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FUNCTIONS WITH FINITE DIRICHLET SUM OF ORDER p AND QUASI-MONOMORPHISMS OF INFINITE GRAPHS

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Abstract. In this paper, we study some potential theoretic properties of connected infinite networks and then investigate the space of p-Dirichlet finite functions on connected infinite graphs, via quasi-monomorphisms. A main result shows that if a connected infinite graph of bounded degrees possesses a quasi-monomorphism into the hyperbolic space form of dimension n and it is not p-parabolic for $p > n-1$, then it admits a lot of p-harmonic functions with finite Dirichlet sum of order p.

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§1. Introduction

A map $\psi : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is said to be a *quasi-isometric embedding* if there exist $a \geq 1$, $b > 0$ such that $a^{-1} d_X(x, x') - b \leq d_Y(\psi(x), \psi(x')) \leq a d_X(x, x') + b$ for all $x, x' \in X$. In addition, ψ is called a *quasi-isometry* if $\psi(X)$ is c-dense in Y for some $c > 0$; namely, for any $y \in Y$, there exists $x \in X$ such that $d_Y(y, \psi(x)) \leq c$.

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We consider the collection BG (bounded geometry) of connected infinite graphs of bounded degrees endowed with the graph distance and complete Riemannian manifolds with Ricci curvature bounded below and with a uniform lower bound on the volume of balls of radius 1. Any manifold in BG is quasi-isometric to some graph(s) in BG and vice versa. There are several important quasi-isometric invariant properties, such as having a certain type of volume growth, p-parabolicity ($1 < p < +\infty$), the existence of spectral gaps, and so on, which have been studied in many papers (see [22], [23], [20], [34], and the references therein). In addition, the space of p-harmonic functions with finite Dirichlet sum of order p on a graph in BG and the space of p-harmonic functions with finite Dirichlet integral of order p on a manifold in BG possess invariant properties under quasi-isometries, as shown in [21], [31], and [18]. For example, it is known that Euclidean space $\boldsymbol{R}^n$ of dimension n is p-parabolic if and only if $p \geq n$, and there exist no nonconstant p-harmonic functions with finite Dirichlet integral of order p on $\boldsymbol{R}^n$ for all $p > 1$; on the other hand, the hyperbolic space form $\boldsymbol{H}^n$ of constant curvature -1 and dimension n is not p-parabolic for all $p > 1$, and it admits a lot of nonconstant p-harmonic functions with finite Dirichlet integral of order p if $p > n-1$ and no such functions if $p \leq n-1$. These are invariant properties under quasi-isometries.

A map $\psi : X \to Y$ of a metric space (X, d_X) to another (Y, d_Y) is by definition a *quasi-monomorphism* if the following two conditions are satisfied:

(i) there exist constants $a > 0$ and $b \geq 0$ such that $d_Y(\psi(x_1), \psi(x_2)) \leq a d_X(x_1, x_2) + b$ for all $x_1, x_2 \in X$, and
(ii) for any $r > 0$, there exists a constant $c > 0$ such that for every $y \in Y$, the inverse image $\psi^{-1}(B)$ of the open ball $B = B_Y(y, r)$ centered at y with radius r can be covered by c open balls of radius r in X.

When we restrict ourselves to spaces in BG, a quasi-isometric embedding is a quasi-monomorphism, and the composition of quasi-monomorphisms is a quasi-monomorphism.

The notion of quasi-monomorphisms is introduced by Benjamini and Schramm [4]. A space in BG is said to be *almost planar* if there exists a planar graph in BG and a quasi-monomorphism from the space to the planar graph. They study (2-)harmonic functions on planar and almost planar graphs and manifolds, via circle packings, and prove that on an almost planar space in BG, there exists a nonconstant Dirichlet finite harmonic function if it is transient, or not (2-)parabolic.

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In this paper, we investigate some potential theoretic properties of infinite networks, or infinite weighted graphs, and then study the interplay between p-Dirichlet finite, p-harmonic functions, and compactifications of the networks. This paper is an expansion of [19].

Associated to the space of bounded p-Dirichlet finite functions on an infinite network, we have a compactification of the network, called the *Royden p-compactification*. This is not metrizable in general. Our study begins with introducing the notion of p-Dirichlet finite maps and showing that a p-Dirichlet finite map from an infinite network to a proper metric space extends continuously to the Royden p-boundary (see Theorem 3.1). We explore p-Dirichlet finite maps relevant to geometric compactifications, particularly the compactifications of Gromov hyperbolic graphs of bounded degrees and Riemannian manifolds by conformal changes of the metrics (see Sections 6 and 8).

A quasi-monomorphism of an infinite graph to another one in BG pulls back p-Dirichlet finite functions or maps in the target graph to such functions or maps in the domain. From this fact, we can prove, for instance, that it induces a continuous map from the Royden p-boundary of the domain graph to that of the target one. As a consequence, we can observe that if the target graph of a quasi-monomorphism is p-parabolic, then so is the domain. This illustrates Rayleigh's monotonicity law.

A main result of this paper shows that if a graph in BG possesses a quasi-monomorphism into $\boldsymbol{H}^n$ and it is not p-parabolic for $p > n-1$, then it admits a lot of p-harmonic functions with finite Dirichlet sum of order p. For the precise statement, see Theorem 9.5, where we use the harmonic p-boundary in the Royden p-compactification of the graph to express p-harmonic functions with finite Dirichlet sum of order p. For example, regarding $\boldsymbol{R}^{n-1}$ as a horosphere in $\boldsymbol{H}^n$, we have a quasi-monomorphism of the lattice $\boldsymbol{Z}^{n-1}$ into $\boldsymbol{H}^n$, but no such map to $\boldsymbol{H}^{n-1}$. Applying Bonk and Schramm's embedding theorem [8, Theorem 1.1], we see that if a graph in BG has a quasi-monomorphism to a visual Gromov hyperbolic geodesic space whose boundary at infinity is doubling for some visual metric, then the graph carries a lot of p-harmonic functions with finite Dirichlet sum of order p for sufficiently large p unless it is p-parabolic. For example, the Cartesian product of the homogeneous tree of degree $d \geq 3$, and any (infinite) graph in BG has no quasi-monomorphisms into such a Gromov hyperbolic space, because for all $p > 1$, the product graph is not p-parabolic, and it has no nonconstant p-harmonic functions with finite Dirichlet sum of order p.

The paper is organized as follows. In Section 2, we introduce several notions of nonlinear potential theory on networks and then collect some known results. In Section 3, the notion of p-Dirichlet finite maps is introduced and boundary behavior is investigated. Section 4 is devoted to exhibiting infinite networks of Liouville D_p-property. The Kuramochi compactification of an infinite network is constructed in Section 5. In Section 6, we consider a Gromov hyperbolic graph of bounded degrees and relate some properties of the Gromov boundary to existence or nonexistence of p-harmonic functions with finite Dirichlet sum of order p. Section 7 is devoted to showing some basic properties of quasi-monomorphisms between graphs in BG. In Section 8, we discuss discrete approximation of Riemannian manifolds in BG and exhibit some examples of Riemannian manifolds on which existence or nonexistence of p-harmonic functions with finite p-Dirichlet integral can be illustrated along with their geometric structures. In Section 9, we discuss infinite graphs of bounded degrees admitting quasi-monomorphisms into the hyperbolic space forms and prove our main theorem.

§2. Infinite networks and the Royden compactification

In this section, we introduce several notions, such as Dirichlet sum of order p, p-harmonic functions, Royden and harmonic p-boundaries, p-parabolicity, extremal length of order p, and Royden decomposition, and then recall some known results.

To begin with, following [13], we explain how to construct compact boundaries of a countably infinite set V. We are first given a family Φ of bounded functions on V. Let $\Phi^* = \Phi \cup \{\delta_x : x \in V\}$, where $\delta_x(x) = 1$ and $\delta_x(y) = 0$ if $y \neq x$. For $f \in \Phi^*$, there is a constant $M(f)$ such that $|f(x)| \leq M(f)$ for all $x \in V$. Endow the product space

$$\Pi_\Phi = \Pi_{f\in\Phi*}[-M(f), M(f)] = \left\{\xi : \Phi^* \to \mathbf{R} : |\xi(f)| \leq M(f) \text{ for all } f \in \Phi^*\right\}$$

with the product topology, and embed V into Π_Φ by $x \to \xi_x$, where $\xi_x(f) = f(x)$ for $f \in \Phi^*$. This mapping is injective, since $\delta_x \in \Phi^*$ for each $x \in V$. Identify V with its image, and take its closure, denoted by $\mathcal{C}_\Phi$, in the product space. We extend $f \in \Phi$ by writing $f(\xi) = \xi(f)$ for $\xi \in \mathcal{C}_\Phi$. In this way, we get a compact Hausdorff space $\mathcal{C}_\Phi$, unique up to homeomorphisms, satisfying the following properties:

(i) V is topologically embedded in $\mathcal{C}_\Phi$ as an open and dense subset;
(ii) every function of Φ extends to a continuous function on $\mathcal{C}_\Phi$;

(iii) the extended functions separate the points of the boundary $\partial\mathcal{C}_\Phi = \mathcal{C}_\Phi \setminus V$.

The uniqueness is checked by using the following description of convergence of nets: a net $\{x_\alpha\}$ in $\mathcal{C}_\Phi$ converges to a point in $\partial\mathcal{C}_\Phi$ if and only if for every finite $K \subset V$ there is an α_0 such that $x_\alpha \notin K$ for every $\alpha > \alpha_0$, and $\lim_\alpha f(x_\alpha)$ exists for every $f \in \Phi$.

Alternatively, $\mathcal{C}_\Phi$ can be characterized by the minimality property as follows. Let X be a compactification of V such that each $f \in \Phi$ extends to a continuous function on X. Then there is a canonical map from X onto $\mathcal{C}_\Phi$.

We note that for two families Φ and Ψ of bounded functions with $\Phi \subset \Psi$, the identity map induces a continuous map from $\partial\mathcal{C}_\Psi$ *onto* $\partial\mathcal{C}_\Phi$, and further, $\partial\mathcal{C}_\Psi$ is homeomorphic to $\partial\mathcal{C}_\Phi$ if Φ is dense in Ψ with respect to the uniform norm.

Now we consider a graph $G = (V, E)$ with the set of vertices V and the set of edges E that consists of pairs of vertices. In this paper, a graph admits no loops and multiple edges, and the set of vertices is finite or countably infinite. We say that a vertex x is *adjacent* to another y if $\{x, y\}$ belongs to E, and we write $x \sim y$ to indicate it. We also use the notation $|xy|$ for $\{x, y\} \in E$. For each vertex x, the cardinality of the subset $\{y \in V \mid y \sim x\}$ is called the *degree* of G at x. We say that G is *locally finite* if the degree at each vertex is finite. In this paper, the graph G under consideration is assumed to be locally finite.

By a path of length n in G, we mean a sequence of $(n+1)$ vertices $c = (x_0, x_1, \ldots, x_n)$ such that $x_i \sim x_{i+1}$ $(0 \le i \le n-1)$, and we say that c *connects* x_0 to x_n. G is called a *connected graph* if for any pair of vertices x, y, there exist paths connecting them. On a connected graph G, we can introduce a distance d_G on V, called the *graph distance* of G, by assigning to each pair of vertices x and y the minimum of the length of a path connecting them. We say that a subset K of V is connected if any pair of points $x, y \in K$ can be connected by a path in K. In what follows, the graph $G = (V, E)$ under consideration is supposed to be connected unless otherwise stated.

We are now given a weight r on the set of edges E, that is, a positive function on E. We call such a couple (G, r) a *network*. Fix a number $p \in (1, +\infty)$. For a function u on V, we define the pth power of the *gradient* at $x \in V$ and the *p-Laplacian* at $x \in V$, respectively, by

$$|du(x)|^p = \sum_{y\sim x} r(|xy|)\Big(\frac{|u(x)-u(y)|}{r(|xy|)}\Big)^p = \sum_{y\sim x} \frac{|u(x)-u(y)|^p}{r(|xy|)^{p-1}},$$

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$$\mathcal{L}_p u(x) = \sum_{y\sim x} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{r(|xy|)^{p-1}}.$$

The *Dirichlet sum* of order p of u over a subset K of V is given by

$$D_{r;p}(u;K) = \frac{1}{2}\sum_{x\in K} |du(x)|^p.$$

When $K=V$, we write simply $D_{r;p}(u)$ for $D_{r;p}(u;V)$, and further, when $r=1$, we write $D_p(u)$ for $D_{1;p}(u)$. More generally, the omission of r in a notation will mean that $r=1$.

A function $u:V\to\mathbf{R}$ is called *p-harmonic* in a subset K of V if $\mathcal{L}_p u=0$ in K. Given a subset K, we denote by ∂K the set of vertices x such that $x\notin K$ and $x\sim y$ for some $y\in K$. Then for a function $u:V\to\mathbf{R}$ and a finite subset K of V, the following are mutually equivalent (see [28]):

(i) Here u is p-harmonic in K.

(ii) Also, u is a minimizer of $D_{r;p}(u;K)$ among functions on $K\cup\partial K$ with the same values on ∂K; that is,

$$\sum_{x\in K}|du(x)|^p \le \sum_{x\in K}|dv(x)|^p$$

for every function v on $K\cup\partial K$ with $v=u$ on ∂K.

(iii) Here, u satisfies

$$\sum_{x\in K}\sum_{y\sim x}\frac{|u(x)-u(y)|^{p-2}}{r(|xy|)^{p-1}}\big(u(x)-u(y)\big)\big(w(x)-w(y)\big)=0$$

for every function w on $K\cup\partial K$ with $w=0$ on ∂K. (The existence of a minimizer in (ii) is easily verified. The uniqueness is described in the following comparison principle; see [28]).

(iv) Let u and v be functions on V, and suppose that they are p-harmonic in a finite connected subset K. Then $u\le v$ on K if $u\le v$ on ∂K.

Let K be a finite connected subset of V. For a function f on ∂K, there exists a unique function u_f on $K\cup\partial K$ which is p-harmonic in K and equals f on ∂K.

LEMMA 2.1. *Let K be a finite connected subset of V. For functions f, g on ∂K, one has* $\max_K|u_f-u_g|\le\max_{\partial K}|f-g|$.

Proof. Let $f \vee g = \max\{f, g\}$, and let $f \wedge g = \min\{f, g\}$. Then (iv) above implies that $u_{f\wedge g} \le u_f \wedge u_g \le u_f \vee u_g \le u_{f\vee g}$ on $K \cup \partial K$. Let $m = \max_{\partial K} |f - g| = \max_{\partial K}(f \vee g - f \wedge g)$. Then $f \vee g \le f \wedge g + m$, so that $u_{f\vee g} \le u_{f\wedge g+m} = u_{f\wedge g} + m \le u_f \wedge u_g + m$, and thus we get $|u_f - u_g| = u_f \vee u_g - u_f \wedge u_g \le m$. □

We denote by $L^{1,p}(G, r)$ the space of all functions u on V whose Dirichlet sum of order $p > 1$ over V is finite; that is,

$$D_{r;p}(u) = \frac{1}{2} \sum_{x\sim y} \frac{|u(x) - u(y)|^p}{r(|xy|)^{p-1}} < +\infty.$$

Then $L^{1,p}(G, r)$ is a Banach space with respect to the norm $D_{r;p}(u)^{1/p} + |u(o)|$, where o is a fixed point of V (see [30]). Let us denote by $L_0^{1,p}(G, r)$ the closure of the set of functions with finite supports. We write $HL^{1,p}(G, r)$ for the space of p-harmonic functions in $L^{1,p}(G, r)$. The space $BL^{1,p}(G, r)$ of all bounded functions in $L^{1,p}(G, r)$ is a Banach algebra with unit element 1 with respect to the norm $D_{r;p}(u)^{1/p} + \sup_V |u|$. Similarly, we define $BL_0^{1,p}(G, r)$ and $BHL^{1,p}(G, r)$.

Associated to $BL^{1,p}(G, r)$ as described above, we have a compactification $\mathcal{R}_p(G, r)$ called the *Royden p-compactification* of the network (G, r), and the boundary $\partial\mathcal{R}_p(G, r) = \mathcal{R}_p(G, r) \setminus V$ is called the *Royden p-boundary* of (G, r). For $u \in BL^{1,p}(G, r)$, we denote by $\bar{u}$ the continuous extension to $\mathcal{R}_p(G, r)$. It is known that every $u \in L^{1,p}(G, r)$, not necessarily bounded, can be extended continuously to a function defined on $\mathcal{R}_p(G, r)$ with values in $[-\infty, +\infty]$ (see [35, Theorem 6.9], where the case of $p = 2$ is discussed, but the argument of which is valid for any $p > 1$). This extension is also denoted by $\bar{u}$. There is an important part of the Royden p-boundary, called the *harmonic p-boundary* of the network (G, r), which is defined by

$$\Delta_p(G, r) := \big\{x \in \partial\mathcal{R}_p(G, r) \;\big|\; \bar{u}(x) = 0,\ \forall u \in BL_0^{1,p}(G, r)\big\};$$

furthermore, the following duality holds (see [40]):

$$BL_0^{1,p}(G, r) = \big\{u \in BL^{1,p}(G, r) \;\big|\; \bar{u}(x) = 0,\ \forall x \in \Delta_p(G, r)\big\}.$$

Now we introduce a number $R^{(p)}_{(G,r)}(x, y)$ for any pair of vertices x, y by letting

$$R^{(p)}_{(G,r)}(x, y) = \sup\Big\{\frac{|f(x) - f(y)|^p}{D_{r;p}(f)} \;\Big|\; f \in L^{1,p}(G, r), D_{r;p}(f) \ne 0\Big\}.$$

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Since G is connected, $R^{(p)}_{(G,r)}(x,y)$ is finite for any pair of vertices $x,y\in V$ (see (2.1) below), and $R^{(p)1/p}_{(G,r)}$ induces a distance on V (see [14]). For $p=2$, $R^{(2)}_{(G,r)}(x,y)$ is called the *effective resistance* between x and y.

The weight r also gives rise to a distance $d_{(G,r)}$ on V by taking $r(e)$ as the length of an edge e. To be precise, a path $c=(x_0,x_1,\dots,x_n)$ has by definition the length $L_r(c)=\sum_{i=0}^{n-1} r(|x_ix_{i+1}|)$, and for any pair of vertices x and y, $d_{(G,r)}(x,y)$ denotes the infimum of $L_r(c)$ over all paths c joining x and y. Then $d_{(G,r)}: V\times V\to[0,+\infty)$ is called the *geodesic distance* of the network (G,r). For a pair of vertices x and y, we connect x to y by a geodesic path $c=(x=x_0,x_1,\dots,x_n=y)$. Then for a function $u\in L^{1,p}(G,r)$, we have

$$\begin{aligned}|u(x)-u(y)| &\le \sum_{i=0}^{n-1}|u(x_i)-u(x_{i+1})|\\ &\le \Big(\sum_{i=0}^{n-1}\frac{|u(x_i)-u(x_{i+1})|^p}{r(|x_ix_{i+1}|)^{p-1}}\Big)^{1/p}\Big(\sum_{i=0}^{n-1} r(|x_ix_{i+1}|)\Big)^{1-1/p}\\ &\le D_{r;p}(u)^{1/p}L_r(c)^{1-1/p}.\end{aligned}$$

This holds for any geodesic path as above, so that we get the following basic inequality:

$$R^{(p)}_{(G,r)}(x,y)\le d_{(G,r)}(x,y)^{p-1},\quad x,y\in V. \tag{2.1}$$

Let

$$M^{(p)}_{(G,r)}(x)=\sup\Big\{\frac{|g(x)|^p}{D_{r;p}(g)}\;\Big|\; g\in L^{1,p}_0(G,r), D_{r;p}(g)>0\Big\}(\le+\infty),\quad x\in V.$$

We say that a network (G,r) is *p-parabolic* if $M^{(p)}_{(G,r)}(x)$ is infinite for some $x\in V$, and (G,r) is *p-nonparabolic* otherwise. We note that $M^{(p)}_{(G,r)}(x)$ is infinite for every $x\in V$ if the network is p-parabolic, since we have by the definitions and convexity of $x\mapsto x^p$

$$M^{(p)}_{(G,r)}(x)\le 2^{p-1}M^{(p)}_{(G,r)}(y)+2^{p-1}R^{(p)}_{(G,r)}(x,y),\quad x,y\in V. \tag{2.2}$$

It is proved in [38] that the following are mutually equivalent: (i) (G,r) is p-parabolic; (ii) $\Delta_p(G,r)$ is empty; (iii) $L^{1,p}(G,r)=L^{1,p}_0(G,r)$; (iv) $1\in L^{1,p}_0(G,r)$.

The *Royden decomposition* of $L^{1,p}(G,r)$ and the *comparison principle* for $HL^{1,p}(G,r)$ are stated in the following.

THEOREM 2.2 ([39, Theorem 2.1], [40, Theorem 3.2], [29, Proposition 1.1]). *Let (G, r) be a connected infinite network that is p-nonparabolic.*

(1) *Every $u \in L^{1,p}(G, r)$ is uniquely decomposed in the form*

$$u = h + g, \quad h \in HL^{1,p}(G, r), g \in L_0^{1,p}(G, r).$$

Here the function h satisfies $D_{r;p}(h) = \inf\{D_{r;p}(u - f) \mid f \in L_0^{1,p}(G, r)\}$; in particular, $D_{r;p}(h) \leq D_{r;p}(u)$. In addition, h and g are bounded if so is u.

(2) *Given that $h_1, h_2 \in HL^{1,p}(G, r)$, one has $h_1 \leq h_2$ on V if $\overline{h}_1 \leq \overline{h}_2$ on $\Delta_p(G, r)$.*

Now we recall the following.

LEMMA 2.3 ([19, Proposition 4]). *Given a connected infinite network (G, r), the following conditions are mutually equivalent:*

(1) $\sup_{x \in V} M^{(p)}_{(G,r)}(x)$ *is finite;*

(2) *all $g \in L_0^{1,p}(G, r)$ are bounded;*

(3) *for any $g \in L_0^{1,p}(G, r)$, $g(x)$ tends to zero as $x \in V$ goes to infinity (namely, for any $\varepsilon > 0$, there exists a finite subset K of V such that $|g(x)| < \varepsilon$ for all $x \in V \setminus K$);*

(4) $\Delta_p(G, r) = \partial\mathcal{R}_p(G, r)$.

A function $m : E \to [0, +\infty)$ is said to be an L^p-*pseudometric* if

$$\|m\|_{r;p}^p := \frac{1}{2} \sum_{x \sim y} \frac{m(|xy|)^p}{r(|xy|)^{p-1}} < \infty.$$

For our purpose, we permit m to vanish on some edges. For example, we set $m(x, y) = |u(x) - u(y)|$ for $u \in L^{1,p}(G, r)$. Then this gives an L^p-pseudometric, because $\|m\|_{r;p}^p = D_{r;p}(u)$.

An infinite path γ starting from a vertex x of G is by definition a sequence of vertices $\{\gamma(n)\}_{n \geq 0}$ such that $\gamma(n) \sim \gamma(n-1)$ for any $n \geq 1$ and $\gamma(0) = x$. Denote by $\boldsymbol{P}_x$ the family of all infinite paths starting from a vertex x of G, and denote by $\boldsymbol{P}(G)$ the union of $\boldsymbol{P}_x$ for all $x \in V$.

Given an L^p-pseudometric m, we define the m-length $L_m(\gamma)$ of $\gamma \in \boldsymbol{P}(G)$ by

$$L_m(\gamma) = \sum_{n=1}^{\infty} m\big(|\gamma(n-1)\gamma(n)|\big) (\leq \infty).$$

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The *extremal length* $EL_{r;p}(\boldsymbol{P})$ of order p of a subset $\boldsymbol{P}$ of $\boldsymbol{P}(G)$ is defined by

$$EL_{r;p}(\boldsymbol{P}) = \sup\Big\{\frac{\inf\{L_m(\gamma) \mid \gamma \in \boldsymbol{P}\}}{\|m\|_{r;p}^p} \;\Big|\; m \text{ is a nontrivial } L^p\text{-pseudometric}\Big\}.$$

(Extremal length was first investigated by Duffin [16] in the discrete setting, and extremal length of order p was studied in Nakamura and Yamasaki [30]. By simple calculation, we verify that our definition is equivalent to theirs. We refer also to [4] and [6].) We will say that a property *holds for almost every path in* $\boldsymbol{P}(G)$ if the subset of all paths for which the property is not true has extremal length ∞.

For an L^p-pseudometric m, we denote by $\boldsymbol{P}_{m,\infty}$ the family of all paths $\gamma \in \boldsymbol{P}(G)$ with $L_m(\gamma) = +\infty$. Obviously, $EL_{r;p}(\boldsymbol{P}_{m,\infty}) = +\infty$, and hence $L_m(\gamma)$ is finite for almost every path γ in $\boldsymbol{P}(G)$.

We recall a property of extremal length in the following.

LEMMA 2.4 ([27, Lemma 2.2]). *Let $\{\boldsymbol{P}_n\}$ be a countable family of subsets of $\boldsymbol{P}(G)$. Then*

$$EL_{r;p}\Big(\bigcup_{n=1}^{\infty} \boldsymbol{P}_n\Big)^{-1} \le \sum_{n=1}^{\infty} EL_{r;p}(\boldsymbol{P}_n)^{-1}.$$

Boundary behavior of p-Dirichlet finite functions is studied in [27], [39], and [40]; some of the results are stated in the following.

THEOREM 2.5 ([27, Theorem 3.1], [39, Theorem 3.2], [40, Lemma 5.3]). *Let (G, r) be a connected infinite network. The following assertions hold.*

(1) *Let $f \in L^{1,p}(G, r)$. Then the sequence $\{f(\gamma(n))\}$ has a limit as n tends to ∞ for almost every path γ in $\boldsymbol{P}(G)$.*
(2) *Let $h \in HL^{1,p}(G, r)$ be nonconstant. Then there is no constant c such that $\lim_{n\to\infty} h(\gamma(n)) = c$ for almost every path γ in $\boldsymbol{P}(G)$.*
(3) *Let $g \in L^{1,p}(G, r)$. Then $g \in L_0^{1,p}(G, r)$ if and only if $\lim_{n\to\infty} g(\gamma(n)) = 0$ for almost every path γ in $\boldsymbol{P}(G)$.*
(4) *For any closed subset F in $\partial\mathcal{R}_p(G, r) \setminus \Delta_p(G, r)$, if it is not empty, there is a function $g \in L_0^{1,p}(G, r)$ such that $g(x)$ tends to $+\infty$ as $x \in V \to F$.*

For a subset B of $\mathcal{R}_p(G, r)$, we denote by $\overline{B}$ the closure of B in $\mathcal{R}_p(G, r)$. The set of *extreme points* for $\gamma \in \boldsymbol{P}(G)$ is given by $EP_{r;p}(\gamma) := \overline{\{\gamma(n)\}} \cap \partial\mathcal{R}_p(G, r)$.

THEOREM 2.6 ([40, Theorem 6.4]). *Let* $\boldsymbol{P}_\infty$ *be a subset of* $\boldsymbol{P}(G)$ *with* $EL_{r;p}(\boldsymbol{P}_\infty) = +\infty$. *Then the set* $\overline{\cup\{EP_{r;p}(\gamma); \gamma \in \boldsymbol{P}(G) \setminus \boldsymbol{P}_\infty\}}$ *contains* $\Delta_p(G, r)$.

§3. p-Dirichlet finite maps

In this section, we introduce the notion of p-Dirichlet finite maps from an infinite network to a metric space and study the boundary behavior of such a map.

Let (G, r) be a connected infinite network. Let (X, d_X) and f be a metric space and a map from V to X, respectively. The Dirichlet sum of order p with weight r of the map f is defined by

$$D_{r;p}(f) = \frac{1}{2} \sum_{x \sim y} \frac{d_X(f(x), f(y))^p}{r(|xy|)^{p-1}},$$

and f is said to be p-*Dirichlet finite* if $D_{r;p}(f) < \infty$.

For a p-Dirichlet finite map $f : V \to X$, we have

$$d_X\big(f(y), f(z)\big)^p \le (2\delta)^{p-1} \sum_{w \in B_{(G,r)}(x, 2\delta)} \sum_{w' \sim w} \frac{d_X(f(w), f(w'))^p}{r(|ww'|)^{p-1}} \tag{3.1}$$

for any $\delta > 0$, every $x \in V$, and all $y, z \in B_{(G,r)}(x, \delta)$, where $B_{(G,r)}(x, \delta)$ stands for the metric ball around x with radius δ with respect to the geodesic distance $d_{(G,r)}$. In fact, we connect y to z by a geodesic path $c = (y = x_0, x_1, \ldots, x_n = z)$. Then we have

$$\begin{aligned} d_X\big(f(y), f(z)\big) &\le \sum_{i=0}^{n-1} d_X\big(f(x_i), f(x_{i+1})\big) \\ &\le \Big(\sum_{i=0}^{n-1} \frac{d_X(f(x_i), f(x_{i+1}))^p}{r(|x_i x_{i+1}|)^{p-1}}\Big)^{1/p} \Big(\sum_{i=0}^{n-1} r(|x_i x_{i+1}|)\Big)^{1-1/p} \\ &\le \Big(\sum_{w \in B_{(G,r)}(x, 2\delta)} \sum_{w \sim w'} \frac{d_X(f(w), f(w'))^p}{r(|ww'|)^{p-1}}\Big)^{1/p} L_r(c)^{1-1/p}. \end{aligned}$$

We remark also that if f is a p-Dirichlet finite map, then for any Lipschitz continuous function η with Lipschitz constant L on X, the composition $\eta \circ f$ belongs to $L^{1,p}(G, r)$ and

$$D_{r;p}(\eta \circ f) \le L^p D_{r;p}(f).$$

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Therefore, $\eta \circ f$ uniquely extends to a continuous function $\overline{\eta \circ f} : \mathcal{R}_p(G,r) \to [-\infty, +\infty]$.

For a p-Dirichlet finite map f, we define a function on the set of edges E by $m_f(|xy|) = d_X(f(x), f(y))$, $|xy| \in E$. Note that $\|m_f\|_{r;p}^p = D_{r;p}(f) < \infty$, and hence m_f is an L^p-pseudometric on E. We put $\boldsymbol{P}_{m_f,\infty} = \{\gamma \in \boldsymbol{P}(G) \mid L_{m_f}(\gamma) = \infty\}$.

Suppose that (X, d_X) is complete. Then for $\gamma \in \boldsymbol{P}(G) \setminus \boldsymbol{P}_{m_f,\infty}$, we have by the definition of $\boldsymbol{P}_{m_f,\infty}$

$$L_{m_f}(\gamma) = \sum_{n=1}^{\infty} d_X\big(f(\gamma(n-1)), f(\gamma(n))\big) < \infty,$$

and hence $\{f(\gamma(n))\}$ converges since it is a Cauchy sequence in X. The limit of the sequence $\{f(\gamma(n))\}$ in X is denoted by $(f \circ \gamma)(\infty)$. Then for any Lipschitz continuous function η on X,

$$\eta\big((f \circ \gamma)(\infty)\big) = \lim_{n\to\infty} \eta\big(f(\gamma(n))\big) = \lim_{n\to\infty} \eta \circ f\big(\gamma(n)\big) = \overline{\eta \circ f}(\xi)$$

for all $\gamma \in \boldsymbol{P}(G) \setminus \boldsymbol{P}_{m_f,\infty}$ and $\xi \in EP_{r;p}(\gamma)$.

THEOREM 3.1. *Let $f : (G,r) \to (X, d_X)$ be a p-Dirichlet finite map from a connected infinite network (G,r) to a proper metric space (X, d_X), that is, a metric space such that any bounded closed subset is compact. Let $\bar{X} = X \cup \{\infty_X\}$ be the 1-point compactification of X. Then f extends to a continuous map $\bar{f} : \mathcal{R}_p(G,r) \to \bar{X}$ from the Royden compactification of (G,r) to $\bar{X}$. Moreover, suppose that (G,r) is p-nonparabolic. Then there exists a family $\boldsymbol{P}_\infty$ in $\boldsymbol{P}(G)$ with $EL_{r;p}(\boldsymbol{P}_\infty) = \infty$, including $\boldsymbol{P}_{m_f,\infty}$, such that*

$$\bar{f}\big(\Delta_p(G,r)\big) = \overline{\big\{(f \circ \gamma)(\infty) \in X \;\big|\; \gamma \in \boldsymbol{P}(G) \setminus (\boldsymbol{P}'_\infty \cup \boldsymbol{P}_\infty)\big\}}$$

for any family $\boldsymbol{P}'_\infty$ in $\boldsymbol{P}(G)$ with $EL_{r;p}(\boldsymbol{P}'_\infty) = \infty$.

Proof. For a point $x \in X$, we denote by η_x the distance function to x in X. Let $\Sigma_f = \{\xi \in \partial\mathcal{R}_p(G,r) \mid \overline{\eta_x \circ f}(\xi) = +\infty\}$. This closed subset is independent of the choice of a reference point x. Now we take a countably infinite dense subset $\{x_i\}$ of X. Let ξ and $\{v_n\}$ be, respectively, a point of $\partial\mathcal{R}_p(G,r) \setminus \Sigma_f$ and a sequence in V converging to ξ. Then $f(v_n)$ stays in a compact subspace in X. Since $d_X(x_i, f(v_n))$ tends to $\overline{\eta_{x_i} \circ f}(\xi)$ as $n \to \infty$ for all x_i which are densely distributed in X, we can deduce that as n tends to infinity, $f(v_n)$ converges to a point, $\bar{f}(\xi)$, in X. By setting $\bar{f}(\xi)=\infty_X$ for $\xi\in\Sigma_f$, we obtain a continuous map $\bar{f}$ from $\mathcal{R}_p(G,r)$ to $\bar{X}$.

Suppose now that (G,r) is p-nonparabolic. For any $j=1,2,\dots,$ let $\bar{f}(\Delta_p(G,r))_j=\{x\in X \mid d_X(x,\bar{f}(\Delta_p(G,r)))<1/j\}$ and $A_j=\bar{f}(\partial\mathcal{R}_p(G,r))\setminus\bar{f}(\Delta_p(G,r))_j$. Since $\bar{f}^{-1}(A_j)$ is disjoint from $\Delta_p(G,r)$, by Theorem 2.5(4), we have a function $g_j\in L_0^{1,p}(G,r)$ such that $\bar{g}_j=+\infty$ on $\bar{f}^{-1}(A_j)\cap\partial\mathcal{R}_p(G,r)$. On the other hand, it follows from Theorem 2.5(3) that $\lim_{n\to\infty}\bar{g}_j(\gamma(n))=0$ for almost every path γ in $\boldsymbol{P}(G)$. This shows that $EL_{r;p}(\{\gamma\in\boldsymbol{P}(G)\setminus\boldsymbol{P}_{m_f,\infty}\mid(f\circ\gamma)(\infty)\in A_j\})=+\infty$, and hence, letting $\boldsymbol{P}_{f,\infty}=\{\gamma\in\boldsymbol{P}(G)\setminus\boldsymbol{P}_{m_f,\infty}\mid(f\circ\gamma)(\infty)\in\cup_jA_j\}$, we have by Lemma 2.4 $EL_{r;p}(\boldsymbol{P}_{f,\infty})=+\infty$. Hence, $(f\circ\gamma)(\infty)\in\bar{f}(\Delta_p(G,r))$ for all $\gamma\in\boldsymbol{P}(G)\setminus(\boldsymbol{P}_{f,\infty}\cup\boldsymbol{P}_{m_f,\infty})$. Moreover, in view of Theorem 2.6, we see that the assertion holds true. $\square$

COROLLARY 3.2. *Let $f:(G,r)\to(X,d_X)$ be a p-Dirichlet finite map from a connected infinite network (G,r) to a proper metric space (X,d_X), and let $\bar{f}$ be its continuous extension to $\mathcal{R}_p(G,r)$ with values in $\bar{X}$. Suppose that (G,r) is p-nonparabolic and that, for any $\xi\in\bar{f}(\Delta_p(G,r))$, $EL_{r;p}(\{\gamma\in\boldsymbol{P}(G)\mid(f\circ\gamma)(\infty)=\xi\})=\infty$. Then $\bar{f}(\Delta_p(G,r))$ is a perfect subspace of $\bar{X}$.*

Proof. In view of the Cantor-Bendixson theorem, $\bar{f}(\Delta_p(G,r))$ can be uniquely written as $\bar{f}(\Delta_p(G,r))=\mathcal{P}\cup\mathcal{C}$, with $\mathcal{P}$ a perfect subset of $\bar{f}(\Delta_p(G,r))$ and $\mathcal{C}$ countably open. Suppose that $\mathcal{C}$ is not empty, and take a point ξ of it. Let U be a neighborhood of ξ in $\bar{f}(\Delta_p(G,r))$ such that $U\cap\mathcal{P}=\emptyset$, and set $\boldsymbol{P}_U=\{\gamma\in\boldsymbol{P}(G)\setminus\boldsymbol{P}_{m_f,\infty}\mid(f\circ\gamma)(\infty)\in U\}$. Since U is at most countably infinite, it follows from the assumption and Lemma 2.4 that $EL_{r;p}(\boldsymbol{P}_U)=\infty$. Therefore, by Theorem 3.1, we have

$$\bar{f}\big(\Delta_p(G,r)\big)=\overline{\big\{(f\circ\gamma)(\infty)\,\big|\,\gamma\in\boldsymbol{P}(G)\setminus(\boldsymbol{P}_U\cup\boldsymbol{P}_\infty)\big\}},$$

which excludes the point ξ. This is a contradiction. $\square$

COROLLARY 3.3. *Let $f:(G,r)\to X$ be a p-Dirichlet finite map from a connected infinite network (G,r) to a proper metric space (X,d_X), and let $\bar{f}$ be its continuous extension to $\mathcal{R}_p(G,r)$. Suppose that (G,r) is p-nonparabolic. Then for a continuous function v in $\bar{f}(\Delta_p(G,r))$ which is Lipschitz continuous in $\bar{f}(\Delta_p(G,r))\cap X$, there exists a unique $h\in HL^{1,p}(G,r)$ such that*

(1) *$\bar{h}=v\circ\bar{f}$ on $\Delta_p(G,r)$; in particular, if $\partial\mathcal{R}_p(G,r)=\Delta_p(G,r)$, then $h(x_n)$ tends to $v(\xi)$ for any sequence $\{x_n\}$ in V such that $f(x_n)$ converges to ξ as $n\to\infty$;*

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(2) $\lim_{n\to\infty} h(\gamma(n)) = v((f\circ\gamma)(\infty))$ *for almost every* $\gamma \in \boldsymbol{P}(G)$.

Proof. We extend v to a Lipschitz function $\overline{v}$ in X (e.g., letting $\overline{v}(x) = \inf\{v(y) + Ld_X(y,x) \mid y \in \bar{f}(\Delta_p(G,r))\}$ at $x \in V$, where L is a Lipschitz constant of v). Then $\overline{v}\circ f$ belongs to $L^{1,p}(G,r)$, so that the p-harmonic part of $\overline{v}\circ f$ in the Royden decomposition gives a unique solution required in (1). As far as (2) is concerned, we use Theorem 2.5(3). ∎

Before ending this section, we consider n-dimensional sphere packings in the Euclidean sphere $\boldsymbol{S}^n$. Let $\mathcal{B} = \{B_x \mid x \in V\}$ be a collection of closed balls indexed by a countably infinite set V with disjoint interiors in $\boldsymbol{S}^n$. Let $K(\mathcal{B})$ be the set of accumulation points of the centers of the balls in $\mathcal{B}$. Associated to the collection, we have a graph $G = (V,E)$ with the set of vertices V and the set of edges E defined by $|xy| \in E$ if and only if B_x and B_y are tangent. We assume that G is connected, of bounded degrees, and of weight $r = 1$. Assigning to a vertex $x \in V$ the center of B_x, we obtain a map $f: V \to \boldsymbol{S}^n$. For any $x, y \in V$, the distance between $f(x)$ and $f(y)$ is not more than the sum of the radii of B_x and B_y. Then it is easy to see that f is a p-Dirichlet finite map from G into $\boldsymbol{S}^n$ for $p \geq n$. Hence, f extends to a continuous map $\bar{f}$ from the Royden p-compactification of G into $\boldsymbol{S}^n$, and we have $\bar{f}(\partial\mathcal{R}_p(G)) = K(\mathcal{B})$. Moreover, it is proved in [4, Theorem 4.1], and [6, Theorem 7], that

$$EL_{r;p}\big(\{\gamma \in \boldsymbol{P}(G) \,\big|\, (f\circ\gamma)(\infty) = \xi\}\big) = +\infty \quad \text{for } \forall\xi \in K(\mathcal{B}).$$

(This is verified for the case $p = n$, but the argument is valid for any $p \geq n$.) As a result of Corollary 3.2, we see that $\bar{f}(\Delta_p(G))$ is a perfect subspace of $K(\mathcal{B})$ if $p \geq n$ and G is p-nonparabolic.

§4. Networks of Liouville D_p-property

In this section, we are concerned with networks admitting no nonconstant p-harmonic functions of finite Dirichlet sum of order p, and we show two propositions which are mentioned in [19] without proof.

The Cartesian product $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is by definition the graph with vertex set $V = V_1 \times V_2$ and adjacency relation given by

$$(a,x) \sim (b,y) \Longleftrightarrow a \sim b \text{ and } x = y \qquad \text{or} \qquad a = b \text{ and } x \sim y.$$

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For two networks (G_1, r_1) and (G_2, r_2), the Cartesian product is endowed with the weight r defined by

$$r\big(|(a,x)(b,y)|\big) = r_1(|ab|)\delta_x(y) + r_2(|xy|)\delta_a(b), \quad (a,x),(b,y) \in V_1 \times V_2.$$

First we prove the following.

PROPOSITION 4.1. *Let (G_1, r_1) and (G_2, r_2) be connected infinite networks. Suppose that $M^{(p)}_{(G_1,r_1)}$ is bounded for an exponent $p > 1$. Then the Royden p-boundary and the harmonic p-boundary of the Cartesian product $(G, r) = (G_1, r_1) \times (G_2, r_2)$ of them coincide and consist of a single point.*

Proof. Given $u \in L^{1,p}(G, r)$, we define a family $\{u_x \mid x \in V_2\}$ of functions on V_1 by $u_x(a) = u(a, x)$ $(a \in V_1)$. Then we have

$$\begin{aligned}\sum_{x\in V_2} D_{r_1;p}(u_x) &= \frac{1}{2}\sum_{x\in V_2}\sum_{a,b\in V_1;a\sim b}\frac{|u_x(a)-u_x(b)|^p}{r_1(|ab|)^{p-1}}\\ &= \frac{1}{2}\sum_{x\in V_2}\sum_{a\sim b}\frac{|u(a,x)-u(b,x)|^p}{r(|(a,x)(b,x)|)^{p-1}}\\ &\le D_{r;p}(u).\end{aligned}$$

Hence, $u_x \in L^{1,p}(G_1, r_1)$, and we have

$$\sum_{x\in V_2} D_{r_1;p}(u_x) \le D_{r;p}(u). \tag{4.1}$$

This shows in particular that $D_{r_1;p}(u_x)$ tends to zero as $x \in V_2$ goes to infinity. Similarly, for $u \in L^{1,p}(G, r)$ and $a \in V_1$, let $u_a(x) = u(a, x)$ $(x \in V_2)$. Then we have

$$\sum_{a\in V_1} D_{r_2;p}(u_a) \le D_{r;p}(u).$$

Hence, we see that $u_a \in L^{1,p}(G_2, r_2)$ and $D_{r_2;p}(u_a)$ tends to zero as $a \in V_1$ goes to infinity. Now we fix a point $x_0 \in V_2$. For a point x of V_2, we join x_0 to x by a path $c = (x_0, x_1, \ldots, x_n = x)$. Then we get

$$\begin{aligned}|u_{x_0}(a) - u_x(a)| &\le \sum_{i=0}^{n-1}|u(a,x_i)-u(a,x_{i+1})|\\ &\le \Big(\sum_{i=0}^{n-1}\frac{|u(a,x_i)-u(a,x_{i+1})|^p}{r_2(|x_ix_{i+1}|)^{p-1}}\Big)^{1/p}\Big(\sum_{i=0}^{n-1} r_2(|x_ix_{i+1}|)\Big)^{1-1/p}\end{aligned}$$

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$$= \Big(\sum_{i=0}^{n-1} \frac{|u(a,x_i) - u(a,x_{i+1})|^p}{r_2(|x_i x_{i+1}|)^{p-1}}\Big)^{1/p} L_{r_2}(c)^{1-1/p}$$
$$\le D_{r_2;p}(u_a)^{1/p} L_{r_2}(c)^{1-1/p},$$

and hence we have

$$\lim_{a \in V_1 \to \infty} |u_x(a) - u_{x_0}(a)| = 0. \tag{4.2}$$

According to the Royden decomposition of $L^{1,p}(G_1, r_1)$, we write $u_x = h_x + g_x$, where $h_x \in HL^{1,p}(G_1, r_1)$ and $g_x \in L_0^{1,p}(G_1, r_1)$. By the assumption and Lemma 2.3, $g_x(a)$ tends to zero as $a \in V_1$ goes to infinity, and hence by (4.2), we see that $h_{x_0}(a) - h_x(a)$ tends to zero as $a \in V_1$ goes to infinity. For this, we can deduce from Lemma 2.3 that $h_{x_0} = h_x$ for all $x \in V_2$. Let $h = h_{x_0}$. Then $D_{r_1;p}(h) \le D_{r_1;p}(u_x)$ for any $x \in V_2$, so that by (4.1), we have $\sum_{x \in V_2} D_{r_1;p}(h) \le D_{r;p}(u) < +\infty$. In this way, we see that $D_{r_1;p}(h) = 0$; that is, h must be constant, say, $h = c$. Then it follows that

$$|u_x(a) - c| = |g_x(a)| \le M^{(p)}_{(G_1,r_1)}(a) D_{r_1;p}(g_x)$$
$$= M^{(p)}_{(G_1,r_1)}(a) D_{r_1;p}(u_x)$$
$$\le \sup_{b \in V_1} M^{(p)}_{(G_1,r_1)}(b) D_{r_1;p}(u_x),$$

which shows that $\sup_{a \in V_1} |u_x(a) - c|$ converges to zero as $x \in V_2$ goes to infinity. Moreover, in view of Lemma 2.3, as $a \in G_1$ goes to infinity, $|u_x(a) - c|$ also tends to zero for any x fixed. Thus, we can deduce that for any $u \in L^{1,p}(G, r)$, $u(a, x)$ converges to a constant as $(a, x) \in V_1 \times V_2$ tends to infinity. This proves the assertion of the proposition. $\square$

We now give another sufficient condition under which the harmonic p-boundary of $(G, r) = (V, E, r)$ is empty or consists of a single point. Let $\{V_n\}$ be an increasing family of finite subsets of V whose union coincides with V. We suppose that we can take a sequence of connected subsets B_n of V in such a way that $\partial {V_n}^c (= \partial(V \setminus V_n)) \subset B_n$, and furthermore, B_n tends to infinity as $n \to \infty$; that is, for any m, B_n does not intersect with V_m if n is sufficiently large. Let $G_n = (B_n, E_n)$ be the connected subgraph of G generated by B_n, and denote by r_n the restriction of r to the set of edges E_n of G_n.

PROPOSITION 4.2. *Let* (G, r) *and* (G_n, r_n) *be as above. Suppose that for an exponent* $p > 1$,

$$\liminf_{n\to\infty} \max_{(x,y)\in\partial V_n{}^c\times\partial V_n{}^c} R^{(p)}_{(G_n,r_n)}(x,y) < +\infty.$$

Then the harmonic p*-boundary of* (G, r) *is empty or consists of a single point; that is,* $HL^{1,p}(G, r) = \boldsymbol{R}$.

Proof. We take a subsequence $\{G_{n_k}\}$ in such a way that

$$\max_{\partial V_{n_k}{}^c\times\partial V_{n_k}{}^c} R^{(p)}_{(G_{n_k},r_{n_k})} \le c \quad \text{for some } c > 0$$

and all k. Then for $u \in L^{1,p}(G, r)$, we have $\max_{(x,y)\in\partial V_{n_k}{}^c\times\partial V_{n_k}{}^c} |u(x) - u(y)|^p \le cD_{r_{n_k};p}(u_{|B_{n_k}})$ for all k. Since the right-hand side of the last inequality tends to zero as $k \to \infty$, we see that $\max\{|u(x) - u(y)| \mid (x, y) \in \partial V_{n_k}{}^c \times \partial V_{n_k}{}^c\}$ goes to zero as $k \to \infty$. On the other hand, in the case where u is p-harmonic, the maximum principle stated in Lemma 2.1 yields that $\max_{V_{n_k}} u - \min_{V_{n_k}} u \le \max_{\partial V_{n_k}{}^c} u - \min_{\partial V_{n_k}{}^c} u$. Thus, we can deduce that u must be constant. ∎

In the proposition above, if $r = 1$ and $\{G_n\}$ is a family of expanders, that is, a family of finite graphs with a uniform upper bound on the degrees such that the cardinality of G_n goes to infinity as $n \to \infty$ and the Cheeger constant $h(G_n)$ is uniformly bounded from below by a positive constant, then the assumption holds for all $p > 1$ (and so does the assertion). In fact, in this case, it can be shown that $R^{(p)}_{G_n} \le Ch(G_n)^{-p}$ for some constant C depending only on p and an upper bound of the degrees (see, e.g., [2]).

We refer the reader to [14], [33], [34], [37], and the references therein for graphs of Liouville D_p-property.

§5. The Kuramochi compactification

Let $G = (V, E)$ be a connected infinite graph of bounded degrees, and let r be a weight on E. In this section, we are concerned with compactifications of (G, r) associated with subsets of $BL^{1,p}(G, r)$ which have countable dense subsets.

We are given a subset Φ in $BL^{1,p}(G, r)$ which separates points of V. Then the identity map I of V extends to a continuous map $\bar{I}$ of $\mathcal{R}_p(G, r)$ onto the compactification $\mathcal{C}_\Phi$ associated to Φ (as explained in Section 2). Suppose that Φ includes a countable subfamily Φ_0 which is dense in Φ with respect

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to the uniform norm. Let $\Phi_0 = \{u_n\}$. Then $\mathcal{C}_{\Phi_0} = \mathcal{C}_\Phi$, and $\mathcal{C}_\Phi$ is metrizable. In fact, we take a sequence of positive numbers $\mu(n)$ in such a way that

$$\sum_{n=1}^{\infty} \mu(n)\Big(D_{r;p}(u_n) + \sup_{x\in V}|u_n|^p\Big) < +\infty,$$

and we define a distance d_Φ on V by

$$d_\Phi(z,w) = \Big\{\sum_{n=1}^{\infty} \mu(n)|u_n(z) - u_n(w)|^p\Big\}^{1/p}.$$

Then d_Φ can be extended to a metric on the compactification $\mathcal{C}_\Phi$, and the completion $(\overline{V}^{d_\Phi}, d_\Phi)$ of (V, d_Φ) is homeomorphic to $\mathcal{C}_\Phi$. In fact, let $d_\Phi^{(m)}(z,w)^p = \sum_{n=1}^{m} \mu(n)|\bar{u}_n(z) - \bar{u}_n(w)|^p$, where $\bar{u}_n$ stands for the continuous extension of u_n to $\mathcal{C}_\Phi$. Since

$$\sup_{z,w\in\mathcal{C}_\Phi} |d_\Phi^{(m)}(z,w)^p - d_\Phi^{(m+k)}(z,w)^p| \le 2^p \sum_{n=m+1}^{m+k} \mu(n) \sup_{x\in V}|u_n(x)|^p,$$

$d_\Phi^{(m)}$ uniformly converges to d_Φ, so that d_Φ is continuous on $\mathcal{C}_\Phi \times \mathcal{C}_\Phi$. It is easy to see that $d_\Phi(z,w) = 0$ implies that $z = w$, because Φ_0 is dense in Φ separating points of V.

Now we set $m_\Phi(|xy|) = d_\Phi(x,y)$, $|xy| \in E$. Then m_Φ is an L^p-pseudometric on E and $\|m_\Phi\|_{r;p}^p = \sum_{n=1}^{\infty} \mu(n) D_{r;p}(u_n)$. This shows that the inclusion map $I : V \to (\mathcal{C}_\Phi, d_\Phi)$ is a p-Dirichlet finite map. Let $\Delta_p^\Phi(G,r) = \bar{I}(\Delta_p(G,r))$. Then it follows from Theorem 3.1 that

$$\Delta_p^\Phi(G,r) = \overline{\big\{\gamma(\infty) \bigm| \gamma \in \boldsymbol{P}(G) \setminus (\boldsymbol{P}_\infty \cup \boldsymbol{P}'_\infty)\big\}}$$

for some $\boldsymbol{P}_\infty \subset \boldsymbol{P}(G)$ with $EL_{r;p}(\boldsymbol{P}_\infty) = \infty$ and any $\boldsymbol{P}'_\infty \subset \boldsymbol{P}(G)$ with $EL_{r;p}(\boldsymbol{P}'_\infty) = \infty$.

In what follows, we specify a subset of $BL^{1,p}(G,r)$. We denote by $\Phi_{\mathcal{K}}$ the set of all bounded functions f on V with finite p-Dirichlet sum such that for some finite subset K of V, $D_{r;p}(f) \le D_{r;p}(u)$ if $u \in L^{1,p}(G,r)$ coincides with f on K. The compactification relative to $\Phi_{\mathcal{K}}$ is called the *Kuramochi p-compactification* of (G,r) and is denoted by $\mathcal{K}_p(G,r)$.

We want to take a countable subset of $\Phi_{\mathcal{K}}$ which is dense in $\Phi_{\mathcal{K}}$ with respect to the uniform norm. For this purpose, we need two lemmas.

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Let K be a (nonempty) finite subset of V. For any function f on K, we set

$$\mathcal{F}_f = \{u \in L^{1,p}(G,r) \mid u = f \text{ on } K\}.$$

$\mathcal{F}_f$ is not empty, because K is finite.

LEMMA 5.1. *There exists a unique $D_{r;p}$-minimizer u_f in $\mathcal{F}_f$.*

Proof. To begin with, we recall Clarkson's inequalities (see [1, p. 36]): in the case where $p \geq 2$, it holds that

(5.1)
$$D_{r;p}(u+v;K) + D_{r;p}(u-v;K) \leq 2\big(D_{r;p}(u;K)^{p^*/p} + D_{r;p}(v;K)^{p^*/p}\big)^{p/p^*}$$

for functions u, v on a subset K, where $1/p^* + 1/p = 1$; in the case where $1 < p \leq 2$, it holds that

(5.2)
$$D_{r;p}(u+v;K)^{p^*/p} + D_{r;p}(u-v;K)^{p^*/p} \leq 2\big(D_{r;p}(u;K) + D_{r;p}(v;K)\big)^{p^*/p}$$

for functions u, v on a subset K.

Let $\{V_n\}$ be an increasing family of finite subsets V_n whose union coincides with V, and let $\{G_n = (V_n, E_n)\}$ be a sequence of finite subgraphs generated by V_n. We consider a sequence of subnetworks (G_n, r_n), where r_n is the restriction of the weight r to E_n. We suppose that $K \subset V_1$. Let u_n be a unique $D_{r_n;p}$-minimizer among functions v on V_n with $v = f$ on K. We would like to show that u_n converges to a unique $D_{r;p}$-minimizer u in $\mathcal{F}_f$ as $n \to \infty$. Take m and n with $m > n$. Let v be a function on V_m with $v = f$ in K, and set $v_n = v_{|V_n}$. Consider the case where $p \geq 2$. Then we have

$$\begin{aligned}
D_{r_n;p}(u_n) &\leq D_{r_n;p}\Big(\frac{u_n + v_n}{2}\Big) \\
&\leq D_{r_n;p}\Big(\frac{u_n + v_n}{2}\Big) + D_{r_n;p}\Big(\frac{u_n - v_n}{2}\Big) \\
&\leq 2\big(D_{r_n;p}(u_n)^{p^*/p} + D_{r_n;p}(v_n)^{p^*/p}\big)^{p/p^*} \\
&= 2^{1-p}\big(D_{r_n;p}(u_n)^{p^*/p} + D_{r_n;p}(v_n)^{p^*/p}\big)^{p/p^*} \\
&\leq 2^{1-p+p/p^*} D_{r_n;p}(v_n) = D_{r_n;p}(v_n) \\
&\leq D_{r_m;p}(v),
\end{aligned}$$

where we have used (5.1) in the third inequality. In particular, taking $v = u_m$, we get

$$D_{r_n;p}(u_n) \leq D_{r_m;p}(u_m) \quad \text{if } n < m.$$

Thus, $\{D_{r_n;p}(u_n)\}$ is nondecreasing. Let $\alpha = \lim_{n\to\infty} D_{r_n;p}(u_n)$. Then for any $g \in \mathcal{F}_f$, $\alpha \le D_{r;p}(g)$, because $D_{r_n;p}(u_n) \le D_{r_n;p}(g_{|V_n})$ for all n. Moreover, it follows from the above inequalities (with $v = u_m$) that $\lim_{n\to\infty;m>n} D_{r_n;p}((u_n + u_{m|V_n})/2) = \lim_{n\to\infty;m>n} D_{r_n;p}(u_{m|V_n}) = \alpha$ and $\lim_{n\to\infty;m>n} D_{r_n;p}((u_n - u_{m|V_n})/2) = 0$. This shows that for any $x \in V$, $\{u_n(x)\}$ is a Cauchy sequence. Let $u(x) = \lim_{n\to\infty} u_n(x)$, $x \in V$. Then we have $D_{r_n;p}(u_{|V_n}) = \lim_{m\to\infty} D_{r_n;p}(u_{m|V_n}) \le \alpha$, and hence $D_{r;p}(u) = \lim_{n\to\infty} D_{r_n;p}(u_{|V_n}) \le \alpha$. Thus, u turns out to be a $D_{r;p}$-minimizer in $\mathcal{F}_f$. Let v be another minimizer in $\mathcal{F}_f$. Then applying the above inequalities again, we see that u_n converges pointwise to v as $n \to \infty$, and thus $v = u$. In the case where $1 < p \le 2$, the same argument as above together with (5.2) yields the same conclusion. This completes the proof of Lemma 5.1. ∎

Lemma 5.2. *Let K be a nonempty finite subset in V.*

(1) *For functions f, g on K, $u_f \le u_g$ on V if $f \le g$ on K.*
(2) *For functions f, g on K, $\sup_V |u_f - u_g| \le \max_K |f - g|$.*

Proof. Let u_n (resp., v_n) be the $D_{r_n;p}$-minimizer among functions w on V_n with $w = f$ (resp., $w = g$) on K as in the proof of Lemma 5.1. If $f \le g$ on K, then by the comparison principle (iv) mentioned before Lemma 2.1, we have $u_n \le v_n$ on V_n and hence $u_f \le u_g$ on V. By Lemma 2.1, we have $\sup_{V_n} |u_n - v_n| \le \max_K |f - g|$, and hence $\sup_V |u_f - u_g| \le \max_K |f - g|$. This completes the proof of Lemma 5.2. ∎

Let $\{V_n\}$ be an increasing family of finite subsets of V whose union coincides with V. Let $\Phi_n = \{u_f \in L^{1,p}(G,r) \mid f : V_n \to \mathbf{Q}\}$ and $\Phi_0 = \bigcup_n \Phi_n$. Then in view of Lemma 5.2(2), we see that Φ_0 is dense in $\Phi_{\mathcal{K}}$ with respect to the uniform norm.

Proposition 5.3. *Let (G,r) be a connected infinite network. The Kuramochi compactification $\mathcal{K}_p(G,r)$ is metrizable and admits a compatible metric such that the inclusion map I of V to $\mathcal{K}_p(G,r)$ is a p-Dirichlet finite map. Moreover, letting $\Delta_p^{\mathcal{K}}(G,r)$ be the image of $\Delta_p(G,r)$ by the continuous extension $\bar{I}$ of $\mathcal{R}_p(G,r)$ onto $\mathcal{K}_p(G,r)$, one has*

$$\Delta_p^{\mathcal{K}}(G,r) = \overline{\{\gamma(\infty) \mid \gamma \in \boldsymbol{P}(G) \setminus (\boldsymbol{P}_\infty \cup \boldsymbol{P}'_\infty)\}},$$

where $\boldsymbol{P}_\infty$ is some subset in $\boldsymbol{P}(G)$ with $EL_{r;p}(\boldsymbol{P}_\infty) = +\infty$ and $\boldsymbol{P}'_\infty$ is any of such subsets.

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The Kuramochi p-compactification of an infinite weighted tree topologically coincides with the end compactification. We have another example, though it is simple, of the Kuramochi p-compactification of an infinite network, which is different from the end compactification (see [25], [19] for details).

Our definition of the Kuramochi p-boundary of an infinite network is adapted to that of a Riemannian manifold given in [36]. On the other hand, in [29], the Kuramochi p-boundary of an infinite network is introduced in a different manner. A family of certain kernel functions that is smaller than ours is used to define the compactification. It is not clear whether both boundaries coincide except for the case $p = 2$. In fact, it is verified that this is true when $p = 2$ (see [25]). In addition, we have an analogue to Proposition 5.3 with $p = 2$ expressed in terms of the random walk of an infinite network (see [26] for details).

§6. Gromov hyperbolic graphs

The purpose of this section is to study projections of the Royden p-boundaries of Gromov hyperbolic graphs of bounded degrees onto their Gromov boundaries for sufficiently large p.

To begin with, we recall some basic facts on the spectral gap of a connected, infinite graph $G = (V, E)$ of bounded degrees with weight $r = 1$ on E. Let

$$\ell^p(V) := \left\{ u : V \to \boldsymbol{R} \,\middle|\, \sum_{x \in V} |u(x)|^p < +\infty \right\}$$

and

$$\|u\|_{\ell^p} := \left(\sum_{x \in V} |u(x)|^p \right)^{1/p}, \quad u \in \ell^p(V).$$

If a positive constant d is an upper bound of the degrees of G, then we have $D_p(u) \le 2^{p-1} d \|u\|_{\ell^p}{}^p$ for all $u \in \ell^p(V)$, so that all $u \in \ell^p(V)$ belong to the Banach space $L_0^{1,p}(G)$ and the inclusion of $\ell^p(V)$ into $L_0^{1,p}(G)$ is bounded. Let $\lambda_p(G)$ be the largest real λ such that

$$\lambda \|u\|_{\ell^p}{}^p \le D_p(u), \quad \forall u \in \ell^p(V).$$

Then it follows from the bounded inverse theorem that $\lambda_p(G) > 0$ if and only if $\ell^p(V) = L_0^{1,p}(G)$; moreover, in this case, we have

$$\sup_{x \in V} M_G^{(p)}(x) \le \frac{1}{\lambda_p(G)} < +\infty,$$

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since

$$|u(x)|^p \le \|u\|_{\ell^p}{}^p \le \lambda_p(G)^{-1} D_p(u), \quad u \in L_0^{1,p}(G), x \in V.$$

We remark that if $\lambda_p(G)$ is positive for some $p > 1$, then this is the case for all $p > 1$ (see [34]).

Let $G = (V, E)$ be a connected, infinite graph of bounded degrees. Regarding each edge $e \in E$ as a segment $[0, 1]$, we obtain a Riemannian polyhedron of dimension 1. We call it the *metric graph* of G and denote it by $|G|$. In the metric graph, we write $d_{|G|}$ for the geodesic distance. Then $(|G|, d_{|G|})$ is a connected, locally compact, geodesic space. It is assumed that (V, d_G) is isometrically embedded in $(|G|, d_{|G|})$. The metric graph is considered hyperbolic in the sense of Gromov if there is a nonnegative number δ such that every geodesic triangle is δ-slim, that is, if each of its sides is contained in the δ-neighborhood of the union of the other two sides. We say that G is *Gromov hyperbolic* if the metric graph is hyperbolic in the sense of Gromov.

In what follows, we focus on a Gromov hyperbolic graph $G = (V, E)$ with bounded degrees.

A ray in $|G|$ is by definition an isometric map from the interval $[0, +\infty)$ into $|G|$, and two rays are equivalent if the Hausdorff distance of their images in $|G|$ are finite. The Gromov boundary $\partial_\infty G$ is defined to be the set of equivalence classes of rays in $|G|$. We say that a ray γ *ends at* $a \in \partial_\infty G$ if γ represents the point a. There is a natural topology on $|G| \cup \partial_\infty G$ making it a compact metrizable space, and there is a natural family of visual metrics on $\partial_\infty G$ (see, e.g., [11] for details). We write $\overline{V}^H$ for $V \cup \partial_\infty G$ in the compactification $\overline{G}^H$ of $|G|$.

Now we introduce another compactification of V. Let

$$\begin{aligned} S(n) &= \{x \in V \mid d_G(x, o) = n\}, \\ E(n) &= \{|xy| \in E \mid x, y \in S(n) \text{ or } x \in S(n), y \in S(n-1)\}, \end{aligned}$$

where $n = 1, 2, \ldots$ and o is a fixed vertex of G. For a positive constant ε, we define a function r_ε on E by letting $r_\varepsilon(e) = \exp(-\varepsilon n)$ if $e \in E(n)$. We denote by d_ε the geodesic distance of the network (G, r_ε). Let $\overline{V}^{d_\varepsilon}$ and $\partial_\varepsilon V$ be the completion of V with respect to d_ε and the boundary $\overline{V}^{d_\varepsilon} \setminus V$, respectively. It is proved in [7, Proposition 4.13] that, for any $\varepsilon > 0$, $\overline{V}^{d_\varepsilon}$ is compact, and there is $\varepsilon_0(G) \in (0, +\infty]$ such that, for $\varepsilon < \varepsilon_0(G)$, $\overline{V}^{d_\varepsilon}$ is homeomorphic to $\overline{V}^H$ and the distance d_ε induces a visual metric on $\partial_\infty G$. For every equivalence class of rays, there is a unique point $a \in \partial_\varepsilon V$ such that $\gamma(t)$ converges to a as $t \to \infty$ for each ray γ in the equivalence class. We note that the visual metric is doubling. For this assertion, we refer to [11, Theorem 8.3.9] or [8, Theorem 9.2].

Here we recall an observation from [19]. Let I be the inclusion map of V into the compact metric space $(\overline{V}^{d_\varepsilon}, d_\varepsilon)$. Then we have $D_p(I) \leq \sum_{n=1}^{\infty} \#E(n) \times \exp(-\varepsilon pn)(\leq +\infty)$, and hence letting

$$e(G) = \limsup_{n\to\infty} \frac{1}{n} \log\bigl(\#E(n)\bigr)(\in [0,\infty)),$$

we see that $D_p(I)$ is finite; that is, I is a p-Dirichlet finite map if $p\varepsilon > e(G)$. Let $\mathrm{Lip}(V, d_\varepsilon)$ be the set of Lipschitz functions on V relative to the distance d_ε, that is, identified with the set of Lipschitz functions on $\overline{V}^{d_\varepsilon}$. When $p\varepsilon > e(G)$, $D_p(I)$ is finite, so that $\mathrm{Lip}(V, d_\varepsilon)$ is included in $BL^{1,p}(G)$. Thus in this case, $\overline{V}^{d_\varepsilon}$ can be identified with the compactification $\mathcal{C}_{\mathrm{Lip}(V,d_\varepsilon)}$ associated with $\mathrm{Lip}(V, d_\varepsilon)$ explained in Section 2, and the inclusion map I can be extended to a continuous map $\bar{I}$ from $\mathcal{R}_p(G)$ onto $(\overline{V}^{d_\varepsilon}, d_\varepsilon)$.

Now we state some important results on Gromov hyperbolic graphs of bounded degrees. We refer to [11, Theorem 5.2.17] for (i) below, and [11, Theorems 6.4.1, 7.1.2] for (ii) below.

(i) If $\Psi : G' \to G$ is a quasi-isometric embedding of Gromov hyperbolic graphs of bounded degrees G' to G, then there exists a power quasi-symmetric (continuous) map $\partial_\infty\Psi$ from $\partial_\infty G'$ to $\partial_\infty G$.

(ii) For any closed subspace A of the Gromov boundary $\partial_\infty G$ of a Gromov hyperbolic graph G of bounded degrees, there exists a visual Gromov hyperbolic graph G_A whose Gromov boundary $\partial_\infty G_A$ coincides with A, and a roughly isometric map $\Psi : G_A \to G$ such that $\partial_\infty\Psi$ is nothing but the inclusion map from A into $\partial_\infty G$. (Here a map Ψ of a metric space (X, d_X) to another one (Y, d_Y) is called a *roughly isometric* map if it satisfies $|d_Y(\Psi(x), \Psi(y)) - d_X(x, y)| \leq C$ for some positive constant C and all $x, y \in X$.) We remark that $e(G_A) \leq e(G)$ and $\varepsilon_0(G_A) \geq \varepsilon_0(G)$.

We are now in a position to prove the following.

THEOREM 6.1. *Let $G = (V, E)$ be a Gromov hyperbolic graph with bounded degrees.*

(1) *If $p > \max\{e(G)/\varepsilon_o(G), 1\}$ and $\sup_{x\in V} M_G^{(p)}(x) < +\infty$, then $\partial_\infty G = \bar{I}(\Delta_p(G))$, where $\bar{I} : \mathcal{R}_p(G) \to \bar{G}$ is the continuous extension of the identity of V which maps $\partial\mathcal{R}_p(G)$ onto $\partial_\infty G$.*

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(2) *Let $G' = (V', E')$ be a Gromov hyperbolic graph with bounded degrees, and suppose that there exists a quasi-isometric embedding Ψ from G' to G. Then $\partial_\infty \Psi(\partial_\infty G')$ is included in $\bar{I}(\Delta_p(G))$ if $p > \max\{1, e(G)/\varepsilon_o(G), e(G')/\varepsilon_o(G')\}$ and $\sup_{x\in V'} M_{G'}^{(p)}(x) < +\infty$.*

(3) *Let A be a closed subset of $\partial_\infty G$, and suppose that A is uniformly perfect (i.e., there exists a constant $\mu \in (0,1)$ so that for every $x \in A$ and any $\varepsilon > 0$, we have $B_A(x,\varepsilon) \setminus B_A(x,\mu\varepsilon) \neq \emptyset$ unless A is included in the metric ball $B_A(x,\mu\varepsilon)$), or A is connected and of positive diameter. Then A is included in $\bar{I}(\Delta_p(G))$ for any $p > \max\{1, e(G)/\varepsilon_0(G)\}$.*

Proof. Since the condition of $\sup_{x\in V} M_G^{(p)}(x)$ being finite is equivalent to the condition that $\partial\mathcal{R}_p(G) = \Delta_p(G)$ by Lemma 2.3, the first assertion can be deduced by choosing $\varepsilon > 0$ in such a way that $p\varepsilon > e(G)$ and $\varepsilon < \varepsilon_0(G)$.

We prove the second assertion. We note that Ψ extends to a continuous map $\bar{\Psi}$ from $\mathcal{R}_p(G')$ to $\mathcal{R}_p(G)$ which maps $\partial\mathcal{R}_p(G')$ (resp., $\Delta_p(G')$) to $\partial\mathcal{R}_p(G)$ (resp., $\Delta_p(G)$). This will be verified later in Lemma 7.3 for the more general case of quasi-monomorphisms of connected, infinite graphs of bounded degrees. Let $\overline{I'}$ be the continuous extension of the identity map I' of V' to $\mathcal{R}_p(G')$ which maps $\partial\mathcal{R}_p(G')$ onto $\partial_\infty G'$. Then we see that $\partial_\infty\Psi \circ \overline{I'} = \bar{I} \circ \bar{\Psi}$ on $\partial\mathcal{R}_p(G')$, from which it follows that

$$\partial_\infty\Psi\big(\overline{I'}(\Delta_p(G'))\big) = \bar{I}\big(\bar{\Psi}(\Delta_p(G'))\big) \subset \bar{I}\big(\Delta_p(G)\big).$$

Since $\partial\mathcal{R}_p(G') = \Delta_p(G')$ by the assumption that $\sup_{x\in V'} M_{G'}^{(p)}(x) < +\infty$, we see that $\overline{I'}(\Delta_p(G')) = \partial_\infty G'$. This proves the second assertion of the theorem.

To verify the last one, let $G_A = (V_A, E_A)$ be a visual Gromov hyperbolic graph as described in (iii) above. Then in view of [9, théorème 3.1], we can deduce that if A is uniformly perfect, then $\ell^p(V_A) = L_0^{1,p}(G_A)$ for any $p > 1$, and hence $\lambda_p(G_A)$ is positive for any $p > 1$. It is also proved by Cao [12] that if A is connected and of positive diameter, then $\lambda_p(G_A)$ is positive for any $p > 1$. In any case, $\sup_{x\in V_A} M_{G_A}^{(p)}(x)$ is finite, so that $A = \partial_\infty G_A = \overline{I_A}(\Delta_p(G_A)) \subset \bar{I}(\Delta_p(G))$, where $\overline{I_A}$ is the continuous extension of the identity of V_A which maps $\partial\mathcal{R}_p(G_A)$ onto $\partial_\infty G_A$. ∎

Corollary 6.2. *A Gromov hyperbolic graph G of bounded degrees admits a lot of p-harmonic functions with finite Dirichlet sum of order p for p large enough, provided that there is a bi-Lipschitz embedding of the binary tree into G.*

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This is proved in Tessera [37, Theorem 2.8], where a different method is employed for the proof. In fact, it is enough to take p greater than $\max\{e(G)/\varepsilon_0(G), 1\}$.

Benjamini and Schramm [5] show that a connected, infinite graph G of bounded degrees with positive Cheeger constant (or equivalently positive spectral gap) contains a tree T with positive Cheeger constant such that the inclusion map $T \to G$ is a bi-Lipschitz embedding and that there is a bi-Lipschitz embedding of the binary tree into G.

§7. Quasi-monomorphisms

A map $\psi : X \to Y$ of a metric space (X, d_X) to another (Y, d_Y) is called a *quasi-monomorphism* if the following two conditions are satisfied:

(i) there exist constants $a > 0$ and $b \geq 0$ such that $d_Y(\psi(x_1), \psi(x_2)) \leq a d_X(x_1, x_2) + b$ for all $x_1, x_2 \in X$;

(ii) for any $r > 0$, there exists a constant $c > 0$ such that for every $y \in Y$, the inverse image $\psi^{-1}(B)$ of the open ball $B = B_Y(y, r)$ centered at y with radius r can be covered by c open balls of radius r in X.

For a graph in BG, the inclusion map of a subgraph to the graph is a quasi-monomorphism. If we take two copies of a graph in BG and identify them along a subset of the graph, then the canonical projection of the resulting graph onto the given one is a quasi-monomorphism. When we consider a finitely generated discrete group acting isometrically and properly on a proper metric space, by taking a finite symmetric generating set of the group and fixing a point of the metric space, the map sending each element of the group to the corresponding point in the orbit of the fixed point turns out to be a quasi-monomorphism from the Cayley graph into the metric space. Following [15], we say that a graph $G = (V, E)$ can be drawn in a *civilized manner* in Euclidean space $\boldsymbol{R}^n$ if there exists an embedding ψ of the set of vertices V into $\boldsymbol{R}^n$ such that for some $r < +\infty$, $s > 0$, $|\psi(x) - \psi(y)| \leq r$ for any edge $|xy| \in E$, and $|\psi(v) - \psi(w)| \geq s$ for any distinct two vertices v and w. In this definition, $\boldsymbol{R}^n$ can be replaced by any metric space. For a graph in BG, such an embedding of a graph into a metric space (X, d_X) is obviously a quasi-monomorphism of the graph to X.

In this section, we consider connected, infinite graphs of bounded degrees with weight $r = 1$ on E and study some properties of quasi-monomorphisms between them.

To begin with, we prove the following.

LEMMA 7.1. *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be connected infinite graphs with bounded degrees. Suppose that there exists a quasi-monomorphism ψ from the graph G_1 to the graph G_2. Then there exists a constant $C>0$ such that*

$$D_p(f\circ\psi)\le CD_p(f)$$

for any p-Dirichlet finite map f of G_2 to a metric space (X,d_X).

Proof. From the definition of a quasi-monomorphism, we see that there exists a constant κ such that $d_{G_2}(\psi(x),\psi(y))\le\kappa$ for any edge $|xy|\in E_1$, and the cardinality of $\psi^{-1}(x')$ is bounded by κ for all $x'\in V_2$. For any edge $|xy|\in E_1$, we can take a path $\{\gamma^{x,y}(n)\}_{n=0}^{N_{x,y}}$ connecting $\psi(x)$ and $\psi(y)$ with length $N_{x,y}\le\kappa$. Let f be a p-Dirichlet finite map of G_2 to a metric space (X,d_X). Then we have

$$\begin{aligned}
D_p(f\circ\psi)&=\frac{1}{2}\sum_{x\in V_1}\sum_{y\sim x}d_X\big(f(\psi(y)),f(\psi(x))\big)^p\\
&\le\kappa^{p-1}\frac{1}{2}\sum_{x\in V_1}\sum_{y\sim x}\sum_{i=1}^{N_{x,y}}d_X\big(f(\gamma^{x,y}(i)),f(\gamma^{x,y}(i-1))\big)^p\\
&\le\kappa^{p-1}\frac{1}{2}\sum_{x\in V_1}\sum_{y\sim x}\sum_{i=1}^{N_{x,y}}\sum_{z'\sim\gamma^{x,y}(i)}d_X\big(f(\gamma^{x,y}(i)),f(z')\big)^p\\
&\le M_1\kappa^{p-1}\frac{1}{2}\sum_{x\in V_1}\sum_{y'\in B_2(\psi(x),\kappa)}\sum_{z'\sim y'}d_X\big(f(z'),f(y')\big)^p\\
&\le M_1\kappa^{p}\frac{1}{2}\sum_{x'\in V_2}\sum_{y'\in B_2(x',\kappa)}\sum_{z'\sim y'}d_X\big(f(y'),f(z')\big)^p\\
&\le M_1M_2^{\kappa}\kappa^pD_p(f),
\end{aligned}$$

where M_1 (resp., M_2) is an upper bound of the degrees of G_1 (resp., G_2), and $B_2(x,r)$ is the metric ball centered at x with radius r in G_2. □

We remark that in Lemma 7.1, for any $g\in L_0^{1,p}(G_2)$, the composition $g\circ\psi$ belongs also to $L_0^{1,p}(G_1)$.

For any two disjoint subsets A and B of V, we define the p-capacity of the pair (A,B) by

$$\mathrm{Cap}_{(G)}^{(p)}(A,B)=\inf\big\{D_p(u)\;\big|\;u\in L^{1,p}(G),u=1\text{ in }A\text{ and }u=0\text{ in }B\big\}.$$

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In the case when $A = \{x\}$ and $B = \{y\}$, the reciprocal of $\mathrm{Cap}_G^{(p)}(\{x\}, \{y\})$ is equal to $R_G^{(p)}(x, y)$.

Now we take functions $f \in L^{1,p}(G_2)$ in Lemma 7.1 and apply the estimate to obtain the following.

COROLLARY 7.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected infinite graphs with bounded degrees. If there exists a quasi-monomorphism ψ form G_1 to G_2, then there exists a constant $C > 0$ such that*

(1) *$\mathrm{Cap}_{G_1}^{(p)}(\psi^{-1}(A), \psi^{-1}(B)) \le C\,\mathrm{Cap}_{G_2}^{(p)}(A, B)$ for all subsets A, B of V_2 with $A \cap B = \emptyset$, where one understands $\mathrm{Cap}_{G_1}^{(p)}(\psi^{-1}(A), \psi^{-1}(B)) = 0$ if $\psi^{-1}(A)$ or $\psi^{-1}(B)$ is empty;*

(2) *$R_{G_2}^{(p)}(\psi(x), \psi(y)) \le C R_{G_1}^{(p)}(x, y)(< +\infty)$ for all $x, y \in V_1$;*

(3) *$M_{G_2}^{(p)}(\psi(x)) \le C M_{G_1}^{(p)}(x)(\le +\infty)$ for all $x \in V_1$.*

In particular, if G_2 is p-parabolic, then so is G_1.

It should be remarked that when G_1 is a subgraph of G_2, Lemma 7.1 and Corollary 7.2 hold with constant $C = 1$. These illustrate Rayleigh's monotonicity law.

LEMMA 7.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected infinite graphs of bounded degrees. If there exists a quasi-monomorphism $\psi : G_1 \to G_2$, then ψ extends to a continuous map $\bar{\psi}$ of $\mathcal{R}_p(G_1)$ to $\mathcal{R}_p(G_2)$ such that $\bar{\psi}(\Delta_p(G_1)) \subset \Delta_p(G_2)$. Moreover, for a p-Dirichlet finite map f of G_2 to a proper metric space (X, d_X), $f \circ \psi$ is also a p-Dirichlet finite map of G_1 to X and satisfies $\overline{f \circ \psi} = \bar{f} \circ \bar{\psi}$.*

Proof. For a sequence $\{x_i\}$ in V_1 converging to a point $z \in \partial\mathcal{R}_p(G_1)$ and a function $u \in L^{1,p}(G_2)$, if a subsequence $\{\psi(x_{i_k})\}$ of $\{\psi(x_i)\}$ tends to a point $w \in \partial\mathcal{R}_p(G_2)$, then we have $\bar{u}(w) = \lim_{k\to\infty} u(\psi(x_{i_k})) = \overline{u \circ \psi}(z) = \lim_{i\to\infty} u \circ \psi(x_i)$. This shows that the sequence $\{\psi(x_i)\}$ converges to w. Letting $\bar{\psi}(z) = w$, we get a continuous map $\bar{\psi} : \mathcal{R}_p(G_1) \to \mathcal{R}_p(G_2)$ such that $\overline{u \circ \psi} = \bar{u} \circ \bar{\psi}$ for all $u \in L^{1,p}(G_2)$. It is easy to see that $\bar{\psi}$ maps $\Delta_p(G_1)$ into $\Delta_p(G_2)$, since for any $g \in L_0^{1,p}(G_2)$, the composition $g \circ \psi$ belongs to $L_0^{1,p}(G_1)$. The second assertion also follows. $\square$

COROLLARY 7.4. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected infinite graphs of bounded degrees. Suppose that there exists a quasi-monomorphism ψ from G_1 to G_2, and suppose that $\sup_{x \in V_1} M_{G_1}^{(p)}(x) < +\infty$ and $\psi(G_1)$ is κ-dense in G_2 for some $\kappa > 0$; that is, for any $v \in V_2$ there*

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exists $w \in V_1$ such that $d_{G_2}(v, \psi(w)) < \kappa$. Then $\sup_{y \in V_2} M^{(p)}_{G_2}(y) < +\infty$ *and* $\bar{\psi}(\Delta_p(G_1)) = \Delta_p(G_2)$.

Proof. For any $y \in V_2$, we take $x \in V_1$ in such a way that $d_{G_2}(\psi(x), y) < \kappa$. Then by (2.1), (2.2), and Corollary 7.2(3), we get

$$M^{(p)}_{G_2}(y) \leq 2^{p-1} M^{(p)}_{G_2}\big(\psi(x)\big) + 2^{p-1} R^{(p)}_{G}\big(\psi(x), y\big) \leq C 2^{p-1} M^{(p)}_{G_1}(x) + 2^{p-1}\kappa^{p-1},$$

and thus $\sup_{y \in V_2} M^{(p)}_{G_2}(y) < +\infty$. In particular, we have by Lemma 2.3 $\partial\mathcal{R}_p(G_1) = \Delta_p(G_1)$ and $\partial\mathcal{R}_p(G_2) = \Delta_p(G_2)$.

Now to complete the proof, let ξ be a point of $\Delta_p(G_2)$, and let $\{y_n\}$ be a sequence in V_2 converging to ξ as $n \to \infty$. Take $x_n \in V_1$ in such a way that $d_{G_2}(\psi(x_n), y_n) < \kappa$. Then $\psi(x_n)$ also converges to ξ as $n \to \infty$. In fact, for all $u \in BL^{1,p}(G_2)$, we have

$$\big|u\big(\psi(x_n)\big) - u(y_n)\big|^p \leq (2\kappa)^{p-1} \sum_{y \in B_{G_2}(y_n, 2\kappa)} \sum_{z \sim y} |u(z) - u(y)|^p,$$

and the right-hand side tends to zero as $n \to \infty$. Passing to a subsequence, we may assume that x_n converges to a point $\eta \in \Delta_p(G_1)$ as $n \to \infty$. Then we have $\bar{\psi}(\eta) = \lim_{n\to\infty} \bar{\psi}(x_n) = \xi$. In this way, we obtain $\bar{\psi}(\Delta_p(G_1)) = \Delta_p(G_2)$. ∎

When ψ in Corollary 7.2 or Lemma 7.3 is a quasi-isometry, these results are restated as follows.

THEOREM 7.5 ([19, Theorem 6]). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected infinite graphs with bounded degrees. Suppose that they are quasi-isometric, and let $\psi : (V_1, d_{G_1}) \longrightarrow (V_2, d_{G_2})$ be a quasi-isometry. Then there exist constants $C \geq 1$ and $C' \geq 0$ such that*

$$\frac{1}{C} R^{(p)}_{G_1}(x, y) - C' \leq R^{(p)}_{G_2}\big(\psi(x), \psi(y)\big) \leq C R^{(p)}_{G_1}(x, y), \quad x, y \in V_1.$$
$$\frac{1}{C} M^{(p)}_{G_1}(x) - C' \leq M^{(p)}_{G_2}\big(\psi(x)\big) \leq C M^{(p)}_{G_1}(x), \quad x \in V_1.$$

In particular, G_1 is p-parabolic if and only if G_2 is p-parabolic.

THEOREM 7.6 ([18, Theorem 1]). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected, infinite graphs with bounded degrees. Suppose that G_1 and G_2 are quasi-isometric, and let $\psi : (V_1, d_{G_1}) \longrightarrow (V_2, d_{G_2})$ be a quasi-isometry. Then ψ extends to a continuous map $\bar{\psi}$ of $\mathcal{R}_p(G_1)$ to $\mathcal{R}_p(G_2)$ whose restriction to*

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$\partial\mathcal{R}_p(G_1)$ induces a homeomorphism between $\partial\mathcal{R}_p(G_1)$ and $\partial\mathcal{R}_p(G_2)$ such that $\bar{\psi}(\Delta_p(G_1)) = \Delta_p(G_2)$. Moreover, assigning to a function h of $BHL^{1,p}(G_2)$ the unique function $\eta(h)$ of $BHL^{1,p}(G_1)$ such that $\overline{\eta(h)} = \bar{h} \circ \bar{\psi}$ on $\Delta_p(G_1)$ is a bijective process, there exists a constant $C > 0$ such that

$$C^{-1} D_p(h) \le D_p\big(\eta(h)\big) \le C D_p(h)$$

for all $h \in BHL^{1,p}(G_2)$. When $p = 2$, η induces a linear isomorphism between $BHL^{1,2}(G_1)$ and $BHL^{1,2}(G_2)$.

Let $G = (V, E)$ be a connected infinite graph of bounded degrees. Notice that for $1 < p < q$, $D_q(f)^{1/q} \le D_p(f)^{1/p}$ for all $f \in L^{1,p}(G)$. In fact, we have (7.1)

$$D_q(f) = \sum_{e \in E} |df(e)|^p \big(|df(e)|^p\big)^{(q-p)/p} \le D_p(f) D_p(f)^{(q-p)/p} = D_p(f)^{q/p}.$$

As consequences, we get $L_0^{1,p}(G) \subset L_0^{1,q}(G)$, $L^{1,p}(G) \subset L^{1,q}(G)$, and moreover,

$$R_G^{(p)}(x,y)^{1/p} \le R_G^{(q)}(x,y)^{1/q}, \quad x, y \in V;$$
$$M_G^{(p)}(x)^{1/p} \le M_G^{(q)}(x)^{1/q} (\le +\infty), \quad x \in V.$$

As a result of the last inequality, G is q-parabolic if it is p-parabolic. On account of this fact, we are able to introduce the *parabolic index* of G by $\mathrm{ind}(G) := \inf\{p \mid G \text{ is } p\text{-parabolic}\}$ (see [38]).

As a direct consequence of Corollary 7.2(3), we have the following.

COROLLARY 7.7. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected infinite graphs with bounded degrees. If there exists a quasi-monomorphism $\psi : G_1 \to G_2$ from G_1 to G_2, then $\mathrm{ind}(G_1) \le \mathrm{ind}(G_2)$.*

PROPOSITION 7.8. *Let G be a connected infinite graph with bounded degrees. For $1 < p < q$, there exists a surjective continuous map $j_{p,q}$ from the Royden q-boundary $\partial\mathcal{R}_q(G)$ to the Royden p-boundary $\partial\mathcal{R}_p(G)$ such that for every $f \in L^{1,p}(G) \subset L^{1,q}(G)$,*

$$tr_p(f) \circ j_{p,q} = tr_q(f),$$

where $tr_p(f)$ stands for $\bar{f}_{|\partial\mathcal{R}_p(G)}$; moreover if $1 < p < q < r < \infty$, then it holds that $j_{p,q} \circ j_{q,r} = j_{p,r}$. In particular, if $\partial\mathcal{R}_q(G) = \Delta_q(G)$, then there exists a continuous surjective map $j_{p,q} : \Delta_q(G) \to \Delta_p(G)$.

Proof. For every $z \in \partial\mathcal{R}_q(G)$ and any sequence $\{v_n\} \subset V$ which converges to z as $n \to \infty$, there exists a subsequence $\{v_{n_k}\} \subset \{v_n\}$ which converges to a point $w \in \partial\mathcal{R}_p(G)$. For any $f \in L^{1,p}(G) \subseteq L^{1,q}(G)$, it holds that

$$tr_p(f)(w) = \lim_{k\to\infty} f(v_{n_k}) = tr_q(f)(z).$$

Thus, we define the continuous map $j_{p,q}$ such that $j_{p,q}(z) = w$, and then $j_{p,q}$ is surjective. From the equality above, it holds that $tr_p(f) \circ j_{p,q}(z) = tr_q(f)(w)$. □

§8. Discrete approximation of Riemannian manifolds and p-Dirichlet finite maps

In this section, we discuss discrete approximation of Riemannian manifolds and p-Dirichlet finite maps, and then we exhibit some examples of Hadamard manifolds with bounded geometry on which existence or nonexistence of p-harmonic functions with finite p-Dirichlet integral can be illustrated along with their geometric structure.

Let $M = (M, g_M)$ be a connected, complete, noncompact Riemannian manifold of dimension n. We assume that M belongs to the collection BG; that is, the Ricci curvature of M is bounded below by a negative constant, and the volume of any ball of radius 1 is bounded below by a positive constant. Given a positive constant κ, let V be a maximal κ-separated subset of M, and define a graph $G = (V, E)$, called a *κ-net* of M, as follows. Two vertices x and y are adjacent; that is, $\{x, y\} \in E$ if and only if $d_M(x, y) \le 3\kappa$. A κ-net of M is of bounded degrees and quasi-isometric to M.

We recall here a result in [18]. Let $HL^{1,p}(M)$ be the space of p-harmonic functions h on M with finite p-Dirichlet integral $D_p(h) = \int_M \|dh\|^p \, dv$. It is known that for any $h \in HL^{1,p}(M)$,

$$|h(x) - h(y)|^p \le C_1 \int_{B_M(z,2\delta)} \|dh\|^p \, dv \, d_M(x, y)^{\alpha p}$$

for every $x \in M$ and any $y, z \in B_M(x, \delta)$, where C_1 is a positive constant depending only on M, p, and a given positive constant δ, and α is a positive constant less than 1 depending only on M and p. For $h \in HL^{1,p}(M)$, we denote by $\nu(h)$ the restriction of h to V. Then $\nu(h)$ belongs to $L^{1,p}(G)$, and in fact, we have $D_p(\nu(h)) \le C_2 D_p(h)$, where C_2 is a positive constant depending only on M and p. When G is p-nonparabolic, letting $\sigma(h)$ be the

p-harmonic part of $\nu(h)$ in the Royden decomposition, we obtain a bijective correspondence σ between $HL^{1,p}(M)$ and $HL^{1,p}(G)$ such that

$$C_3^{-1}D_p(h) \le D_p\big(\sigma(h)\big) \le C_3 D_p(h)$$

for some positive constant C_3 and all $h \in HL^{1,p}(M)$. In the case of $p=2$, σ is a linear isomorphism between $HL^{1,2}(G)$ and $HL^{1,2}(M)$. Moreover, for $p>n$, a quasi-isometry $\phi:(V,d_G)\to(M,d_M)$ induces a homeomorphism between the Royden p-boundary of G and that of M which sends the harmonic p-boundary of G to that of M (see [18] for details).

Let f be a smooth map from M as above to another Riemannian manifold N which is not necessarily complete. We assume that

$$D_p(f) = \int_M \|df\|^p\,dv < +\infty,$$

and for a given positive constant δ, there exists a positive constant C_4 such that

$$d_N\big(f(y),f(z)\big)^p \le C_4 \int_{B_M(x,2\delta)} \|df\|^p\,dv \tag{8.1}$$

for every $x\in M$ and any $y,z\in B_M(x,\delta)$. Let $G=(V,E)$ be a connected, infinite graph of bounded degrees, and suppose that G admits a quasi-monomorphism ψ into M. Then the composition $f\circ\psi$ is a p-Dirichlet finite map from G into N (see (3.1)), and hence $f\circ\psi$ extends to a continuous map from $\mathcal{R}_p(G)$ to the completion $\overline{N}$ of N if $\overline{N}$ is compact (see Theorem 3.1).

Let σ be a positive smooth function on M, and suppose that $\int_M \sigma^p\,dv < +\infty$ and, further, that $\sup_{x\in M}\max_{B(x,1)}\sigma/\min_{B(x,1)}\sigma < +\infty$. Consider the identity map I of M as a map from the Riemannian manifold (M,g_M) to a Riemannian manifold endowed with metric $\sigma^2 g_M$. Then $D_p(I)=\int_M\sigma^p\,dv < +\infty$ and I satisfies condition (8.1).

EXAMPLE 8.1. Let η be a smooth function on $[0,\infty)$ such that $\eta(t)>0$ for $t>0$, $\eta(0)=0$, and $\eta'(0)=1$. We consider a rotationally symmetric metric g_η on $\boldsymbol{R}^n$ written as $g_\eta = dr^2+\eta(r)^2\,d\theta_n^2$ in polar coordinates (r,θ_n), where $d\theta_n^2$ stands for the Riemannian metric of the unit sphere $S^{n-1}(1)$ in $\boldsymbol{R}^n$. We denote by M_η the Riemannian manifold $(\boldsymbol{R}^n, g_\eta)$.

It is proved in [17, Proposition 4] that M_η is p-nonparabolic if and only if

$$\int_1^\infty \frac{dt}{\eta(t)^{(n-1)/(p-1)}} < +\infty. \tag{8.2}$$

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Suppose that $\int_1^\infty 1/\eta(t)\,dt < +\infty$, and let

$$F_\eta(r,\theta) = \Big(R^{-1}\exp\int_1^r \frac{dt}{\eta(t)}, \theta\Big),$$

where $R = \exp\int_1^\infty 1/\eta(t)\,dt$. Then F_η induces a conformal diffeomorphism of M_η onto the Euclidean unit open ball $\mathrm{B}^n(1)$ around the origin, and the pullback of the Euclidean metric $g_0 = dt^2 + t^2 d\theta^2$ by F_η is given by

$$F_\eta^* g_0 = \Big(\frac{1}{R\eta(r)}\exp\int_1^r \frac{dt}{\eta(t)}\Big)^2 g_\eta.$$

Since

$$\int_{M_\eta} \|dF_\eta\|^p\,dv = n^{p/2}R^{-p}\,\mathrm{Vol}\big(S^{n-1}(1)\big)\int_0^\infty \eta(r)^{-p+n-1}\exp\Big(p\int_1^r \frac{dt}{\eta(t)}\Big)\,dr,$$

$\int_{M_\eta} \|dF_\eta\|^p\,dv < +\infty$ if and only if

$$\int_1^\infty \eta(t)^{-p+n-1}\,dt < +\infty. \tag{8.3}$$

In what follows, to keep our manifold M_η in BG, we assume that the radial curvature $-\eta''(r)/\eta(r)$ of M_η is pinched by $-a^2$ and 0 for some positive constant a. We note that (8.1) is satisfied for F_η. Then M_η is a Hadamard manifold of bounded sectional curvature. We remark that $1/r \le \eta'(r)/\eta(r) \le a\cosh ar/\sinh ar$ on $(0,+\infty)$. Let G_η be a κ-net of M_η. Under condition (8.3), the restriction f_η of F_η to G_η gives rise to a p-Dirichlet finite map of G_η into $\mathrm{B}^n(1)$, so that f_η extends to a continuous map $\overline{f_\eta}$ from the Royden p-compactification $\mathcal{R}_p(G_\eta)$ of G_η to $\overline{\mathrm{B}^n(1)}$ which maps $\partial\mathcal{R}_p(G_\eta)$ onto $\mathrm{S}^{n-1}(1)$ (see Theorem 3.1). We notice that for any element τ of the orthogonal group $O(n)$ acting isometrically on both M_η and $\overline{\mathrm{B}^n(1)}$, τ induces a homeomorphism $\bar\tau$ of the Royden p-boundary $\partial\mathcal{R}_p(G_\eta)$ of G_η in such a way that $\overline{f_\eta}\circ\bar\tau = \tau\circ\overline{f_\eta}$ on $\partial\mathcal{R}_p(G_\eta)$. Since $\bar\tau(\Delta_p(G_\eta)) = \Delta_p(G_\eta)$, we can deduce that $\overline{f_\eta}(\Delta_p(G_\eta))$ is invariant under the action of $O(n)$, and hence $\overline{f_\eta}(\Delta_p(G_\eta)) = \mathrm{S}^{n-1}(1)$ if $\Delta_p(G_\eta)$ is not empty; namely, G_η is p-nonparabolic.

Now we can deduce from the arguments of [24] that M_η admits no nonconstant p-harmonic functions with finite Dirichlet integral of order p if

$$1 + (n-1-p)\frac{r\eta'(r)}{\eta(r)} \ge 0, \quad \forall r > 0. \tag{8.4}$$

Clearly, (8.4) implies the divergence of the integral in (8.3). But there are cases where (8.3) and (8.4) are mutually complementary.

(i) In the case where $r\eta'(r)/\eta(r) \le \delta$ for some $\delta \ge 1$ and all $r > 0$, and as $r \to \infty$, $r\eta'(r)/\eta(r)$ goes to δ, (8.2) is equivalent to the condition $p < 1 + \delta(n-1)$; (8.3) is equivalent to the condition $p > n - 1 + (1/\delta)$; and (8.4) is equivalent to the condition $1 < p \le n - 1 + (1/\delta)$.

(ii) In the case where $\lim_{r\to\infty} r\eta'(r)/\eta(r) = \infty$, (8.2) holds for any $p > 1$; (8.3) is equivalent to the condition $p > n - 1$; and (8.4) is equivalent to the condition $1 < p \le n - 1$.

EXAMPLE 8.2. We consider a warped product $M = (M, g)$ of Euclidean spaces $(\boldsymbol{R}^{n-k}, g_E)$ $(n \ge 3)$ and $(\boldsymbol{R}^k, g_E)$ $(1 \le k \le n-2)$ with a warping function $\cosh r$, where r denotes the distance to a fixed point o_1 of $\boldsymbol{R}^{n-k}$. The metric is written as

$$g = g_E + (\cosh r)^2 g_E = dr^2 + r^2\, d\theta_{n-k}{}^2 + (\cosh r)^2\, dt^2$$

for the case of $k = 1$, and

$$g = g_E + (\cosh r)^2 g_E = dr^2 + r^2\, d\theta_{n-k}{}^2 + (\cosh r)^2 (dt^2 + t^2\, d\theta_k{}^2)$$

for the case of $k \ge 2$, where we use polar coordinates (r, θ_{n-k}) and (t, θ_k) of $\boldsymbol{R}^{n-k}$ and $\boldsymbol{R}^k$, respectively. We denote by ρ the Riemannian distance in M to a fixed point $o = (o_1, o_2)$. Let N be a Riemannian manifold endowed with a Riemannian metric $\bar{g} = (\cosh \rho)^{-2} g$, and consider the identity map of M as a map from M onto N, which will be denoted by I. We note here that $\sqrt{r^2 + t^2} \le \rho \le r + t$, where the first inequality is verified by comparing the metric g with the product metric and the second one follows from the fact that the distance is realized by the minimal geodesic. We further observe that $\alpha r + \sqrt{1 - \alpha^2}\, t \le \sqrt{r^2 + t^2}$ for all $t, r \ge 0$ and $0 \le \alpha \le 1$. Then we have

$$\begin{aligned} \int_M \big(\cosh(r+t)\big)^{-p}\, dv \le D_p(I) &= \int_M (\cosh \rho)^{-p}\, dv \\ &\le \int_M (\cosh \sqrt{r^2 + t^2})^{-p}\, dv \\ &\le \int_M \big(\cosh(\alpha r + \sqrt{1-\alpha^2}\, t)\big)^{-p}\, dv. \end{aligned}$$

Since the volume element dv of M is given by $C_{n,k} (\cosh r)^k r^{n-k-1} \times t^{k-1}\, dr\, d\theta_{n-k}\, dt\, d\theta_k$ in the coordinates $(r, \theta_{n-k}, t, \theta_k)$, where $C_{n,k}$ is a positive constant depending only on n, k, we can deduce that $D_p(I)$ is finite if and only if $p > k$. It is not hard to see that (8.1) is satisfied.

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Now applying [10, Theorem 3.10], to the completion $\overline{N}$ of N, we can conclude that the boundary ∂N of $\overline{N}$ is homeomorphic to a closed interval if $k=1$, and $S^k(1)$ if $k \geq 2$.

We remark that the Laplacian $\Delta_M r$ of r on M is equal to $(n-k-1)r^{-1}+k\sinh r(\cosh r)^{-1}$, and hence $\Delta_M\sqrt{r^2+1} > \beta > 0$, where we set $\beta = \inf\{(r^2+1)^{-3/2}+(n-k-1)(r^2+1)^{-1/2}+kr(r^2+1)^{-1/2}\sinh r(\cosh r)^{-1}\}$. This implies that

$$\beta\,\mathrm{Vol}_n(\Omega) < \mathrm{Vol}_{n-1}(\partial\Omega)$$

for any bounded smooth domain Ω in M. In fact, we have

$$\beta\,\mathrm{Vol}_n(\Omega) \leq \int_\Omega \Delta_M\sqrt{r^2+1}\,dv \leq \int_{\partial\Omega}\frac{r}{\sqrt{r^2+1}}\left|\frac{\partial r}{\partial\nu}\right| dv_{\partial\Omega} \leq \mathrm{Vol}_{n-1}(\partial\Omega).$$

It is known that

$$\inf_u \frac{\int_M |\nabla u|\,dv}{\int_M |u|\,dv} = \inf_\Omega \frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}_n(\Omega)},$$

where u (resp., Ω) ranges over $C_0(M)$ (resp., bounded smooth domains of M). This shows that

$$\beta\int_M |u|\,dv \leq \int_M |\nabla u|\,dv, \quad u\in C_0^\infty(M).$$

Therefore, we have

$$\beta\int_M |u|^p\,dv \leq \int_M p|u|^{p-1}|\nabla u|\,dv \leq p\Big(\int_M |u|^p\,dv\Big)^{1-1/p}\Big(\int_M |\nabla u|^p\,dv\Big)^{1/p},$$

which implies that

$$\left(\frac{\beta}{p}\right)^p\int_M |u|^p\,dv \leq \int_M |\nabla u|^p\,dv, \quad u\in C_0^\infty(M). \tag{8.5}$$

Let $G=(V,E)$ be a κ-net of M. Then it follows from (8.5) that $\lambda_p(G)>0$ for all $p>1$ (see [22], [23], [34]). Thus we see that $\partial\mathcal{R}_p(G)=\Delta_p(G)$ for all $p>1$, so that $\overline{I_{|V}}(\Delta_p(G))=\partial N$ if $p>k$. It follows that for any Lipschitz continuous functions ϕ on ∂N, there exists uniquely a p-harmonic function $h\in L^{1,p}(G)$ such that $h(x)$ converges to $\phi(\xi)$ as $x\in V$ tends to $\xi\in\partial N$ (see Corollary 3.3); correspondingly, we have a unique p-harmonic function H with finite Dirichlet integral of order p on M such that $H(x)$ converges to $\phi(\xi)$ as $x\in M$ tends to $\xi\in\partial N$. Thus, M admits a lot of p-harmonic functions with finite Dirichlet integral of order p if $p>k$, and for any $p>1$ if $k=1$.

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EXAMPLE 8.3. We consider a warped product $M_a = (M_a, g_a)$ of the hyperbolic space form $(\boldsymbol{H}^{n-k}, g_H)$ $(n \geq 3)$ and Euclidean space $(\boldsymbol{R}^k, g_E)$ $(1 \leq k \leq n-2)$ with a warping function $\cosh ar$, where a is a constant $\in [1, \infty)$ and r denotes the distance to a fixed point o_H of $\boldsymbol{H}^{n-k}$. The metric is written as

$$g_a = g_H + (\cosh ar)^2 g_E = dr^2 + (\sinh r)^2 \, d\theta_{n-k}{}^2 + (\cosh ar)^2 \, dt^2$$

for the case of $k = 1$, and

$$g_a = g_H + (\cosh ar)^2 g_E = dr^2 + (\sinh r)^2 \, d\theta_{n-k}{}^2 + (\cosh ar)^2 (dt^2 + t^2 \, d\theta_k{}^2)$$

for the case of $k \geq 2$, where we use polar coordinates (r, θ_{n-k}) and (t, θ_k) of $\boldsymbol{H}^{n-k}$ and $\boldsymbol{R}^k$, respectively. We denote by ρ the Riemannian distance in M_a to a fixed point $o = (o_H, o_E)$. Let N_ε be a Riemannian manifold endowed with a Riemannian metric $\bar{g}_\varepsilon = (\cosh \varepsilon\rho)^{-2} g_a$, where a positive constant ε will be chosen later, and we consider the identity map of M_a as a map from M_a onto N_ε, which will be denoted by I_ε. Since $\alpha r + \sqrt{1-\alpha^2}\, t \leq \sqrt{r^2 + t^2} \leq \rho \leq r + t$, we have

$$\begin{aligned}
\int_{M_a} \bigl(\cosh \varepsilon(r+t)\bigr)^{-p} \, dv \leq D_p(I_\varepsilon) &= \int_{M_a} (\cosh \varepsilon\rho)^{-p} \, dv \\
&\leq \int_{M_a} (\cosh \varepsilon \sqrt{r^2 + t^2})^{-p} \, dv \\
&\leq \int_{M_a} \bigl(\cosh \varepsilon(\alpha r + \sqrt{1-\alpha^2}\, t)\bigr)^{-p} \, dv.
\end{aligned}$$

The volume element dv is given by $C_{n,k} (\cosh ar)^k (\sinh r)^{n-k-1} \times t^{k-1} \, dr \, d\theta_{n-k} \, dt \, d\theta_k$ in the coordinates $(r, \theta_{n-k}, t, \theta_k)$, and hence we can deduce that $D_p(I_\varepsilon)$ is finite if and only if $\varepsilon p > ak + n - k - 1$. It is not hard to see that (8.1) is satisfied.

Now applying [10, Theorem 3.10] to the completion $\overline{N_\varepsilon}$ of N_ε, we can conclude that, for the case where $0 < \varepsilon \leq 1$, the boundary ∂N_ε of $\overline{N_\varepsilon}$ is homeomorphic to $S^{n-1}(1)$ if $k = 1$ and the gluing of $S^{n-k-1}(1) \times S^k(1)$ is onto a point N of $S^k(1)$ along $S^{n-k-1}(1) \times \{N\}$, using the projection map; for the case where $1 < \varepsilon \leq a$, ∂N_ε is homeomorphic to a closed interval if $k = 1$ and $S^k(1)$ if $k \geq 2$.

We remark that the Laplacian $\Delta_{M_a} r$ of r on M_a is equal to $(n-k-1) \cosh r (\sinh r)^{-1} + ak \sinh ar (\cosh ar)^{-1}$, and hence $\Delta_{M_a} \sqrt{r^2 + 1} > \gamma > 0$,

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where we set $\gamma = \inf\{(r^2+1)^{-3/2} + (n-k-1)r(r^2+1)^{-1/2}\cosh r(\sinh r)^{-1} + kar(r^2+1)^{-1/2}\sinh ar(\cosh ar)^{-1}\}$. This implies that

$$\gamma \operatorname{Vol}_n(\Omega) < \operatorname{Vol}_{n-1}(\partial\Omega)$$

for any bounded smooth domain Ω in M_a, so that we obtain

$$\left(\frac{\gamma}{p}\right)^p \int_{M_a} |u|^p \, dv \le \int_{M_a} |\nabla u|^p \, dv, \quad u \in C_0^\infty(M_a). \tag{8.6}$$

Let $G_a = (V, E)$ be a κ-net of M_a. Then it follows from (8.6) that $\lambda_p(G_a) > 0$ for all $p > 1$. Thus, we see that $\partial\mathcal{R}_p(G_a) = \Delta_p(G_a)$ for all $p > 1$, so that $\overline{I_{a|V}}(\Delta_p(G_a)) = \partial N_1$ if $p > ak + (n-k-1)$ and $\overline{I_{a|V}}(\Delta_p(G_a)) = \partial N_a$ if $p > k + (n-k-1)/a$. It follows that for any Lipschitz continuous functions ϕ on ∂N_a, there exists uniquely a p-harmonic function $h \in L^{1,p}(G)$ such that $h(x)$ converges to $\phi(\xi)$ as $x \in V$ tends to $\xi \in \partial N_a$ (see Corollary 3.3); correspondingly, we have a unique p-harmonic function H with finite Dirichlet integral of order p on M_a such that $H(x)$ converges to $\phi(\xi)$ as $x \in M_a$ tends to $\xi \in \partial N_a$. Thus, M_a admits a lot of p-harmonic functions with finite Dirichlet integral of order $p > k + (n-k-1)/a$. When $n = 3$, $k = 1$, and $p = 2$, this is verified by Anderson [3] in a different manner. It should be remarked that when $k = 1$, the sectional curvature of M_a is pinched by $-a^2$ and -1, and M_a admits no nonconstant p-harmonic functions with finite Dirichlet integral of order p for $1 < p < (n-1)/a$. This is true for a complete simply connected manifold M of dimension n whose curvature is pinched by $-a^2$ and -1, by a result of Pansu [31] (see also [24]). On the other hand, it admits a lot of nonconstant p-harmonic functions with finite Dirichlet integral of order p for $p > a(n-1)$. In fact, for a Gromov hyperbolic graph $G = (V, E)$ of bounded degrees approximating M in the above manner, we see that $e(G) \le a(n-1)$ and $\varepsilon_0(G) \ge 1$. Also using the comparison arguments, we can show that the map $F_\eta : M \to \mathrm{B}^n(1)$ defined above with $\eta = \sinh r$ is a p-Dirichlet finite map satisfying (8.1) for $p > a(n-1)$. This explains a nonvanishing theorem due to Pansu [31, théorème 5]. We also refer to [9] for related results.

§9. Quasi-monomorphisms to hyperbolic space forms

In this section, we consider a connected infinite graph of bounded degrees admitting a quasi-monomorphism into the hyperbolic space form $\boldsymbol{H}^n$. In view of the above example, Theorem 3.1, and Lemma 7.3, a quasi-monomorphism extends continuously to the Royden p-boundary which is sent to the

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hyperbolic boundary $\partial_\infty \boldsymbol{H}^n$ of $\boldsymbol{H}^n$ if $p > n-1$. Theorem 9.5 below shows that if the graph is p-nonparabolic for $p > n-1$, then the image of the harmonic p-boundary $\Delta_p(G)$ is a perfect subspace of $\partial_\infty \boldsymbol{H}^n$.

To begin with, we recall some basic properties of $\boldsymbol{H}^n$ which are used in Pansu [32] to estimate the Hölder invariant.

We denote by g_H and d_H, respectively, the Riemannian metric and the distance of $\boldsymbol{H}^n$. Fix a point o of $\boldsymbol{H}^n$, and let $\ell(t), -\infty < t < +\infty$, be a line on $\boldsymbol{H}^n$, that is, a unit speed geodesic defined on $\boldsymbol{R}$, such that $\ell(0) = o$. In the case where $n \geq 3$, we employ cylindrical coordinates (r, θ, t) of $\boldsymbol{H}^n$ around ℓ to express the Riemannian metric g_H as

$$g_H = dr^2 + (\sinh r)^2\, d\theta^2 + (\cosh r)^2\, dt^2, \quad (r, \theta, t) \in [0, +\infty) \times S^{n-2}(1) \times \boldsymbol{R},$$

where r is the distance to the line ℓ and $d\theta^2$ stands for the Riemannian metric of the unit sphere $S^{n-2}(1)$ in $\boldsymbol{R}^{n-1}$. In the case where $n = 2$, we use coordinates (s, t) of $\boldsymbol{H}^2$ such that

$$g_H = ds^2 + (\cosh s)^2\, dt^2, \quad (s, t) \in \boldsymbol{R}^2,$$

where $r = |s|$ is the distance to the line ℓ. Given $R > 0$, we define a function $\xi_{l,R}$ on $\boldsymbol{H}^n$ by

$$\xi_{\ell,R}(r, \theta, t) = \begin{cases} R & (t \geq R), \\ t & (0 \leq t \leq R) \\ 0 & (t \leq 0) \end{cases}$$

if $n \geq 3$, and by

$$\xi_{\ell,R}(s, t) = \begin{cases} R & (t \geq R), \\ t & (0 \leq t \leq R) \\ 0 & (t \leq 0) \end{cases}$$

if $n = 2$. Let $U(\ell, R) = \{x \in \boldsymbol{H}^n \mid \xi_{\ell,R}(x) = R\}$, and let $L(\ell, R) = \{x \in \boldsymbol{H}^n \mid \xi_{\ell,R}(x) = 0\}$. Then in $\boldsymbol{H}^n \setminus U(\ell, R) \cup L(\ell, R)$, we have $\|d\xi_{\ell,R}\| = 1/\cosh r$ when $n \geq 3$ and $\|d\xi_{\ell,R}\| = 1/\cosh s$ when $n = 2$. For $p > n-1$, the p-Dirichlet integral $D_p(\xi_{\ell,R})$ of $\xi_{\ell,R}$ is finite and given by

$$D_p(\xi_{\ell,R}) = \mathrm{Vol}\big(S^{n-2}(1)\big) \int_0^\infty (\cosh r)^{1-p} (\sinh r)^{n-2}\, dr R$$

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if $n \geq 3$, and by

$$D_p(\xi_{\ell,R}) = \int_0^\infty (\cosh r)^{1-p}\, dr R$$

if $n = 2$. Thus, we have the following.

LEMMA 9.1 ([32, section 4]). *For any $p > n-1$, there exists a constant $C_1 > 0$ depending only on n and p such that*

$$D_p(\xi_{l,R}) < C_1 R.$$

LEMMA 9.2. *Let δ be a positive constant. Then there exists a constant $C_2 > 0$ depending only on n, p, and δ such that*

$$|\xi_{\ell,R}(y) - \xi_{\ell,R}(z)|^p \leq C_2 \int_{B(x,2\delta)} \|d\xi_{\ell,R}\|^p\, dv \tag{9.1}$$

for every $x \in \boldsymbol{H}^n$ and any $y, z \in B(x,\delta)$, where $B(x,a)$ denotes the geodesic ball centered at x with radius a in $\boldsymbol{H}^n$.

Proof. We assume that $B(x,2\delta)$ is included in $\boldsymbol{H}^n \setminus U(\ell,R) \cup L(\ell,R)$. Then we have

$$\begin{aligned}\int_{B(x,2\delta)} \|d\xi_{\ell,R}\|^p\, dv &= \int_{B(x,2\delta)} \Big(\frac{1}{\cosh r}\Big)^p dv\\ &\geq \mathrm{Vol}\big(B(x,2\delta)\big)\Big(\frac{1}{\cosh(r(x)+\delta)}\Big)^p.\end{aligned}$$

On the other hand, let $c(t) : [0, d_H(y,z)] \to \boldsymbol{H}^n$ be the unit speed geodesic joining y to z. Then we have

$$\begin{aligned}|\xi_{\ell,R}(y) - \xi_{\ell,R}(x)| &\leq \int_0^{d_H(y,z)} \Big|\frac{d}{dt}\xi_{\ell,R}\big(c(t)\big)\Big|\, dt\\ &\leq \int_0^{d_H(y,z)} \|d\xi_{\ell,R}\|\big(c(t)\big)\, dt\\ &\leq \int_0^{d_H(y,z)} \frac{1}{\cosh r(c(t))}\, dt\\ &\leq \frac{d_H(y,z)}{\cosh(r(x)-\delta)}.\end{aligned}$$

From these inequalities, we get

$$|\xi_{\ell,R}(y)-\xi_{\ell,R}(z)|^p \leq \Big(\frac{\cosh\big(r(x)+\delta\big)}{\cosh\big(r(x)-\delta\big)}\Big)^p \frac{d_H(y,z)^p}{\mathrm{Vol}(B(x,2\delta))}\int_{B(x,2\delta)} \|d\xi_{\ell,R}\|^p\, dv.$$

Similarly, we can derive (9.1) in the case where $B(x,2\delta)$ intersects $U(\ell,R)$ or $L(\ell,R)$. ∎

PROPOSITION 9.3. *Let $G=(V,E)$ be a connected infinite graph with bounded degrees. Suppose that there exists a quasi-monomorphism ψ from G to $\boldsymbol{H}^n$. For $p>n-1$, there exists a constant $C_3>0$ such that*

$$R_G^{(p)}(x,y)^{1/p} \geq C_3 d_H\big(\psi(x),\psi(y)\big)^{1-1/p}, \quad x,y\in V.$$

In particular, $R_G^{(p)}$ is not bounded for $p>n-1$.

Proof. For any $x,y\in V$, let ℓ be a line such that $\ell(0)=\psi(x)$, $\ell(R)=\psi(y)$, where we put $R=d_H(\psi(x),\psi(y))$. Let $f=\xi_{\ell,R}\circ\psi$. Then it follows from the proofs of Lemmas 7.1 and 9.2 that $D_p(f)\leq C_4D_p(\xi_{l,R})$ for some constant C_4 depending only on n, p, and ψ. Hence, by Lemma 9.1, we obtain

$$R_G^{(p)}(x,y) \geq \frac{|f(x)-f(y)|^p}{D_p(f)} \geq \frac{1}{C_1C_4}R^{p-1} = \frac{1}{C_1C_4}d_H\big(\psi(x),\psi(y)\big)^{p-1}$$

for all $x,y\in V$. ∎

In view of (2.1), we see the following.

COROLLARY 9.4. *Let $G=(V,E)$ be a connected infinite graph with bounded degrees. Suppose that there exists a quasi-isometric embedding ψ from G to $\boldsymbol{H}^n$. For any $p>n-1$, there exist constants $C_5>0$ and $C_5'\geq 0$ such that*

$$C_5 d_G(x,y)^{1-1/p} - C_5' \leq R_G^{(p)}(x,y)^{1/p} \leq d_G(x,y)^{1-1/p}, \quad x,y\in V.$$

We refer to [14] and [32] for related results on $R_G^{(p)}$.

Now we are in a position to prove the following.

THEOREM 9.5. *Let $G=(V,E)$ be a connected infinite graph with bounded degrees. Suppose that there exists a quasi-monomorphism ψ from G to the hyperbolic space form $\boldsymbol{H}^n$ of constant curvature -1 and dimension n. Then for $p>n-1$, ψ extends to a continuous map $\bar\psi$ from $\mathcal{R}_p(G)$ to $\overline{\boldsymbol{H}^n}=\boldsymbol{H}^n\cup\partial_\infty\boldsymbol{H}^n$, and moreover, if G is p-nonparabolic, that is, the harmonic p-boundary $\Delta_p(G)$ is not empty, then $\bar\psi(\Delta_p(G))$ is a perfect subspace of $\overline{\psi(V)}\cap\partial_\infty\boldsymbol{H}^n$.*

Proof. It follows from Example 8.1(ii) (or Example 8.2 ($k = 1$)), Lemma 7.3, and Theorem 3.1 that for $p > n-1$, ψ extends to a continuous map $\bar{\psi}$ from $\mathcal{R}_p(G)$ to $\overline{\boldsymbol{H}^n} = \boldsymbol{H}^n \cup \partial_\infty \boldsymbol{H}^n$. Suppose that G is p-nonparabolic. Then it follows from Theorem 3.1 again that

$$\bar{\psi}\big(\Delta_p(G)\big) = \overline{\big\{(\psi \circ \gamma)(\infty); \gamma \in \boldsymbol{P}(G) \setminus (\boldsymbol{P}_\infty \cup \boldsymbol{P}'_\infty)\big\}}$$

for some $\boldsymbol{P}_\infty \subset \boldsymbol{P}(G)$ with $EL_p(\boldsymbol{P}_\infty) = +\infty$ and any $\boldsymbol{P}'_\infty \subset \boldsymbol{P}(G)$ with $EL_p(\boldsymbol{P}'_\infty) = +\infty$. Given points $v \in V$ and $w \in \partial_\infty \boldsymbol{H}^n$, let $\ell_w(t), -\infty < t < +\infty$, be the line on $\boldsymbol{H}^n$ such that $\ell_w(0) = \psi(v)$ and $\ell_w(\infty) = \lim_{t\to\infty} \ell_w(t) = w$. For any $R > 0$, we set $f_R := \xi_{\ell_w,R}/R$, $K(w,R) = \overline{U(\ell_w,R)} \cap \partial_\infty \boldsymbol{H}^n$, and $\mathbf{Q}_{w,R} = \{\gamma \in \boldsymbol{P}(G) \setminus \boldsymbol{P}_\infty \mid \psi \circ \gamma(\infty) \in K(w,R)\}$. Since $L_{m_{f_R}}(\gamma) \geq 1$ for all $\gamma \in \mathbf{Q}_{w,R} \cap \boldsymbol{P}_v$, we have

$$\begin{aligned} EL_p(\mathbf{Q}_{w,R} \cap \boldsymbol{P}_v)^{-1} &\leq \frac{\|m_{f_R}\|_p^p}{\inf\{L_{m_{f_R}}(\gamma) \mid \gamma \in \mathbf{Q}_{w,R} \cap \boldsymbol{P}_v\}} \\ &\leq D_p(f_R) \\ &\leq \frac{C}{R^{p-1}}. \end{aligned}$$

Now in view of the Cantor-Bendixson theorem, $\bar{\psi}(\Delta_p(G))$ can be uniquely written as $\bar{\psi}(\Delta_p(G)) = \mathcal{P} \cup \mathcal{C}$, with $\mathcal{P}$ a perfect subset of $\bar{\psi}(\Delta_p(G))$ and $\mathcal{C}$ countably open. If $\mathcal{C}$ is not empty, we write $\mathcal{C} = \{w_k \mid k = 1, 2, \ldots\}$. Then we first choose positive numbers R_k, $k = 1, 2, \ldots$ in such a way that $\mathcal{P} \cap \big(\bigcup K(w_k, R_k)\big) = \emptyset$. Let $\mathbf{Q} = \bigcup \mathbf{Q}_{w_k,R_k}$. Then for positive numbers $R'_k > R_k$, we have $\mathbf{Q} = \bigcup \mathbf{Q}_{w_k,R'_k}$. Since

$$EL_p(\mathbf{Q}_{w_k,R'_k} \cap \boldsymbol{P}_v)^{-1} \leq \frac{C}{{R'_k}^{p-1}},$$

we obtain by using Lemma 2.4

$$EL_p(\mathbf{Q} \cap \boldsymbol{P}_v)^{-1} \leq \sum_{k=1}^{\infty} EL_p(\mathbf{Q}_{w_k,R'_k} \cap \boldsymbol{P}_v)^{-1} \leq C \sum_{k=1}^{\infty} \frac{1}{{R'_k}^{p-1}}.$$

This holds for all R'_k with $R'_k > R_k$. Hence, we get $EL_p(\mathbf{Q} \cap \boldsymbol{P}_v) = +\infty$, so that $EL_p(\mathbf{Q}) = +\infty$. This implies that

$$\bar{\psi}\big(\Delta_p(G)\big) = \overline{\big\{\psi \circ \gamma(\infty) \,\big|\, \gamma \in \boldsymbol{P}(G) \setminus (\boldsymbol{P}_\infty \cup \mathbf{Q})\big\}} \subset \mathcal{P}.$$

In this way, we can deduce that $\mathcal{C}$ is empty. $\square$

In Theorem 9.5, we remark that in general, $\bar{\psi}(\Delta_p(G))$ does not coincide with $\overline{\psi(V)} \cap \partial_\infty \boldsymbol{H}^n$ and also with $\bar{\psi}(\Delta_q(G))$ for $q > p > n-1$ even if G is q-nonparabolic.

Now using [8, Theorem 1.1] and Theorem 9.5, we obtain the following.

Corollary 9.6. *Let G be a connected infinite graph of bounded degrees. Suppose that G is p-nonparabolic for all $p > 1$ and that it admits a quasi-monomorphism into a visual Gromov hyperbolic geodesic space whose boundary at infinity is doubling for some visual metric. Then G possesses a lot of nonconstant p-harmonic functions with finite Dirichlet sum of order p for all sufficiently large p.*

Example 9.7. We consider an upper half-space model of the hyperbolic space form of dimension 3, $\boldsymbol{H}^3 = (\{(x, y, z) \mid z > 0\}, g_H = (dx^2 + dy^2 + dz^2)/z^2)$. Let $\tau(t)$ be a smooth positive function on an interval $(0, a]$ such that $\tau'(t) > 0$ on $(0, b]$ $(b < a)$, $\lim_{t \to 0} \tau(t) > 0$, and $\tau(a) = 0$. Consider a surface of revolution in $\boldsymbol{H}^3$ defined by

$$S_\tau = \bigl\{ \bigl(\tau(z)\cos\theta, \tau(z)\sin\theta, z\bigr) \bigm| 0 < z \le a, 0 \le \theta \le 2\pi \bigr\},$$

where it is assumed that S_τ is smooth at $(0, 0, a)$. Let g_τ be the induced metric of S_τ. In the coordinates (z, θ), g_τ can be expressed as

$$g_\tau = \frac{\tau'(z)^2 + 1}{z^2}\, dz^2 + \frac{\tau(z)^2}{z^2}\, d\theta^2.$$

Changing the coordinates (z, θ) to (r, θ) by $dr = -(\sqrt{\tau'(z)^2 + 1}/z)\, dz$ on $(0, b]$ and letting $\eta(r) = \tau(z(r))/z(r)$, we get

$$g_\tau = dr^2 + \eta(r)^2\, d\theta^2, \quad r \in [r(b), +\infty), \theta \in [0, 2\pi].$$

Now we take $\tau(z)$ in such a way that $\tau(z) = Az^\alpha + B$ on $(0, b]$, where A, B are positive constants and α is a constant in $(0, 1)$. Then there is a constant $C > 1$ such that

$$C^{-1}(r^\delta + r) \le \eta(r) \le C(r^\delta + r), \quad r \in [r(b), +\infty),$$

where $\delta = 1/(1-\alpha) (> 1)$. Let $\tilde{\eta}(r) = r^\delta + r$. Then the resulting 2-dimensional Riemannian manifold $M_{\tilde{\eta}}$ as in Example 8.1 is quasi-isometric to our surface S_τ. Thus, we can apply Example 8.1(i) to S_τ with δ. The inclusion map of S_τ into $\boldsymbol{H}^3$ is a quasi-monomorphism. In other words, $M_{\tilde{\eta}}$ admits a

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quasi-monomorphism ψ into $\boldsymbol{H}^3$ such that $\psi(M_{\tilde{\eta}})$ touches on $\partial_\infty \boldsymbol{H}^3 (= \boldsymbol{R}^2)$ along a circle. Notice that there exist no quasi-monomorphisms from $M_{\tilde{\eta}}$ to $\boldsymbol{H}^2$ because $M_{\tilde{\eta}}$ is p-nonparabolic for $p < 1+\delta$ and admits no nonconstant p-harmonic functions of finite p-Dirichlet sum for $1 < p \leq 1 + 1/\delta$.

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