t^{n-1}/\omega_t^n)$ are uniformly bounded, so we only need to control the L^2 -norm of the first term $\langle d\rho_t, d\chi \rangle_{\omega_t}$:

$$\begin{aligned} \int_{U_2} |\langle d\rho_t, d\chi \rangle_{\omega_t}|^2 \omega_0^n &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|_{\omega_0}^2 \right) \int_{U_2} |d\rho_t|_{\omega_t}^2 \omega_t^n \\ &= C_{U_2}^2 \left(\sup_{U_2} |d\chi|_{\omega_0}^2 \right) \int_{M_{U_2,t}} |d\rho_t|_{\omega_t}^2 \omega_t^n \\ &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|_{\omega_0}^2 \right) \int_{X_t} |d\rho_t|_{\omega_t}^2 \omega_t^n \\ &\leq C_{U_2}^2 \left(\sup_{U_2} |d\chi|_{\omega_0}^2 \right) \frac{nB}{2} \int_{X_t} \rho_t^2 \omega_t^n. \end{aligned}$$

Here the first line is by Cauchy–Schwarz inequality and (3.1). The fourth line follows from an argument similar to that used in (2.1) (just replace $-p$ by p). As $1 \leq \rho_t \leq C_G$ and $\text{Vol}_{\omega_t}(X_t) \leq C_V$, we find a uniform bound of $\|P_t(\chi\rho_t)\|_{L^2(U_2, \omega_0)}$. Hence, $\|\rho_t\|_{L^2_s(U_1, \omega_0)}$ is uniformly bounded by some uniform constant $C(U_1, U_2)$.

For higher-order estimates, we apply higher-order Gårding inequalities on the fixed domains $U_1 \Subset U_2 \Subset X_0^{\text{reg}}$:

$$\|u\|_{L^2_{s+2}(U_1, \omega_0)} \leq C_{s, U_1, U_2} (\|P_t u\|_{L^2_s(U_2, \omega_0)} + \|u\|_{L^2(U_2, \omega_0)})$$

for all $u \in C_c^\infty(U_2)$. Let $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ be a relatively compact exhaustion of X_0^{reg} . Differentiating (3.3) on both sides and using a bootstrapping argument, we obtain $\|\rho_t\|_{L^2_s(U_1, \omega_0)} < C(s, \mathcal{U})$ where $C(s, \mathcal{U})$ does not depend on t . By Rellich’s theorem, there exists a subsequence $(\rho_{t_j})_{j \in \mathbb{N}}$ such that ρ_{t_j} converges to ρ in $C^k(\overline{U_1})$ for all $k \in \mathbb{N}$ when $t_j \rightarrow 0$. Therefore, $\text{dd}_0^c(\rho\omega_0^{n-1}) = \lim_{j \rightarrow +\infty} \text{dd}_{t_j}^c(\rho_{t_j}\omega_{t_j}^{n-1}) = 0$ on U_1 . Using a diagonal argument, we can infer that there is a smooth function ρ on X_0^{reg} which is bounded between 1 and C_G , and satisfies $\text{dd}_0^c(\rho\omega_0^{n-1}) = 0$ on X_0^{reg} . \square

3.2 Proof of Theorem B

In this subsection, we always assume that X is an irreducible reduced compact complex space and ω_G is a bounded Gauduchon metric on X . Before proving the uniqueness result and extension property, we recall the existence of cutoff functions with small L^2 -gradients. From a classical property in Riemannian geometry, these cutoff functions do exist on so-called *parabolic* manifolds (see, e.g., [Gla83, EG92]). In our case, one can easily construct explicit cutoff functions by Hironaka’s desingularization and log log-potentials (cf. [Ber12, Lemma 2.2] and [CGP13, § 9]):

LEMMA 3.2. *Suppose that (X, ω) is a compact hermitian variety. There exist cutoff functions $(\chi_\varepsilon)_{\varepsilon > 0} \subset C_c^\infty(X^{\text{reg}})$ satisfying the following properties:*

- (i) χ_ε is increasing to the characteristic function of X^{reg} when ε decreases to zero;
- (ii) $\int_X |\text{dd}^c \chi_\varepsilon \wedge \omega^{n-1}| \rightarrow 0$ when $\varepsilon \rightarrow 0$;
- (iii) $\int_X \text{d}\chi_\varepsilon \wedge \text{d}^c \chi_\varepsilon \wedge \omega^{n-1} \rightarrow 0$ when $\varepsilon \rightarrow 0$.

These functions allow us to perform integration by parts as on a compact manifold. Then we argue that the desired results hold when ε tends to zero. Now, we give a proof of Theorem B:

Proof of Theorem B. We divide the proof in two parts.

Part 1: uniqueness of singular Gauduchon metrics. Assume that ω_G and ω'_G are both bounded Gauduchon metrics in the same conformal class. We write ρ to be the bounded Gauduchon factor satisfying $\rho^{1/(n-1)}\omega_G = \omega'_G$. Let $(\chi_\varepsilon)_{\varepsilon > 0}$ be cutoff functions given in Lemma 3.2. From Stokes formula and direct computations, we derive

$$\int_{X^{\text{reg}}} \chi_\varepsilon |\text{d}\rho|_{\omega_G}^2 \omega_G^n = \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 \text{dd}^c \chi_\varepsilon \wedge \omega_G^{n-1}. \tag{3.4}$$

Indeed,

$$\begin{aligned}
 \int_{X^{\text{reg}}} \chi_\varepsilon |d\rho|_{\omega_G}^2 \omega_G^{n-1} &= \int_{X^{\text{reg}}} \chi_\varepsilon d\rho \wedge d^c \rho \wedge \omega_G^{n-1} = - \int_{X^{\text{reg}}} \rho d(\chi_\varepsilon d^c \rho \wedge \omega_G^{n-1}) \\
 &= - \int_{X^{\text{reg}}} \rho (d\chi_\varepsilon \wedge d^c \rho \wedge \omega_G^{n-1} + \chi_\varepsilon dd^c \rho \wedge \omega_G^{n-1} - \chi_\varepsilon d^c \rho \wedge d\omega_G^{n-1}) \\
 &= -\frac{1}{2} \int_{X^{\text{reg}}} d\rho^2 \wedge d^c \chi_\varepsilon \wedge \omega_G^{n-1} + \int_{X^{\text{reg}}} \rho \chi_\varepsilon d\rho \wedge d^c \omega_G^{n-1} \\
 &= \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 d(d^c \chi_\varepsilon \wedge \omega_G^{n-1}) - \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 d(\chi_\varepsilon d^c \omega_G^{n-1}) \\
 &= \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 dd^c \chi_\varepsilon \wedge \omega_G^{n-1} - \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 d^c \chi_\varepsilon \wedge d\omega_G^{n-1} \\
 &\quad - \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 d\chi_\varepsilon \wedge d^c \omega_G^{n-1} \\
 &= \frac{1}{2} \int_{X^{\text{reg}}} \rho^2 dd^c \chi_\varepsilon \wedge \omega_G^{n-1}.
 \end{aligned}$$

By Lemma 3.2 and the boundedness of ω_G , we can see that $\int_X |dd^c \chi_\varepsilon \wedge \omega_G^{n-1}|$ converges to zero when ε tends to zero. As ρ is bounded, the right-hand side of (3.4) goes to zero and, hence,

$$\int_{X^{\text{reg}}} |d\rho|_{\omega_G}^2 \omega_G^n = 0.$$

Because X^{reg} is connected by the irreducible assumption, ρ is a constant.

Part 2: extension to a pluriclosed current. We already have a smooth pluriclosed bounded positive $(n-1, n-1)$ -form ω_G^{n-1} on X^{reg} . Note that ω_G^{n-1} can extend trivially as a bounded positive $(n-1, n-1)$ -current T on X . If ω_G^{n-1} is closed, so would be T by the Skoda–El Mir extension result [Sko82, EM84]. Some results for plurisubharmonic currents do exist in the literature (see, e.g., [AB93, DEEM03]), but we could not find the one of interest for us here. Therefore, we provide a quick proof.

Again, we are going to use good cutoff functions $(\chi_\varepsilon)_{\varepsilon>0}$ in Lemma 3.2 to prove that T is pluriclosed in the sense of currents. Fix a smooth function f on X . We need to show that

$$\langle f, dd^c T \rangle := \int_X dd^c f \wedge T = 0.$$

Using the cutoff functions constructed in Lemma 3.2, we can write

$$\int_X dd^c f \wedge T = \underbrace{\int_{R_\varepsilon} \chi_\varepsilon dd^c f \wedge T}_{:=I_\varepsilon} + \underbrace{\int_{S_\varepsilon} (1 - \chi_\varepsilon) dd^c f \wedge T}_{:=II_\varepsilon},$$

where R_ε is a small open neighborhood of the closure of $\{\chi_\varepsilon > 0\}$ contained in X^{reg} and similarly S_ε is a small neighborhood of the closure of $\{\chi_\varepsilon < 1\}$. According to Lemma 3.2, as $\varepsilon \rightarrow 0$, R_ε tends to X^{reg} and S_ε shrinks to X^{sing} . Therefore, II_ε converges to zero when ε goes to zero.

We compute the term I_ε by the Stokes formula:

$$\begin{aligned} I_\varepsilon &= \int_{R_\varepsilon} f \, dd^c(\chi_\varepsilon \omega_G^{n-1}) = \int_{R_\varepsilon} f (dd^c \chi_\varepsilon \wedge \omega_G^{n-1} + 2d\omega_G^{n-1} \wedge d^c \chi_\varepsilon) \\ &= - \underbrace{\int_{R_\varepsilon} f \, dd^c \chi_\varepsilon \wedge \omega_G^{n-1}}_{:=III_\varepsilon} - 2 \underbrace{\int_{R_\varepsilon} df \wedge d^c \chi_\varepsilon \wedge \omega_G^{n-1}}_{:=IV_\varepsilon}. \end{aligned}$$

By Cauchy–Schwarz inequality, the term IV_ε can be bounded by

$$|IV_\varepsilon| \leq \underbrace{\int_{R_\varepsilon} \mathbf{1}_{\{d\chi_\varepsilon \neq 0\}} df \wedge d^c f \wedge \omega_G^{n-1}}_{:=V_\varepsilon} + \underbrace{\int_{R_\varepsilon} d\chi_\varepsilon \wedge d^c \chi_\varepsilon \wedge \omega_G^{n-1}}_{:=VI_\varepsilon}.$$

Using the dominated convergence theorem, V_ε converges to zero when ε goes to zero, because the set $\{d\chi_\varepsilon \neq 0\}$ is contained in S_ε and S_ε shrinks to X^{sing} . Applying Lemma 3.2, III_ε and VI_ε converge to zero as ε tending to zero. These yield that $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. All in all, we have $\int_X dd^c f \wedge T = 0$ for all $f \in C^\infty(X)$; hence, $dd^c T = 0$ in the sense of currents. \square

ACKNOWLEDGEMENTS

The author is grateful to his thesis advisors Vincent Guedj and Henri Guenancia for their support, suggestions and encouragement. The author thanks Tat Dat Tô for pointing out the reference [ACS17] and Tsung-Ju Lee for helpful comments. The author would like to thank the anonymous referees for useful comments and suggestions. This work has benefited from state aid managed by the ANR under the ‘PIA’ program bearing the reference ANR-11-LABX-0040 (research project HERMETIC). The author is also partially supported by the ANR project PARAPLUI and the EUR MINT project ANR-18-EURE-0023.

REFERENCES

AB93 L. Alessandrini and G. Bassanelli, *Plurisubharmonic currents and their extension across analytic subsets*, Forum Math. **5** (1993), 577–602.

ACS17 D. Angella, S. Calamai and C. Spotti, *On the Chern-Yamabe problem*, Math. Res. Lett. **24** (2017), 645–677.

Ber12 B. Berndtsson, *L^2 -extension of $\bar{\partial}$ -closed form*, Illinois J. Math. **56** (2012), 21–31 (2013).

CGP13 F. Campana, H. Guenancia and M. Păun, *Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), 879–916.

Chu12 M.-T. Chuan, *Existence of Hermitian-Yang-Mills metrics under conifold transitions*, Comm. Anal. Geom. **20** (2012), 677–749.

Cle83 C. H. Clemens, *Double solids*, Adv. Math. **47** (1983), 107–230.

CPY21 T. C. Collins, S. Picard and S.-T. Yau, *Stability of the tangent bundle through conifold transitions*, Preprint (2021), [arXiv:2102.11170](https://arxiv.org/abs/2102.11170).

DEEM03 K. Dabbek, F. Elkhadhra and H. El Mir, *Extension of plurisubharmonic currents*, Math. Z. **245** (2003), 455–481.

Dem85 J.-P. Demailly, *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. Fr. (N.S.) **19** (1985), 124.

DGG20 E. Di Nezza, V. Guedj and H. Guenancia, *Families of singular Kähler-Einstein metrics*, J. Eur. Math. Soc., to appear. Preprint (2020), [arXiv:2003.08178](https://arxiv.org/abs/2003.08178).

- EM84 H. El Mir, *Sur le prolongement des courants positifs fermés*, Acta Math. **153** (1984), 1–45.
- EG92 L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992).
- FU13 A. Fino and L. Ugarte, *On generalized Gauduchon metrics*, Proc. Edinb. Math. Soc. (2) **56** (2013), 733–753.
- Fri86 R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986), 671–689.
- Fri91 R. Friedman, *On threefolds with trivial canonical bundle*, in *Complex geometry and Lie theory (Sundance, UT, 1989)*, Proceedings of Symposia in Pure Mathematics, vol. 53 (American Mathematical Society, Providence, RI, 1991), 103–134.
- FLY12 J. Fu, J. Li and S.-T. Yau, *Balanced metrics on non-Kähler Calabi–Yau threefolds*, J. Differential Geom. **90** (2012), 81–129.
- FWW13 J. Fu, Z. Wang and D. Wu, *Semilinear equations, the γ_k function, and generalized Gauduchon metrics*, J. Eur. Math. Soc. (JEMS) **15** (2013), 659–680.
- FY08 J. Fu and S.-T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge–Ampère equation*, J. Differential Geom. **78** (2008), 369–428.
- Gau77 P. Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris **285** (1977), A387–A390.
- Gla83 M. Glasner, *Stokes' theorem and parabolicity of Riemannian manifolds*, Proc. Amer. Math. Soc. **87** (1983), 70–72.
- Har74 R. Hartshorne, *Topological conditions for smoothing algebraic singularities*, Topology **13** (1974), 241–253.
- Hul86 C. M. Hull, *Superstring compactifications with torsion and spacetime supersymmetry*, Superunification and Extra Dimensions (Torino, 1985) (World Scientific, Singapore, 1986), 347–375.
- Li21 P. Li, *On some applications of Gauduchon metrics*, Geom. Dedicata **213** (2021), 473–486.
- LY87 J. Li and S.-T. Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, Advanced Series in Mathematical Physics, vol. 1 (World Scientific, Singapore, 1987), 560–573.
- LT94 P. Lu and G. Tian, *The complex structure on a connected sum of $S^3 \times S^3$ with trivial canonical bundle*, Math. Ann. **298** (1994), 761–764.
- MS73 J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of R^n* , Comm. Pure Appl. Math. **26** (1973), 361–379.
- PPZ18 D. H. Phong, S. Picard and X. Zhang, *Geometric flows and Strominger systems*, Math. Z. **288** (2018), 101–113.
- Rei87 M. Reid, *The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible*, Math. Ann. **278** (1987), 329–334.
- Ros06 M. Rossi, *Geometric transitions*, J. Geom. Phys. **56** (2006), 1940–1983.
- RZ11 W.-D. Ruan and Y. Zhang, *Convergence of Calabi–Yau manifolds*, Adv. Math. **228** (2011), 1543–1589.
- Sko82 H. Skoda, *Prolongement des courants, positifs, fermés de masse finie*, Invent. Math. **66** (1982), 361–376.
- Str86 A. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274** (1986), 253–284.
- STW17 G. Székelyhidi, V. Tosatti and B. Weinkove, *Gauduchon metrics with prescribed volume form*, Acta Math. **219** (2017), 181–211.
- Tia92 G. Tian, *Smoothing 3-folds with trivial canonical bundle and ordinary double points*, in *Essays on mirror manifolds* (International Press, Hong Kong, 1992), 458–479.

- TW10 V. Tosatti and B. Weinkove, *The complex Monge–Ampère equation on compact Hermitian manifolds*, J. Amer. Math. Soc. **23** (2010), 1187–1195.
- Uen75 K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, vol. 439 (Springer, Berlin, 1975). Notes written in collaboration with P. Cherenack.
- UY86 K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, in *Proceedings of the symposium on Frontiers of the Mathematical Sciences: 1985*, Communications on Pure and Applied Mathematics, vol. 39 (Wiley, New York, 1986), S257–S293.
- Yau78 S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), 339–411.
- Yos97 K.-i. Yoshikawa, *Degeneration of algebraic manifolds and the spectrum of Laplacian*, Nagoya Math. J. **146** (1997), 83–129.

Chung-Ming Pan Chung_Ming.Pan@math.univ-toulouse.fr

Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS,
118 route de Narbonne, F-31062 Toulouse Cedex 9, France