Bull. Aust. Math. Soc. **105** (2022), 372–384
doi:10.1017/S0004972721000903

TIGHT UNIVERSAL TRIANGULAR FORMS

MINGYU KIM

(Received 22 June 2021; accepted 23 September 2021; first published online 18 November 2021)

Abstract

For a subset S of nonnegative integers and a vector $\mathbf{a} = (a_1, \ldots, a_k)$ of positive integers, define the set $V'_S(\mathbf{a}) = \{a_1s_1 + \cdots + a_ks_k : s_i \in S\} - \{0\}$. For a positive integer n, let $\mathcal{T}(n)$ be the set of integers greater than or equal to n. We consider the problem of finding all vectors $\mathbf{a}$ satisfying $V'_S(\mathbf{a}) = \mathcal{T}(n)$ when S is the set of (generalised) m-gonal numbers and n is a positive integer. In particular, we completely resolve the case when S is the set of triangular numbers.

2020 *Mathematics subject classification*: primary 11E12; secondary 11E20, 11P99.

Keywords and phrases: tight universal form, polygonal number.

1. Introduction

For a positive integer m greater than or equal to 3, the polynomial

$$P_m(x) = \frac{(m-2)x^2 - (m-4)x}{2}$$

is an integer-valued quadratic polynomial and $P_m(s)$ is the sth m-gonal number for a nonnegative integer s. For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ of positive integers, a polynomial of the form

$$p_m(\mathbf{a}) = p_m(\mathbf{a})(x_1, \ldots, x_k) = a_1P_m(x_1) + \cdots + a_kP_m(x_k)$$

in variables $x_1, x_2, \ldots, x_k$ is called *a k-ary m-gonal form* (or *a k-ary sum of generalised m-gonal numbers*). We say that an integer N is represented by an m-gonal form $p_m(\mathbf{a})$ if the equation

$$p_m(\mathbf{a})(x_1, \ldots, x_k) = N$$

has an integer solution. The *minimum of* $p_m(\mathbf{a})$, denoted by $\min(p_m(\mathbf{a}))$, is the smallest positive integer represented by $p_m(\mathbf{a})$. We call an m-gonal form *tight universal* if it represents every positive integer greater than its minimum. A tight universal m-gonal

This work was supported by a National Research Foundation of Korea (NRF) grant funded by the government of Korea (MSIT) (NRF-2021R1C1C2010133).

form having minimum 1 is simply called *universal*. There are several results on the classification of universal m-gonal forms (see, for example, [2, 6, 7, 8]). Note that $P_4(x) = x^2$ and the classification of universal diagonal quadratic forms can be easily done by using the Conway–Schneeberger 15 theorem (see [1, 3]).

Recently, the author and Oh [10] studied (positive definite integral) quadratic forms which represent every positive integer greater than the minimum of the form. We called such a quadratic form f *tight* $\mathcal{T}(n)$*-universal*, where n is the minimum of the quadratic form f. We classified 'diagonal' tight universal quadratic forms, which gives the classification of tight universal m-gonal forms in the case of $m = 4$.

We follow the notation and terminologies used in [10]. For $n = 1, 2, 3, \ldots,$ we denote by $\mathcal{T}(n)$ the set of integers greater than or equal to n. We say that an m-gonal form is tight $\mathcal{T}(n)$-universal if it is tight universal with minimum n. In Section 3, we resolve the classification problem of tight $\mathcal{T}(n)$-universal m-gonal forms in the following cases:

$$\text{(i) } m = 5,\ n \geq 7; \quad \text{(ii) } m = 7,\ n \geq 11; \quad \text{(iii) } m \geq 8,\ n \geq 2m - 5.$$

In fact, it will be proved that there are 'essentially' two tight $\mathcal{T}(n)$-universal m-gonal forms in the cases (ii) and (iii). It will also be shown that there is a unique tight $\mathcal{T}(n)$-universal pentagonal form for any $n \geq 7$. In addition, we classify tight $\mathcal{T}(n)$-universal sums of m-gonal numbers (for the definition, see Section 3). In Section 4, we classify tight universal triangular forms by finding all tight $\mathcal{T}(n)$-universal triangular forms for every integer $n \geq 3$. Universal triangular forms were classified in [2] and tight $\mathcal{T}(2)$-universal triangular forms were found by Ju ['*Almost universal sums of triangular numbers with one exception*', submitted for publication]. To classify tight universal triangular forms, we use the theory of quadratic forms and adapt the geometric language of quadratic spaces and lattices, generally following [11, 12]. Some basic notation and terminologies will be given in Section 2.

2. Preliminaries

Let R be the ring of rational integers $\mathbb{Z}$ or the ring of p-adic integers $\mathbb{Z}_p$ for a prime p and let F be the field of fractions of R. An R*-lattice* is a finitely generated R-submodule of a quadratic space (W, Q) over F. We let $B : W \times W \to F$ be the symmetric bilinear form associated to the quadratic map Q so that $B(x, x) = Q(x)$ for every $x \in W$. For an element a in R and an R-lattice L, we say that *a is represented by L over R* and write $a \longrightarrow L$ over R if $Q(\mathbf{x}) = a$ for some vector $\mathbf{x} \in L$.

Let L be a $\mathbb{Z}$-lattice on a quadratic space W over $\mathbb{Q}$. The genus of L, denoted gen(L), is the set of all $\mathbb{Z}$-lattices on W which are locally isometric to L. The number of isometry classes in gen(L) is called the class number of L and denoted by $h(L)$. If an integer a is represented by L over $\mathbb{Z}_p$ for all primes p (including ∞), then there is a $\mathbb{Z}$-lattice K in gen(L) such that $a \longrightarrow K$ (see [12, 102:5 Example]). In this case, we say that a is represented by the genus of L and write $a \longrightarrow \text{gen}(L)$. For a $\mathbb{Z}$-basis

$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of L, the corresponding quadratic form f_L is defined by

$$f_L = \sum_{i,j=1}^{k} B(\mathbf{v}_i, \mathbf{v}_j) x_i x_j.$$

If L admits an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$, then we simply write

$$L \simeq \langle Q(\mathbf{w}_1), Q(\mathbf{w}_2), \ldots, Q(\mathbf{w}_k)\rangle.$$

We abuse the notation and the diagonal quadratic form $a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2$ will also be denoted by $\langle a_1, a_2, \ldots, a_k\rangle$. The scale of L is denoted by $\mathfrak{s}(L)$. Throughout, we always assume that every $\mathbb{Z}$-lattice is positive definite and primitive in the sense that $\mathfrak{s}(L) = \mathbb{Z}$. Any unexplained notation and terminologies on the representation of quadratic forms can be found in [11] or [12].

Throughout this section, S always denotes a set of nonnegative integers containing 0 and 1, unless otherwise stated. For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$, we define

$$V_S(\mathbf{a}) = \{a_1s_1 + a_2s_2 + \cdots + a_ks_k : s_i \in S\}$$

and define $V'_S(\mathbf{a}) = V_S(\mathbf{a}) - \{0\}$. For example, if S is the set of squares of integers, then

$$V'_S(1, 1, 1, 1) = \mathbb{N}, \quad V'_S(1, 1, 1) = \mathbb{N} - \{4^a(8b + 7) : a, b \in \mathbb{N}_0\}$$

by Lagrange's four-square theorem and Legendre's three-square theorem, respectively. We denote the set of nonnegative integers by $\mathbb{N}_0$ for simplicity. For two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_r) \in \mathbb{N}^r$ and $\mathbf{v} = (v_1, v_2, \ldots, v_s) \in \mathbb{N}^s$, we write

$$\mathbf{u} \preceq \mathbf{v} \quad (\mathbf{u} \prec \mathbf{v})$$

if $\{u_i\}_{1\le i\le r}$ is a subsequence (proper subsequence, respectively) of $\{v_j\}_{1\le j\le s}$. Let n be a positive integer and let $\mathbf{a}$ be a vector of positive integers. We say that $\mathbf{a}$ *is tight $\mathcal{T}(n)$-universal with respect to S* if $V'_S(\mathbf{a}) = \mathcal{T}(n)$. When $n = 1$, we simply say that $\mathbf{a}$ *is universal with respect to S*. We say that $\mathbf{a}$ *is new tight $\mathcal{T}(n)$-universal with respect to S* if $V'_S(\mathbf{a}) = \mathcal{T}(n)$ and $V'_S(\mathbf{b}) \subsetneq \mathcal{T}(n)$ whenever $\mathbf{b} \prec \mathbf{a}$. For $n_1, n_2, \ldots, n_r \in \mathbb{N}$ and $e_1, e_2, \ldots, e_r \in \mathbb{N}_0$, we denote by $\mathbf{n_1}^{e_1}\mathbf{n_2}^{e_2}\cdots\mathbf{n_r}^{e_r}$ the vector

$$(n_1, \ldots, n_1, n_2, \ldots, n_2, \ldots, n_r, \ldots, n_r) \in \mathbb{Z}^{e_1+e_2+\cdots+e_r},$$

where each n_i is repeated e_i times for $i = 1, 2, \ldots, r$. The first lemma is straightforward.

Lemma 2.1. *Let $\mathbf{a}, \mathbf{b}$ be vectors of positive integers such that $\mathbf{a} \preceq \mathbf{b}$ and let S, S' be sets of nonnegative integers containing 0 and 1 such that $S \subseteq S'$. Then:*

(i) $V_S(\mathbf{a}) \subseteq V_S(\mathbf{b})$;
(ii) $V_S(\mathbf{a}) \subseteq V_{S'}(\mathbf{a})$;
(iii) $V_S(u + v) \subset V_S(u, v)$ *for any* $u, v \in \mathbb{N}$;
(iv) $\min(V'_S(\mathbf{a})) = \min\{a_i : 1 \le i \le k\}$, *where* $\mathbf{a} = (a_1, a_2, \ldots, a_k)$.

Lemma 2.2. *Let $\mathbf{a} = \mathbf{1}^{e_1}\mathbf{2}^{e_2}\mathbf{3}^{e_3}$ be a vector with a positive integer e_1 and nonnegative integers e_2 and e_3. Assume that $V_S(\mathbf{a}) = \mathbb{N}_0$. Then, for any integer $n \ge 2e_3 + 3$,*

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

the vector

$$\mathbf{b} = \mathbf{n}^{e_1}\mathbf{n+1}^1\mathbf{n+2}^1\cdots\mathbf{2n-1}^1\mathbf{2n}^{e_2}$$

is tight $\mathcal{T}(n)$-universal with respect to S.

PROOF. Let n be an integer with $n \geq 2e_3 + 3$ and let m be an integer greater than or equal to n. Then m can be written in the form $un + v$ for a nonnegative integer u and an integer v with $n \leq v \leq 2n - 1$. To prove the lemma, it suffices to show that $un + v \in V_S(\mathbf{b})$. Since

$$u \in \mathbb{N}_0 = V_S(\mathbf{a}) = V_S(\mathbf{1}^{e_1}\mathbf{2}^{e_2}\mathbf{3}^{e_3}),$$

we have

$$un \in V_S(\mathbf{n}^{e_1}\mathbf{2n}^{e_2}\mathbf{3n}^{e_3}).$$

Since the other cases can be dealt with in a similar manner, we only provide the proof when

$$n + 1 \leq v \leq e_3 + 1 \quad \text{or} \quad 2n - e_3 - 1 \leq v \leq 2n - 1.$$

By applying Lemma 2.1(iii) e_3 times,

$$V_S(\mathbf{3n}^{e_3}) \subseteq V_S(n+1, 2n-1, n+2, 2n-2, \ldots, \widehat{v}, \widehat{3n-v}, \ldots, n+e_3+1, 2n-e_3-1),$$

where the hat symbol ^ indicates that the component is omitted. It follows that

$$\begin{aligned} un &\in V_S(\mathbf{n}^{e_1}\mathbf{2n}^{e_2}\mathbf{3n}^{e_3}) \\ &\subseteq V_S(\mathbf{n}^{e_1}\mathbf{2n}^{e_2}\mathbf{n+1}^1\mathbf{2n-1}^1\cdots\widehat{\mathbf{v}}^1\widehat{\mathbf{3n-v}}^1\cdots\mathbf{n+e_3+1}^1\mathbf{2n-e_3-1}^1). \end{aligned}$$

Therefore,

$$\begin{aligned} un + v &\in V_S(\mathbf{n}^{e_1}\mathbf{2n}^{e_2}\mathbf{n+1}^1\mathbf{2n-1}^1\cdots\mathbf{v}^1\mathbf{3n-v}^1\cdots\mathbf{n+e_3+1}^1\mathbf{2n-e_3-1}^1) \\ &\subseteq V_S(\mathbf{n}^{e_1}\mathbf{n+1}^1\mathbf{n+2}^1\cdots\mathbf{2n-1}^1\mathbf{2n}^{e_2}). \end{aligned}$$

This completes the proof. □

For $n = 1, 2, 3, \ldots$, we define vectors $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{Z}^{n+1}$ by

$$\mathbf{x}_n = (n, n, n+1, n+2, \ldots, 2n-1), \quad \mathbf{y}_n = (n, n+1, n+2, \ldots, 2n).$$

LEMMA 2.3. *Let n be a positive integer and let $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ with $a_1 \leq a_2 \leq \cdots \leq a_k$ such that $V'_S(\mathbf{a}) = \mathcal{T}(n)$. Then $(n, n+1, n+2, \ldots, 2n-1) \preceq \mathbf{a}$. Furthermore, if $2 \notin S$, then $\mathbf{x}_n \preceq \mathbf{a}$ or $\mathbf{y}_n \preceq \mathbf{a}$.*

PROOF. Since $V'_S(\mathbf{a}) = \mathcal{T}(n)$,

$$n = a_1 \leq a_2 \leq \cdots \leq a_k. \tag{2.1}$$

To prove the first assertion, it suffices to show that for any integer v with $n + 1 \leq v \leq 2n - 1$, there is an integer j_v with $1 \leq j_v \leq k$ such that $a_{j_v} = v$. Let v be an integer such that $n + 1 \leq v \leq 2n - 1$. Since $v \in V'_S(\mathbf{a})$, we have $v = a_1s_1 + a_2s_2 + \cdots + a_ks_k$ for some

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

$s_1, s_2, \ldots, s_k \in S$. Since $v > 0$, there is an integer j_v with $1 \le j_v \le k$ such that $s_{j_v} > 0$. If $s_l > 0$ for some l different from j_v, then

$$v = a_1 s_1 + a_2 s_2 + \cdots + a_k s_k \ge a_{j_v} s_{j_v} + a_l s_l \ge a_{j_v} + a_l \ge 2n$$

by (2.1) and this is absurd since $v \le 2n - 1$. It follows that $s_{j_v} = 1$ and $s_l = 0$ for any $l \ne j_v$. Thus, $v = a_{j_v}$ and the first assertion follows.

Now we assume further that $2 \notin S$. Then clearly

$$2n \in V_S(\mathbf{a}) - V_S(n, n+1, n+2, \ldots, 2n-1).$$

From this, one may easily deduce that

$$(n, n, n+1, n+2, \ldots, 2n-1) \le \mathbf{a} \quad \text{or} \quad (n, n+1, n+2, \ldots, 2n-1, 2n) \le \mathbf{a}.$$

This completes the proof. □

3. Tight $\mathcal{T}(n)$-universal sums of (generalised) m-gonal numbers

Let m be an integer greater than or equal to 3. We denote the set of (generalised) m-gonal numbers by $\mathcal{P}_m$ (respectively, $\mathcal{GP}_m$), that is,

$$\mathcal{P}_m = \left\{ \frac{(m-2)x^2 - (m-4)x}{2} : x \in \mathbb{N}_0 \right\}, \quad \mathcal{GP}_m = \left\{ \frac{(m-2)x^2 - (m-4)x}{2} : x \in \mathbb{Z} \right\}.$$

One may easily check that:

(i) $\{0, 1\} \subset \mathcal{P}_m \subseteq \mathcal{GP}_m$ for any $m \ge 3$;
(ii) $2 \notin \mathcal{P}_m$ for any $m \ge 3$;
(iii) $2 \in \mathcal{GP}_m$ only if $m = 5$;
(iv) $\mathcal{P}_3 = \mathcal{GP}_3 = \mathcal{GP}_6$;
(v) $\mathcal{P}_4 = \mathcal{GP}_4$.

PROPOSITION 3.1. *Let m be an integer greater than or equal to 8. If $n \ge 2m - 5$, then both $\mathbf{x}_n$ and $\mathbf{y}_n$ are tight $\mathcal{T}(n)$-universal with respect to $\mathcal{GP}_m$.*

PROOF. By [13, Theorem 1.1] and [8, Theorem 3.2], $V_{\mathcal{GP}_m}(\mathbf{1}^{m-4}) = \mathbb{N}_0$. From this, one may easily deduce that $V_{\mathcal{GP}_m}(\mathbf{1}^{e_1}\mathbf{2}^{e_2}\mathbf{3}^{m-4}) = \mathbb{N}_0$ for $(e_1, e_2) \in \{(2, 0), (1, 1)\}$. Now the proposition follows immediately from Lemma 2.2. □

THEOREM 3.2. *Let m be an integer greater than or equal to 8. If $n \ge 2m - 5$, then there are exactly two new tight $\mathcal{T}(n)$-universal m-gonal forms.*

PROOF. Note that $2 \notin \mathcal{GP}_m$ since $m \ne 5$. The theorem follows immediately from the second assertion of Lemma 2.3 and Proposition 3.1. □

PROPOSITION 3.3. *There is only one new tight $\mathcal{T}(n)$-universal pentagonal form for any $n \ge 7$.*

PROOF. Note that the vector $(1, 3, 3)$ is universal with respect to $\mathcal{GP}_5$ (see [4]). By Lemma 2.2, the vector $(n, n+1, n+2, \ldots, 2n-1)$ is tight $\mathcal{T}(n)$-universal with respect

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

to $\mathcal{GP}_5$ for any $n \geq 7$. Now the proposition follows immediately from the first assertion of Lemma 2.3. □

PROPOSITION 3.4. *There are exactly two new tight $\mathcal{T}(n)$-universal heptagonal forms for any $n \geq 11$.*

PROOF. Note that $V_{\mathcal{GP}_7}(1,1,1,1) = \mathbb{N}_0$ (see [13] or [8, Theorem 1.2]). It follows that $V_{\mathcal{GP}_7}(\mathbf{1}^{e_1}\mathbf{2}^{e_2}\mathbf{3}^4) = \mathbb{N}_0$ for $(e_1, e_2) \in \{(2,0),(1,1)\}$. The proposition follows immediately from Lemma 2.2 and the second assertion of Lemma 2.3. □

Let n be a positive integer. Now we define (new) tight $\mathcal{T}(n)$-universal sums of m-gonal numbers. For an integer $m \geq 3$ and a vector $\mathbf{a}$ of positive integers, we call the pair $(\mathcal{P}_m, \mathbf{a})$ *a sum of m-gonal numbers*. We say that $(\mathcal{P}_m, \mathbf{a})$ is tight $\mathcal{T}(n)$-universal if $V'_{\mathcal{P}_m}(\mathbf{a}) = \mathcal{T}(n)$. A tight $\mathcal{T}(n)$-universal sum of m-gonal numbers $(\mathcal{P}_m, \mathbf{a})$ is called *new* if $(\mathcal{P}_m, \mathbf{b})$ is not $\mathcal{T}(n)$-universal whenever $\mathbf{b} \prec \mathbf{a}$ or, equivalently, $V'_{\mathcal{P}_m}(\mathbf{b}) \subsetneq \mathcal{T}(n)$ whenever $\mathbf{b} \prec \mathbf{a}$.

PROPOSITION 3.5. *Let m be an integer greater than or equal to 3. If $n \geq 2m+3$, then both $(\mathcal{P}_m, \mathbf{x}_n)$ and $(\mathcal{P}_m, \mathbf{y}_n)$ are tight $\mathcal{T}(n)$-universal.*

PROOF. Fermat's polygonal number theorem says that $V_{\mathcal{P}_m}(\mathbf{1}^m) = \mathbb{N}_0$. From this, one may easily deduce that $V_{\mathcal{P}_m}(\mathbf{1}^{e_1}\mathbf{2}^{e_2}\mathbf{3}^m) = \mathbb{N}_0$ for $(e_1, e_2) \in \{(2,0),(1,1)\}$. Now the tight $\mathcal{T}(n)$-universalities (with respect to $\mathcal{P}_m$) of $\mathbf{x}_n$ and $\mathbf{y}_n$ follow immediately from Lemma 2.2. □

THEOREM 3.6. *Let m be an integer greater than or equal to 3. If $n \geq 2m+3$, then there are exactly two new tight $\mathcal{T}(n)$-universal sums of m-gonal numbers.*

PROOF. Note that $2 \notin \mathcal{P}_m$. The theorem follows immediately from the second assertion of Lemma 2.3 and Proposition 3.5. □

4. Tight universal triangular forms

In this section, we classify tight universal triangular forms. As noted in the introduction, for $n = 1, 2$, tight $\mathcal{T}(n)$-universal triangular forms were already classified. We first prove that there are exactly 12 new tight $\mathcal{T}(3)$-universal triangular forms as listed in Table 1. We also prove that there are exactly two new tight $\mathcal{T}(n)$-universal triangular forms

$$X_n = p_3(n, n, n+1, n+2, \ldots, 2n-1) \quad \text{and} \quad Y_n = p_3(n, n+1, n+2, \ldots, 2n-1, 2n)$$

for any $n \geq 4$. We introduce some notation which will be used throughout this section. Recall that a triangular form is a polynomial of the form

$$p_3(a_1, a_2, \ldots, a_k) = a_1 \frac{x_1(x_1+1)}{2} + \cdots + a_k \frac{x_k(x_k+1)}{2},$$

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

TABLE 1. New tight $\mathcal{T}(3)$-universal triangular forms $p_3(a_1, a_2, \ldots, a_k)$.

a_1	a_2	a_3	a_4	a_5	Conditions on a_5
3	3	4	5		
3	4	4	5	6	
3	4	5	5	6	
3	4	5	6	a_5	$6 \le a_5 \le 16,\ a_5 \ne 14, 15$

where $(a_1, a_2, \ldots, a_k)$ is a vector of positive integers. For a nonnegative integer g and a triangular form $p_3(a_1, a_2, \ldots, a_k)$, we write

$$g \longrightarrow p_3(a_1, a_2, \ldots, a_k)$$

if g is represented by $p_3(a_1, a_2, \ldots, a_k)$. For a positive integer u and a diagonal quadratic form $\langle a_1, a_2, \ldots, a_k\rangle$, we write

$$u \xrightarrow{2} \langle a_1, a_2, \ldots, a_k\rangle$$

if there is a vector $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k$ with $(2, x_1x_2 \cdots x_k) = 1$ such that

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2 = u.$$

One may easily see the following observation, which will be used to show the tight universality of triangular forms: a nonnegative integer g is represented by a triangular form $p_3(a_1, a_2, \ldots, a_k)$ if and only if

$$8g + a_1 + a_2 + \cdots + a_k \xrightarrow{2} \langle a_1, a_2, \ldots, a_k\rangle.$$

A ternary triangular form $p_3(a, b, c)$ is called *regular* if, for every nonnegative integer g, the following holds: if $8g + a + b + c \longrightarrow \langle a, b, c\rangle$ over $\mathbb{Z}_p$ for every odd prime p, then $8g + a + b + c \xrightarrow{2} \langle a, b, c\rangle$. For more information about regular ternary triangular forms, we refer the reader to [9].

PROPOSITION 4.1. *The quaternary triangular form $X_3 = p_3(3, 3, 4, 5)$ is tight $\mathcal{T}(3)$-universal.*

PROOF. One may directly check that X_3 represents all integers from 3 to 14. Let g be a positive integer greater than 14 and put $g' = 8g + 15$. To show that g is represented by X_3, it suffices to show that $g' \xrightarrow{2} \langle 3, 3, 4, 5\rangle$.

Define sets A and B by

$$A = \{u \in \mathbb{N} : u \equiv 1 \pmod 3 \text{ or } u \equiv 3, 6 \pmod 9\},$$
$$B = \{u \in \mathbb{N} : u \equiv 2 \pmod 8,\ u \ge 10\}.$$

We assert that $v \xrightarrow{2} \langle 3, 3, 4\rangle$ for any $v \in A \cap B$. To show the assertion, let $v \in A \cap B$. One may easily check that every positive integer in A is represented by the diagonal

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

quadratic form $\langle 3,3,4\rangle$ over $\mathbb{Z}_3$. Note that $\langle 3,3,4\rangle$ represents all elements in $\mathbb{Z}_p$ over $\mathbb{Z}_p$ for any prime $p \geq 5$. Thus, $v \longrightarrow \langle 3,3,4\rangle$ over $\mathbb{Z}_p$ for all odd primes p. Furthermore, $v = 8v' + 10$ for some nonnegative integer v' since $v \in B$. From these statements and the fact that the ternary triangular form $p_3(3,3,4)$ is regular (see [9]), it follows that $v \xrightarrow{2} \langle 3,3,4\rangle$. So, we have the assertion.

If we define an odd positive integer d by

$$d = \begin{cases} 1 & \text{if } g' \equiv 0 \pmod 3 \ \text{ or } \ g' \equiv 2, 8 \pmod 9, \\ 3 & \text{if } g' \equiv 1 \pmod 3, \\ 5 & \text{if } g' \equiv 5 \pmod 9, \end{cases}$$

then one may easily check that $g' - 5d^2 \in A \cap B$. Thus, $g' - 5d^2 \xrightarrow{2} \langle 3,3,4\rangle$. Since d is odd, it follows that $g' \xrightarrow{2} \langle 3,3,4,5\rangle$. This completes the proof. □

We use the following lemma proved by B. W. Jones in his unpublished thesis [5].

LEMMA 4.2 (Jones). *Let p be an odd prime and k be a positive integer not divisible by p such that the Diophantine equation $x^2 + ky^2 = p$ has an integer solution. If the Diophantine equation*

$$x^2 + ky^2 = N \quad (N > 0)$$

has an integer solution, then it also has an integer solution (x_0, y_0) satisfying

$$\gcd(x_0, y_0, p) = 1.$$

PROPOSITION 4.3. *Let g be a positive integer congruent to 5 modulo 8. Assume that g is congruent to 1 modulo 3 or is a multiple of 9. Then g is represented by the diagonal ternary quadratic form $3x^2 + 4y^2 + 6z^2$.*

PROOF. Let $L = \langle 3,4,6\rangle$. The class number $h(L)$ of L is 2 and the genus mate is $\langle 1,6,12\rangle$. From the assumptions, one may easily check that $g \longrightarrow \mathrm{gen}(\langle 3,4,6\rangle)$. We may assume that $g \longrightarrow \langle 1,6,12\rangle$ since otherwise we are done. Thus, there is a vector $(x_1, y_1, z_1) \in \mathbb{Z}^3$ such that

$$g = x_1^2 + 6y_1^2 + 12z_1^2.$$

First, assume that $g \equiv 0 \pmod 9$. One may easily check that $x_1 \equiv 0 \pmod 3$ and that $y_1 \equiv 0 \pmod 3$ if and only if $z_1 \equiv 0 \pmod 3$. By changing the sign of z_1 if necessary, we may further assume that $y_1 \equiv z_1 \pmod 3$. Thus, $x_1 = 3x_2$ and $y_1 = z_1 - 3y_2$ with integers x_2 and y_2. Now

$$\begin{aligned} g &= x_1^2 + 6y_1^2 + 12z_1^2 \\ &= (3x_2)^2 + 6(z_1 - 3y_2)^2 + 12z_1^2 \\ &= 3(x_2 + 2y_2 - 2z_1)^2 + 4(3y_2)^2 + 6(x_2 - y_2 + z_1)^2. \end{aligned}$$

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

Second, assume that $g \equiv 1 \pmod 3$. If $y_1^2 + 2z_1^2 = 0$, then $g = x_1^2$ and this is absurd since $g \equiv 5 \pmod 8$. Hence, $y_1^2 + 2z_1^2 \neq 0$ and thus, by Lemma 4.2, there are integers y_3 and z_3 with $\gcd(y_3, z_3, 3) = 1$ such that

$$y_1^2 + 2z_1^2 = y_3^2 + 2z_3^2.$$

Note that $x_1 \not\equiv 0 \pmod 3$ since $g \equiv 1 \pmod 3$. After changing the signs of y_3 and z_3 if necessary, we may assume that $x_1 + y_3 + 2z_3 \equiv 0 \pmod 3$. Then

$$\begin{aligned} g &= x_1^2 + 6y_3^2 + 12z_3^2 \\ &= 3\Big(\frac{x_1 - 2y_3 - 4z_3}{3}\Big)^2 + 4(y_3 - z_3)^2 + 6\Big(\frac{x_1 + y_3 + 2z_3}{3}\Big)^2. \end{aligned}$$

Since $x_1 - 2y_3 - 4z_3 \equiv x_1 + y_3 + 2z_3 \equiv 0 \pmod 3$, we have $g \longrightarrow L$. This completes the proof. □

PROPOSITION 4.4. *The quaternary triangular form $Y_3 = p_3(3, 4, 5, 6)$ represents all positive integers except 1, 2 and 16.*

PROOF. One may directly check that Y_3 represents all integers from 3 to 29 except 16. Let g be an integer greater than 29 and put $g' = 8g + 18$. If we define an odd positive integer d by

$$d = \begin{cases} 1 & \text{if } g' \equiv 0 \pmod 3 \ \text{ or } \ g' \equiv 5 \pmod 9, \\ 3 & \text{if } g' \equiv 1 \pmod 3, \\ 5 & \text{if } g' \equiv 8 \pmod 9, \\ 7 & \text{if } g' \equiv 2 \pmod 9, \end{cases}$$

then one may easily check that $g' - 5d^2 \equiv 1 \pmod 3$ or $g' - 5d^2 \equiv 0 \pmod 9$. Furthermore, $g' - 5d^2 \equiv 5 \pmod 8$ since d is odd. Hence, $g' - 5d^2 \longrightarrow \langle 3, 4, 6\rangle$ by Proposition 4.3. Thus, there is a vector $(x, y, z) \in \mathbb{Z}^3$ such that $g' - 5d^2 = 3x^2 + 4y^2 + 6z^2$. One may easily deduce from $g' - 5d^2 \equiv 5 \pmod 8$ that $xyz \equiv 1 \pmod 2$. Thus, $g' \overset{2}{\longrightarrow} \langle 3, 4, 5, 6\rangle$. This completes the proof. □

COROLLARY 4.5. *All of the quinary triangular forms in Table 1 are tight $\mathcal{T}(3)$-universal.*

PROOF. Let $Z = p_3(a_1, a_2, a_3, a_4, a_5)$ be any quinary triangular form in Table 1. One may see that

$$(3, 4, 5, 6) \prec (a_1, a_2, a_3, a_4, a_5).$$

From this and Proposition 4.4, it follows that Z represents every integer greater than or equal to 3 except 16. One may directly check that Z also represents 16. This completes the proof. □

PROPOSITION 4.6. *Every new tight $\mathcal{T}(3)$-universal triangular form appears in Table 1.*

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

PROOF. Let $p_3 = p_3(a_1, a_2, \ldots, a_k)$ be a new tight $\mathcal{T}(3)$-universal triangular form. By Lemma 2.3, we have $X_3 \preceq p_3$ or $Y_3 \preceq p_3$.

First, assume that $X_3 \preceq p_3$. From the fact that X_3 is tight $\mathcal{T}(3)$-universal and the assumption that p_3 is new tight $\mathcal{T}(3)$-universal, it follows that $p_3 = X_3$.

Second, assume that $Y_3 \preceq p_3$. Since Y_3 is not $\mathcal{T}(3)$-universal, it follows that $k > 4$ and there is a vector $(j_1, j_2, j_3, j_4) \in \mathbb{Z}^4$ with $(j_1, j_2, j_3, j_4) \prec (1, 2, \ldots, k)$ such that $(a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}) = (3, 4, 5, 6)$. We put $A = \{u \in \mathbb{N} : 3 \le u \le 16, u \ne 14, 15\}$. If $a_j \notin A$ for every $j \in \{1, 2, \ldots, k\} \setminus \{j_1, j_2, j_3, j_4\}$, then one may easily show that p_3 cannot represent 16, which is absurd. Thus, there is an integer j with

$$j \in \{1, 2, \ldots, k\} \setminus \{j_1, j_2, j_3, j_4\}$$

such that $a_j \in A$. One may check that $p_3(a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}, a_j)$ is in Table 1 and thus it is tight $\mathcal{T}(3)$-universal. It follows that $k = 5$ and $p_3 = p_3(a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}, a_j)$ since otherwise p_3 is not new. This completes the proof. □

THEOREM 4.7. *For any integer n greater than or equal to* 3, *the triangular form* $X_n = p_3(n, n, n+1, n+2, \ldots, 2n-1)$ *is tight* $\mathcal{T}(n)$*-universal.*

PROOF. First, assume that $n \ge 6$. Let g be an integer greater than or equal to n. Then g can be written in the form $g = un + v$ for some nonnegative integer u and an integer v with $n \le v \le 2n-1$. Note that the ternary triangular form $p_3(1, 1, 4)$ is universal and thus it represents u. Thus, un is represented by $p_3(n, n, 4n)$. It follows that un is represented by $p_3(n, n, n+1, n+2, 2n-3)$. Thus, if $v \notin \{n+1, n+2, 2n-3\}$, then $un + v$ is represented by $p_3(n, n, n+1, n+2, 2n-3, v)$ and thus by X_n. On the other hand, the ternary triangular form $p_3(1, 1, 5)$ is also universal. Hence, un is represented by $p_3(n, n, 5n)$ and thus also represented by $p_3(n, n, n+3, 2n-2, 2n-1)$. From this we deduce that if $v \notin \{n+3, 2n-2, 2n-1\}$, then $un + v$ is represented by $p_3(n, n, n+3, 2n-2, 2n-1, v)$ and thus by X_n.

Second, assume that $n = 5$. Let g be an integer greater than or equal to 236. We write $g = 15u + v$, where u is a positive integer and v is an integer such that $0 \le v \le 14$. Note that the ternary triangular form $p_3(1, 1, 3)$ is regular. For any nonnegative integer w, both $8 \cdot 3w + 5$ and $8(3w+1) + 5$ are represented by $\langle 1, 1, 3 \rangle$ over $\mathbb{Z}_3$. Thus, $p_3(1, 1, 3)$ represents every nonnegative integer not equivalent to 2 modulo 3. It follows that $p_3(5, 5, 6+9)$ represents every nonnegative integer congruent to 0 or 5 modulo 15. Hence, if $v \in \{0, 5\}$, then $g = 15u + v \longrightarrow p_3(5, 5, 6+9)$ and so $g \longrightarrow p_3(5, 5, 6, 9)$. One may directly check that the binary triangular form $p_3(7, 8)$ represents all integers in the set

$$\{31, 122, 48, 94, 80, 231, 7, 8, 24\}.$$

If $v \notin \{0, 5\}$, then one may easily see that there is a positive integer a in the above set such that $g - a$ is a nonnegative integer congruent to 0 or 5 modulo 15. Thus, we have $g - a \longrightarrow p_3(5, 5, 6+9, 7, 8)$. One may directly check that $p_3(5, 5, 6, 7, 8, 9)$ represents all integers from 5 to 235.

Third, assume that $n = 4$. Note that the ternary triangular form $p_3(2, 2, 3)$ is regular. From this, one may easily show that it represents every nonnegative integer not

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

congruent to 1 modulo 3. Thus, $p_3(4,4,6)$ represents every nonnegative integer of the form $6u$ and $6u+4$, where $u \in \mathbb{Z}_{\geq 0}$. Note that $p_3(5,7)$ represents 5, 7, 15 and 26 as

$$5 = 5\cdot 1 + 7\cdot 0,\ 7 = 5\cdot 0 + 7\cdot 1,\ 15 = 5\cdot 3 + 7\cdot 0,\ 26 = 5\cdot 1 + 7\cdot 3.$$

From this and the fact that $p_3(4,4,6)$ represents every nonnegative integer of the form $6u$, it follows that $p_3(4,4,5,6,7)$ represents every nonnegative integer of the form

$$6u+7, \quad 6u+26, \quad 6u+15 \quad \text{and} \quad 6u+5.$$

One may directly check that $p_3(4,4,5,6,7)$ represents all integers from 4 to 25.

The case of $n = 3$ was already proved in Proposition 4.1. This completes the proof. □

THEOREM 4.8. *For any integer n greater than or equal to 4, the triangular form $Y_n = p_3(n, n+1, n+2, \ldots, 2n)$ is tight $\mathcal{T}(n)$-universal.*

PROOF. First, assume that the integer n is greater than 4. Let g be an integer greater than or equal to n. We write $g = un + v$ for some nonnegative integer u and an integer v with $n \leq v \leq 2n-1$. Since $n \geq 5$, there is an integer n_1 with $1 \leq n_1 \leq [n/2]$ such that the three integers $n+n_1$, $2n-n_1$ and v are all distinct. Since the ternary triangular form $p_3(1,2,3)$ is universal, every nonnegative integer which is a multiple of n is represented by $p_3(n,2n,3n)$ and thus also by $p_3(n,2n,n+n_1,2n-n_1)$. It follows that $g = un+v$ is represented by $p_3(n,2n,n+n_1,2n-n_1,v)$. From this and the choice of v, it follows that g is represented by Y_n.

Now we assume that $n = 4$. Let g_1 be an integer greater than or equal to 830. If we define two odd positive integers α and β as

$$(\alpha,\beta) = \begin{cases} (1,1) & \text{if } g_1 \equiv 0 \pmod 6, \\ (1,17) & \text{if } g_1 \equiv 1 \pmod 6, \\ (3,43) & \text{if } g_1 \equiv 2 \pmod 6, \\ (3,27) & \text{if } g_1 \equiv 3 \pmod 6, \\ (1,33) & \text{if } g_1 \equiv 4 \pmod 6, \\ (5,37) & \text{if } g_1 \equiv 5 \pmod 6, \end{cases}$$

then one may easily check that $8g_1 + 30 - 5\alpha^2 - 7\beta^2$ is a nonnegative integer congruent to 18 modulo 48. Put

$$s = 8g_1 + 30 - 5\alpha^2 - 7\beta^2$$

and let $L = \langle 4,6,8\rangle$. We assert that $s \xrightarrow{2} L$. One may easily check that s is locally represented by L. Note that the class number of L is 2 and the genus mate is $M = \langle 2,4,24\rangle$. If $s \longrightarrow L$, then we have $s \xrightarrow{2} L$ since $s \equiv 2 \pmod{16}$. Hence, we may assume that $s \longrightarrow M$. Thus, there is a vector $(x,y,z) \in \mathbb{Z}^3$ such that

$$s = 2x^2 + 4y^2 + 24z^2.$$

https://doi.org/10.1017/S0004972721000903 Published online by Cambridge University Press

Since $s \equiv 0 \pmod 3$, either $xy \not\equiv 0 \pmod 3$ or $x \equiv y \equiv 0 \pmod 3$ holds. After changing the sign of y if necessary, we may assume that $x \equiv y \pmod 3$. If we put $x = y - 3x_1$, then

$$\begin{aligned} s &= 2x^2 + 4y^2 + 24z^2 \\ &= 2(y - 3x_1)^2 + 4y^2 + 24z^2 \\ &= 4(x_1 + 2z)^2 + 6(x_1 - y)^2 + 8(x_1 - z)^2. \end{aligned}$$

In the above equation, one may easily deduce that

$$x_1 + 2z \equiv x_1 - y \equiv x_1 - z \equiv 1 \pmod 2$$

from the fact that $s \equiv 2 \pmod{16}$. Thus, we have $s \xrightarrow{2} L$. It follows immediately from this that

$$8g_1 + 30 \xrightarrow{2} \langle 4, 5, 6, 7, 8 \rangle,$$

which is equivalent to $g_1 \longrightarrow Y_4$. On the other hand, one may directly check that Y_4 represents all integers from 4 to 829. This completes the proof. □

THEOREM 4.9. *For any integer n exceeding 3, there are exactly two new tight $\mathcal{T}(n)$-universal triangular forms X_n and Y_n.*

PROOF. The result follows immediately from Lemma 2.3 and Theorems 4.7 and 4.8. □

References

[1] M. Bhargava, 'On the Conway–Schneeberger fifteen theorem', *Contemp. Math.* **272** (2000), 27–38.
[2] W. Bosma and B. Kane, 'The triangular theorem of eight and representation by quadratic polynomials', *Proc. Amer. Math. Soc.* **141**(5) (2013), 1473–1486.
[3] J. H. Conway, 'Universal quadratic forms and the fifteen theorem', *Contemp. Math.* **272** (2000), 23–26.
[4] F. Ge and Z.-W. Sun, 'On some universal sums of generalized polygonal numbers', *Colloq. Math.* **145** (2016), 149–155.
[5] B. W. Jones, *Representation by Positive Ternary Quadratic Forms* (PhD Thesis, University of Chicago, 1928).
[6] J. Ju, 'Universal sums of generalized pentagonal numbers', *Ramanujan J.* **51**(3) (2020), 479–494.
[7] J. Ju and B.-K. Oh, 'Universal sums of generalized octagonal numbers', *J. Number Theory* **190** (2018), 292–302.
[8] D. Kim, 'Weighted sums of generalized polygonal numbers with coefficients 1 or 2', *Acta Arith.*, to appear.
[9] M. Kim and B.-K. Oh, 'Regular ternary triangular forms', *J. Number Theory* **214** (2020), 137–169.
[10] M. Kim and B.-K. Oh, 'Tight universal quadratic forms', Preprint, arXiv:2104.02440 (2021).
[11] Y. Kitaoka, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Mathematics, 106 (Cambridge University Press, Cambridge, 1993).

[12] O. T. O'Meara, *Introduction to Quadratic Forms* (Springer, New York, 1963).
[13] Z.-W. Sun, 'A result similar to Lagrange's theorem', *J. Number Theory* **162** (2016), 190–211.

MINGYU KIM, Department of Mathematics, Sungkyunkwan University,
Suwon 16419, Korea
e-mail: kmg2562@skku.edu