OPTIMAL REINSURANCE WITH LIMITED CEDED RISK: A STOCHASTIC DOMINANCE APPROACH

BY

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ABSTRACT

An optimal reinsurance problem from the perspective of an insurer is studied in this paper, where an upper limit is imposed on a reinsurer's expected loss over a prescribed level. In order to reduce the moral hazard, we assume that both the insurer and the reinsurer are obligated to pay more as the amount of loss increases in a typical reinsurance treaty. We further assume that the optimization criterion preserves the convex order. Such a criterion is very general as most of the criteria for optimal reinsurance problems in the literature preserve the convex order. When the reinsurance premium is calculated as a function of the actuarial value of coverage, we show via a stochastic dominance approach that any admissible reinsurance policy is dominated by a stop-loss reinsurance or a two-layer reinsurance, depending upon the amount of the reinsurance premium. Moreover, we obtain a similar result to Mossin's Theorem and find that it is optimal for the insurer to cede a loss as much as possible under the net premium principle. To further examine the reinsurance premium for the optimal piecewise linear reinsurance policy, we assume the expected value premium principle and derive the optimal reinsurance explicitly under (1) the criterion of minimizing the variance of the insurer's risk exposure, and (2) the criterion of minimizing the risk-adjusted value of the insurer's liability where the liability valuation is carried out using the cost-of-capital approach based on the conditional value at risk.

KEYWORDS

Conditional value at risk, convex order, cost of capital, Mossin's Theorem, stop-loss reinsurance, two-layer reinsurance.

1. INTRODUCTION

Reinsurance, as one of the important risk management tools, allows an insurer to cede large losses to a reinsurer and to reduce the insurer's risk exposure. As a

Astin Bulletin 44(1), 103-126. doi: 10.1017/asb.2013.28

result, the insurer will have a manageable insurance portfolio. It also enables the insurer to free up additional capital to issue more policies. However, the effective use of reinsurance depends highly on reinsurance designs. It is imperative to develop an optimal reinsurance scheme for the insurer to minimize its risk exposure and smooth its surplus.

Since the seminal work of Borch (1960), the study of optimal insurance/reinsurance problems has attracted a lot of attention. Borch (1960) shows that the stop-loss reinsurance is optimal under the criterion of minimizing the variance of an insurer's retained loss when the reinsurance premium is calculated by the expected value principle. Arrow (1963) considers the criterion that maximizes the expected utility of the terminal wealth of a risk-averse insurer but reaches a similar result to Borch (1960). Arrow's result has been extended in many papers. See e.g., Raviv (1979), Van Heerwaarden *et al.* (1989), Gollier and Schlesinger (1996), Young (1999), Cummins and Mahul (2004), Zhou *et al.* (2010), and references therein. Recently, due to the popularity of risk measures in quantifying financial and insurance risks, risk measure-based optimal reinsurance problems have been studied by many researchers (see Gajek and Zagrodny, 2004; Balbás *et al.*, 2009; Guerra and Centeno, 2012; Asimit *et al.*, 2013; Chi and Tan, 2013; Lu *et al.*, 2013 and references therein). A common characteristic of most of the optimization criteria used in the afore-mentioned studies is the preservation of the convex order, which may allow the use of a unified approach to tackle optimal reinsurance problems. See Van Heerwaarden *et al.* (1989) and Gollier and Schlesinger (1996) for unified treatments of the problems.

In the literature, it is often assumed that the ceded loss from a reinsurance treaty is smaller than the indemnity. Again, see Borch (1960), Arrow (1963), Gajek and Zagrodny (2004), Balbás *et al.* (2009), Bernard and Tian (2009), Zhou *et al.* (2010), and references therein. However, as pointed out by Huberman *et al.* (1983), this assumption is insufficient as it neglects the moral hazard that may arise in a reinsurance treaty. In order to reduce the moral hazard, it is necessary to require that both the insurer and the reinsurer are obligated to pay more for a larger loss, as described in Huberman *et al.* (1983). Mathematically, it is equivalent to that the ceded loss function is increasing and Lipschitz continuous. In addition, more constraints may be imposed on a reinsurance contract in practice. For instance, Cummins and Mahul (2004) point out that the real-world insurance and reinsurance markets typically impose limits on coverage. For this reason, Zhou and Wu (2008) impose an upper limit on the reinsurer's expected loss over a prescribed level, while Cummins and Mahul (2004) and Zhou *et al.* (2010) set an upper bound on ceded losses. It is worthwhile noting that different constraints may lead to different optimal reinsurance policies as discussed in Chi and Tan (2011). On the other hand, more constraints imposed on a reinsurance contract may lead to more difficulties in solving the respective optimal reinsurance problem.

Optimal control theory and calculus of variation are traditional approaches for the study of optimal reinsurance. See e.g., Cummins and Mahul (2004), Gajek and Zagrodny (2004), Zhou and Wu (2008), Balbás *et al.* (2009), Zhou *et al.* (2010), and references therein. However, these methods have two significant shortcomings. One is that they are inapplicable for certain monotonic constraints on reinsurance contracts. The other is that they require objective functions under consideration to be smooth. For instance, in Cummins and Mahul (2004) the utility function is assumed to be twice differentiable. Thus, the approaches in these papers can not be applied to optimal reinsurance problems under a general optimization criterion that is only known to preserve a stochastic order. On the other hand, a stochastic dominance approach could be applied to solve optimal reinsurance problems of this kind when the reinsurance premium only depends on the actuarial value of the coverage. As demonstrated in Gollier and Schlesinger (1996), this approach not only widens the accessibility of Arrow's result to a large number of criteria but also sheds light on why the deductible reinsurance is optimal for a risk-averse insurer.

In this paper, we study an optimal reinsurance problem from the perspective of an insurer and under a general criterion that preserves the convex order. We extend the studies of Van Heerwaarden *et al.* (1989) and Gollier and Schlesinger (1996) by imposing an upper limit on a reinsurer's expected loss above a prescribed level. In order to reduce the moral hazard, we follow Huberman *et al.* (1983) to assume that both the insurer and the reinsurer are obligated to pay more as the amount of loss increases in a typical reinsurance treaty. When the reinsurance premium depends only on the actuarial value of the coverage, we show via a stochastic dominance approach that any feasible reinsurance contract is dominated by a stop-loss reinsurance or a two-layer reinsurance, depending upon the level of the reinsurance premium. A surprising finding from this research is that given a fixed reinsurance premium, the optimal reinsurance policy is independent of the form of the criterion. Moreover, we obtain a similar result to Mossin's Theorem, and find that it is optimal for the insurer to cede a loss as much as possible under the net premium principle. To further examine the reinsurance premium for the optimal piecewise linear reinsurance policy, we assume the expected value premium principle and derive the optimal reinsurance explicitly under (1) the criterion of minimizing the variance of the insurer's risk exposure, and (2) the criterion of minimizing the risk-adjusted value of the insurer's liability where the liability valuation is carried out using the cost-of-capital approach and the capital at risk is calculated by conditional value at risk.

The rest of the paper is organized as follows. In Section 2, we introduce the reinsurance model and its optimization criterion. In Section 3, we show that any admissible reinsurance policy is dominated by a stop-loss reinsurance or a two-layer reinsurance depending on the amount of the reinsurance premium, and obtain a similar result to Mossin's Theorem. To illustrate the applicability of the results established in Section 3, we assume the expected value premium principle, and derive the optimal reinsurance policy explicitly under the criterion of minimizing the variance of an insurer's risk exposure in Section 4 and the criterion of minimizing the risk-adjusted value of an insurer's liability in Section 5. Finally, some concluding remarks are given in Section 6.

2. Mathematical description of a reinsurance model

Suppose that X denotes the amount of loss an insurer faces over a given time period. We assume X is a nonnegative random variable defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with cumulative distribution function $F_X(x) = \mathbb{P}(X \leq x),\ x \geq 0$. Insurability implies that the mean of the loss $0 < \mathbb{E}[X] < \infty$. An optimal reinsurance problem concerns with an optimal partition of X into two parts: $f(X)$ and $R_f(X)$, where $f(X)$, satisfying $0 \leq f(X) \leq X$, represents the portion of the loss that is ceded to a reinsurer and $R_f(X) = X - f(X)$ is the loss retained by the insurer. The functions $f(x)$ and $R_f(x)$ are usually called the ceded and retained loss functions, respectively. In order to reduce the moral hazard, we assume that both the insurer and the reinsurer are obligated to pay more when the amount of loss increases in a typical reinsurance treaty. In other words, both $f(x)$ and $R_f(x)$ are increasing functions. As shown in Chi and Tan (2011), this monotonic property is equivalent to

$$0 \leq f(x_2) - f(x_1) \leq x_2 - x_1, \quad \forall\, 0 \leq x_1 \leq x_2, \tag{2.1}$$

and hence $f(x)$ is Lipschitz continuous.

Similar to that in Zhou and Wu (2008), we assume the reinsurance premium is a function of the net premium. More precisely, let $\pi(\cdot)$ represent a reinsurance premium principle, then we have

$$\pi(Y) = h(\mathbb{E}[Y]) \quad \text{for any nonnegative random variable} \quad Y,$$

where $h(\cdot)$ is a strictly increasing function with $h(0) = 0$ and $h(y) \geq y$. If $h(y) = y$, we recover the net premium principle. If $h(y) = (1 + \rho)y$ for some positive safety loading ρ, we recover the expected value premium principle. Further, let $L_f(X)$ be the net loss to the reinsurer. That is,

$$L_f(X) = f(X) - \pi(f(X)). \tag{2.2}$$

Denote by $T_f(X)$ the total risk exposure of the insurer, and it is easy to see

$$T_f(X) = R_f(X) + \pi(f(X)).$$

As mentioned earlier, the real-world insurance and reinsurance markets typically impose limits on coverage. Hence, we assume that in addition to reducing the moral hazard, the ceded losses are subject to the reinsurer's risk tolerance. In particular, the reinsurer wants to restrict the risk exposure by imposing an upper limit on its expected loss over a threshold. Mathematically, it may be expressed as

$$\mathbb{E}\left[\left(L_f(X) - L_0\right)_+\right] \leq \epsilon \tag{2.3}$$

for a threshold of $L_0 > 0$ and an upper bound of $\epsilon \geq 0$, where $(x)_+ \triangleq \max\{x, 0\}$. If $\epsilon \geq \mathbb{E}[(X - L_0)_+]$, the above constraint is redundant, and the problem has been analyzed in Van Heerwaarden *et al.* (1989). For this reason, we assume

$$0 \le \epsilon < \mathbb{E}[(X - L_0)_+].$$

When $\epsilon > 0$, (2.3) coincides with the constraint in Zhou and Wu (2008); if $\epsilon = 0$, it is equivalent to $L_f(X) \le L_0$, $a.s.$, which is assumed in Zhou *et al.* (2010). The set of admissible ceded loss functions is now given by

$$\mathfrak{C} \triangleq \Big\{0 \le f(x) \le x : f \text{ satisfies constraints (2.1) and (2.3)}\Big\}. \tag{2.4}$$

The optimization criterion we are choosing is only based on the convex order. Specifically, we let $\Psi(Y)$ be an objective function to optimize, and assume that it preserves the convex order (cx), i.e.,

$$\Psi(Y_1) \le \Psi(Y_2), \quad \text{if} \quad Y_1 \le_{cx} Y_2,$$

where $Y_1 \le_{cx} Y_2$ if and only if

$$\mathbb{E}[Y_1] = \mathbb{E}[Y_2] \quad \text{and} \quad \mathbb{E}[(Y_1 - d)_+] \le \mathbb{E}[(Y_2 - d)_+], \quad \text{for any real number } d. \tag{2.5}$$

For the convex order and its properties, see Shaked and Shanthikumar (2007). We are seeking the optimal reinsurance scheme f such that

$$\min_{f \in \mathfrak{C}} \Psi(T_f(X)). \tag{2.6}$$

This criterion is very general, and many optimization criteria for reinsurance models including the maximization of the expected utility of the terminal wealth of a risk-averse insurer, the minimization of the variance of an insurer's total risk exposure, the minimization of the insurance premium, to name a few, are special cases.

3. Optimal reinsurance policies

In this section, we investigate the optimization problem described in Section 2. The optimal reinsurance scheme is identified using a stochastic dominance approach, and especially it is derived explicitly for the net premium principle.

To proceed, we define

$$d_X(t) \triangleq \inf\{d \ge 0 : \mathbb{E}[(X - d)_+] \le t\}, \ t \ge 0, \tag{3.1}$$

where $\inf \emptyset = \infty$. Moreover, let

$$\mathcal{M} \triangleq \{0 \le \mu \le \mathbb{E}[X] : \mathbb{E}[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+] \ge \mu \quad \text{or} \quad w(\mu) \ge 0\}, \tag{3.2}$$

https://doi.org/10.1017/asb.2013.28 Published online by Cambridge University Press

where

$$w(\mu) \triangleq \int_0^{h(\mu)+L_0} S_X(t)\mathrm{d}t - \mu + \epsilon. \tag{3.3}$$

Here, $S_X(t) \triangleq 1 - F_X(t)$. Also denote the layer of reinsurance loss Y between a and b where $0 \leq a < b$ as

$$\mathcal{L}_{(a,b]}(Y) \triangleq \min\{(Y-a)_+, b-a\} = (Y-a)_+ - (Y-b)_+. \tag{3.4}$$

In order to solve the optimal reinsurance problem (2.6), we need the following lemma.

Lemma 3.1. *For any* $0 \leq \mu \leq \mathbb{E}[X]$*, define a piecewise linear reinsurance policy*

$$f_\mu(x) \triangleq \begin{cases} (x - d_X(\mu))_+, & \mathbb{E}\left[\left(X - (d_X(\epsilon) - L_0 - h(\mu))_+\right)_+\right] \geq \mu; \\ \mathcal{L}_{(a_\mu, a_\mu + L_0 + h(\mu)]}(x) + (x - d_X(\epsilon))_+, & \text{otherwise}, \end{cases} \tag{3.5}$$

where $0 \leq a_\mu \leq d_X(\epsilon) - L_0 - h(\mu)$ *is determined by solving the equation* $\mathbb{E}[f_\mu(X)] = \mu$*. Then the following three conditions are equivalent:*
(i) μ *is a member of set* $\mathcal{M}$*; (ii)* f_μ *is well defined; and (iii)* f_μ *is admissible, i.e.,* $f_\mu \in \mathfrak{C}$*.*

Proof. First, we show that $\mu \in \mathcal{M}$ if and only if $f_\mu(x)$ is well defined, which is further equivalent to the existence of a_μ. Specifically, if $\mathbb{E}\left[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+\right] < \mu$, (3.5) implies the existence of a_μ is equivalent to that equation

$$\mathfrak{M}(a) \triangleq \int_a^{a+h(\mu)+L_0} S_X(t)\mathrm{d}t - \mu + \epsilon = 0,\ 0 \leq a \leq d_X(\epsilon) - h(\mu) - L_0 \tag{3.6}$$

has solutions. Obviously, $\mathfrak{M}(a)$ is a decreasing function with

$$\mathfrak{M}\left(d_X(\epsilon) - h(\mu) - L_0\right) = \mathbb{E}\left[(X - (d_X(\epsilon) - h(\mu) - L_0))_+\right] - \mu < 0.$$

Thus, equation (3.6) has solutions if and only if $w(\mu) = \mathfrak{M}(0) \geq 0$.

Next, if $f_\mu \in \mathfrak{C}$, the existence of f_μ would imply $\mu \in \mathcal{M}$. On the other hand, if $\mu \in \mathcal{M}$, the previous analysis implies that f_μ is well defined and is an increasing and Lipschitz continuous function. Thus, to show $f_\mu \in \mathfrak{C}$, it is only necessary to prove that $f_\mu(x)$ satisfies the constraint (2.3). The proof is divided into three cases:

(i) If $\mu = 0$, we have $f_\mu(X) = 0$, $a.s.$ and obviously (2.3) is satisfied;
(ii) If $\mathbb{E}\left[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+\right] \geq \mu > 0$, (3.1) implies

$$d_X(\mu) \geq d_X(\epsilon) - L_0 - h(\mu).$$

https://doi.org/10.1017/asb.2013.28 Published online by Cambridge University Press

In this case, the definition of $f_\mu(x)$ in (3.5) implies

$$\begin{aligned}\mathbb{E}[(f_\mu(X) - h(\mu) - L_0)_+] &\leq \mathbb{E}[((X - (d_X(\epsilon) - L_0 - h(\mu)))_+ - h(\mu) - L_0)_+] \\ &\leq \mathbb{E}[(X - d_X(\epsilon))_+] \leq \epsilon.\end{aligned}$$

(iii) Otherwise, $\mathbb{E}\left[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+\right] < \mu$. It follows from (3.5) that

$$\mathbb{E}[(f_\mu(X) - h(\mu) - L_0)_+] = \mathbb{E}[(X - d_X(\epsilon))_+] \leq \epsilon.$$

Collecting all the above arguments yields that $f_\mu \in \mathfrak{C}$ if and only if $\mu \in \mathcal{M}$. The proof is finally complete. □

With the help of the above lemma, we obtain the main result of this paper in the following theorem.

Theorem 3.1. *For any admissible policy $f \in \mathfrak{C}$, let $\mu = \mathbb{E}[f(X)]$, then f_μ exists and the inequality*

$$\Psi(T_{f_\mu}(X)) \leq \Psi(T_f(X)) \tag{3.7}$$

holds. As a result, we have

$$\min_{f\in\mathfrak{C}} \Psi(T_f(X)) = \min_{\mu\in\mathcal{M}} \Psi(T_{f_\mu}(X)). \tag{3.8}$$

In other words, the optimal reinsurance policy is of the piecewise linear form defined in (3.5).

Proof. For any ceded loss function $f \in \mathfrak{C}$ with $\mu = \mathbb{E}[f(X)]$, define

$$x_f \triangleq \inf\{x \geq 0 : f(x) \geq L_0 + h(\mu)\}.$$

As $0 \leq f(x) \leq x$, we know from the above definition that $x_f \geq L_0 + h(\mu)$.

Recall that $f(x)$ is increasing and Lipschitz continuous as stated in (2.1). We can construct a ceded loss function

$$f_1(x) \triangleq \begin{cases} (x - d_0)_+ + L_0 + h(\mu), & x > x_f; \\ f(x), & 0 \leq x \leq x_f, \end{cases}$$

where $d_0 \geq x_f$ is determined by $\mathbb{E}[f_1(X)] = \mu$. It follows from $\mathbb{E}[f(X)] = \mathbb{E}[f_1(X)] = \mu$ that

$$\mathbb{E}[(L_{f_1}(X) - L_0)_+] = \mathbb{E}[(X - d_0)_+] = \mathbb{E}[(f(X) - L_0 - h(\mu))_+] \leq \epsilon, \tag{3.9}$$

where $L_f(X)$ is defined in (2.2) and the last inequality is implied by (2.3). We thus have $f_1 \in \mathfrak{C}$.

Building upon $f_1(x)$, we construct another ceded loss function

$$f_2(x) \triangleq \mathcal{L}_{(a,\, a+L_0+h(\mu)]}(x) + (x - d_0)_+,\ x \geq 0,$$

https://doi.org/10.1017/asb.2013.28 Published online by Cambridge University Press

where $a \in [0, x_f - L_0 - h(\mu)]$ is determined by $\mathbb{E}[f_2(X)] = \mu$. It is easy to see that f_2 satisfies (2.1). Moreover, the definition of $f_2(x)$, together with (3.9), implies

$$\mathbb{E}[(L_{f_2}(X) - L_0)_+] = \mathbb{E}[(X - d_0)_+] \leq \epsilon.$$

Thus, we have $f_2 \in \mathfrak{C}$.

Using (2.1) again, it is easy to see that $R_f(x)$ up-crosses $R_{f_1}(x)$ and $R_{f_1}(x)$ up-crosses $R_{f_2}(x)$.[1] Consequently, it follows from Lemma 3 in Ohlin (1969) that

$$R_{f_2}(X) \leq_{cx} R_{f_1}(X) \leq_{cx} R_f(X). \tag{3.10}$$

We now proceed to show the existence of f_μ and compare the reinsurance scheme f_μ with f_2. The following analysis is divided into two cases:

(i) If $\mathbb{E}\left[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+\right] \geq \mu$, we have $f_\mu(x) = (x - d_X(\mu))_+$. It is easy to see from (2.1) and Lemma 3 in Ohlin (1969) that $R_{f_\mu}(X) \leq_{cx} R_{f_2}(X)$.

(ii) Otherwise, $\mathbb{E}\left[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+\right] < \mu$. Noting that $\mathbb{E}[f_2(X)] = \mu$, we have

$$a + L_0 + h(\mu) < d_X(\epsilon) \leq d_0,$$

where the last inequality is implied by (3.9). Therefore, for this case, $f_\mu(x)$ in (3.5) is well defined with $a \leq a_\mu \leq d_X(\epsilon) - L_0 - h(\mu)$. Furthermore, it is easy to see that $R_{f_2}(.)$ up-crosses $R_{f_\mu}(.)$ such that $R_{f_\mu}(X) \leq_{cx} R_{f_2}(X)$.

Collecting all the above arguments, together with Lemma 3.1 and (3.10), yields $\mu \in \mathcal{M}$ and

$$T_{f_\mu}(X) = R_{f_\mu}(X) + h(\mu) \leq_{cx} T_f(X).$$

Recall that $\Psi(.)$ preserves the convex order. Then we have (3.7). Furthermore, using Lemma 3.1 again, we have $\{f_\mu : \mu \in \mathcal{M}\} \subset \mathfrak{C}$ and hence (3.8) holds. The proof is finally complete. □

Noting that the reinsurance premium is a function of the net premium, the above result shows that under a fixed reinsurance premium, any feasible reinsurance policy is dominated by a stop-loss reinsurance policy or a two-layer reinsurance policy, depending upon the level of the reinsurance premium, and the optimal reinsurance policy does not depend on the specific form of the objective function Ψ. Since our criterion is more general than the maximization of the expected utility of the terminal wealth of a risk-averse insurer, we generalize the results in Proposition 3 in Zhou and Wu (2008) and Proposition 1 in Zhou *et al.* (2010). Furthermore, our proof uses a constructive approach that provides more insights on why the piecewise linear reinsurance policy is optimal than the traditional methods such as the optimal control theory and the calculus of variation.

By the above theorem, the analysis of optimal reinsurance model (2.6) is simplified to solving an optimization problem of one variable in (3.8). Intuitively, the optimal f_μ would rely on the specific form of the objective function Ψ. Surprisingly, we find that it is not the case under the net premium principle as in the following proposition.

Proposition 3.1. *When the reinsurance premium is calculated by the net premium principle, i.e., $h(x) = x$, the optimal ceded loss function for the reinsurance model* (2.6) *is given by*

$$f^*(x) = \begin{cases} x, & \mathbb{E}[X] \geq d_X(\epsilon) - L_0; \\ \mathcal{L}_{(0,\, \mu^*+L_0]}(x) + (x - d_X(\epsilon))_+, & \textit{otherwise}, \end{cases} \tag{3.11}$$

where μ^* is determined by $\mathbb{E}[f^*(X)] = \mu^*$.

Proof. Since $h(x) = x$, we have that $\mathbb{E}[(X - (d_X(\epsilon) - L_0 - h(\mu))_+)_+] \geq \mu$ is equivalent to $\phi(d_X(\mu)) \geq d_X(\epsilon) - L_0$, where

$$\phi(d) \triangleq d + \int_d^\infty S_X(t)\mathrm{d}t,\ d \geq 0.$$

It is easy to see that $\phi(d)$ is an increasing continuous function on $\mathbb{R}_+$. Thus, if $\mathbb{E}[X] \geq d_X(\epsilon) - L_0$, we have

$$\phi(d) \geq \phi(0) = \mathbb{E}[X] \geq d_X(\epsilon) - L_0,\ \forall d \geq 0.$$

In this case, Lemma 3.1 implies $f_\mu(x) = (x - d_X(\mu))_+$ for any $0 \leq \mu \leq \mathbb{E}[X]$. Further, it follows from Lemma A.2 in Chi (2012) that

$$T_{f_\mu}(X) = \mathcal{L}_{(0,d_X(\mu)]}(X) - \mathbb{E}[\mathcal{L}_{(0,d_X(\mu)]}(X)] + \mathbb{E}[X]$$

is increasing of $d_X(\mu)$ in the convex order. Recall that $\Psi(.)$ preserves the convex order. Then it follows from (3.8) that $f^*(x) = x$ is a solution to the optimal reinsurance problem (2.6) under the net premium principle.

If $\mathbb{E}[X] < d_X(\epsilon) - L_0$, for any $d_X(\mu) \geq d_M$ where

$$d_M \triangleq \inf\{d \geq 0 : \phi(d) \geq d_X(\epsilon) - L_0\},$$

we have $\phi(d_X(\mu)) \geq d_X(\epsilon) - L_0$. Then it follows from Lemma 3.1 that $f_\mu(x) = (x - d_X(\mu))_+$. Using a similar analysis, we have

$$\Psi(T_{f_{\mu_M}}(X)) \leq \Psi(T_{f_\mu}(X)),\ \forall 0 \leq \mu \leq \mu_M,$$

where $\mu_M \triangleq \mathbb{E}[(X - d_M)_+]$. Consequently, the minimum of $\Psi(T_{f_\mu}(X))$ would appear on $\mu \geq \mu_M$ in this case.

Further, when $\mu > \mu_M$, we have $\mathbb{E}[(X - (d_X(\epsilon) - L_0 - h(\mu)))_+] < \mu$. Hence, if $w(\mu) \geq 0$ where $w(\mu)$ is defined in (3.3), Lemma 3.1 implies

$$f_\mu(x) = \mathcal{L}_{(a_\mu, a_\mu+L_0+\mu]}(x) + (x - d_X(\epsilon))_+$$

and equality $\mathbb{E}[f_\mu(X)] = \mu$ leads to

$$\int_{a_\mu}^{a_\mu+L_0+\mu} S_X(t)\mathrm{d}t = \mu - \epsilon. \tag{3.12}$$

We demonstrate that $R_{f_\mu}(X) - \mathbb{E}[R_{f_\mu}(X)]$ is decreasing in the convex order. Specifically, according to (2.5), it suffices to show that for any $\mu_1 > \mu_2 > \mu_M$,

$$\psi(\mu_1, t) \le \psi(\mu_2, t), \forall t \in \mathbb{R} \tag{3.13}$$

holds true, where

$$\psi(\mu, t) \triangleq \mathbb{E}[(R_{f_\mu}(X) - \mathbb{E}[R_{f_\mu}(X)] - t)_+]$$
$$= \begin{cases} 0, & t \ge d_X(\epsilon) - L_0 - \mathbb{E}[X]; \\ \int_{t+\mathbb{E}[X]+L_0}^{d_X(\epsilon)} S_X(y)\mathrm{d}y, & a_\mu + \mu - \mathbb{E}[X] \le t < d_X(\epsilon) - L_0 - \mathbb{E}[X]; \\ \int_{t+\mathbb{E}[X]-\mu}^{\infty} S_X(y)\mathrm{d}y - \mu, & t < a_\mu + \mu - \mathbb{E}[X]. \end{cases}$$

Taking the derivative of a_μ with respect to μ in (3.12) yields

$$\frac{\partial a_\mu}{\partial \mu} = -\frac{F_X(a_\mu + L_0 + \mu)}{S_X(a_\mu) - S_X(a_\mu + L_0 + \mu)}, \ a.s.$$

Then $a_\mu + \mu$ is decreasing in μ. Thus, the proof of (3.13) can be divided into three cases as follows.

(i) If $t \ge a_{\mu_2} + \mu_2 - \mathbb{E}[X]$, we have $\psi(\mu_1, t) - \psi(\mu_2, t) = 0$;

(ii) If $a_{\mu_1} + \mu_1 - \mathbb{E}[X] \le t < a_{\mu_2} + \mu_2 - \mathbb{E}[X]$, we have

$$\begin{aligned} \psi(\mu_1, t) - \psi(\mu_2, t) &= -\int_{t+\mathbb{E}[X]-\mu_2}^{t+\mathbb{E}[X]+L_0} S_X(y)\mathrm{d}y + \mu_2 - \epsilon \\ &\le -\int_{a_{\mu_2}}^{a_{\mu_2}+\mu_2+L_0} S_X(y)\mathrm{d}y + \mu_2 - \epsilon = 0, \end{aligned}$$

where the inequality is derived by the fact $t < a_{\mu_2} + \mu_2 - \mathbb{E}[X]$ and the last equality is implied by (3.12);

(iii) If $t < a_{\mu_1} + \mu_1 - \mathbb{E}[X]$, we have

$$\psi(\mu_1, t) - \psi(\mu_2, t) = \int_{t+\mathbb{E}[X]-\mu_1}^{t+\mathbb{E}[X]-\mu_2} S_X(y)\mathrm{d}y - (\mu_1 - \mu_2) \le 0.$$

As a consequence, noting that $T_{f_\mu}(X) = R_{f_\mu}(X) - \mathbb{E}[R_{f_\mu}(X)] + \mathbb{E}[X]$, we have that the minimum of $\Psi(T_{f_\mu}(X))$ is attainable at $\mu = \mu^*$, where

$$\mu^* = \max\{\mu_M \le \mu \le \mathbb{E}[X] : \ w(\mu) \ge 0\}.$$

The remaining task is to show $a_{\mu^*} = 0$. Since $\frac{\partial a_\mu}{\partial \mu} \le 0$, it suffices to prove that equation

$$w(\mu) = 0, \ \mu_M \le \mu \le \mathbb{E}[X] \tag{3.14}$$

has a solution. Specifically, since it is assumed that $\mathbb{E}[X] < d_X(\epsilon) - L_0$, then we have $w(\mathbb{E}[X]) = -\int_{\mathbb{E}[X]+L_0}^{d_X(\epsilon)} S_X(t)\mathrm{d}t < 0$. On the other hand, if $d_M < \infty$, the definition of d_M implies

$$a_{\mu_M} = d_X(\mu_M) = d_X(\epsilon) - L_0 - \mu_M,$$

then $w(\mu_M) \ge \int_{a_{\mu_M}}^{a_{\mu_M}+\mu_M+L_0} S_X(t)\mathrm{d}t + \epsilon - \mu_M = 0$; otherwise, we have $\mu_M = 0$ such that $w(\mu_M) > 0$. Recall that $w(\mu)$ is a decreasing continuous function for $h(x) = x$. Then equation (3.14) has solutions. The final result follows from Theorem 3.1 and the proof is complete. □

Note that the condition $\mathbb{E}[X] \ge d_X(\epsilon) - L_0$ is equivalent to

$$\mathbb{E}\left[(X - \mathbb{E}[X] - L_0)_+\right] \le \epsilon.$$

Hence, under this condition, the full reinsurance is feasible. By the above proposition, the optimal reinsurance strategy for the insurer is to cede the full loss to the reinsurer under the net premium principle, which is consistent with Mossin's Theorem (Mossin, 1968). Further, if the full reinsurance becomes inadmissible, the above proposition shows that it remains optimal for the insurer to cede a loss as much as possible. As a by-product, the above proof provides an example to illustrate that f_μ may be nonexistent for some $0 \le \mu \le \mathbb{E}[X]$. Specifically, if $\mathbb{E}[X] < d_X(\epsilon) - L_0$, noting that $\mu^* = \sup\{0 \le \mu \le \mathbb{E}[X] : \mu \in \mathcal{M}\} < \mathbb{E}[X]$, f_μ is not well defined for any $\mu^* < \mu \le \mathbb{E}[X]$ according to Lemma 3.1.

It is obvious that μ^* is a function of the threshold L_0 and the upper bound ϵ. It is interesting to carry out sensitivity analysis to explore the effect of these two factors on the optimal reinsurance scheme.

Proposition 3.2. *μ^* is strictly increasing and concave in both L_0 and ϵ.*

Proof. If $\mathbb{E}[X] < d_X(\epsilon) - L_0$, noting that $\mathbb{E}[f^*(X)] = \mu^*$ in Proposition 3.1, we have

$$\int_0^{\mu^*+L_0} S_X(t)\mathrm{d}t = \mu^* - \epsilon.$$

Taking the derivatives of μ^* with respect to L_0 and ϵ yields

$$\frac{\partial \mu^*}{\partial L_0} = \frac{1}{F_X(\mu^* + L_0)} - 1 > 0, \ a.s. \quad \text{and} \quad \frac{\partial \mu^*}{\partial \epsilon} = \frac{1}{F_X(\mu^* + L_0)} > 1, a.s.$$

Moreover, it follows from the above equation that both $\frac{\partial \mu^*}{\partial L_0}$ and $\frac{\partial \mu^*}{\partial \epsilon}$ are decreasing in L_0 and ϵ respectively, and hence μ^* is concave in L_0 and ϵ. The proof is thus complete. □

From Proposition 3.1, we have a clear picture on the optimal reinsurance scheme under the net premium principle. Generally speaking, the insurer will cede a loss as much as possible. The above proposition further indicates that when the reinsurer increases its risk tolerance, i.e., increasing the value of L_0 or ϵ, for the expected tail loss, the insurer would cede more loss. However, when we move away from the net premium principle, the specification of the optimization criterion is needed. In the next two sections, we discuss two risk measure-based criteria when the expected value premium principle is considered.

4. Minimal variance criterion

Theorem 3.1 shows that any feasible reinsurance policy is suboptimal to a piecewise linear reinsurance policy that is in the form of stop-loss or two-layer depending upon the level of the reinsurance premium. In this and next sections, we further derive the optimal piecewise linear reinsurance policy under some specific form of the objective function.

The criterion under consideration in this section is to minimize the variance of an insurer's risk exposure, as variance is a commonly used risk measure in insurance to quantify the fluctuation risk. This criterion has been considered in several papers on optimal reinsurance models. See Borch (1960) and Kaluszka (2001) for example. We assume that the reinsurance premium is calculated by the expected value principle, i.e., $h(x) = (1+\rho)x$, where $\rho > 0$ is the relative safety loading. Hence, the optimal reinsurance problem (2.6) has the objective function

$$\Psi(Y) = var(Y),$$

where $var(Y)$ is the variance of the loss Y. We have the following result.

Proposition 4.1. *Under the criterion of minimizing the variance of an insurer's risk exposure, when the reinsurance premium is calculated by the expected value principle, the optimal ceded loss function for the reinsurance model* (2.6) *is given by*

$$f^*(x) = \begin{cases} x, & (1+\rho)\mathbb{E}[X] + L_0 \geq d_X(\epsilon); \\ \mathcal{L}_{(0,(1+\rho)\tilde{\mu}+L_0]}(x) + (x - d_X(\epsilon))_+, & otherwise, \end{cases} \tag{4.1}$$

where

$$\tilde{\mu} \triangleq \max\left\{0 \leq \mu \leq \mathbb{E}[X] : \mathbb{E}[\mathcal{L}_{(0,(1+\rho)\mu+L_0]}(X)] = \mu - \epsilon\right\}. \tag{4.2}$$

Proof. It is easy to see that

$$\mathbb{E}[(X - (d_X(\epsilon) - L_0 - (1+\rho)\mu))_+] \geq \mu \quad \text{if and only if} \quad \nu(d_X(\mu)) \geq 0 \tag{4.3}$$

for any $0 \leq \mu \leq \mathbb{E}[X]$, where

$$\nu(d) \triangleq d + (1+\rho)\int_d^{\infty} S_X(t)\mathrm{d}t + L_0 - d_X(\epsilon),\ d \geq 0. \tag{4.4}$$

If $\nu(0) \geq 0$, i.e., $(1+\rho)\mathbb{E}[X]+L_0 \geq d_X(\epsilon)$, then Lemma 3.1 implies that $f(x) = x$ is a feasible ceded loss function, and we have $T_f(X) = (1+\rho)\mathbb{E}[X]$ such that $var(T_f(X)) = 0$. Consequently, $f(x) = x$ is the optimal ceded loss function in this case.

If $\nu(0) < 0$, when $d_X(\epsilon) < \infty$, it is easy to see that equation

$$\nu(d) = 0,\ 0 \leq d < \infty \tag{4.5}$$

has a unique solution d_ν as $\nu(d)$ is a convex function. For $d_X(\epsilon) = \infty$, we let $d_\nu = \infty$. Thus, we have $\nu(d) > 0$ for $d > d_\nu$ and $\nu(d) < 0$ for $d < d_\nu$. Hence, for any $0 \leq \mu \leq \mathbb{E}[(X-d_\nu)_+]$, we have $\nu(d_X(\mu)) \geq 0$ and Lemma 3.1 implies $f_\mu(x) = (x - d_X(\mu))_+$. In this case, we obtain

$$var(T_{f_\mu}(X)) = var\left(\mathcal{L}_{(0,d_X(\mu)]}(X) - \mathbb{E}[\mathcal{L}_{(0,d_X(\mu)]}(X)]\right).$$

Since the variance preserves the convex order, it follows from Lemma A.2 in Chi (2012) that

$$var(T_{f_\mu}(X)) \geq var(T_{f_{\mathbb{E}[(X-d_\nu)_+]}}(X)).$$

On the other hand, if $\mathbb{E}[(X-d_\nu)_+] \leq \mu \leq \mathbb{E}[X]$, it follows from Lemma 3.1 that

$$f_\mu(x) = \mathcal{L}_{(a_\mu, a_\mu+(1+\rho)\mu+L_0]}(x) + (x - d_X(\epsilon))_+,\ \text{if} \quad w(\mu) \geq 0, \tag{4.6}$$

where $w(\mu)$ is given in (3.3) and $0 \leq a_\mu \leq d_X(\epsilon) - (1+\rho)\mu - L_0$ is determined by

$$\int_{a_\mu}^{a_\mu+(1+\rho)\mu+L_0} S_X(t)\mathrm{d}t - \mu + \epsilon = 0. \tag{4.7}$$

Simple calculation leads to

$$\frac{\partial a_\mu}{\partial \mu} = \frac{(1+\rho)S_X(a_\mu + (1+\rho)\mu + L_0) - 1}{S_X(a_\mu) - S_X(a_\mu + (1+\rho)\mu + L_0)},\ a.s. \tag{4.8}$$

and

$$\mathbb{P}(R_{f_\mu}(X) > t) = \begin{cases} 0, & t \geq d_X(\epsilon) - (1+\rho)\mu - L_0; \\ S_X((1+\rho)\mu + L_0 + t), & a_\mu \leq t < d_X(\epsilon) - (1+\rho)\mu - L_0; \\ S_X(t), & t < a_\mu. \end{cases}$$

Further, it is well known that $\mathbb{E}[X^2] = 2\int_0^\infty tS_X(t)\mathrm{d}t$ and $\mathbb{E}[R_{f_\mu}(X)] = \mathbb{E}[X] - \mu$. Then we have

$$\frac{\partial var(T_{f_\mu}(X))}{\partial \mu} = -2a_\mu - 2(1+\rho)\int_{a_\mu+(1+\rho)\mu+L_0}^{d_X(\epsilon)} S_X(t)\mathrm{d}t + 2(\mathbb{E}[X]-\mu)$$
$$= -2\int_0^{a_\mu} F_X(t)\mathrm{d}t - 2\rho\int_{a_\mu+(1+\rho)\mu+L_0}^{d_X(\epsilon)} S_X(t)\mathrm{d}t \le 0, a.s.$$

where the second equality is obtained by (4.7). Consequently, if $\nu(0) < 0$, the optimal ceded loss function is $f_{\hat{\mu}}(x)$ in the form of (4.6), where

$$\hat{\mu} \triangleq \max\left\{\mu \in \left[\mathbb{E}[(X-d_\nu)_+], \mathbb{E}[X]\right] : w(\mu) \ge 0\right\}.$$

Finally, it is easy to see that $w(\mu)$ for $h(x) = (1+\rho)x$ is a concave function with

$$w(\mathbb{E}[X]) < 0 \quad \text{and} \quad w(\mathbb{E}[(X-d_\nu)_+]) \ge \int_{d_\nu}^{d_X(\epsilon)} S_X(t)\mathrm{d}t - \mathbb{E}[(X-d_\nu)_+] + \epsilon = 0.$$

Thus, $\tilde{\mu}$ defined in (4.2) exists and $\hat{\mu} = \tilde{\mu}$ and the proof is complete. □

The above proposition shows that the optimal reinsurance policy under the minimal variance criterion is similar to that in Proposition 3.1. Specifically, if the full reinsurance is admissible, then the optimal strategy for an insurer is to cede the full loss; otherwise, the insurer cedes the loss as much as possible.

In the following, we present an example in which we consider two loss distributions with the same mean and variance: one is a Pareto and hence heavy tailed and the other is a Gamma and light tailed. We identify the optimal reinsurance policy for each distribution and examine the impact of the tail heaviness on the optimal policy.

Example 4.1. *Suppose that the threshold* $L_0 = 20$ *and the relative safety loading* $\rho = 10\%$.

(i) Let a loss random variable X follow a Pareto distribution with probability density function

$$p_1(x) = \frac{3\times 10^6}{(x+100)^4}, \ x > 0. \tag{4.9}$$

Then we have

$$S_X(t) = 10^6/(t+100)^3, \quad \mathbb{E}[X] = 50, \quad var(X) = 7500$$
$$\text{and} \quad d_X(\mu) = \frac{10^3}{\sqrt{2\mu}} - 100, \forall 0 < \mu \le 50.$$

The following analysis is divided into two cases: the upper bound $\epsilon = 0$ *and* $\epsilon = 10$.

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- *If* $\epsilon = 0$, *we have* $(1+\rho)\mathbb{E}[X]+L_0-d_X(\epsilon) = -\infty$. *By solving the equation*

$$\mathbb{E}\left[\mathcal{L}_{(0,\,(1+\rho)\mu+L_0]}(X)\right] = \mu,\ 0 \le \mu \le \mathbb{E}[X],$$

we have $\tilde{\mu} = 28.01$.

- *If* $\epsilon = 10$, *we have* $(1+\rho)\mathbb{E}[X] + L_0 - d_X(\epsilon) = 175 - 10^3/\sqrt{2\epsilon} < 0$. *By solving the equation*

$$0 = \int_0^{(1+\rho)\mu+L_0} S_X(t)\mathrm{d}t + \epsilon - \mu,\ \epsilon \le \mu \le \mathbb{E}[X], \tag{4.10}$$

we have $\tilde{\mu} = 41.86$.

Consequently, Proposition 4.1 implies that the optimal reinsurance scheme under the criterion of minimizing the variance of an insurer's risk exposure is given by

$$f^*(x) = \begin{cases} \mathcal{L}_{(0,\,50.81]}(x), & \epsilon = 0; \\ \mathcal{L}_{(0,\,66.05]}(x) + (x - 123.61)_+, & \epsilon = 10. \end{cases}$$

(ii) If the loss X follows a Gamma distribution with probability density function

$$p_2(x) = \frac{1}{150^{\frac{1}{3}}\Gamma(\frac{1}{3})} x^{-\frac{2}{3}} e^{-\frac{x}{150}},\ x > 0, \tag{4.11}$$

where $\Gamma(t) \triangleq \int_0^\infty x^{t-1}e^{-x}\mathrm{d}x,\ t \ge 0$, *then it has both the same mean and variance as that of the Pareto distribution given in* (4.9). *While* $S_X(t)$ *and* $d_X(\mu)$ *have no explicit expressions under the Gamma distribution, they can be calculated numerically via the function*

$$G_{a,b}(t) = \int_0^t \frac{1}{b^a\Gamma(a)} x^{a-1}e^{-x/b}\mathrm{d}x,\ t \ge 0 \tag{4.12}$$

for $a > 0, b > 0$, *and many softwares such as Excel and Matlab have package for* $G_{a,b}(t)$. *In particular, we have* $S_X(t) = 1 - G_{\frac{1}{3},150}(t)$ *and*

$$\mathbb{E}[(X-t)_+] = \int_t^\infty (y-t)p_2(y)\mathrm{d}y = 50\left(1 - G_{\frac{4}{3},150}(t)\right) - t\left(1 - G_{\frac{1}{3},150}(t)\right) \tag{4.13}$$

for $t \ge 0$.

- *If* $\epsilon = 0$, *we have* $d_X(\epsilon) = \infty$ *such that* $(1+\rho)\mathbb{E}[X] + L_0 - d_X(\epsilon) < 0$. *By solving the equation*

$$\begin{aligned}\mu &= \int_0^{(1+\rho)\mu+L_0} S_X(t)\mathrm{d}t \\ &= 50 \times G_{\frac{4}{3},150}((1+\rho)\mu + L_0) + ((1+\rho)\mu + L_0) \\ &\quad \times \left(1 - G_{\frac{1}{3},150}((1+\rho)\mu + L_0)\right)\end{aligned}$$

for $0 \le \mu \le \mathbb{E}[X]$*, we have* $\tilde{\mu} = 19.68$.

- *If* $\epsilon = 10$*, we have* $d_X(\epsilon) = 161.33$ *by solving* $\epsilon = \mathbb{E}[(X - d_X(\epsilon))_+]$ *with the help of* (4.13)*. Then* $(1+\rho)\mathbb{E}[X] + L_0 - d_X(\epsilon) = 75 - d_X(\epsilon) < 0$*. Similarly, solving equation* (4.10) *numerically, we have* $\tilde{\mu} = 34.24$.

As a consequence, Proposition 4.1 implies that the optimal reinsurance scheme under the criterion of minimizing the variance of an insurer's risk exposure is given by

$$f^*(x) = \begin{cases} \mathcal{L}_{(0,\,41.65]}(x), & \epsilon = 0; \\ \mathcal{L}_{(0,\,57.66]}(x) + (x - 161.33)_+, & \epsilon = 10. \end{cases}$$

From the above numerical results, we find that when $\epsilon = 0$*, it is optimal for an insurer to cede all the loss with an upper limit. Moreover, when the loss distribution changes from the light tailed to the heavy tailed, the coverage limit increases by more than* 20% *and from* 41.65 *to* 50.81*. On the other hand, when* $\epsilon = 10$*, it is optimal for the insurer to retain a layer of loss with the retention larger than the mean of loss for both loss distributions. Moreover, when the Gamma loss distribution is changed to the Pareto distribution, the layer is compressed and the insurer would retain less loss. Consequently, we can conclude that the optimal reinsurance policy is sensitive to the tail of a loss distribution and to the reinsurer's risk tolerance.*

In this section, we study optimal reinsurance problem (2.6) under the criterion of minimizing the variance of an insurer's risk exposure. However, in practice, an insurer often tries to not only reduce the risk exposure but also to seek the higher profit. Therefore, the insurer may want to make a trade-off between the loss it cedes and the premium it would like to pay. For this reason, the criterion of only minimizing the variance of risk exposure might not be most desirable. In the next section, we will study this optimal reinsurance problem using an alternative but interesting criterion.

5. Minimization of risk-adjusted liability

Recently, the cost-of-capital approach was introduced by the Swiss insurance supervisor(see Swiss Federal Office of Private Insurance (2006)) to assess an insurer's liability. Under such an approach, the risk-adjusted value of the insurer's liability, which is also known as a market-consistent price of liability, is composed of two parts: best estimate and risk margin. The best estimate is represented by the expected liability, $\mathbb{E}[T_f(X)]$, and the insurer is required to hold additional capital to partly cover the unexpected loss, $T_f(X) - \mathbb{E}[T_f(X)]$, the difference between the risk and its expectation. The unexpected loss is usually quantified by the value at risk (VaR) or the conditional VaR (CVaR) . VaR and CVaR can be defined formally as follows:

Definition 5.1. *The VaR of a random variable Z at a confidence level* $1-\alpha$ *where* $0<\alpha<1$ *is defined as*

$$VaR_{\alpha}(Z) \triangleq \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq \alpha\}. \tag{5.1}$$

Based upon the definition of VaR, CVaR of Z at a confidence level $1-\alpha$ *is defined as*

$$CVaR_{\alpha}(Z) \triangleq \frac{1}{\alpha}\int_0^{\alpha} VaR_s(Z)\mathrm{d}s. \tag{5.2}$$

From the above definition of $VaR_{\alpha}(Z)$, we have

$$VaR_{\alpha}(Z) \leq z \quad \text{if and only if} \quad S_Z(z) \leq \alpha \tag{5.3}$$

for any $z \in \mathbb{R}$. Moreover, for any increasing continuous function $H(x)$, we have (see theorem 1 in Dhaene *et al.* (2002))

$$VaR_{\alpha}(H(Z)) = H(VaR_{\alpha}(Z)). \tag{5.4}$$

It is well known that VaR is not a coherent risk measure as it fails to satisfy the sub-additive condition. On the other hand, CVaR is a coherent risk measure and hence is widely used in industry. We refer to Artzner *et al.* (1999) and Föllmer and Schied (2004) for more detailed discussions on the properties of VaR and CVaR.

Due to the nice properties of CVaR, it is employed to calculate the *capital at risk* in this paper, i.e.,

$$CVaR_{\alpha}\left(T_f(X) - \mathbb{E}[T_f(X)]\right).$$

In practice, the return from a capital investment is much smaller than that required for shareholders. We denote by $\delta \in (0, 1)$ the return difference, which is known as the cost-of-capital rate. The risk margin is now set to be the product of the cost-of-capital rate and the capital at risk. Consequently, using $\mathscr{L}_f(X)$ to denote the risk-adjusted value of the insurer's liability, we have

$$\mathscr{L}_f(X) = \mathbb{E}[T_f(X)] + \delta \times CVaR_{\alpha}\left(T_f(X) - \mathbb{E}[T_f(X)]\right). \tag{5.5}$$

The optimal reinsurance model in this section is now formulated by

$$\min_{f \in \mathfrak{C}} \mathscr{L}_f(X). \tag{5.6}$$

It is worthwhile noting that Chi (2012) similarly studies this optimal reinsurance problem except that his discussion does not take into account the reinsurer's risk constraint.

Proposition 5.1. *When the reinsurance premium is calculated by the expected value principle, the optimal ceded loss function for the reinsurance model* (5.6)

https://doi.org/10.1017/asb.2013.28 Published online by Cambridge University Press

is given by

$$f^*(x) = \begin{cases} 0, & \text{if} \quad \alpha \geq \frac{\delta}{\delta+\rho}; \\ \left(x - VaR_{\frac{\delta}{\delta+\rho}}(X)\right)_+, & \text{if} \quad \alpha < \frac{\delta}{\delta+\rho} \quad \text{and} \quad \nu\left(VaR_{\frac{\delta}{\delta+\rho}}(X)\right) \geq 0; \\ \left(x - d_\nu^+\right)_+, & \text{if} \quad \alpha < \frac{\delta}{\delta+\rho},\ \nu\left(VaR_{\frac{\delta}{\delta+\rho}}(X)\right) < 0 \\ & \quad \text{and} \quad d_X(\epsilon) \leq VaR_\alpha(X), \end{cases} \tag{5.7}$$

where $\nu(d)$ *is defined in* (4.4) *and*

$$d_\nu^+ \triangleq \sup\left\{d \geq VaR_{\frac{\delta}{\delta+\rho}}(X) : \nu(d) \leq 0\right\}. \tag{5.8}$$

Proof. As pointed out in Föllmer and Schied (2004), CVaR preserves the convex order. Then Theorem 3.1, together with theorem 6.1 in Van Heerwaarden *et al.* (1989), implies

$$\min_{\mu\in\mathcal{M}} \mathcal{L}_{f_\mu}(X) = \min_{f\in\mathfrak{C}} \mathcal{L}_f(X) \geq \min_{d\geq 0} \mathcal{L}_{(x-d)_+}(X), \tag{5.9}$$

where $\mathcal{M}$ and $f_\mu(x)$ are given in (3.2) and (3.5), respectively.

For any $0 \leq d \leq VaR_\alpha(X)$, it follows from (5.4) that

$$CVaR_\alpha((X-d)_+) = \frac{1}{\alpha}\int_0^\alpha (VaR_s(X) - d)_+ \mathrm{d}s = CVaR_\alpha(X) - d.$$

On the other hand, for any $d > VaR_\alpha(X)$, we have

$$CVaR_\alpha((X-d)_+) = \frac{1}{\alpha}\int_0^1 (VaR_s(X) - d)_+ \mathrm{d}s = \frac{1}{\alpha}\mathbb{E}[(X-d)_+],$$

where the last equality is derived using the fact that X and $VaR_U(X)$ have the same distribution. Here, U is the uniform random variable on $(0, 1)$. Consequently, we get

$$\mathcal{L}_{(x-d)_+}(X) - (1-\delta)\mathbb{E}[X] = \begin{cases} \delta d + (\rho+\delta)\int_d^\infty S_X(t)\mathrm{d}t, & 0 \leq d \leq VaR_\alpha(X); \\ \delta CVaR_\alpha(X) + (\rho + \delta - \frac{\delta}{\alpha})\int_d^\infty S_X(t)\mathrm{d}t, & d > VaR_\alpha(X). \end{cases} \tag{5.10}$$

If $\alpha \geq \frac{\delta}{\delta+\rho}$, it follows from the above equation that $\mathcal{L}_{(x-d)_+}(X)$ is decreasing in d for $d \geq VaR_\alpha(X)$. Moreover, for $0 \leq d < VaR_\alpha(X)$, we have

$$\frac{\partial \mathcal{L}_{(x-d)_+}(X)}{\partial d} = (\delta+\rho)\left(\frac{\delta}{\delta+\rho} - S_X(d)\right) \leq (\delta+\rho)\left(\frac{\delta}{\delta+\rho} - \alpha\right) \leq 0,\ a.s. \tag{5.11}$$

where the first inequality is derived by (5.3). In this case, as $0 \in \mathfrak{C}$, (5.9) implies that $f(x) = 0$ is a solution to the optimal reinsurance model (5.6).

If $\alpha < \frac{\delta}{\delta+\rho}$, a similar analysis leads to

$$\mathcal{L}_{(x-d)_+}(X) \geq \mathcal{L}_{(x-VaR_{\frac{\delta}{\delta+\rho}}(X))_+}(X),\ \forall d \geq 0.$$

The following analysis is divided into two cases:

(i) If $\nu(VaR_{\frac{\delta}{\delta+\rho}}(X)) \geq 0$, Lemma 3.1 together with (4.3) implies $(x - VaR_{\frac{\delta}{\delta+\rho}}(X))_+ \in \mathfrak{C}$, then it follows from (5.9) that $f(x) = (x - VaR_{\frac{\delta}{\delta+\rho}}(X))_+$ is the optimal ceded loss function in this case.

(ii) If

$$\nu(VaR_{\frac{\delta}{\delta+\rho}}(X)) < 0 \quad \text{and} \quad d_X(\epsilon) \leq VaR_\alpha(X),$$

noting that $\nu(d)$ defined in (4.4) is a convex function with $\nu(\infty) = \infty$, we have $\nu(d_\nu^+) = 0$ and

$$\nu(d) \begin{cases} \leq 0, & d_\nu^- \leq d \leq d_\nu^+; \\ > 0, & \text{otherwise,} \end{cases}$$

where

$$d_\nu^- \triangleq \inf\left\{0 \leq d \leq VaR_{\frac{\delta}{\delta+\rho}}(X) : \nu(d) \leq 0\right\}. \tag{5.12}$$

In this case, Lemma 3.1 and (4.3) imply

$$f_\mu(x) = (x - d_X(\mu))_+, \ \forall \mu > \mathbb{E}[(X - d_\nu^-)_+] \quad \text{or} \quad \mu \leq \mathbb{E}[(X - d_\nu^+)_+].$$

As $d_\nu^- \leq VaR_{\frac{\delta}{\delta+\rho}}(X) < d_\nu^+$, it follows from (5.10) and (5.11) that $\mathscr{L}_{f_\mu}(X) \geq \mathscr{L}_{f_{\mathbb{E}[(X-d_\nu^-)_+]}}(X)$ for any $\mu \geq \mathbb{E}[(X - d_\nu^-)_+]$ and $\min_{\mu \leq \mathbb{E}[(X-d_\nu^+)_+]} \mathscr{L}_{f_\mu}(X) = \mathscr{L}_{f_{\mathbb{E}[(X-d_\nu^+)_+]}}(X)$. Consequently, the minimum value of $\mathscr{L}_{f_\mu}(X)$ must appear on $\left[\mathbb{E}[(X - d_\nu^+)_+], \mathbb{E}[(X - d_\nu^-)_+]\right]$.

Further, for $\mu \in \mathcal{M}$ and $\nu(d_X(\mu)) \leq 0$, $f_\mu(x)$ is given in (4.6) where $a_\mu + (1+\rho)\mu + L_0 \leq d_X(\epsilon) \leq VaR_\alpha(X)$, then simple calculation leads to

$$\begin{aligned} \mathscr{L}_{f_\mu}(X) - (1-\delta)\mathbb{E}[X] &= \delta CVaR_\alpha(X) + (\rho+\delta)\mu - \delta CVaR_\alpha(f_\mu(X)) \\ &= \delta d_X(\epsilon) - \delta L_0 + \rho(1-\delta)\mu, \end{aligned}$$

which is increasing in μ. Thus, the minimum of $\mathscr{L}_{f_\mu}(X)$ is attainable at $\mu = \mathbb{E}[(X - d_\nu^+)_+]$. As a consequence, $f(x) = (x - d_\nu^+)_+$ is a solution to optimal reinsurance model (5.6) in this case.

□

We now proceed to study the optimal reinsurance model (5.6) for the case:

$$\alpha < \delta/(\delta+\rho), \quad \nu\left(VaR_{\frac{\delta}{\delta+\rho}}(X)\right) < 0 \quad \text{and} \quad VaR_\alpha(X) < d_X(\epsilon). \tag{5.13}$$

Using a similar proof to that of Proposition 5.1, we have

$$\min_{f \in \mathfrak{C}} \mathscr{L}_f(X) = \min_{\substack{\mathbb{E}[(X-d_\nu^+)_+] \leq \mu \leq \mathbb{E}[(X-d_\nu^-)_+] \\ \mu \in \mathcal{M}}} \mathscr{L}_{f_\mu}(X), \tag{5.14}$$

where $f_\mu(x)$ is given in (4.6) and d_v^+ and d_v^- are defined in (5.8) and (5.12) respectively. Further, for $\mu \in \mathcal{M} \cap \left[\mathbb{E}[(X-d_v^+)_+], \mathbb{E}[(X-d_v^-)_+]\right]$, simple calculation leads to

$$\begin{aligned}
&\mathscr{L}_{f_\mu}(X) - (1-\delta)\mathbb{E}[X] \qquad (5.15)\\
&= \delta CVaR_\alpha(X) + (\rho+\delta)\mu - \delta CVaR_\alpha(f_\mu(X))\\
&= \begin{cases} \delta CVaR_\alpha(\min\{X, d_X(\epsilon)\}) - \delta L_0 + \rho(1-\delta)\mu, & a_\mu + (1+\rho)\mu + L_0 \le VaR_\alpha(X);\\ \delta CVaR_\alpha(X) + (\rho + \delta - \frac{\delta}{\alpha})\mu, & a_\mu \ge VaR_\alpha(X);\\ \frac{\delta}{\alpha}\mathbb{E}[X] + (\rho+\delta-\frac{\delta}{\alpha})\mu + \delta\left(a_\mu - \int_0^{a_\mu} S_X(t)\mathrm{d}t/\alpha\right), & a_\mu < VaR_\alpha(X) < a_\mu + (1+\rho)\mu + L_0. \end{cases}
\end{aligned}$$

It is easy to see from the above equation that $\mathscr{L}_{f_\mu}(X)$ increases in μ for $a_\mu + (1+\rho)\mu + L_0 \le VaR_\alpha(X)$ and decreases in μ for $a_\mu \ge VaR_\alpha(X)$ under the assumption (5.13). However, it is unclear for $a_\mu < VaR_\alpha(X) < a_\mu + (1+\rho)\mu + L_0$. Thus, it seems impossible to derive the optimal reinsurance explicitly for this case, and we resort to numerical analysis in the following example.

Example 5.1. *In addition to the assumptions in Example 4.1, we further assume*

$$\delta = 6\% \quad \textit{and} \quad \alpha = 5\%,$$

then we have $\alpha < \frac{\delta}{\delta+\rho}$*. In this example, we investigate the optimal reinsurance problem* (5.6) *when a loss X follows a heavy-tailed distribution or a light-tailed distribution.*

(i) Let the loss X follow a Pareto distribution with probability density function (4.9)*. Then we have*

$$VaR_\alpha(X) = 100/\sqrt[3]{\alpha} - 100 = 171.44, \quad d_X(\epsilon) = 10^3/\sqrt{2\epsilon} - 100$$

and $\nu(d)$ *in* (4.4) *can be rewritten as*

$$\nu(d) = d + 55 \times 10^4/(100+d)^2 - 10^3/\sqrt{2\epsilon} + 120.$$

We analyze the optimal reinsurance policies for two cases: $\epsilon = 10$ *and* $\epsilon = 0$.

- *If* $\epsilon = 0$*, we have* $d_X(\epsilon) = \infty > VaR_\alpha(X)$ *and* $\nu(d) < 0$ *for any* $0 \le d < \infty$*, then* $0 = d_v^- < d_v^+ = \infty$ *and* (5.13) *is satisfied. By* (4.3)*,* $\mu \in \mathcal{M}$ *if and only if* $\int_0^{(1+\rho)\mu+L_0} S_X(t)\mathrm{d}t \ge \mu - \epsilon$*, which is equivalent to* $0 \le \mu \le 28.01$*. Thus,* (5.14) *implies that it is only necessary to investigate the optimal* $f_\mu(x)$ *in* (4.6) *over* $[0, 28.01]$*. Further, for each* $\mu \in [0, 28.01]$*, we could derive the* a_μ *numerically from* (4.7)*. Consequently, we have*

$$\begin{cases} a_\mu + (1+\rho)\mu + L_0 \le VaR_\alpha(X), & \mu \ge 1.2;\\ a_\mu > VaR_\alpha(X), & \mu < 0.95;\\ a_\mu \le VaR_\alpha(X) < a_\mu + (1+\rho)\mu + L_0, & \textit{otherwise} \end{cases}$$

and the minimum of $\mathscr{L}_{f_\mu}(X)$ *is attainable at* $\mu = 1.18$ *and* $a_\mu = 151.95$*. Thus, the optimal ceded loss function for the reinsurance model* (5.6) *is* *given by*

$$f^*(x) = \mathcal{L}_{(151.95,\,173.25]}(x).$$

- *If* $\epsilon = 10$*, we have* $d_X(\epsilon) = 123.61 < VaR_\alpha(X)$ *and* $\nu(VaR_{\frac{\delta}{\delta+\rho}}(X)) = -36.33 < 0$*. Further, it follows from* (5.8) *that* $d_v^+ = 88.05$*. In this case, Proposition 5.1 implies that the optimal ceded loss function for the reinsurance model* (5.6) *is given by*

$$f^*(x) = (x - 88.05)_+.$$

(ii) Let the loss X follow a Gamma distribution with probability density function (4.11)*. Using the function* $G_{a,b}(x)$ *in* (4.12)*, we have*

$$VaR_\alpha(X) = 220.99 \quad and \quad VaR_{\frac{\delta}{\delta+\rho}}(X) = 30.18.$$

- *If* $\epsilon = 0$*, we have* $d_X(\epsilon) = \infty$ *such that* $\nu(d) < 0$ *for any* $0 \le d < \infty$*, then* (5.13) *is satisfied and* $0 = d_v^- < d_v^+ = \infty$*. Similarly, we can show that* $\mathcal{M} = [0,\, 19.68]$*. Further, for any* $\mu \in [0,\, 19.68]$*, we can derive* a_μ *numerically using* (4.7)*. Consequently, we have*

$$\begin{cases} a_\mu + (1+\rho)\mu + L_0 \le VaR_\alpha(X), & \mu \ge 1.17; \\ a_\mu > VaR_\alpha(X), & \mu < 0.97; \\ a_\mu \le VaR_\alpha(X) < a_\mu + (1+\rho)\mu + L_0, & otherwise \end{cases}$$

and the minimum of $\mathscr{L}_{f_\mu}(X)$ *is attainable at* $\mu = 1.15$ *and* $a_\mu = 201.68$*. Thus, the optimal ceded loss function is given by*

$$f^*(x) = \mathcal{L}_{(201.68,\,222.94]}(x).$$

- *If* $\epsilon = 10$*, we have* $d_X(\epsilon) = 161.32 < VaR_\alpha(X)$ *and* $\nu(d)$ *in* (4.4) *can be rewritten by*

$$\nu(d) = d + 1.1 \times \mathbb{E}[(X - d)_+] - 141.32,$$

where $\mathbb{E}[(X - d)_+]$ *is given in* (4.13)*. Consequently, we have* $\nu(VaR_{\frac{\delta}{\delta+\rho}}(X)) = -73.45 < 0$ *and* $d_v^+ = 126.4$ *according to* (5.8)*. In this case, it follows from Proposition 5.1 that the solution to the optimal reinsurance model* (5.6) *is given by*

$$f^*(x) = (x - 126.4)_+.$$

In summary, when the upper bound ϵ *is set to be* 10*, the stop-loss reinsurance is optimal under the reinsurance model* (5.6) *in contrast to the optimality of two-layer reinsurance under the minimal variance criterion. Further, in this case, the insurer would cede more risk for a Pareto loss distribution than for a Gamma loss distribution even though both loss distributions have the same mean and variance. On the other hand, when* $\epsilon = 0$*, the numerical result shows that the insurer would cede a higher layer for the Gamma distribution than for the Pareto distribution.*

For these reasons, we could say that the loss tail heaviness plays an important role in optimal reinsurance design for an insurer.

6. Concluding remarks

In this paper, we consider an optimal reinsurance problem under a criterion that preserves the convex order, when the reinsurance premium is a function of the net premium and the ceded loss is subject to the restriction that reduces the moral hazard and to an upper limit on the reinsurer's expected tail loss. We show via a stochastic dominance approach that any feasible reinsurance contract is dominated by a stop-loss policy or a two-layer reinsurance policy, depending upon the amount of reinsurance premium. The main contributions of this paper are three-fold. First, we extend Arrow's result in Zhou and Wu (2008) and Zhou *et al*. (2010) to a broader class of criteria including minimizing the variance and the risk-adjusted value of the insurer's liability, and intuitively explain why the piecewise linear reinsurance policy is optimal. Second, we complement the study of optimal reinsurance problems in Van Heerwaarden *et al*. (1989) and Gollier and Schlesinger (1996) by showing that the stochastic dominance approach remains powerful even when an upper limit of the reinsurer's expected tail loss is imposed. Third, we obtain a generalized Mossin's Theorem and study the effect of the reinsurer's risk constraint on the optimal reinsurance design under the net premium principle.

We recognize the shortcomings that the optimal reinsurance model in this paper assumes a specific reinsurance premium principle and the reinsurer's risk constraint is in the form of the expected tail loss. We intend to extend our work in this paper to incorporate other premium principles and other types of reinsurer's risk constraints in future research.

Acknowledgements

The authors thank two reviewers for their valuable comments that have significantly improved this paper. This work was supported by grants from the Natural Sciences and Engineering Research Council of Canada and the National Science Foundation of China (grant No. 11001283). The work was completed when Chi was a Postdoctoral Fellow in the Department of Statistical Sciences at the University of Toronto.

Note

1. An increasing function $g_1(x)$ is said to up-cross an increasing function $g_2(x)$, if there exists an $x_0 \in \mathbb{R}$ such that

$$\begin{cases} g_1(x) \leq g_2(x), & x < x_0; \\ g_1(x) \geq g_2(x), & x > x_0. \end{cases}$$

https://doi.org/10.1017/asb.2013.28 Published online by Cambridge University Press

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