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The L_p Minkowski problem for q-capacity

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In the present paper, we first introduce the concepts of the L_p q-capacity measure and L_p mixed q-capacity and then prove some geometric properties of L_p q-capacity measure and a L_p Minkowski inequality for the q-capacity for any fixed $p \geqslant 1$ and $q > n$. As an application of the L_p Minkowski inequality mentioned above, we establish a Hadamard variational formula for the q-capacity under p-sum for any fixed $p \geqslant 1$ and $q > n$, which extends results of Akman *et al.* (*Adv. Calc. Var.* (in press)). With the Hadamard variational formula, variational method and L_p Minkowski inequality mentioned above, we prove the existence and uniqueness of the solution for the L_p Minkowski problem for the q-capacity which extends some beautiful results of Jerison (1996, *Acta Math.* **176**, 1–47), Colesanti *et al.* (2015, *Adv. Math.* **285**, 1511–588), Akman *et al.* (*Mem. Amer. Math. Soc.* (in press)) and Akman *et al.* (*Adv. Calc. Var.* (in press)). It is worth mentioning that our proof of Hadamard variational formula is based on L_p Minkowski inequality rather than the direct argument which was adopted by Akman (*Adv. Calc. Var.* (in press)). Moreover, as a consequence of L_p Minkowski inequality for q-capacity, we get an interesting isoperimetric inequality for q-capacity.

Keywords: q-Capacity; L_p q-capacity measure; L_p Minkowski inequality; L_p Minkowski problem; Variational method; Hadamard variational formula

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1. Introduction

A non-empty compact, convex subset of $\mathbb{R}^n$ with non-empty interior is called a convex body. Let $\mathcal{K}^n$ be the set of all convex bodies of $\mathbb{R}^n$ and $\mathcal{K}_0^n$ be the set of all convex bodies of $\mathbb{R}^n$ containing the origin in their interior.

The support function $h_\Omega : \mathbb{R}^n \mapsto \mathbb{R}$ of a convex body Ω is defined as follows:

$$h_\Omega(u) = \sup_{x\in\Omega} x \cdot u, \forall u \in \mathbb{R}^n.$$

For any nonnegative real numbers α, β and any convex bodies Ω_1 and Ω_2, the Minkowski linear combination of Ω_1 and Ω_2 is defined by:

$$\alpha\Omega_1 + \beta\Omega_2 = \{\alpha x + \beta y : x \in \Omega_1, y \in \Omega_2\}.$$

The support function of $\alpha\Omega_1 + \beta\Omega_2$ is given by

$$h_{\alpha\Omega_1+\beta\Omega_2} = \alpha h_{\Omega_1} + \beta h_{\Omega_2}. \tag{1.1}$$

Inversely, the Minkowski linear combination $\alpha\Omega_1 + \beta\Omega_2$ is also defined by (1.1) (see Schneider [58]).

For a smooth convex domain Ω of $\mathbb{R}^n$, the *Gauss map* $g_\Omega(x) \in \mathbb{S}^{n-1}$ is the unique unit outer normal vector at x for any $x \in \partial\Omega$. From a well-known result of Aleksandrov, convex function is twice differentiable almost everywhere (see Schneider [58] or Evans–Gariepy [31]). Therefore, for a convex body Ω of $\mathbb{R}^n$ without the assumption of smoothness, the Gauss map g_Ω of Ω is well-defined a.e. on $\partial\Omega$ with respect to $(n-1)$-dimensional Hausdorff measure. Let Ω be a convex body of $\mathbb{R}^n$ and g_Ω be the Gauss map of Ω. The *reverse Gauss map* $g_\Omega^{-1} : \mathbb{S}^{n-1} \mapsto \partial\Omega$ is defined by:

$$g_\Omega^{-1}(E) = \{x \in \partial\Omega : g_\Omega(x) \text{ is well-defined and } g_\Omega(x) \in E\}$$

for any set $E \subseteq \mathbb{S}^{n-1}$. If E is a Borel set, then $g_\Omega^{-1}(E)$ is $\mathcal{H}^{n-1}$-measurable (see Schneider [58] or Colesanti *et al.* [29]).

The *volume* V of convex body $\Omega \in \mathcal{K}_0^n$ can be described by

$$V(\Omega) = \frac{1}{n}\int_{\partial\Omega} h_\Omega(g_\Omega(x))\mathrm{d}\mathcal{H}^{n-1}$$

where g_Ω is the Gauss map on Ω and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

The *surface area measure* $S_{1,\Omega}$ of $\Omega \in \mathcal{K}_0^n$ is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$ defined by the following beautiful formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\Omega + t\Omega_1)|_{t=0} = \int_{\partial\Omega} h_{\Omega_1}(g_\Omega(x))\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} h_{\Omega_1}(g_\Omega(r_\Omega(\theta)))\,\mathrm{d}S_{1,\Omega}(\theta), \tag{1.2}$$

for any $\Omega_1 \in \mathcal{K}_0^n$ where g_Ω is the Gauss map and r_Ω is the radial map on Ω defined in § 2 (see Schneider [58]). From the definition of the surface area measure $S_{1,\Omega}$ of Ω, we see that

$$S_{1,\Omega}(E) = \int_E \mathrm{d}S_{1,\Omega}(\theta) = \int_{g_\Omega^{-1}(E)} \mathrm{d}\mathcal{H}^{n-1} \tag{1.3}$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$. If we multiply by $1/n$ on integral on the right-hand side of (1.2), we have the well-known Minkowski's *mixed volume* of Ω and Ω_1, $V_{1,1}(\Omega, \Omega_1)$,

$$V_{1,1}(\Omega, \Omega_1) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{\Omega_1}(g_\Omega(r_\Omega(\theta)))\,\mathrm{d}S_{1,\Omega}(\theta), \tag{1.4}$$

(see Schneider [58]). The relationship between *volume* and *mixed volume* can be described by the following well-known inequality.

The classical Minkowski inequality: *For any $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$, we have*

$$V_{1,1}(\Omega_1, \Omega_2) \geqslant V^{1-1/n}(\Omega_1)V^{1/n}(\Omega_2), \tag{1.5}$$

equality in (1.5) holds if and only if Ω_1 and Ω_2 are homothetic.

The classical Minkowski problem can be stated as follows:

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The classical Minkowski Problem: *Given a positive finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, under what necessary and sufficient conditions does there exist a unique(up to a translation) convex body Ω of $\mathbb{R}^n$ such that $S_{1,\Omega} = \mu$?*

Minkowski [54, 55], Aleksandrov [1–3] and Fenchel–Jessen [32] showed that

THEOREM A.1. *Let μ be a positive and finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$. Then there exists a unique (up to a translation) convex body $\Omega \in \mathcal{K}_0^n$ such that $S_{1,\Omega} = \mu$ if and only if μ satisfies the following two conditions:*

(A.1.1) the measure μ is not concentrated on any closed hemisphere, that is,

$$\inf_{e\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ \,\mathrm{d}\mu(\theta) > 0,$$

where $(e\cdot\theta)_+ = \max\{e\cdot\theta, 0\}$.

(A.1.2) the centroid of the measure μ is at the origin, that is,

$$\int_{\mathbb{S}^{n-1}} \xi \,\mathrm{d}\mu(\xi) = O.$$

The uniqueness of solution of classical Minkowski problem followed from the Minkowski inequality (1.5) directly. Comparing to the uniqueness, the existence of solution of classical Minkowski problem seems to be more complicated and more interesting.

Minkowski, Aleksandrov and Fenchel–Jessen adopted the powerful variational argument to solve the problem. More precisely, they first transformed the solvability of the classical Minkowski problem into the solvability of an associated variational problem and then proved that the variational problem had a solution. In particular, Minkowski dealt with the original problem for discrete measure and extended the result to the measures whose density function is continuous via the approximation argument. Aleksandrov and Fenchel–Jessen extended the results of Minkowski to arbitrary Borel measure on the unit sphere $\mathbb{S}^{n-1}$. The details can be found in pp. 317–320 of Aleksandrov [4], pp. 108–112 of Aleksandrov [3], pp. 121–131 of Bonnesen–Fenchel [12], pp. 60–67 of Busemann [16], pp. 75–86 of Bakelman [8], pp. 22–32 of Pogorelov [57] and pp. 455–459 of Schneider [58].

A basic but important property of the surface area measure is its weak continuity in the sense of Hausdorff metric (see p. 510 of [23] or pp. 208–223 of [58]). This means that we can solve the Minkowski problem in the smooth frame and then achieve the goal via the approximation argument. This route was adopted by Cheng and Yau [23] for the classical Minkowski problem and Jerison [45] for a Minkowski-type problem.

In smooth frame, the classical Minkowski problem can be formulated as follows. Let Ω be a $C^{2,\alpha}$-smooth and strictly convex domain of $\mathbb{R}^n$, (that is $\partial\Omega$ is $C^{2,\alpha}$-smooth and the Gauss curvature $K(x) > 0$ for any $x \in \partial\Omega$), it follows from (1.3) that

$$S_{1,\Omega}(E) = \int_{g_\Omega^{-1}(E)} \mathrm{d}\sigma = \int_E \frac{1}{K(\theta)} \,\mathrm{d}\theta = \int_E \det(h_{ij}(\theta) + \delta_{ij}h(\theta)) \,\mathrm{d}\theta \tag{1.6}$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$ where h is the support function of Ω, h_{ij} is the second-order covariant derivatives of h on $\mathbb{S}^{n-1}$ and δ_{ij} is the Kronecker delta. We let $f \in C^{\alpha}(\mathbb{S}^{n-1})$ be the density function of the positive Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ for some $\alpha \in (0,1)$, that is,

$$\mu(E) = \int_E f(\theta)\,\mathrm{d}\theta \tag{1.7}$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$. From (1.6) and (1.7), we can see that, in order to solve the Minkowski problem, it suffices to analyse the existence and uniqueness of convex solution for the following Monge–Ampère equation,

$$\det(h_{ij}(\theta) + \delta_{ij}h(\theta)) = f(\theta), \forall\theta \in \mathbb{S}^{n-1}. \tag{1.8}$$

There are many beautiful results in this direction, see for example Lewy [**50**], Nirenberg [**56**], Cheng–Yau [**23**], Pogorelov [**57**] and Caffarelli [**17**–**20**]. Moreover, apart from the existence and uniqueness of solution for the Minkowski problem, we also get the following regularity result from the studies of [**18**, **19**, **23**, **50**, **56**, **57**].

THEOREM A.2. *Let f be the positive density function of the Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, that is $\mathrm{d}\mu(\theta) = f(\theta)\,\mathrm{d}\theta$ and $\inf_{\theta\in\mathbb{S}^{n-1}} f(\theta) > 0$. Let $\Omega \in \mathcal{K}_0^n$ such that $S_{1,\Omega} = \mu$. If $f \in C^{k,\alpha}(\mathbb{S}^{n-1})$, the boundary of Ω is of $C^{k+2,\alpha}$ class.*

After the great studies of Minkowski, Aleksandrov, Fenchel, Jessen, Lewy, Nirenberg, Cheng–Yau and Caffarelli, there are many subsequent researches in this topic. On the one hand, similar problems have been solved for other important geometric measures in convex geometry, such as *curvature measure*, *dual curvature measure* and their L_p's generalizations, see for example [**10**, **11**, **14**, **15**, **22**, **24**, **25**, **27**, **34**–**36**, **36**, **38**, **40**–**44**, **44**, **48**, **51**–**53**, **59**, **63**, **64**].

In order to formulate the so-called L_p version of the classical Minkowski problem, we need to state the L_p versions of mixed volume and surface area measure.

We first recall the L_p *surface area measure* and L_p *mixed volume* proposed by Lutwak [**51**]. The well-known p-sum of Firey [**33**] for $p \geqslant 1$ was formulated as follows. We let Ω_1 and Ω_2 be two convex bodies of $\mathbb{R}^n$ and h_{Ω_i} be the support function of $\Omega_i (i = 1, 2)$ respectively. For any nonnegative real number α, β, the p-linear combination of h_{Ω_1} and h_{Ω_2} is defined by

$$h_{\alpha\Omega_1 +_p \beta\Omega_2} = (\alpha h_{\Omega_1}^p + \beta h_{\Omega_2}^p)^{1/p}.$$

In particular, when $p = 1$, Firey's p-linear combination is the classical Minkowski linear combination. The convex body with support function $h_{\alpha\Omega_1 +_p \beta\Omega_2}$ is denoted by $\alpha\Omega_1 +_p \beta\Omega_2$ (see Schneider [**58**]). In [**51**], Lutwak proposed the p-sum of a support function h_{Ω_1} to a convex body Ω_1 and a continuous function f, which is defined by:

$$h_{p,t} \triangleq h_{\Omega_1} +_p tf = (h_{\Omega_1}^p + tf^p)^{1/p}.$$

for any sufficiently small real number t. In particular, if f is a support function of a convex body $\Omega(f)$ and $t \geqslant 0$, we see that $h_{p,t} = h_{\Omega_1 +_p t\Omega(f)}$. In the same paper, Lutwak also built up the following Hadamard variational formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\Omega_1 +_p t\Omega_2)|_{t=0} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h^p_{\Omega_2}(g_{\Omega_1}(r_{\Omega_1}(\theta)))h^{1-p}_{\Omega_1}(g_{\Omega_1}(r_{\Omega_1}(\theta)))\,\mathrm{d}S_{1,\Omega_1}(\theta)$$

for any convex bodies $\Omega_1, \Omega_2 \in \mathcal{K}^n_0$. This leaded him to introduce the so-called L_p *surface area measure* $S_{p,\Omega}$ of a convex body Ω and L_p *mixed volume* $V_{p,1}$ of two convex bodies, which are defined as follows:

$$\mathrm{d}S_{p,\Omega} = h^{1-p}_{\Omega}\,\mathrm{d}S_{1,\Omega} \tag{1.9}$$

and

$$V_{p,1}(\Omega_1, \Omega_2) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h^p_{\Omega_2}(g_{\Omega_1}(r_{\Omega_1}(\theta)))\,\mathrm{d}S_{p,\Omega_1}(\theta)$$

provided the right-hand side of (1.9) is a finite measure. In particular, if $p = 1$, the L_p *surface area measure* $S_{p,\Omega}$ and L_p *mixed volume* $V_{p,1}$ are the classical *surface area measure* and *mixed volume* defined in (1.2) and (1.4). For any fixed $p \geqslant 1$, by the definition of L_p *mixed volume* and Hölder inequality, Lutwak [**51**] proved the following interesting inequality.

L_p Minkowski inequality for volume: *For any fixed $p > 1$ and $\Omega_1, \Omega_2 \in \mathcal{K}^n_0$, we have*

$$V_{p,1}(\Omega_1, \Omega_2) \geqslant V^{1-p/n}(\Omega_1)V^{p/n}(\Omega_2), \tag{1.10}$$

equality in (1.10) holds if and only if Ω_1 and Ω_2 are dilatates.

Lutwak [**51**] proposed the following L_p Minkowski problem:

The L_p Minkowski problem. *For any fixed $p > 1$, given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, under what necessary and sufficient conditions does there exist a unique convex body Ω of $\mathbb{R}^n$ such that $S_{p,\Omega} = \mu$?*

In the same paper, Lutwak [**51**] gave a positive answer to the L_p Minkowski problem for any fixed $p > 1$ and $p \neq n$ when μ is even and satisfies (A.1.1). Later, Lutwak and Oliker [**52**] resolved the L_p Minkowski problem via the method of continuity. As the consequence of the main result of [**52**], Lutwak–Oliker built up a regularity result for the solution to the L_p Minkowski problem. Moreover, Chou and Wang [**27**] extended their result to more general p via the theory of PDEs. One of the beautiful results of Lutwak–Oliker and Chou–Wang can be stated as follows:

THEOREM A.4. *For any fixed $p > -n$. Let f be the positive density function of the Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, that is $\mathrm{d}\mu(\theta) = f(\theta)\,\mathrm{d}\theta$ and $\inf_{\theta\in\mathbb{S}^{n-1}} f(\theta) > 0$. Let $\Omega \in \mathcal{K}^n_0$ such that $S_{p,\Omega} = \mu$. If $f \in C^{k,\alpha}(\mathbb{S}^{n-1})$, the boundary of Ω is of $C^{k+2,\alpha}$ class.*

For more results about the regularity result to the L_p Minkowski problem readers can be referred to Chou and Wang [**27**], Bianchi *et al.* [**9**] and Bianchi *et al.* [**10**].

On the other hand, similar problems have also been solved for other important Borel measures in physics, such as *Harmonic measure*, *capacity measure*, *$\mathcal{A}$-capacity measure*, *the first Dirichlet eigenvalue measure* and *the torsion measure* of the Laplacian and some of their L_p generalizations, see for example [**5–7**, **26**, **28–30**, **39**, **45–47**, **62**, **65**].

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To the best knowledge of the author, there is no research on the L_p Brunn–Minkowski theory for q-capacity for any fixed $p > 1$ and $q > n$. This leads us to focus on the L_p Brunn–Minkowski theory for q-capacity for any fixed $p > 1$ and $q > n$ in the present paper.

From the analysis of classical Brunn–Minkowski theory, we can see that important concepts in this topic are the surface area measure, volume, mixed volume and Minkowski inequality. This means that we need to formulate the so-called L_p q-capacity measure, L_p mixed q-capacity and associated Minkowski inequality.

We first recall the *q-capacity*, *mixed q-capacity* and *q-capacity measure* for any fixed $q > n$. Let Ω be a convex domain of $\mathbb{R}^n$ and U_Ω be the unique solution of the following problem:

$$\begin{cases} \Delta_q U_\Omega = 0, x \in \mathbb{R}^n \backslash \Omega, \\ U_\Omega = 0, x \in \partial\Omega, U_\Omega(x) \simeq |x|^{(q-n)/(q-1)}, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.11}$$

that is U_Ω is the so-called q-Green function on $\mathbb{R}^n \backslash \Omega$ whose pole is at infinity. From a result of Akman *et al.* [6], we see that

$$U_\Omega(x) - F(x) \to a,$$

as $|x| \to \infty$ where a is a constant depending only on the convex domain Ω and

$$F(x) \simeq |x|^{(q-n)/(q-1)}.$$

The *q-capacity* $\mathcal{C}(\Omega)$ of Ω is defined as follows:

$$\mathcal{C}(\Omega) \triangleq (-a)^{q-1}.$$

Akman *et al.* [6] formulated the following beautiful formulas:

$$\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_0) = \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} h_{\Omega_0}(g_{\Omega_0}(r_{\Omega_0}(\theta)))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_0}(\theta)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_0 + t\Omega_1)|_{t=0} = \int_{\mathbb{S}^{n-1}} h_{\Omega_1}(g_{\Omega_0}(r_{\Omega_0}(\theta)))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_0}(\theta) \tag{1.12}$$

for any convex bodies $\Omega_0, \Omega_1 \in \mathcal{K}^n_0$ and

$$\mu^{\mathcal{C}}_{1,\Omega_0}(E) = \int_{g^{-1}_{\Omega_0}(E)} |\nabla U_{\Omega_0}|^q \,\mathrm{d}\mathcal{H}^{n-1}$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$ (see (10.37) of p. 57 and (10.2) of p. 49 of [6]). Combining (1.12) and the definition of the *surface area measure* $S_{1,\Omega}$, we say $\mu^{\mathcal{C}}_{1,\Omega_0}$ is the *q-capacity measure* of a convex domain Ω_0. The so-called *mixed q-capacity* $\mathcal{C}_{1,1}$ of Ω_0 and Ω_1 can be defined by:

$$\mathcal{C}_{1,1}(\Omega_0, \Omega_1) \triangleq \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} h_{\Omega_1}(g_{\Omega_0}(r_{\Omega_0}(\theta)))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_0}(\theta).$$

Combining the definition of the mixed q-capacity and an interesting Brunn–Minkowski inequality proved by Akman *et al.* [6], we have the following interesting inequality:

Minkowski inequality for q***-capacity:*** *For any fixed* $n \geqslant 2$, $q > n$ *and* $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$, *we have*

$$\mathcal{C}_{1,1}(\Omega_1, \Omega_2) \geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-1/(q-n)}(\Omega_1)\mathcal{C}^{1/(q-n)}(\Omega_2), \tag{1.13}$$

equality in (1.13) *holds if and only if* Ω_1 *and* Ω_2 *are homothetic.*

For any fixed $p \geqslant 1$, adopting a similar argument of Lutwak [**51**], if we replace the Minkowski linear combination by the Firey's p-linear combination in (1.12), it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_1 +_p t\Omega_2)|_{t=0} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h^p_{\Omega_2}(g_{\Omega_1}(r_{\Omega_1}(\theta))h^{1-p}_{\Omega_1}(g_{\Omega_1}(r_{\Omega_1}(\theta))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_1}(\theta) \tag{1.14}$$

for any two $C^{2,\alpha}$-smooth and strictly convex bodies $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$. By the weak continuity of the *q-capacity measure* proved by Akman *et al.* in [**6**] in the sense of Hausdorff metric, we can extend formula (1.14) to the general convex bodies $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$ without the assumption of smoothness via the approximation argument. Thus, we define the so-called L_p *q-capacity measure* $\mu^{\mathcal{C}}_{p,\Omega_1}$ and L_p *mixed q-capacity* $\mathcal{C}_{p,1}$ by:

$$\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_1} \triangleq h^{1-p}_{\Omega_1}\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_1}$$

and

$$\mathcal{C}_{p,1}(\Omega_1, \Omega_2) \triangleq \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} h^p_{\Omega_2}(g_{\Omega_1}(r_{\Omega_1}(\theta)))\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_1}(\theta) \tag{1.15}$$

for all $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$. In particular, if Ω is $C^{2,\alpha}$-smooth and strictly convex, it is easy to see that

$$\mu^{\mathcal{C}}_{p,\Omega}(E) = \int_E h^{1-p}|\nabla U_\Omega(g^{-1}(\theta))|^q \det(h_{ij}(\theta) + \delta_{ij}h(\theta))\,\mathrm{d}\theta$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$ where $g^{-1}(\theta) = \nabla h(\theta)$. That is, L_p *q-capacity* measure $\mu^{\mathcal{C}}_{p,\Omega}$ is absolutely continuous with respect to spherical Lebesgue measure and its density function is

$$h^{1-p}|\nabla U_\Omega(g^{-1}(\theta))|^q \det(h_{ij}(\theta) + \delta_{ij}h(\theta))$$

provided Ω is $C^{2,\alpha}$-smooth and strictly convex.

The main results of the present paper can be stated as follows:

Theorem 1.1. *For any fixed* $n \geqslant 2$, $p \geqslant 1$ *and* $q > n$, *any* $\{\Omega_i\}_{i=0}^\infty \in \mathcal{K}_0^n$ *and any fixed* $i \in \{0, 1, \ldots\}$, *we let* $\mu^{\mathcal{C}}_{p,\Omega_i}$ *be the* L_p *q-capacity measure of* Ω_i *and* h_{Ω_i} *be the support function of* Ω_i. *Then the following statements hold:*

(a) $\mu^{\mathcal{C}}_{p,\Omega_0}$ *is absolutely continuous with respect to the surface area measure* S_{1,Ω_0}.

(b) *if* $\Omega_i \to \Omega_0$ *in the sense of Hausdorff metric as* $i \to \infty$, *then*

$$\mu^{\mathcal{C}}_{p,\Omega_i} \to \mu^{\mathcal{C}}_{p,\Omega_0}$$

weakly as $i \to \infty$.

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(c) *For any* $\Omega_0 \in \mathcal{K}_0^n$, $\mu^{\mathcal{C}}_{p,\Omega_0}$ *is not concentrated on any closed hemisphere, that is,*

$$\inf_{e\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) > 0,$$

where $(e\cdot\theta)_+ = \max\{e\cdot\theta, 0\}$.

THEOREM 1.2. *For any fixed* $n \geqslant 2$, $p \geqslant 1$, $q > n$ *and* $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$, *we let* $\mathcal{C}_{p,1}(\Omega_1, \Omega_2)$ *be the* L_p *mixed q-capacity of* Ω_1 *and* Ω_2. *Then, the following statements hold:*

(a)

$$\mathcal{C}_{p,1}(\Omega_1,\Omega_2) \geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1)\mathcal{C}^{p/(q-n)}(\Omega_2), \tag{1.16}$$

equality in (1.16) *holds if and only if* Ω_1 *and* Ω_2 *are homothetic.*

(b) *if* $p > 1$ *and*

$$\mathcal{C}_{p,1}(\Omega_1,\Omega) = \mathcal{C}_{p,1}(\Omega_2,\Omega), \quad \forall\, \Omega \in \mathcal{K}_0^n,$$

then, $\Omega_1 = \Omega_2$.

With theorem 1.2, we have the following isoperimetric inequality, which has independent interests.

COROLLARY 1.3. *For any fixed* $n \geqslant 2$, $p \geqslant 1$, $q > n$ *and* $\Omega \in \mathcal{K}_0^n$, *we have*

$$\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega) \leqslant \frac{q(q-1)}{(q-n)\mathcal{C}^{p/(q-n)}(B_1)} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega}(\theta), \tag{1.17}$$

equality in (1.17) *holds if and only if* Ω *is a ball of* $\mathbb{R}^n$ *where* B_1 *is the unit ball of* $\mathbb{R}^n$.

We let $C_+(\mathbb{S}^{n-1})$ be the set of positive continuous functions on the unit sphere $\mathbb{S}^{n-1}$. For any $f \in C_+(\mathbb{S}^{n-1})$, the so-called Aleksandrov body Ω_f associated with the function f is defined by:

$$\Omega_f = \bigcap_{u\in\mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x\cdot u \leqslant f(u)\}.$$

From the Minkowski inequality (1.16) and the upper-lower limit argument, we have,

THEOREM 1.4. *For any fixed* $n \geqslant 2$, $p \geqslant 1$, $q > n$. *Let* h_Ω *be the support function of a convex body* $\Omega \in \mathcal{K}_0^n$. *For any* $f \in C(\mathbb{S}^{n-1})$ *and sufficiently small* t, *we let* $\Omega_{p,t}$ *be the Aleksandrov body associated with the function* $h_\Omega +_p tf$. *Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_{p,t})|_{t=0} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} f^p(\theta)\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega}(\theta).$$

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REMARK 1.5. If $p = 1$ and f is a support function of convex body Ω_f, theorem 1.4 was proved by Akman *et al.* [6] via the so-called direct argument.

Now, we propose the following L_p Minkowski problem:

The L_p Minkowski problem for q-capacity. *For any fixed $n \geqslant 2$, $q > n$ and $p > 1$, given a positive and finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, under what necessary and sufficient conditions does there exist a unique convex body Ω of $\mathbb{R}^n$ such that $\mu^{\mathcal{C}}_{p,\Omega} = \mu$?*

The last result of the present paper is,

THEOREM 1.6. *For any fixed $n \geqslant 2$, $q > n$ and $p > 1$. Let μ be a positive and finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$ satisfying (A.1.1). Then there exists a unique convex body $\Omega \in \mathcal{K}^n_0$ such that $\mu^{\mathcal{C}}_{p,\Omega} = \mu$.*

REMARK 1.7. If $1 < q < n$, the L_p Minkowski problem for the q-capacity can be referred to Borell [13], Jerison [46], Caffarelli *et al.* [21], Colesanti *et al.* [29], Akman *et al.* [5], Hong *et al.* [39], Zou and Xiong [65] and Xiong *et al.* [62].

The paper is organized as follows: § 2 is devoted to some knowledges about q-capacity. In § 3, we show the proof of theorems 1.1, 1.2, corollary 1.3, theorems 1.4 and 1.6.

2. Some preliminaries

Section 2 is devoted to some basic knowledges.

For any $\Omega \in \mathcal{K}^n_0$, the *radial function* of Ω $\varrho_\Omega : \mathbb{R}^n \backslash \{O\} \mapsto \mathbb{R}$ is defined as follows:

$$\varrho_\Omega(x) = \max\{\lambda : \lambda x \in \Omega\}, \quad \forall x \in \Omega \backslash \{O\}. \tag{2.1}$$

The *radial map* of Ω $r_\Omega : \mathbb{S}^{n-1} \mapsto \partial\Omega$ is defined as follows:

$$r_\Omega(\theta) = \varrho_\Omega(\theta)\theta, \quad \forall \theta \in \mathbb{S}^{n-1}, \tag{2.2}$$

that is, $r_\Omega(\theta)$ is the unique point on $\partial\Omega$ satisfying the direction $Or_\Omega(\theta)$ is parallel to the direction θ. It follows from p. 336 of [43], we have,

$$h_\Omega(v) = \sup_{u \in \mathbb{S}^{n-1}} (u \cdot v)\varrho_\Omega(u), \quad \forall v \in \mathbb{S}^{n-1} \tag{2.3}$$

and

$$\frac{1}{\varrho_\Omega(u)} = \sup_{v \in \mathbb{S}^{n-1}} \frac{u \cdot v}{h_\Omega(v)}, \quad \forall u \in \mathbb{S}^{n-1}. \tag{2.4}$$

Let Ω_1 and Ω_2 be two convex bodies of $\mathbb{R}^n$ and $\mathbb{B}^n$ be the unit ball of $\mathbb{R}^n$, the Hausdorff distance $d(\Omega_1, \Omega_2)$ between Ω_1 and Ω_2 is defined as follows:

$$d(\Omega_1, \Omega_2) = \min\{\lambda \geqslant 0 : \Omega_1 \subseteq \Omega_2 + \lambda\mathbb{B}^n, \Omega_2 \subseteq \Omega_1 + \lambda\mathbb{B}^n\} \tag{2.5}$$

where $\mathbb{B}^n$ is the unit ball of $\mathbb{R}^n$. We let $C_+(\mathbb{S}^{n-1})$ be the set of positive continuous functions on the unit sphere $\mathbb{S}^{n-1}$.

From (2.2), (2.4) and (2.5), it is easy to see that,

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LEMMA 2.1. *For any fixed $i \in \{0, 1, 2, \ldots\}$, let h_{Ω_i} be the support function of $\Omega_i \in \mathcal{K}_0^n$ and $d(\Omega_i, \Omega_0)$ be the Hausdorff distance between Ω_i and Ω_0. Then $d(\Omega_i, \Omega_0) \to 0$ as $i \to \infty$ if and only if*

$$\sup_{u \in \mathbb{S}^{n-1}} |h_{\Omega_i}(u) - h_{\Omega_0}(u)| \to 0$$

as $i \to \infty$. If in addition $h_{\Omega_0} \in C_+(\mathbb{S}^{n-1})$, then $d(\Omega_i, \Omega_0) \to 0$ as $i \to \infty$ if and only if

$$\sup_{u \in \mathbb{S}^{n-1}} |\varrho_{\Omega_i}(u) - \varrho_{\Omega_0}(u)| \to 0$$

or

$$\sup_{u \in \mathbb{S}^{n-1}} |r_{\Omega_i}(u) - r_{\Omega_0}(u)| \to 0$$

as $i \to \infty$.

The following convergence lemma is due to Aleksandrov (see p. 102 of [3]).

LEMMA 2.2 (Aleksandrov's convergence lemma). *For any fixed $i \in \{0, 1, 2, \ldots\}$, we assume that $h_i \in C_+(\mathbb{S}^{n-1})$ and let Ω_i be Aleksandrov body associated with the function h_i and d be the Hausdorff distance. If*

$$h_i \to h_0, \textit{uniformly on } \mathbb{S}^{n-1}$$

as $i \to \infty$, then $d(\Omega_i, \Omega_0) \to 0$ as $i \to \infty$.

The following lemma is due to Aleksandrov, see also Jerison [46] or Colesanti *et al.* [29]. A direct proof can be referred to lemma 2.9 of Huang *et al.* [43] and lemma 2.1 mentioned above.

LEMMA 2.3. *For any $\Omega \in \mathcal{K}_0^n$, we let ϱ_Ω, r_Ω and h_Ω be the radial function, radial map and support function of Ω respectively. We also let*

$$J_\Omega(\theta) = \frac{\varrho_\Omega^n(\theta)}{h_\Omega(g_\Omega(r_\Omega(\theta)))}$$

for any $\theta \in \mathbb{S}^{n-1}$. Then, the following statements hold:

(a) *J_Ω is defined $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ and there exists a positive constant c, depending only on the inner radius and the diameter of Ω, such that*

$$0 < c^{-1} \leqslant J_\Omega(\theta) \leqslant c < \infty$$

for $\mathcal{H}^{n-1}$-a.e. $\theta \in \mathbb{S}^{n-1}$.

(b) *Let $f : \partial\Omega \mapsto \mathbb{R}$ be $\mathcal{H}^{n-1}$-integrable. Then,*

$$\int_{\partial\Omega} f(x)\, \mathrm{d}\sigma = \int_{\mathbb{S}^{n-1}} f(r_\Omega(\theta)) J_\Omega(\theta)\, \mathrm{d}\theta.$$

(c) *Suppose that there exists a sequence* $\{\Omega_i\}_{i=0}^{\infty} \subseteq \mathcal{K}_0^n$ *such that* $\Omega_i \to \Omega_0$ *in the sense of Hausdorff distance as* $i \to \infty$. *We let*

$$J_{\Omega_i}(\theta) = \frac{\varrho_{\Omega_i}^n(\theta)}{h_{\Omega_i}(g_{\Omega_i}(r_{\Omega_i}(\theta)))}$$

where ϱ_{Ω_i}, r_{Ω_i} *and* h_{Ω_i} *be the radial function, radial map and support function defined on* Ω_i *respectively. Then, for sufficiently large* i, J_{Ω_i} *is bounded from above and below, uniformly with respect to* θ *and* i, *and*

$$J_{\Omega_i} \to J_{\Omega_0}$$

for almost every $\theta \in \mathbb{S}^{n-1}$ *as* $i \to \infty$ *with respect to the spherical Lebesgue measure.*

For any fixed $i \in \{0, 1, \ldots\}$, *we suppose* $\Omega_i \in \mathcal{K}_0^n$ *and we let* U_{Ω_i} *be the* q*-Green function of* $\mathbb{R}^n \backslash \Omega_i$ *whose pole is at infinity. We also denote*

$$J_{\Omega_i}(\theta) = \frac{\varrho_i^n(\theta)}{h_{\Omega_i}(g_{\Omega_i}(r_{\Omega_i}(\theta)))}, H_i^q(\theta) = |\nabla U_{\Omega_i}(r_{\Omega_i}(\theta))|^q J_{\Omega_i}(\theta) \tag{2.6}$$

for any fixed $i \in \{0, 1, \ldots\}$. *For any* $Q \in \partial\Omega$, *the so-called non-tangential cone* $\Gamma(Q)$ *is defined by:*

$$\Gamma(Q) = \{y \in \Omega : |y - Q| \leqslant bd(y, \partial\Omega)\}$$

for some constant b.

The following lemma is due to Akman *et al.* [6].

Lemma 2.4. *For any* $\{\Omega_i\}_{i=0}^{\infty} \subseteq \mathcal{K}_0^n$, *we let* d *be the Hausdorff distance. For any fixed* $i \in \{0, 1, \ldots\}$, *let* h_{Ω_i}, g_{Ω_i} *and* r_{Ω_i} *the support function, Gauss map and radial map on* Ω_i *respectively and let* U_{Ω_i} *be* q*-Green function on* $\mathbb{R}^n \backslash \Omega_i$ *whose pole is at infinity. We also let* H_i *be the function defined in* (2.6), $\mathcal{C}(\Omega_i)$ *and* $\mu_{1,\Omega_i}^{\mathcal{C}}$ *be the* q*-capacity and the* q*-capacity measure on* Ω_i *respectively, then the following statements hold:*

(a) *For any* $\Omega_0 \subseteq \mathcal{K}_0^n$, *there exists a set* $E_2 \subseteq \partial\Omega_0$ *such that* $\mathcal{H}^{n-1}(E_2) = 0$ *and for any* $Q \in \partial\Omega_0 \backslash E_2$ *and* $x \in \Gamma(Q)$, *the non-tangential limit*

$$\lim_{x \to Q} \frac{\partial U_{\Omega_0}(x)}{\partial \bar{n}_Q} \text{ exists,}$$

we denote the limit by $\partial U_{\Omega_0}(Q)/\partial \bar{n}_Q (= -|\nabla U_{\Omega_0}(Q)|)$.

(b) *For any fixed* $q_1 > q$, *there exists a positive constant* c, *depending only on the diameter and the inner radius of* Ω_0, *such that*

$$\int_{\partial\Omega_0} |\nabla U_{\Omega_0}|^{q_1} \mathrm{d}\sigma \leqslant c.$$

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(c) *If* $d(\Omega_i, \Omega_0) \to 0$ *as* $i \to \infty$, *then*

$$\mu^{\mathcal{C}}_{1,\Omega_i} \to \mu^{\mathcal{C}}_{1,\Omega_0}$$

weakly as $i \to \infty$.

(d) $\mathcal{C}(\cdot)$ *is homogeneous of order* $q-n$. *That is,*

$$\mathcal{C}(t\Omega_0) = t^{q-n}\mathcal{C}(\Omega_0)$$

for any $t > 0$.

(e) $\mathcal{C}(\cdot)$ *is translation invariant. That is,*

$$\mathcal{C}(\Omega_0 + x_0) = \mathcal{C}(\Omega_0)\ \forall x_0 \in \mathbb{R}^n,$$

for any $\Omega_0 \in \mathcal{K}^n_0$.

(f) *If* $d(\Omega_i, \Omega_0) \to 0$ *as* $i \to \infty$, *then*

$$\lim_{i\to\infty} \mathcal{C}(\Omega_i) = \mathcal{C}(\Omega_0).$$

(g)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_1 + t\Omega_2)|_{t=0} = \int_{\mathbb{S}^{n-1}} h_2(g_{\Omega_1}(r_{\Omega_1}(\theta)))H^n_1(\theta)\,\mathrm{d}\theta.$$

(h) *For any* $t \in [0,1]$, *we have*

$$\mathcal{C}^{1/(q-n)}(t\Omega_1 + (1-t)\Omega_2) \geqslant t\mathcal{C}^{1/(q-n)}(\Omega_1) + (1-t)\mathcal{C}^{1/(q-n)}(\Omega_2), \qquad (2.7)$$

equality in (2.7) *holds if and only if* Ω_1 *and* Ω_2 *are homothetic. Equivalently, we have the following statement: for any* $t \in [0,1]$, *we have*

$$\mathcal{C}^{1/(q-n)}(\Omega_1 + t\Omega_2) \geqslant \mathcal{C}^{1/(q-n)}(\Omega_1) + t\mathcal{C}^{1/(q-n)}(\Omega_2), \qquad (2.8)$$

equality in (2.8) *holds if and only if* Ω_1 *and* Ω_2 *are homothetic.*
Adopting some similar arguments of Jerison [46], Colesanti et al. [29], Lewis and Nyström [49] and Akman et al. [6], we have,

LEMMA 2.5. *For any* $\{\Omega_i\}_{i=0}^{\infty} \subseteq \mathcal{K}^n_0$ *and any fixed* $i \in \{0, 1, \dots\}$, *we let* H_i *be the function defined in* (2.6), *if* $\Omega_i \to \Omega_0$ *in the sense of Hausdorff metric as* $i \to \infty$, *then*

$$\int_{\mathbb{S}^{n-1}} |H^q_i(\theta) - H^q_0(\theta)|\mathrm{d}\theta \to 0$$

as $i \to \infty$.

REMARK 2.5.1. If $q = n$, lemma 2.5 was proved by Xiao [61] or its early version in arXiv [60] (see (v) of theorem 4.2 in p. 966 of [61] or (v) of Theorem 6.2 in p. 20 of [60]). It is easy to see that the proof of lemma 2.5 between the case $q > n$ and $q = n$ is the same and their crucial ingredients are the conformal invariance

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of q-Laplace operator and the boundary behaviour of q-Green function. Since the boundary behaviour of q-Green function was given by Akman *et al.* [**6**] and the proof of lemma 2.5 can be referred to Colesanti *et al.* [**29**], Lewis and Nyström [**49**] and Xiao [**60**, **61**], we omit the details here.

For any $f \in C_+(\mathbb{S}^{n-1})$, the so-called Aleksandrov body associated with the function f is defined by:

$$\Omega_f = \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot \theta \leqslant f(\theta)\}.$$

For any fixed $p \geqslant 1$ and $f \in C_+(\mathbb{S}^{n-1})$, we define a functional $\mathcal{G} : C_+(\mathbb{S}^{n-1}) \times \mathbb{R}^n \mapsto \mathbb{R}$ as follows:

$$\mathcal{G}_{\Omega_f}(f) = \int_{\mathbb{S}^{n-1}} f^p(\theta) \, \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_f}(\theta). \tag{2.9}$$

Moreover, for any fixed $\Omega \in \mathcal{K}^n_0$, the support function of convex body Ω is denoted by h_Ω. We define a functional $\mathcal{F} : \mathcal{K}^n_0 \mapsto \mathbb{R}$ as follows:

$$\mathcal{F}(\Omega) = \int_{\mathbb{S}^{n-1}} h^p_\Omega(\theta) \, \mathrm{d}\mu(\theta) \tag{2.10}$$

for any fixed Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$. We let h_{Ω_f} be the support function of Ω_f and

$$\mathcal{E}_2 = \{\theta \in \mathbb{S}^{n-1} : h^p_{\Omega_f}(\theta) \neq f^p(\theta)\}.$$

From a result of Aleksandrov, $\mathcal{E}_2$ is a set of measure zero on $\mathbb{S}^{n-1}$ with respect to the spherical Lebesgue measure (see p. 411 of [**58**] or p. 143 of [**51**]). Then, we have,

Lemma 2.6. *For any $p \geqslant 1$, $q > n$ and any $f \in C_+(\mathbb{S}^{n-1})$, we let $\mu^{\mathcal{C}}_{p,\Omega}$ be the L_p q-capacity measure, $\mathcal{G}$ and $\mathcal{F}$ be the functionals defined in* (2.9) *and* (2.10) *respectively. Suppose that Ω_f is the Aleksandrov body associated with the function f, then*

$$\frac{1}{q} \mathcal{C}^{1/(q-1)}(\Omega_f) = \frac{q-1}{q-n} \mathcal{G}_{\Omega_f}(f)$$

and

$$\mathcal{F}(\Omega_f) = \int_{\mathbb{S}^{n-1}} f^p(\theta) \, \mathrm{d}\mu(\theta).$$

3. Final proofs of the main results

Section 3 is devoted to the proof of theorems 1.1, 1.2, corollary 1.3, theorems 1.4 and 1.6.

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Proof of theorem 1.1. (a) Since $h_{\Omega_0} \in C_+(\mathbb{S}^{n-1})$, it follows from the compactness of $\mathbb{S}^{n-1}$ that there exists a constant $c > 0$ such that

$$0 < c^{-1} \leqslant h_{\Omega_0}(\theta) \leqslant c < \infty, \quad \forall \theta \in \mathbb{S}^{n-1}.$$

This, together with lemma 2.4(b) and Hölder inequality, implies that

$$\begin{aligned}
\mu^{\mathcal{C}}_{p,\Omega_0}(E) =& \int_{g^{-1}_{\Omega_0}(E)} h^{1-p}_{\Omega_0} |\nabla U_{\Omega_0}|^q \, \mathrm{d}\mathcal{H}^{n-1} \\
\leqslant & c^{1-p} \int_{g^{-1}(E)} |\nabla U_{\Omega_0}|^q \, \mathrm{d}\mathcal{H}^{n-1} \\
\leqslant & c^{1-p} \left(\int_{g^{-1}_{\Omega_0}(E)} |\nabla U_{\Omega_0}|^{qq_1} \, \mathrm{d}\mathcal{H}^{n-1} \right)^{1/q_1} \left(\int_{g^{-1}_{\Omega_0}(E)} \mathrm{d}\mathcal{H}^{n-1} \right)^{1-1/q_1} \\
\leqslant & c \left(\int_{g^{-1}_{\Omega_0}(E)} \mathrm{d}\mathcal{H}^{n-1} \right)^{1-1/q_1} = c S^{1-1/q_1}_{1,\Omega_0}(E).
\end{aligned}$$

for any $q_1 > 1$ and any Borel set $E \subseteq \mathbb{S}^{n-1}$. Therefore, we have

$$S_{1,\Omega_0}(E) = 0 \Rightarrow \mu^{\mathcal{C}}_{p,\Omega_0}(E) = 0$$

for any Borel set $E \subseteq \mathbb{S}^{n-1}$, which implies the desired conclusion of (a).

(b) To get the conclusion of (b), it suffices to show that

$$\int_{\mathbb{S}^{n-1}} f(\theta) \, \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_i}(\theta) \to \int_{\mathbb{S}^{n-1}} f(\theta) \, \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) \tag{3.1}$$

as $i \to \infty$ for any $f \in C(\mathbb{S}^{n-1})$. Indeed, for any $\Omega_0 \in \mathcal{K}^n_0$, it follows from the definition of the L_p q-capacity measure and lemma 2.3 that

$$\int_{\mathbb{S}^{n-1}} f(\theta) \, \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) = \int_{\mathbb{S}^{n-1}} f(g_{\Omega_0}(r_{\Omega_0}(\theta))) h^{1-p}_{\Omega_0}(g_{\Omega_0}(r_{\Omega_0}(\theta))) H^q_0(\theta) \, \mathrm{d}\theta \tag{3.2}$$

for any $f \in C(\mathbb{S}^{n-1})$. We let

$$h_i(\cdot) = h_{\Omega_i}(\cdot), h(\cdot) = h_{\Omega_0}(\cdot), r_i(\cdot) = r_{\Omega_i}(\cdot), r(\cdot) = r_{\Omega_0}(\cdot) \tag{3.3}$$

and

$$g_i(\cdot) = g_{\Omega_i}(\cdot), g(\cdot) = g_{\Omega_0}(\cdot), \alpha_i(\cdot) = g_i(r_i(\cdot)), \alpha(\cdot) = g(r(\cdot)). \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}
&\left|\int_{\mathbb{S}^{n-1}} f(\theta)\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_i}(\theta) - \int_{\mathbb{S}^{n-1}} f(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta)\right| \\
&\quad = \left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))h_i^{1-p}(\alpha_i(\theta))H_i^q(\theta)\,\mathrm{d}\theta\right. \\
&\qquad \left. - \int_{\mathbb{S}^{n-1}} f(\alpha(\theta))h^{1-p}(\alpha(\theta))H_0^q(\theta)\,\mathrm{d}\theta\right| \\
&\quad \leqslant \left|\int_{\mathbb{S}^{n-1}} (f(\alpha_i(\theta)) - f(\alpha(\theta)))h^{1-p}(\alpha(\theta))H_0^q(\theta)\,\mathrm{d}\theta\right| \\
&\qquad + \left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))(h_i^{1-p}(\alpha_i(\theta)) - h^{1-p}(\alpha_i(\theta)))H_i^q(\theta)\,\mathrm{d}\theta\right| \\
&\qquad + \left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))(h^{1-p}(\alpha_i(\theta)) - h^{1-p}(\alpha(\theta)))H_i^q(\theta)\,\mathrm{d}\theta\right| \\
&\qquad + \left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))h^{1-p}(\alpha(\theta)))(H_i^q(\theta) - H_0^q(\theta))\,\mathrm{d}\theta\right|.
\end{aligned} \tag{3.5}$$

Since $d(\Omega_i, \Omega_0) \to 0$ as $i \to \infty$, we see that

$$r_i \to r \text{ uniformly on } \mathbb{S}^{n-1}$$

$$g_i \to g \text{ a.e. on } \mathbb{S}^{n-1}$$

and thus

$$\alpha_i \to \alpha \quad \text{a.e. on } \mathbb{S}^{n-1} \tag{3.6}$$

with respect to the spherical Lebesgue measure as $i \to \infty$. Since $f \in C(\mathbb{S}^{n-1})$, we see that

$$f(\alpha_i) \to f(\alpha) \quad \text{a.e. on } \mathbb{S}^{n-1} \tag{3.7}$$

as $i \to \infty$. Since $h \in C_+(\mathbb{S}^{n-1})$, it follows from the compactness of $\mathbb{S}^{n-1}$ that there exists a positive constant c_0 such that

$$0 < c_0^{-1} \leqslant \inf_{u\in\mathbb{S}^{n-1}} h(u) \leqslant \sup_{u\in\mathbb{S}^{n-1}} h(u) \leqslant c_0. \tag{3.8}$$

This implies that

$$0 < c_0^{1-p} \leqslant \inf_{u\in\mathbb{S}^{n-1}} h^{1-p}(u) \leqslant \sup_{u\in\mathbb{S}^{n-1}} h^{1-p}(u) \leqslant c_0^{p-1} \tag{3.9}$$

and thus,

$$h^{1-p}(\alpha(\theta)) \leqslant c_0^{1-p} \tag{3.10}$$

for almost every $\theta \in \mathbb{S}^{n-1}$ and any fixed $p \geqslant 1$ with respect to the spherical Lebesgue measure. This, together with (3.7), (3.10), lemma 2.4(b), and

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Lebesgue dominated convergence theorem, implies that

$$\left| \int_{\mathbb{S}^{n-1}} (f(\alpha_i(\theta)) - f(\alpha(\theta)))h^{1-p}(\alpha(\theta))H_0^q(\theta)\,\mathrm{d}\theta \right| \to 0 \tag{3.11}$$

as $i \to \infty$.

From (3.7), we see that there exists a constant c, independent of i, such that

$$|f(\alpha_i(\theta))| \leqslant c \tag{3.12}$$

for almost every $\theta \in \mathbb{S}^{n-1}$. This imply that

$$\begin{aligned} &\left| \int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))(h_i^{1-p}(\alpha_i(\theta)) - h^{1-p}(\alpha_i(\theta)))H_i^q(\theta)\,\mathrm{d}\theta \right| \\ &\quad \leqslant c \sup_{u\in\mathbb{S}^{n-1}} |h_i^{1-p}(u) - h^{1-p}(u)| \mu_{1,\Omega_i}^{\mathcal{C}}(\mathbb{S}^{n-1}). \end{aligned} \tag{3.13}$$

Lemma 2.1 and the assumption that

$$d(\Omega_i, \Omega) \to 0$$

as $i \to \infty$ imply that

$$\sup_{u\in\mathbb{S}^{n-1}} |h_i(u) - h(u)| \to 0 \tag{3.14}$$

as $i \to \infty$. This implies that

$$\sup_{u\in\mathbb{S}^{n-1}} |h_i^{p-1}(u) - h^{p-1}(u)| \to 0 \tag{3.15}$$

as $i \to \infty$ for any fixed $p \geqslant 1$. Since $h \in C_+(\mathbb{S}^{n-1})$, it follows from (3.14) that there exists a positive constant c_1, independent of i such that

$$0 \leqslant c_1^{-1} \leqslant h_i(u) \leqslant c_1 \tag{3.16}$$

and thus

$$0 < c_1^{1-p} \leqslant h_i^{1-p}(u) \leqslant c_1^{p-1}, \quad \forall u \in \mathbb{S}^{n-1} \tag{3.17}$$

for any fixed $i \in \{1, 2, \ldots\}$ and $p \geqslant 1$. (3.15), (3.9) and (3.16) imply that

$$\sup_{u\in\mathbb{S}^{n-1}} |h_i^{1-p}(u) - h^{1-p}(u)| \to 0 \tag{3.18}$$

as $i \to \infty$ for any fixed $p > 1$. From (3.6) and (3.18), we see that

$$h_i^{1-p}(\alpha_i) \to h^{1-p}(\alpha_i) \text{ a.e. on } \mathbb{S}^{n-1}, \tag{3.19}$$

as $i \to \infty$ for any fixed $p \geqslant 1$ with respect to the spherical Lebesgue measure. Since $d(\Omega_i, \Omega_0) \to 0$ as $i \to \infty$, it follows from the weak continuity

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of q-capacity measure and lemma 2.4(b) that there exists a constant c, independent of i, such that

$$\int_{\mathbb{S}^{n-1}} H_i^q(\theta)\,\mathrm{d}\theta \leqslant c \tag{3.20}$$

for any fixed $i \in \{1, 2, \ldots, \infty\}$. Equations (3.12), (3.19) and (3.20) yield that

$$\left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))(h_i^{1-p}(\alpha_i(\theta)) - h^{1-p}(\alpha_i(\theta)))H_i^q(\theta)\,\mathrm{d}\theta\right| \to 0 \tag{3.21}$$

as $i \to \infty$ for any fixed $p \geqslant 1$.
From (3.6), (3.9) and lemma 2.1, we see that

$$h^{1-p}(\alpha_i) \to h^{1-p}(\alpha) \quad \text{a.e. on } \mathbb{S}^{n-1} \tag{3.22}$$

with respect to the spherical Lebesgue measure as $i \to \infty$. It follows from (3.12), (3.20), (3.22) and Lebesgue dominated convergence theorem that

$$\left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))(h^{1-p}(\alpha_i(\theta)) - h^{1-p}(\alpha(\theta)))H_i^q(\theta)\,\mathrm{d}\theta\right| \to 0 \tag{3.23}$$

as $i \to \infty$ for any fixed $p \geqslant 1$.
Equations (3.12), (3.10) and lemma 2.5 yield that

$$\left|\int_{\mathbb{S}^{n-1}} f(\alpha_i(\theta))h^{1-p}(\alpha(\theta)))(H_i^q(\theta) - H_0^q(\theta))\,\mathrm{d}\theta\right| \to 0 \tag{3.24}$$

as $i \to \infty$ for any fixed $p \geqslant 1$.
Putting (3.11), (3.21), (3.23) and (3.24) into (3.5), we obtain (3.1). This is the desired conclusion of (b).

(c) For any $\Omega_0 \in \mathcal{K}_0^n$, we let R_0 be the diameter of the domain Ω_0. We first suppose that $\Omega_0 \in \mathcal{K}_0^n$ is smooth. Adopting a similar idea of lemma 2.18 of Colesanti *et al.* [**29**], we can show that there exists a constant c, depending only on q and n such that

$$|\nabla U_{\Omega_0}(x)| \geqslant \frac{c}{R_0}, \quad \forall x \in \partial\Omega_0, \tag{3.25}$$

where U_{Ω_0} is the q-Green function of $\mathbb{R}^n \backslash \Omega_0$ whose pole is at infinity. For the general convex domain Ω_0 without the assumption of smoothness, we consider a smooth sequence $\{\Omega_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_0^n$ such that

$$\Omega_i \to \Omega_0 \tag{3.26}$$

as $i \to \infty$ in the sense of Hausdorff metric. For any fixed $i \in \{0, 1, \ldots\}$, we let R_i be the diameter of Ω_i. We can see that

$$\lim_{i\to\infty} R_i = R_0 > 0$$

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and

$$\sup_{u\in\mathbb{S}^{n-1}} |h_{\Omega_i}(u) - h_{\Omega_0}(u)| \to 0$$

as $i \to \infty$. Since $h_{\Omega_0} \in C_+(\mathbb{S}^{n-1})$, it follows from the compactness of $\mathbb{S}^{n-1}$ that there exists a constant $c > 0$ such that

$$0 < c^{-1} \leqslant h_{\Omega_0}(\theta) \leqslant c < \infty, \quad \forall\theta \in \mathbb{S}^{n-1}.$$

This means that there exists a positive constant c, independent of i, such that

$$\min\left\{R_i^{-q}, \inf_{u\in\mathbb{S}^{n-1}} h_{\Omega_i}^{1-p}(u)\right\} \geqslant c$$

for sufficiently large i due to the facts: $R_0 > 0$ and $h_{\Omega_0} \in C_+(\mathbb{S}^{n-1})$. Therefore,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_i}(\theta) &= \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ h_{\Omega_i}^{1-p} |\nabla U_{\Omega_i}|^q J_{\Omega_i}(\theta)\,\mathrm{d}\theta \\ &\geqslant \min\{R_i^{-q}, \inf_{u\in\mathbb{S}^{n-1}} h_{\Omega_i}^{1-p}(u)\} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_i}(\theta)\,\mathrm{d}\theta \\ &\geqslant c\int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_i}(\theta)\,\mathrm{d}\theta\end{aligned} \tag{3.27}$$

for any $e \in \mathbb{S}^{n-1}$ and sufficiently large i. For any fixed $e \in \mathbb{S}^{n-1}$, we define a function $f_0 : \mathbb{S}^{n-1} \mapsto \mathbb{R}$ as follows:

$$f_0(\theta) = (e\cdot\theta)_+ = \max\{e\cdot\theta, 0\}, \quad \forall\theta \in \mathbb{S}^{n-1}.$$

It is easy to see that

$$f_0 \in C(\mathbb{S}^{n-1}). \tag{3.28}$$

We conclude from (3.27), (3.28), the weak continuity of L_p q-capacity measure $\mu^{\mathcal{C}}_{p,\Omega}$ and the weak continuity of surface area measure that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) &= \lim_{i\to\infty} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ \mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_i}(\theta) \\ &\geqslant c \lim_{i\to\infty} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_i}(\theta)\,\mathrm{d}\theta \\ &\geqslant c\int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_0}(\theta)\,\mathrm{d}\theta \\ &\geqslant c \inf_{e\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_0}(\theta)\,\mathrm{d}\theta\end{aligned} \tag{3.29}$$

for any $e \in \mathbb{S}^{n-1}$. It follows from lemma 2.3 that

$$\inf_{e\in\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (e\cdot\theta)_+ J_{\Omega_0}(\theta)\,\mathrm{d}\theta > 0.$$

Putting this into (3.29), we get

$$\inf_{e\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}(e\cdot\theta)_+\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}\geqslant c\inf_{e\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}(e\cdot\theta)_+J_{\Omega_0}(\theta)\,\mathrm{d}\theta>0.$$

This implies the desired conclusion of (c) and this completes the proof of theorem 1.1. □

Now, we give the proof of theorem 1.2.

Proof of theorem 1.2. We first show (a). For any $\Omega_1,\Omega_2\in\mathcal{K}^n_0$ and any $s\geqslant 0$, we let

$$f(s)=\mathcal{C}^{1/(q-n)}(\Omega_1+s\Omega_2)-\mathcal{C}^{1/(q-n)}(\Omega_1)-s\mathcal{C}^{1/(q-n)}(\Omega_2).$$

It follows from lemma 2.4(h) that $f(s)$ is a non-negative function for any $s\in[0,1]$ and $f(0)=0$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}s}f(s)\bigg|_{s=0}\geqslant 0.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{C}^{1/(q-n)}(\Omega_1+s\Omega_2)|_{t=0}\geqslant\mathcal{C}^{1/(q-n)}(\Omega_2). \tag{3.30}$$

From the definition of mixed q-capacity, we have

$$\mathcal{C}_{1,1}(\Omega_1,\Omega_2)\geqslant\frac{1}{q}\mathcal{C}^{1/(q-1)-1/(q-n)}(\Omega_1)\mathcal{C}^{1/(q-n)}(\Omega_2). \tag{3.31}$$

Next, we extend (3.31) to its L_p version via the Hölder inequality. We let g_{Ω_1} and r_{Ω_1} be the Gauss map and radial map on Ω_1 and we also denote

$$\alpha_1=g_{\Omega_1}\circ r_{\Omega_1}. \tag{3.32}$$

For any fixed $p>1$, it follows from the definition of $\mathcal{C}_{1,1}$, $\mathcal{C}_{q,1}$ and Hölder inequality that

$$\begin{aligned}\mathcal{C}_{1,1}(\Omega_1,\Omega_2)&=\frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}}h_{\Omega_2}(\alpha_1(\theta))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_1}(\theta)\\&=\frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}}h_{\Omega_2}(\alpha_1(\theta))h^{(1-p)/p}_{\Omega_1}(\alpha_1(\theta))h^{(p-1)/p}_{\Omega_1}(\alpha_1(\theta))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_1}(\theta)\\&\leqslant\left(\frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}}h^p_{\Omega_2}(\alpha_1(\theta))\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_1}(\theta)\right)^{1/p}\\&\quad\cdot\left(\frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}}h_{\Omega_1}(\alpha_1(\theta))\,\mathrm{d}\mu^{\mathcal{C}}_{1,\Omega_1}(\theta)\right)^{(p-1)/p}\\&=\left(\frac{1}{q}\right)^{(p-1)/p}\mathcal{C}^{1/p}_{p,1}(\Omega_1,\Omega_2)\mathcal{C}^{(p-1)/((q-1)p)}(\Omega_1).\end{aligned}$$

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This and (3.31) imply that

$$\begin{aligned}\mathcal{C}_{p,1}(\Omega_1,\Omega_2) &\geqslant \left(\frac{1}{q}\right)^{1-p}\mathcal{C}_{1,1}^{p}(\Omega_1,\Omega_2)\mathcal{C}^{(1-p)/(q-1)}(\Omega_1)\\ &\geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1)\mathcal{C}^{p/(q-n)}(\Omega_2).\end{aligned} \tag{3.33}$$

Since t^p is a strictly convex function on $(0,\infty)$ for any fixed $p>1$, following a similar idea of theorem 9.1.4 of Schneider [58], we can see that the equality in (3.33) holds if and only if Ω_1 and Ω_2 are dilates. This completes the proof of part (a).

Now, we show part (b). We first claim that

$$\mathcal{C}(\Omega_1)=\mathcal{C}(\Omega_2) \tag{3.34}$$

provided

$$\mathcal{C}_{p,1}(\Omega_1,\Omega)=\mathcal{C}_{p,1}(\Omega_2,\Omega),\quad \forall\Omega\in\mathcal{K}_0^n. \tag{3.35}$$

We first take $\Omega=\Omega_1$ in (3.35) and it follows from the definition of L_p mixed q-capacity that

$$\begin{aligned}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_1)=\mathcal{C}_{p,1}(\Omega_1,\Omega_1)&=\mathcal{C}_{p,1}(\Omega_2,\Omega_1)\\ &\geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_2)\mathcal{C}^{p/(q-n)}(\Omega_1).\end{aligned}$$

From this, we know

$$\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_2)\leqslant\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1). \tag{3.36}$$

We then take $\Omega=\Omega_2$ in (3.35) and we get,

$$\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_2)\geqslant\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1). \tag{3.37}$$

From (3.36) and (3.37), we get

$$\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_2)=\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1)$$

and thus we get (3.34) due to $1/(q-1)-p/(q-n)\neq 0$. This is the desired conclusion of the claim.

It follows from the definition of L_p mixed q-capacity and (3.34) that

$$\begin{aligned}\mathcal{C}_{p,1}(\Omega_1,\Omega_2)&=\mathcal{C}_{p,1}(\Omega_2,\Omega_2)\\ &=\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_2)\\ &=\frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_2)\mathcal{C}^{p/(q-n)}(\Omega_2)\\ &=\frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_1)\mathcal{C}^{p/(q-n)}(\Omega_2).\end{aligned} \tag{3.38}$$

This means that the equality in (3.33) holds provided the assumption (3.35) holds. It follows from the conclusion of part (a) that Ω_1 and Ω_2 are dilates. Noting that

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$\mathcal{C}$ is homogeneous of order $q-n$, it follows from (3.34) that $\Omega_1 = \Omega_2$. This is the conclusion of part (b). This completes the proof of theorem 1.2. □

Now, we give the proof of corollary 1.3.

Proof of corollary 1.3. We let $B_r(0)$ be a ball of $\mathbb{R}^n$, whose centre is the origin and radius is r. For any $\Omega \in \mathcal{K}_0^n$, it follows from theorem 1.2 that

$$\begin{aligned} \mathcal{C}_{p,1}(\Omega, B_r(0)) &\geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega)\mathcal{C}^{p/(q-n)}(B_r(0)) \\ &\geqslant \frac{r^p}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega)\mathcal{C}^{p/(q-n)}(B_1(0)) \end{aligned} \tag{3.39}$$

since $\mathcal{C}$ is a homogeneous functional of order $q-n$. From the definition of the L_p mixed q-capacity, we can see that

$$\mathcal{C}_{p,1}(\Omega, B_r(0)) = \frac{q-1}{q-n} r^p \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{p,\Omega}^{\mathcal{C}}(\theta).$$

This, together with (3.39), implies that

$$\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega) \leqslant \frac{q(q-1)}{(q-n)\mathcal{C}^{p/(q-n)}(B_1(0))} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{p,\Omega}^{\mathcal{C}}(\theta). \tag{3.40}$$

Noting that the equality in (3.40) holds if and only if the equality in (3.39) holds, it follows from theorem 1.2 that the equality in (3.39) holds if and only if Ω is a ball of $\mathbb{R}^n$. This completes the proof of corollary 1.3. □

Now, we are in a position to prove theorem 1.4.

Proof of theorem 1.4. The proof is based on the L_p Minkowski inequality for q-capacity and the upper-lower limit argument. Without any special statement, we always assume that $p \geqslant 1$. Since $h_{\Omega_0} \in C_+(\mathbb{S}^{n-1})$, for any $f \in C(\mathbb{S}^{n-1})$ and sufficiently small t, we see that

$$h_{p,t} = h_{\Omega_0} +_p tf = (h_{\Omega_0}^p + tf^p)^{1/p} \in C_+(\mathbb{S}^{n-1}). \tag{3.41}$$

We let $\Omega_{p,t}$ be the Aleksandrov body associated with the function $h_{p,t}$, that is,

$$\Omega_{p,t} = \bigcap_{\theta \in \mathbb{S}^1} \{x \in \mathbb{R}^n : x \cdot \theta \leqslant h_{p,t}(\theta)\}.$$

From the L_p Minkowski inequality for q-capacity, we see that

$$\begin{cases} \frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_{p,t}) - \mathcal{C}_{p,1}(\Omega_{p,t}, \Omega_0) \\ \quad \leqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_{p,t})(\mathcal{C}^{p/(q-n)}(\Omega_{p,t}) - \mathcal{C}^{p/(q-n)}(\Omega_0)) \\ \mathcal{C}_{p,1}(\Omega_0, \Omega_{p,t}) - \frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_0) \\ \quad \geqslant \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_0)(\mathcal{C}^{p/(q-n)}(\Omega_{p,t}) - \mathcal{C}^{p/(q-n)}(\Omega_0)). \end{cases} \tag{3.42}$$

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We also denote

$$g_p(t) = \mathcal{C}^{p/(q-n)}(\Omega_{p,t}).$$

On the one hand, we conclude from (3.42) that

$$\begin{cases} \liminf_{t\to 0^+} \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_{p,t})\frac{g_p(t)-g_p(0)}{t} \geqslant l_1 \\ \limsup_{t\to 0^+} \frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_0)\frac{g_p(t)-g_p(0)}{t} \leqslant l_2 \end{cases} \tag{3.43}$$

where

$$\begin{aligned} l_1 &= \liminf_{t\to 0^+} \frac{(1/q)\mathcal{C}^{1/(q-1)}(\Omega_{p,t}) - \mathcal{C}_{p,1}(\Omega_{p,t},\Omega_0)}{t}, \\ l_2 &= \limsup_{t\to 0^+} \frac{\mathcal{C}_{p,1}(\Omega_0,\Omega_{p,t}) - (1/q)\mathcal{C}^{1/(q-1)}(\Omega_0)}{t}. \end{aligned} \tag{3.44}$$

Now, we claim that

$$l_1 = l_2 = \int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \tag{3.45}$$

Indeed, it follows from the definition of support function that

$$h_{\Omega_{p,t}}(\theta) \leqslant h_{p,t}(\theta), \quad \forall\theta\in\mathbb{S}^{n-1} \tag{3.46}$$

for sufficiently small t. It follows from (3.46) that

$$\limsup_{t\to 0^+} \frac{h^p_{\Omega_{p,t}} - h^p_{\Omega_0}}{t} \leqslant \limsup_{t\to 0^+} \frac{h^p_{p,t} - h^p_{\Omega_0}}{t} = f^p, \quad \text{uniformly on } \mathbb{S}^{n-1}. \tag{3.47}$$

This and Lebesgue dominated convergence theorem yield that

$$\begin{aligned} l_2 &= \limsup_{t\to 0^+} \frac{\mathcal{C}_{p,1}(\Omega_0,\Omega_{p,t}) - (1/q)\mathcal{C}^{1/(q-1)}(\Omega_0)}{t} \\ &= \limsup_{t\to 0^+} \frac{\mathcal{C}_{p,1}(\Omega_0,\Omega_{p,t}) - \mathcal{C}_{p,1}(\Omega_0,\Omega_0)}{t} \\ &= \limsup_{t\to 0^+} \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} \frac{h^p_{\Omega_{p,t}}(\theta) - h^p_{\Omega_0}(\theta)}{t}\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) \\ &\leqslant \limsup_{t\to 0^+} \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} \frac{h^p_{p,t}(\theta) - h^p_{\Omega_0}(\theta)}{t}\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta) \\ &= \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \end{aligned} \tag{3.48}$$

Moreover, it follows from the definition of $h_{p,t}$ that

$$h_{p,t} \to h_{\Omega_0}, \text{ uniformly on } \mathbb{S}^{n-1} \tag{3.49}$$

as $t \to 0$. From lemma 2.1, we can see that

$$\Omega_{p,t} \to \Omega_{h_{\Omega_0}} \tag{3.50}$$

as $t \to 0$ where $\Omega_{h_{\Omega_0}}$ is the Aleksandrov body associated with the function h_{Ω_0}. Since h_{Ω_0} is the support function of Ω_0, we have

$$\Omega_0 = \Omega_{h_{\Omega_0}}$$

(see pp. 1562–1563 of [**29**]). Combining this and (3.50), we see that

$$\Omega_{p,t} \to \Omega_0 \tag{3.51}$$

as $t \to 0$. From (3.51) and theorem 1.1, we have

$$\mu^{\mathcal{C}}_{p,\Omega_{p,t}} \to \mu^{\mathcal{C}}_{p,\Omega_0} \tag{3.52}$$

weakly as $t \to 0$. Lemma 2.6 and (3.52) yield that

$$\begin{aligned} l_1 &= \liminf_{t\to 0^+} \frac{(1/q)\mathcal{C}^{1/(q-1)}(\Omega_{p,t}) - \mathcal{C}_{p,1}(\Omega_{p,t},\Omega_0)}{t} \\ &= \liminf_{t\to 0^+} \frac{((q-1)/(q-n))\mathcal{G}_{\Omega_{p,t}}(h_{p,t}) - \mathcal{C}_{p,1}(\Omega_{p,t},\Omega_0)}{t} \\ &= \liminf_{t\to 0} \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} \frac{h^p_{p,t} - h^p_{\Omega_0}}{t}\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_{p,t}}(\theta) \\ &= \lim_{t\to 0} \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_{p,t}}(\theta) \\ &= \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \end{aligned} \tag{3.53}$$

Equations (3.48) and (3.53) imply that

$$\begin{aligned} \liminf_{t\to 0^+} \frac{(1/q)\mathcal{C}^{1/(q-1)}(\Omega_{p,t}) - \mathcal{C}_{p,1}(\Omega_{p,t},\Omega_0)}{t} &= \limsup_{t\to 0^+} \frac{\mathcal{C}_{p,1}(\Omega_0,\Omega_{p,t}) - (1/q)\mathcal{C}(\Omega_0)}{t} \\ &= \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \end{aligned}$$

This is the desired conclusion of the claim.

It follows from (3.51) and lemma 2.4 (f) that

$$\lim_{t\to 0^+} \mathcal{C}(\Omega_{p,t}) = \lim_{t\to 0} \mathcal{C}(\Omega_{p,t}) = \mathcal{C}(\Omega_0). \tag{3.54}$$

From (3.43)–(3.45) and (3.54), we obtain

$$\frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_0)\lim_{t\to 0^+}\frac{g_p(t)-g_p(0)}{t} = \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \tag{3.55}$$

On the other hand, adopting a similar argument, we also see that

$$\frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_0)\lim_{t\to 0^-}\frac{g_p(t)-g_p(0)}{t} = \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta). \tag{3.56}$$

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Combining (3.55) and (3.56), we have

$$\frac{1}{q}\mathcal{C}^{1/(q-1)-p/(q-n)}(\Omega_0)\lim_{t\to 0}\frac{g_p(t)-g_p(0)}{t}=\frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}}f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta).$$

This, together with the definition of g_p, implies that

$$\frac{\mathcal{C}^{1/(q-1)-1}(\Omega_0)}{q(q-1)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\Omega_{p,t})\bigg|_{t=0}=\frac{1}{p}\int_{\mathbb{S}^{n-1}}f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta),$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_{p,t})\bigg|_{t=0}=\frac{1}{p}\int_{\mathbb{S}^{n-1}}f^p(\theta)\,\mathrm{d}\mu^{\mathcal{C}}_{p,\Omega_0}(\theta).$$

This is the desired conclusion of theorem 1.4. □

The remaining part of this section is devoted to the proof of theorem 1.6. For any fixed positive, finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, we let the functional $\mathcal{F}$ be the functional defined in (2.10). We also let

$$\mathcal{M}=\{\Omega\in\mathcal{K}^n_0 : h_\Omega\in C_+(\mathbb{S}^{n-1}),\mathcal{C}(\Omega)=1\}$$

and consider the following variational problem:

$$m^{\mathcal{C}}_p=\inf\{\mathcal{F}(K) : K\in\mathcal{M}\}. \tag{3.57}$$

Lemma 3.5. *For any fixed $n\geqslant 3$ and $p>1$, we assume that there exists a domain $\Omega_0\in\mathcal{M}$ such that*

$$\mathcal{F}(\Omega_0)=m^{\mathcal{C}}_p. \tag{3.58}$$

Then,

$$\frac{q(q-1)\mathcal{F}(\Omega_0)}{q-n}\mu^{\mathcal{C}}_{p,\Omega_0}=\mu. \tag{3.59}$$

If we let

$$t_0=\left(\frac{q(q-1)\mathcal{F}(\Omega_0)}{q-n}\right)^{\frac{1}{(q-n)/(q-1)-p}},$$

we have

$$\mu^{\mathcal{C}}_{p,t_0\Omega_0}=\mu. \tag{3.60}$$

Proof. It follows from the definition of $\mathcal{M}$ and the assumption that $\Omega_0\in\mathcal{M}$ that

$$h_{\Omega_0}\in C_+(\mathbb{S}^{n-1}).$$

For any $f\in C(\mathbb{S}^{n-1})$ and sufficiently small t,

$$h_{p,t}=(h^p_{\Omega_0}+tf^p)^{1/p}\in C_+(\mathbb{S}^{n-1}).$$

We let $\Omega_{p,t}$ the Aleksandrov body associated with $h_{p,t}$. Since Ω_0 is the minimizer of $m^{\mathcal{C}}_p$, it follows from the method of Lagrange multiplier that there exists a constant

β_0, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\Omega_{p,t})\bigg|_{t=0} = \beta_0 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\Omega_{p,t})\bigg|_{t=0}, \tag{3.61}$$

where β_0 is the so-called Lagrange multiplier. From lemma 2.6, we see that

$$\mathcal{F}(\Omega_{p,t}) = \int_{\mathbb{S}^{n-1}} h_{p,t}^p(\theta)\,\mathrm{d}\mu(\theta).$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\Omega_{p,t})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^{n-1}} h_{p,t}^p(\theta)\,\mathrm{d}\mu(\theta)\bigg|_{t=0} = \int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu(\theta). \tag{3.62}$$

It follows from theorem 1.4 and $\mathcal{C}(\Omega_0) = 1$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(\Omega_{p,t})\bigg|_{t=0} = \frac{q(q-1)}{p}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu_{p,\Omega_0}^{\mathcal{C}}(\theta). \tag{3.63}$$

Putting (3.62) and (3.63) into (3.61), we get

$$\frac{\beta_0}{p\sigma_{n-1}}\int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu_{p,\Omega_0}^{\mathcal{C}}(\theta) = \int_{\mathbb{S}^{n-1}} f^p(\theta)\,\mathrm{d}\mu(\theta).$$

This implies that

$$\frac{\beta_0 q(q-1)}{p}\mu_{p,\Omega_0}^{\mathcal{C}} = \mu \tag{3.64}$$

due to the arbitrariness of f. Recalling the definition of $\mathcal{F}$ and the fact

$$\frac{1}{q} = \frac{1}{q}\mathcal{C}^{1/(q-1)}(\Omega_0) = \frac{q-1}{q-n}\int_{\mathbb{S}^{n-1}} h_{\Omega_0}^p\,\mathrm{d}\mu_{p,\Omega_0}^{\mathcal{C}}(\theta),$$

we have

$$\mathcal{F}(\Omega_0) = \int_{\mathbb{S}^{n-1}} h_{\Omega_0}^p\,\mathrm{d}\mu(\theta) = \frac{\beta_0 q(q-1)}{p}\int_{\mathbb{S}^{n-1}} h_{\Omega_0}^p\,\mathrm{d}\mu_{p,\Omega_0}^{\mathcal{C}}(\theta) = \frac{\beta_0(q-n)}{p}.$$

Combining this and (3.64), we get

$$\frac{q(q-1)\mathcal{F}(\Omega_0)}{q-n}\mu_{p,\Omega_0}^{\mathcal{C}} = \mu. \tag{3.65}$$

From the standard analysis, we see that

$$\mu_{p,t\Omega_0}^{\mathcal{C}} = t^{(q-n)/(q-1)-p}\mu_{p,\Omega_0} \tag{3.66}$$

for any $t > 0$. Therefore, if we take

$$t_0 = \left(\frac{q(q-1)\mathcal{F}(\Omega_0)}{q-n}\right)^{\frac{1}{(q-n)/(q-1)-p}},$$

it follows from (3.65) and (3.66) that

$$\mu_{p,t_0\Omega_0}^{\mathcal{C}} = \mu.$$

This implies that $t_0\Omega_0$ is the desired convex body. This completes the proof of lemma 3.5. □

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Lemma 3.6. *For any fixed $n \geqslant 3$ and $p > 1$, we let μ be a positive and finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$ satisfying (A.1.1). Then there exists a convex body $\Omega_0 \in \mathcal{M}$ such that*

$$\mathcal{F}(\Omega_0) = m_p^{\mathcal{C}} = \inf\{\mathcal{F}(K) : K \in \mathcal{M}\}. \tag{3.67}$$

The proof of lemma 3.6 will be postponed and now, we are in a position to complete the proof of theorem 1.6.

Final proof of theorem 1.6. The proof of the existence part of theorem 1.6 follows from lemmas 3.5 and 3.6. Now, it suffices to prove the uniqueness part of theorem 1.6. Indeed, if there exist two $\Omega_1, \Omega_2 \in \mathcal{K}_0^n$ such that $\mu_{p,\Omega_1}^{\mathcal{C}} = \mu_{p,\Omega_2}^{\mathcal{C}} = \mu$, it follows from the definition of $\mathcal{C}_{p,1}$ that

$$\mathcal{C}_{p,1}(\Omega_1, \Omega) = \int_{\mathbb{S}^{n-1}} h_\Omega^p \, \mathrm{d}\mu_{p,\Omega_1}^{\mathcal{C}}(\theta) = \int_{\mathbb{S}^{n-1}} h_\Omega^p \, \mathrm{d}\mu_{p,\Omega_2}^{\mathcal{C}}(\theta) = \mathcal{C}_{p,1}(\Omega_2, \Omega)$$

for any $\Omega \in \mathcal{K}_0^n$. Then, following from theorem 1.2(b), we can see that

$$\Omega_1 = \Omega_2.$$

This completes the proof of the uniqueness part of theorem 1.6 and completes the proof of theorem 1.6. □

We let

$$f_+ = \max\{f, 0\}, A_1 = \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu(\theta), A_2 = \inf_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} (\xi \cdot \theta)_+ \, \mathrm{d}\mu(\theta). \tag{3.68}$$

Now, we divide the proof of lemma 3.6 into following two lemmas.

Lemma 3.7. *For any fixed $n \geqslant 3$ and $p > 1$, we let $\{P_j\}_{j=1}^{\infty}$ be a minimizing sequence of the extremal problem (3.57), ϱ_{P_j} be the radial function of P_j and A_1, A_2 be defined in (3.68). Then there exists a constant $c = c(n, A_1, A_2)$, independent of j, such that*

$$0 \leqslant R_j = \max_{\theta \in \mathbb{S}^{n-1}} \varrho_{P_j}(\theta) \leqslant c \tag{3.69}$$

for any fixed $j \in \{1, 2, \ldots\}$.

Proof. For any fixed $j \in \{1, 2, \ldots\}$, it follows from the definition of the radial function ϱ_{P_j} that ϱ_{P_j} is a continuous function on $\mathbb{S}^{n-1}$. By the compactness of $\mathbb{S}^{n-1}$, we see that the following extremal problem

$$R_j = \max_{\theta \in \mathbb{S}^{n-1}} \varrho_{P_j}(\theta) \tag{3.70}$$

is achieved and we let the maximizer of R_j be v_j. By the definition of R_j, we get

$$[0, R_j v_j] \subseteq P_j$$

and thus

$$(R_j \theta \cdot v_j)_+ = (\theta \cdot R_j v_j)_+ \leqslant h_{P_j}(\theta), \quad \forall \theta \in \mathbb{S}^{n-1}$$

due to the definition of h_{P_j}. It follows from Hölder inequality, the definition of $\mathcal{F}$ and the monotonicity of t^p that

$$R_j^p A_2^p \leqslant \left(\int_{\mathbb{S}^{n-1}} R_j(\theta\cdot v_j)_+ \,\mathrm{d}\mu(\theta)\right)^p \leqslant \left(\int_{\mathbb{S}^{n-1}} h_{P_j}(\theta)\,\mathrm{d}\mu(\theta)\right)^p$$
$$\leqslant A_1^{p-1}\int_{\mathbb{S}^{n-1}} h_{P_j}^p(\theta)\,\mathrm{d}\mu(\theta) = A_1^{p-1}\mathcal{F}(P_j).$$

Since

$$\lim_{j\to\infty}\mathcal{F}(P_j) = m_p^{\mathcal{C}},$$

we see that

$$0 \leqslant R_j \leqslant \frac{(2m_p^{\mathcal{C}}A_1^{p-1})^{1/p}}{A_2}$$

for sufficiently large j. This implies that there exists a positive constant c, independent of j, such that

$$0 \leqslant R_j \leqslant c(m_p^{\mathcal{C}}, A_1, A_2)$$

for any fixed $j \in \{1, 2, \ldots\}$. This is the desired conclusion of the lemma 3.7.

Let $\{P_j\}_{j=1}^{\infty}$ be a minimizing sequence of the extremal problem (3.57), for any fixed j, it follows from the well-known John's lemma that

$$n^{-3/2}\mathcal{E}_j \subseteq P_j \subseteq \mathcal{E}_j \tag{3.71}$$

where $\mathcal{E}_j$ is the ellipsoid of minimum volume containing P_j centred at the centre of mass of P_j (see p. 29 of Gutiérrez [**37**]). Without loss of generality, we may denote $\mathcal{E}_j$ as follows:

$$\mathcal{E}_j = \left\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n}\frac{x_i^2}{b_{i,j}^2} \leqslant 1\right\}$$

where $\{b_{i,j}\}_{i=1}^n$ satisfy

$$b_{n,j} \geqslant b_{n-1,j} \geqslant \cdots \geqslant b_{1,j}. \tag{3.72}$$

With lemma 3.7 and the definition of $b_{n,j}$, we have,

COROLLARY 3.8. *For any fixed j, let $\{b_{i,j}\}_{i=1}^n$ be a sequence defined in (3.72). Then there exists a constant c, independent of j, such that*

$$b_{n,j} \leqslant c < \infty \tag{3.73}$$

for any fixed j.

Combining the boundary behaviours of q-Green function established by Akman *et al.* [**6**] and adopting a similar blow-up argument, we have the following lemma.

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LEMMA 3.9. *For any fixed j, let $\{b_{i,j}\}_{i=1}^{n}$ be a sequence defined in (3.72). Then there exists a constant c, independent of j, such that*

$$b_{1,j} \geqslant c > 0$$

for any fixed j.

□

Since the proof lemma 3.9 can be referred to Hong *et al.* [**39**], we omit the proof here.

Now, we are ready to give the proof of lemma 3.6.

Proof of lemma 3.6. Let $\{P_j\}_{j=1}^{\infty}$ be a minimizing sequence of the extremal problem (3.57). It follows from corollary 3.8, lemma 3.9 and Blaschke's selection theorem (see theorem 1.8.7 of Schneider [**58**]) that there exists a convex body $\Omega_0 \in \mathcal{K}_0^n$ such that, up to a subsequence,

$$P_j \to \Omega_0 \tag{3.74}$$

in the sense of Hausdorff metric as $j \to \infty$. It follows from 2.1 that

$$h_{P_j} \to h_{\Omega_0}$$

and thus

$$h_{P_j}^p \to h_{\Omega_0}^p$$

uniformly on $\mathbb{S}^{n-1}$ as $j \to \infty$ for any fixed $p > 1$. From the definition of $\mathcal{F}$, we have,

$$\lim_{j\to\infty} \mathcal{F}(P_j) = \mathcal{F}(\Omega_0).$$

It is easy to see that

$$h_{\Omega_0} \in C_+(\mathbb{S}^{n-1}) \tag{3.75}$$

due to $h_{P_j} \in C_+(\mathbb{S}^{n-1})$ and 3.9. It follows from (3.74) and 2.5(f) that

$$\mathcal{C}(\Omega_0) = \lim_{j\to\infty} \mathcal{C}(P_j) = 1. \tag{3.76}$$

We conclude from (3.75) and (3.76) that

$$\Omega_0 \in \mathcal{M}.$$

Summing up, we have,

$$\mathcal{F}(\Omega_0) = \inf\{\mathcal{F}(K) : K \in \mathcal{M}\}.$$

This completes the proof of lemma 3.6. □

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