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On several dynamical properties of shifts acting on directed trees

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Abstract. This article explores the notions of $\mathcal{F}$-transitivity and topological $\mathcal{F}$-recurrence for backward shift operators on weighted ℓ^p-spaces and c_0-spaces on directed trees, where $\mathcal{F}$ represents a Furstenberg family of subsets of $\mathbb{N}_0$. In particular, we establish the equivalence between recurrence and hypercyclicity of these operators on unrooted directed trees. For rooted directed trees, a backward shift operator is hypercyclic if and only if it possesses an orbit of a bounded subset that is weakly dense.

1 Introduction

The study of the dynamical properties of weighted shift operators on sequence spaces of trees has been an active research topic in recent years. Martínez-Avendaño [27], Rivera-Guasco and Martínez-Avendaño [28], and Grosse-Erdmann and Papathanasiou [22, 23] have made significant contributions in this field. See also [6, 7] for some further related research. Jblónski, Jung, and Stochel [26] introduced and studied weighted forward shift operators on trees, while backward shift operators were introduced in [27]. In [22, Theorems 4.3 and 5.2], the authors provided a complete characterization of the hypercyclicity of backward shift operators on weighted ℓ^p-spaces and c_0-spaces on directed trees.

Recall that a bounded linear operator is *hypercyclic* if it has dense orbit. More precisely, let us consider X to be a Banach space and let $\mathcal{L}(X)$ be the space of bounded linear operators from X into itself. An operator $T \in \mathcal{L}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that its orbit under T, denoted by

$$\mathrm{Orb}(x, T) := \{T^n x : n \in \mathbb{N}_0\},$$

is dense in X, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}_0 \backslash \{0\}$. Hypercyclicity is a well-studied concept in linear dynamics, see [8, 24]. In [29], Salas provided a description of hypercyclic weighted backward shift operators on ℓ^p-spaces, which was later extended by Grosse–Erdmann to backward shift operators defined on Fréchet sequence spaces [21]. Additionally, Chan and Seceleanu proved that the hypercyclicity of backward shift operators is equivalent to the existence of an orbit with a nonzero limit point [18]. Some recent improvements to these results have been achieved in [1, 11, 25]. An immediate consequence of Chan and Seceleanu's result is that the notions of recurrence and hypercyclicity coincide for backward shift operators on weighted

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ℓ^p-spaces of sequences indexed by $\mathbb{Z}$. A vector $x \in X$ is called *recurrent vector* for an operator $T \in \mathcal{L}(X)$ if there exists a strictly increasing sequence of integers $(n_k)_{k\in\mathbb{N}_0}$ such that

$$T^{n_k}x \underset{k\to+\infty}{\longrightarrow} x.$$

The operator T is called *recurrent* if its set of recurrent vectors is dense in X, or equivalently (see [15, Proposition 2.1]), for every nonempty open set U of X, there exists an integer $n \in \mathbb{N}$ such that

$$T^n(U) \cap U \neq \varnothing.$$

The study of recurrence for linear operators was initiated by Costakis and Parissis [16] and later developed further by Costakis, Manoussos, and Parissis [15]. For recent contributions, see also [4, 12, 19, 20]. For any nonempty open subsets U and V of X, the return set (or, time set) from U to V will be denoted as

$$N_T(U,V) := N(U,V) = \{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \varnothing\},$$

When there is no ambiguity, the index T will be omitted. Several topological notions in linear dynamics can be expressed by using return sets. The operator T is said to be topologically transitive if for any nonempty open subsets U, V of X, the return set $N(U,V)$ is nonempty (or, equivalently, infinite). Birkhoff's transitivity theorem says that hypercyclicity and transitivity coincide, see [8, Theorem 1.2]. Moreover, T is said to be topologically mixing (resp., weakly mixing, ergodic) if for any nonempty open subsets U, V of X, the return set $N(U,V)$ is cofinite (resp., thick, syndetic). Recall that a subset A of $\mathbb{N}_0$ is called

- *thick* if it contains arbitrarily long intervals, i.e., for every $n \in \mathbb{N}_0$, there exists $m \in A$ such that $[\![m, m+n]\!] := \{k \in \mathbb{N}_0 : m \leq k \leq m+n\} \subset A$.
- *syndetic* if it has bounded gaps, i.e., there exists an integer $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}_0$, we have $[\![m, m+n]\!] \cap A \neq \varnothing$.

Furstenberg families, as defined in Section 2.3, serve to generalize the concepts of topological transitivity, mixing, weakly mixing, and ergodicity in terms of $\mathcal{F}$-transitivity, see [9]. Let $\mathcal{F}$ be a Furstenberg family of subsets of $\mathbb{N}_0$. The operator T is called *$\mathcal{F}$-transitive* if, for any pair of nonempty open subsets U and V in X, the set $N(U,V)$ belongs to $\mathcal{F}$. Moreover, T is said to be topologically $\mathcal{F}$-recurrent if, for any nonempty open subset U in X, the set $N(U,U)$ belongs to $\mathcal{F}$ (see [12, Remark 8.2]). In [9], the authors provided a characterization of $\mathcal{F}$-transitive weighted bilateral backward shift operators on the spaces $\ell^p(\mathbb{Z})$ and $c_0(\mathbb{Z})$, as well as $\mathcal{F}$-transitive weighted unilateral backward shift operators.

This article is devoted to studying $\mathcal{F}$-transitive and topological $\mathcal{F}$-recurrence backward shift operators on trees, which extends the results obtained in [9, 22]. Additionally, we investigate the concept of Γ-supercyclicity, introduced in [14], for these operators. The structure of this article is outlined as follows:

In Section 2, we introduce the necessary notations and definitions for our analysis, including Furstenberg families and a review of backward shift operators on weighted ℓ^p-spaces and c_0-spaces on trees.

In Section 3, we give a variant of the $\mathcal{F}$-transitivity criterion that we will specifically use to describe $\mathcal{F}$-transitive backward shift operators on unrooted directed trees (see Theorem 3.1).

Section 4 focuses on characterizing $\mathcal{F}$-transitive backward shift operators on weighted ℓ^p-spaces and c_0-spaces defined on rooted directed trees. We demonstrate that $\mathcal{F}$-transitivity is equivalent to a property weaker than topological $\mathcal{F}$-recurrence (see Theorem 4.1). Additionally, we derive a corollary indicating that hypercyclicity of a backward shift operator B on a weighted ℓ^p-space or c_0-space defined on a rooted directed tree is equivalent to the existence of a bounded subset whose orbit under B is weakly dense in the underlying space (see Corollary 4.3).

In Section 5, we provide a characterization of $\mathcal{F}$-transitivity and establish its equivalence with topological $\mathcal{F}$-recurrence for backward shift operators on weighted ℓ^p-spaces and c_0-spaces defined on unrooted directed trees (see Theorem 5.1).

In Section 6, we discuss the concept of Γ-supercyclicity for backward shift operators on trees (see Theorem 6.1). This enables us to recover the characterization of hypercyclicity for these operators, as established in [22], and also to provide a characterization of their supercyclicity.

Finally, in the last section, we demonstrate that unlike the cases of the trees $\mathbb{N}_0$ or $\mathbb{Z}$, there exist backward shift operators on weighted spaces on trees that are not hypercyclic but possess orbits with a nonzero limit point (see Examples 7.2 and 7.4). Theorem 7.1 characterizes backward shifts that have an orbit with a nonzero limit point in the case of rooted trees. For unrooted cases, we provide a characterization of backward shifts that have an orbit of some nonnegative function with a nonzero limit point (see Proposition 7.3).

2 Preliminaries

2.1 Directed trees

In what follows, we will recall the necessary terminology related to trees, for more information we refer to [22, 26] and the references therein. A *(directed) graph* is a pair $\mathcal{G} = (V, E)$ which satisfies the two conditions:

(1) V is a nonempty set, its elements are called *vertices*.
(2) E is a subset of $\{(u, v) : u, v \in V, u \neq v\}$, its elements are called *edges*.

An *undirected edge* of $\mathcal{G}$ means an element of the set

$$\tilde{E} := \{\{u, v\} : (u, v) \in E \text{ or } (v, u) \in E\}.$$

If $\{u, v\} \in \tilde{E}$, we say that u and v are adjacent and we denote that by $u \backsim v$.

- The graph $\mathcal{G}$ is *connected* if any two distinct vertices $u \neq v \in V$ are related by an *undirected path*, i.e., there exist $v_1, \dots, v_n \in V$ such that $u = v_1 \backsim v_2 \backsim \cdots \backsim v_n = v$.
- A *circuit* of $\mathcal{G}$ is a sequence $\{v_j\}_{j=1}^n$ of distinct vertices ($n \geqslant 2$) such that, for all $j \in \{1, \dots, n-1\}$,

$$(v_j, v_{j+1}) \in E \quad \text{and} \quad (v_n, v_1) \in E.$$

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Definition 2.1 (Directed trees) A *directed tree* $\mathcal{T} = (V, E)$ is a directed graph such that

(1) (V, E) is connected.
(2) The set E of edges has no circuits.
(3) For any vertex $v \in E$, there exists at most one vertex $u \in V$ such that $(u, v) \in E$.
(4) The set V of vertices is countable.

In what follows, let us assume that (V, E) is a directed tree.

- A *parent* of a vertex $v \in V$ is a vertex $u \in V$ such that $(u, v) \in E$; we denote such vertex by $\mathrm{par}(v)$. Moreover, for any $n \geq 2$ and $v \in V$, $\mathrm{par}^n(v) := \mathrm{par}(\mathrm{par}^{n-1}(v))$, whenever this is well-defined. Furthermore, for any $F \subset V$ and $n \geq 1$, $\mathrm{par}^n(F)$ represents the set of all vertices $\mathrm{par}^n(v)$ (when well-defined) for $v \in F$.
- A *child* of a vertex $v \in V$ is a vertex $u \in V$ whose parent is v. We denote by $\mathrm{Chi}(v)$ the set of children of v, that is,

$$\mathrm{Chi}(v) = \{u \in V : \mathrm{par}(u) = v\}.$$

Moreover, for every $n \geqslant 2$, we denote

$$\mathrm{Chi}^n(v) := \bigcup_{u \in \mathrm{Chi}(v)} \mathrm{Chi}^{n-1}(u),$$

where $\mathrm{Chi}^1(v) = \mathrm{Chi}(v)$ and $\mathrm{Chi}^0(v) = \{v\}$. Furthermore, for any $F \subset V$ and $n \in \mathbb{N}_0$, we define

$$\mathrm{Chi}^n(F) := \bigcup_{v \in F} \mathrm{Chi}^n(v).$$

- A *root* of (V, E) is a vertex without a parent. Any directed tree has at most one root, if such an element exists, it will be denoted simply by $\mathtt{r}$.
- A *leaf* of (V, E) is a vertex without children.

2.2 Backward shift on sequences spaces on directed trees.

Let (V, E) be a directed tree, let $\mu = (\mu_v)_{v \in V}$ be a *weight* on V, that is, a sequence of nonzero numbers and let $\mathbb{K}$ denote the set of real or complex numbers. For $1 \leqslant p < +\infty$, the weighted ℓ^p-space of V is defined by

$$\ell^p(V, \mu) := \Big\{ f \in \mathbb{K}^V : \sum_{v \in V} |f(v)\, \mu_v|^p < +\infty \Big\},$$

and equipped with the norm

$$\|f\|_{p,\mu} := \Big(\sum_{v \in V} |f(v)\, \mu_v|^p \Big)^{1/p},$$

is a Banach space. As usual, $\ell^\infty(V, \mu)$ is the Banach space of functions $f \in \mathbb{K}^V$ such that

$$\|f\|_{\infty,\mu} = \sup_{v \in V} |f(v)\, \mu_v| < +\infty.$$

Finally, the weighted c_0-space on V is defined by

$$c_0(V, \mu) := \Big\{ f \in \mathbb{K}^V : \forall \varepsilon > 0, \exists F \subset V \text{ finite}, \forall v \in V \backslash F, |f(v)\, \mu_v| < \varepsilon \Big\},$$

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endowed with the norm $\|.\|_{\infty,\mu}$ is a closed subspace of $\ell^\infty(V,\mu)$. For $v \in V$, let $e_v = \chi_{\{v\}}$ be the characteristic function of $\{v\}$. Note that the space $\operatorname{span}\{e_v : v \in V\}$ is dense in $X = \ell^p(V,\mu), 1 \le p < +\infty$ or $X = c_0(V,\mu)$. The support of a function $f \in \mathbb{K}^V$ is the set

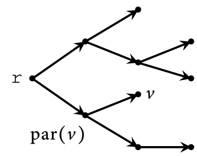
$$\operatorname{supp}(f) := \{v \in V : f(v) \neq 0\}.$$

According to [27], the backward shift B on $\mathbb{K}^V$ is defined by:

$$(Bf)(v) = \sum_{u \in \operatorname{Chi}(v)} f(u), \quad v \in V,$$

where an empty sum is zero. It can be seen as the adjoint of the forward shift operator S on trees, which is naturally defined on $\mathbb{K}^V$ as follows, for any $f \in \mathbb{K}^V$ and $v \in V$:

$$(Sf)(v) = \begin{cases} f(\operatorname{par}(v)) & \text{if } v \neq \mathtt{r} \\ 0 & \text{if } v = \mathtt{r} \end{cases},$$



Under the following pairing of duality:

$$\langle f, g\rangle = \sum_{v \in V} f(v)\overline{g(v)},$$

the dual of $\ell^p(V,\mu)$, for $1 \le p < +\infty$, is $\ell^{p^*}(V, 1/\overline{\mu})$, where p^* is the conjugate exponent of p. Moreover, the dual of $c_0(V,\mu)$ is $\ell^1(V,1/\overline{\mu})$. Now, if B or S defines a bounded linear operator on $X = \ell^p(V,\mu), 1 \le p < +\infty$, or $X = c_0(V,\mu)$ then

$$\langle Bf, g\rangle = \langle f, Sg\rangle,$$

and this is why B has been named the backward shift. Alternatively, we can define the backward shift B as the unique bounded linear operator that satisfies

$$Be_v = \begin{cases} e_{\operatorname{par}(v)} & \text{if } v \neq \mathtt{r} \\ 0 & \text{if } v = \mathtt{r} \end{cases}, \qquad v \in V.$$

In the following proposition, we will recall the necessary and sufficient conditions for the boundedness of the backward shift on weighted ℓ^p-spaces or c_0-spaces on trees, see [22, Proposition 2.3].

Proposition 2.2 *Let (V,E) be a directed tree, let $\mu = (\mu_v)_{v \in V}$ be a weight on V, and let B be the backward shift on $\mathbb{K}^V$.*

(a) *B is a bounded linear operator on $\ell^1(V,\mu)$ if and only if*

$$\sup_{v \in V \setminus \{\mathtt{r}\}} \left| \frac{\mu_{\operatorname{par}(v)}}{\mu_v} \right| < +\infty.$$

In this case, $\|B\| = \sup_{v \in V \setminus \{\mathtt{r}\}} \left| \frac{\mu_{\operatorname{par}(v)}}{\mu_v} \right|$.

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(b) *Let* $1 < p < +\infty$. *B is a bounded linear operator on* $\ell^p(V,\mu)$ *if and only if*

$$\sup_{v\in V}\sum_{u\in \mathrm{Chi}(v)}\left|\frac{\mu_v}{\mu_u}\right|^{p^*} < +\infty.$$

In this case, $\|B\| = \sup_{v\in V}\Big(\sum_{u\in \mathrm{Chi}(v)}\left|\frac{\mu_v}{\mu_u}\right|^{p^*}\Big)^{1/p^*}$.

(c) *B is a bounded linear operator on* $c_0(V,\mu)$ *if and only if*

$$\sup_{v\in V}\sum_{u\in \mathrm{Chi}(v)}\left|\frac{\mu_v}{\mu_u}\right| < +\infty.$$

In this case, $\|B\| = \sup_{v\in V}\sum_{u\in \mathrm{Chi}(v)}\left|\frac{\mu_v}{\mu_u}\right|$.

As mentioned previously, K. Grosse-Erdmann and D. Papathanasiou provided a comprehensive characterization of hypercyclicity for backward shift operators on weighted ℓ^p-spaces and c_0-spaces. In the cases of rooted directed trees, they obtained the following theorem, see [22, Theorem 4.3].

Theorem 2.3 *Let* (V,E) *be a rooted directed tree and let* $\mu = (\mu_v)_{v\in V}$ *be a weight on* V. *Let* $X = \ell^p(V,\mu)$, $1 \le p < +\infty$, *or* $X = c_0(V,\mu)$ *and suppose that the backward shift B is a bounded operator on X. Then the following assertions are equivalent:*

(1) *B is hypercyclic.*
(2) *B is weakly mixing.*
(3) *There is an increasing sequence* $(n_k)_{k\in\mathbb{N}}$ *of positive integers such that, for each* $v \in V$, *we have*

$$\begin{cases} \displaystyle\sum_{u\in \mathrm{Chi}^{n_k}(v)}\frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty,\\ \displaystyle\inf_{u\in \mathrm{Chi}^{n_k}(v)}|\mu_u| \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu),\\ \displaystyle\sum_{u\in \mathrm{Chi}^{n_k}(v)}\frac{1}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu). \end{cases}$$

For the case of unrooted directed trees, we refer to [22, Theorem 5.2]. In what follows, we will require the following lemma, which played a crucial role in establishing the results in [22].

Lemma 2.4 [22, Lemma 4.2] *Let J be a finite or countable set and let* $\mu = (\mu_j)_{j\in J} \in (\mathbb{K}\backslash\{0\})^J$. *Then*

$$\inf_{\|x\|_1=1}\sum_{j\in J}|x_j\mu_j| = \inf_{j\in J}|\mu_j|,$$

$$\inf_{\|x\|_1=1}\Big(\sum_{j\in J}|x_j\mu_j|^p\Big)^{1/p} = \Big(\sum_{j\in J}\frac{1}{|\mu_j|^{p^*}}\Big)^{-1/p^*},\quad 1<p<+\infty,\ \text{and } p^* = \frac{p}{p-1},$$

$$\inf_{\|x\|_1=1}\sup_{j\in J}|x_j\mu_j| = \Big(\sum_{j\in J}\frac{1}{|\mu_j|}\Big)^{-1},$$

where $x \in \mathbb{K}^J$, $\|x\|_1 = \sum_{j \in J} |x_j|$ *and* $\infty^{-1} = 0$. *The same holds when the sequences x are required, in addition, to be of finite support.*

2.3 Furstenberg families

We devote this subsection to recalling the necessary definitions related to Furstenberg families. A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ is a *Furstenberg family*, if for all $A \in \mathcal{F}$ it holds

- A is infinite;
- If $A \subset B \subset \mathbb{N}_0$, then $B \in \mathcal{F}$.

Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N}_0)$ be a Furstenberg family. Following [9], we denote by $\widetilde{\mathcal{F}}$ (resp., $\widetilde{\mathcal{F}}_+$) the Furstenberg family consisting of subsets $A \subset \mathbb{N}_0$ such that for every $N \in \mathbb{N}_0$, there exists $B \in \mathcal{F}$ satisfying

$$(B + [[-N, N]]) \cap \mathbb{N}_0 \subset A \quad (\text{resp., } B + [[0, N]] \subset A).$$

It is clear that $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{F}}_+ \subset \mathcal{F}$; thus, any $\widetilde{\mathcal{F}}$-transitive operator is $\mathcal{F}$-transitive.

A *filter* on $\mathbb{N}_0$ is a Furstenberg family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ that satisfies the property $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$. A nonempty collection $\mathcal{B}$ of subsets of $\mathbb{N}_0$ is called a *filter base* if the following conditions hold:

- Every set in $\mathcal{B}$ is infinite.
- The intersection of any two sets in $\mathcal{B}$ contains a set in $\mathcal{B}$.

It is worth noting that every filter is a filter base. Conversely, if $\mathcal{B}$ is a filter base of subsets of $\mathbb{N}_0$, then the collection of sets

$$\mathcal{F}_{\mathcal{B}} = \{A \subset \mathbb{N}_0 : B \subset A \text{ for some } B \in \mathcal{B}\},$$

forms a filter, called the filter generated by $\mathcal{B}$. For a more detailed discussion on filters and filter bases, see [13, Chapter I, Section 6]. See also [9, 20] for further examples of Furstenberg families and filters.

3 $\mathcal{F}$-transitivity criterion

In [9, Theorem 2.4], Bès, Menet, Peris, and Puig established an $\mathcal{F}$-transitivity criterion. This criterion uses the concept of a limit along the family $\mathcal{F}$. Let $(x_n)_n$ be a sequence in X, and let $x \in X$. We say that

$$\mathcal{F}\text{-}\lim_n x_n = x,$$

if $\{n \in \mathbb{N}_0 : x_n \in U\} \in \mathcal{F}$ holds for every neighborhood U of x. In our analysis, we will require a variant of this criterion, combined with the hypercyclicity criterion [22, Proposition 5.1]. For the sake of completeness, we will include its proof, which is an adaptation of the proof provided in [9, Theorem 2.4].

Theorem 3.1 ($\mathcal{F}$-Transitivity Criterion) *Let X be an infinite-dimensional Banach space, $T \in \mathcal{L}(X)$ and let $\mathcal{F}$ be a Furstenberg family on $\mathbb{N}_0$ such that $\widetilde{\mathcal{F}}$ is a filter. Suppose that there exist two dense subsets X_0 and Y_0 in X and maps $I_n : X_0 \to X$ and*

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$S_n : Y_0 \to X$, $n \in \mathbb{N}_0$, such that, for any $x \in X_0$ and any $y \in Y_0$, the following conditions hold:

(1) $\mathcal{F}\text{-}\lim_n (I_n x, T^n I_n x) = (x, 0)$;
(2) $\mathcal{F}\text{-}\lim_n (S_n y, T^n S_n y) = (0, y)$.

Then T is $\widetilde{\mathcal{F}}$-transitive.

Proof Note that, according to condition (2), T has dense range in X. Now, let U, V be nonempty open subsets of X. There exist nonempty open subsets U', V' of X and a 0-neighborhood W such that

$$U' + W \subset U \quad \text{and} \quad V' + W \subset V.$$

Since $\widetilde{\mathcal{F}}$ is a filter and

$$N(U, V) \supset N(U' + W, V' + W) \supset N(U', W) \cap N(W, V'),$$

it is enough to show that $N(U', W) \in \widetilde{\mathcal{F}}$ and $N(W, V') \in \widetilde{\mathcal{F}}$. Let $N \in \mathbb{N}_0$. Since T has dense range, the sets $T^{-N}U'$ and $T^{-N}V'$ are nonempty open sets. Choose $x \in X_0 \cap T^{-N}U'$ and $y \in Y_0 \cap T^{-N}V'$. We start by showing that $N(T^{-N}U', W) \in \widetilde{\mathcal{F}}_+$. For any $M \in \mathbb{N}_0$, by (1), there exists $A_M \in \mathcal{F}$ such that, for every $n \in A_M$,

$$I_n x \in T^{-N}U' \quad \text{and} \quad T^n I_n x \in \bigcap_{k=0}^{M} T^{-k}W.$$

Note that, for any $n \in A_M + [\![0, M]\!]$, there exists $k \in [\![0, M]\!]$ such that $n - k \in A_M$, hence

$$I_{n-k}x \in T^{-N}U' \quad \text{and} \quad T^{n-k}I_{n-k}x \in T^{-k}W,$$

therefore, $I_{n-k}x \in T^{-N}U'$ and $T^n I_{n-k}x \in W$, so $n \in N(T^{-N}U', W)$. Thus, $A_M + [\![0, M]\!] \subset N(T^{-N}U', W)$ holds for any arbitrary $M \in \mathbb{N}_0$. Consequently, $N(T^{-N}U', W) \in \widetilde{\mathcal{F}}_+$. Thus, there is $B \in \mathcal{F}$ such that $B + [\![0, 2N]\!] \subset N(T^{-N}U', W)$. Therefore,

$$(B + [\![-N, N]\!]) \cap \mathbb{N}_0 \subset (N(T^{-N}U', W) - N) \cap \mathbb{N}_0 \subset N(U', W),$$

and, since N was arbitrary, we deduce that $N(U', W) \in \widetilde{\mathcal{F}}$.

On the other hand, by (2), there is $B \in \mathcal{F}$ such that, for every $n \in B$,

$$S_n y \in \bigcap_{k=0}^{2N} T^{-k}W \quad \text{and} \quad T^n S_n(y) \in T^{-N}V'.$$

In particular, for any $n \in (B + [\![-N, N]\!]) \cap \mathbb{N}_0$, there exists $k \in [\![-N, N]\!]$ such that $n - k \in B$ and $N - k \in [\![0, 2N]\!]$, thus

$$S_{n-k}y \in T^{k-N}W \quad \text{and} \quad T^{n-k}S_{n-k}y \in T^{-N}V',$$

hence

$$T^{N-k}S_{n-k}y \in W \quad \text{and} \quad T^n(T^{N-k}S_{n-k}y) \in T^{-N}V',$$

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from which we deduce that $n \in N(W, V')$. Therefore,

$$(B + [\![-N, N]\!]) \cap \mathbb{N}_0 \subset N(W, V').$$

Since N was arbitrary, we obtain that $N(W, V') \in \widetilde{\mathcal{F}}$. This finishes the proof. ■

4 $\mathcal{F}$-transitivity - rooted case

Our first main result provides a characterization of $\mathcal{F}$-transitivity for backward shifts on rooted directed trees and establishes its equivalence with certain weak properties.

Theorem 4.1 *Let (V, E) be a rooted directed tree, let $\mu = (\mu_v)_{v \in V}$ be a weight on V, and let $\mathcal{F}$ be a Furstenberg family on $\mathbb{N}_0$. Let $X = \ell^p(V, \mu), 1 \le p < +\infty$, or $X = c_0(V, \mu)$ and suppose that the backward shift B is a bounded operator on X. The following assertions are equivalent:*

(1) *B is $\widetilde{\mathcal{F}}$-transitive.*
(2) *B is $\mathcal{F}$-transitive.*
(3) *There exists a bounded subset $C \subset X \backslash \{0\}$ such that for every nonempty weakly open subset $W \subset X$, $N(C, W) \in \mathcal{F}$.*
(4) *For every $N \in \mathbb{N}$ and every finite subset $F \subset V$, we have*

$$\bigcap_{v \in F} \left\{ n \in \mathbb{N}_0 : \sup_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} \in \mathcal{F}, \quad \text{if } X = \ell^1(V, \mu);$$

$$\bigcap_{v \in F} \left\{ n \in \mathbb{N}_0 : \Big(\sum_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*} > N \right\} \in \mathcal{F}, \quad \text{if } X = \ell^p(V, \mu), 1 < p < +\infty;$$

$$\bigcap_{v \in F} \left\{ n \in \mathbb{N}_0 : \sum_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} \in \mathcal{F}, \quad \text{if } X = c_0(V, \mu).$$

If $\mathcal{F}$ satisfies $A \cap [n, +\infty[\in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $n \in \mathbb{N}$, then the above conditions are equivalent to

(5) *For every $g \in X$, for every open neighborhood U of g, for every weakly open neighborhood W of g, $N(U, W) \in \mathcal{F}$.*

If $\mathcal{F}$ is a filter, then the conditions (1)-(4) are equivalent to

(6) *For every $N \in \mathbb{N}$ and every $v \in V$, we have*

$$\left\{ n \in \mathbb{N}_0 : \sup_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} \in \mathcal{F}, \quad \text{if } X = \ell^1(V, \mu);$$

$$\left\{ n \in \mathbb{N}_0 : \Big(\sum_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*} > N \right\} \in \mathcal{F}, \quad \text{if } X = \ell^p(V, \mu), 1 < p < +\infty;$$

$$\left\{ n \in \mathbb{N}_0 : \sum_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} \in \mathcal{F}, \quad \text{if } X = c_0(V, \mu).$$

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Proof We will only prove the equivalences in the case where $X = \ell^p(V,\mu)$ with $1 < p < +\infty$. A similar argument can be made to deduce the cases where $X = \ell^1(V,\mu)$ and $X = c_0(V,\mu)$. It is clear that (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5). Moreover, (1) $\Rightarrow$ (2) since $\widetilde{\mathcal{F}} \subset \mathcal{F}$. Suppose that B is $\mathcal{F}$-transitive; thus, it is transitive (equivalently, hypercyclic). Using [22, Theorem 4.3], we can conclude that B is weakly mixing. Thus, B is both $\mathcal{F}$-transitive and weakly mixing, which is equivalent to saying that B is $\widetilde{\mathcal{F}}$-transitive, according to [9, Lemma 2.3]. Hence, (2) $\Rightarrow$ (1). Moreover, it clear that (4) $\Leftrightarrow$ (6) when $\mathcal{F}$ is a filter.

(3) $\Rightarrow$ (4). Let C be a bounded subset of $\ell^p(V,\mu)\backslash\{0\}$ such that $N(W,C) \in \mathcal{F}$, for every nonempty weakly open subset W of $\ell^p(V,\mu)$. Set $M := \sup\{\|f\|_{p,\mu} : f \in C\} < +\infty$. Let $N \in \mathbb{N}$ and let $F \subset V$ be a finite subset. Then

$$W = \{f \in \ell^p(V,\mu) : |\langle f - (MN+1)\sum_{u\in F} e_u, e_v\rangle| < 1,\ \forall v \in F\},$$

is a weakly open neighborhood of $(MN+1)\sum_{u\in F} e_u$. By the hypothesis, $N(C,W) \in \mathcal{F}$. Since $\mathcal{F}$ is a Furstenberg family, we need only to prove that

$$N(C,W) \subset \bigcap_{v\in F}\Big\{n \in \mathbb{N}_0 : \Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*} > N\Big\}. \tag{4.1}$$

Let $n \in N(C,W)$. There exists then $f \in C$ such that $B^n f \in W$, thus for every $v \in F$

$$|(B^n f)(v) - (MN+1)| < 1,$$

hence, by using Hölder's inequality, we obtain

$$\begin{aligned} MN &< \sum_{u\in \mathrm{Chi}^n(v)} |f(u)| \\ &\le \Big(\sum_{u\in \mathrm{Chi}^n(v)} |f(u)\mu_u|^p\Big)^{1/p}\Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*} \\ &\le \|f\|_{p,\mu}\Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*} \\ &\le M\Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*}, \end{aligned}$$

therefore

$$\Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*} > N, \quad \forall v \in F,$$

and so (4.1) holds.

Let us show that (4) $\Rightarrow$ (2). Assume that (4) holds, that is, for every $N > 0$ and every finite subset $F \subset V$, we have

$$I(F,N) := \bigcap_{v\in F}\Big\{n \in \mathbb{N}_0 : \Big(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}}\Big)^{1/p^*} > N\Big\} \in \mathcal{F}.$$

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Let $\mathcal{B}$ be the filter base consisting of all subsets $I(F,N)$ of $\mathbb{N}_0$, where $N > 0$ and $F \subset V$ is finite. We define

$$\mathcal{F}_{\mathcal{B}} = \{A \subset \mathbb{N}_0 : B \subset A \text{ for some } B \in \mathcal{B}\}.$$

In other words, $\mathcal{F}_{\mathcal{B}}$ is the filter generated by $\mathcal{B}$. We will show that B satisfies the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion, which implies that B is $\mathcal{F}$-transitive since $\mathcal{F}_{\mathcal{B}} \subset \mathcal{F}$. Set $\mathcal{D} = \operatorname{span}\{e_v : v \in V\}$. Note that $\mathcal{D}$ is dense in $\ell^p(V,\mu)$. Let $f = \sum_{u\in F} f(u)e_u \in \mathcal{D}$, with $F \subset V$ finite. Let $\varepsilon > 0$ and $\mathcal{U} := B(0,\varepsilon)$ be the open ball of center 0 and radius ε in $\ell^p(V,\mu)$. Let us start by proving that

$$N(f,\mathcal{U}) := \{n \in \mathbb{N}_0 : \ B^n f \in \mathcal{U}\} \in \mathcal{F}_{\mathcal{B}}.$$

To achieve this, it is enough to find a number $N \geq 1$ such that $I(F,N) \subset N(f,\mathcal{U})$. First, select an integer $n_0 \in \mathbb{N}$ large enough such that $\operatorname{Chi}^n(V) \cap F = \varnothing$ for every integer $n \geq n_0$. Then, we set

$$N := 1 + \max_{v\in F} \max_{0\leq k<n_0} \Big(\sum_{u\in \operatorname{Chi}^k(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*}.$$

Note that, by using Proposition 2.2 and the boundedness of B^k for every $0 \leq k < n_0$, N is finite. Now, suppose that $n \in I(F,N)$. This implies that $n \geq n_0$. Thus, for every $v \in V$, we have $\operatorname{Chi}^n(v) \cap F = \varnothing$, hence, $(B^n f)(v) = 0$. This implies that $n \in N(f,\mathcal{U})$. Therefore, we have shown that $I(F,N) \subset N(f,\mathcal{U})$, and consequently, $N(f,\mathcal{U}) \in \mathcal{F}_{\mathcal{B}}$ since $\mathcal{F}_{\mathcal{B}}$ is a Furstenberg family. Hence, the first condition of the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion holds, where the maps I_n correspond to the identity map.

Now, for every $v \in V$, let $(\delta_{v,n})_n$ be a decreasing sequence of positive real numbers tending to zero. For all $v \in V$ and $n \in \mathbb{N}_0$, by Lemma 2.4, we have

$$\inf_{\|x\|_1=1} \Big(\sum_{u\in \operatorname{Chi}^n(v)} |x_u \mu_u|^p \Big)^{1/p} = \Big(\sum_{u\in \operatorname{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*}.$$

There exists then $g_{v,n} \in \mathbb{K}^V$ nonnegative, of support in $\operatorname{Chi}^n(v)$, such that

$$\sum_{u\in \operatorname{Chi}^n(v)} g_{v,n}(u) = 1 \quad \text{and}$$

$$\Big(\sum_{u\in \operatorname{Chi}^n(v)} |g_{v,n}(u)\,\mu_u|^p \Big)^{1/p} < \Big(\sum_{u\in \operatorname{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*} + \delta_{v,n}. \tag{4.2}$$

Note that $g_{v,n} \in \ell^p(V,\mu)$. Let us define the maps $S_k : \mathcal{D} \to \ell^p(V,\mu)$ linearly, for $k \geq 0$, by

$$S_k e_v = g_{v,k}, \quad \forall v \in V.$$

Let $\varepsilon > 0$ and $g = \sum_{u\in F} g(a)e_a \in \mathcal{D}$, where F is the support of g. Set again $\mathcal{U} := B(0,\varepsilon)$ and $\mathcal{V} := B(g,\varepsilon)$. We want to show that

$$A := \{n \in \mathbb{N}_0 : \ (S_n g, B^n S_n g) \in \mathcal{U} \times \mathcal{V}\} \in \mathcal{F}_{\mathcal{B}}.$$

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Let $n_0 \in \mathbb{N}$ be large enough so that $\frac{\varepsilon}{|g(a)||F|} - \delta_{a,n_0} > 0$, for every $a \in F$ (where $|F|$ stands for the cardinality of the finite set F). Set

$$N = \max\left\{1 + \max_{v \in F} \max_{0 \le k < n_0} \Big(\sum_{u \in \mathrm{Chi}^k(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*}, \frac{1}{\min_{a \in F}\left\{ \frac{\varepsilon}{|g(a)||F|} - \delta_{a,n_0} \right\}} \right\}.$$

We will show that $I(F,N) \subset A$. Note that $B^n S_n e_v = B^n g_{v,n} = e_v$, for every integer $n \ge 0$. Indeed, by (4.2), we have

$$\begin{aligned}(B^n g_{v,n})(a) &= \sum_{b \in \mathrm{Chi}^n(a)} g_{v,n}(b) \\ &= \begin{cases} 1 & \text{if } a = v \\ 0 & \text{if } a \ne v \end{cases} \\ &= e_v(a).\end{aligned}$$

Thus, $B^n S_n g = g \in \mathcal{V}$, for every integer $n \ge 0$. Let $n \in I(F,N)$, so as above and by the definition of N, we have $n \ge n_0$ and

$$\Big(\sum_{u \in \mathrm{Chi}^n(a)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*} < \frac{1}{N}, \ \forall a \in F,$$

thus, by using (4.2), we obtain:

$$\Big(\sum_{u \in \mathrm{Chi}^n(a)} |g_{a,n}(u)\, \mu_u|^p \Big)^{1/p} < \frac{1}{N} + \delta_{a,n}, \ \forall a \in F,$$

by using again the definition of N, we deduce

$$\Big(\sum_{u \in \mathrm{Chi}^n(a)} |g_{a,n}(u)\, \mu_u|^p \Big)^{1/p} < \frac{\varepsilon}{|g(a)||F|}, \ \forall a \in F,$$

therefore

$$\|S_n g\|_{p,\mu} \le \sum_{a \in F} |g(a)| \|g_{a,n}\|_{p,\mu} = \sum_{a \in F} |g(a)| \Big(\sum_{b \in \mathrm{Chi}^n(a)} |g_{a,n}(b)\mu_b|^p \Big)^{1/p} < \varepsilon,$$

hence $S_n g \in \mathcal{U}$. We have shown that $n \in A$ and so $I(F,N) \subset A$. Consequently, $A \in \mathcal{F}_{\mathcal{B}}$. This provides the second condition of the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion. Hence, B is $\mathcal{F}$-transitive and statement (2) holds.

(5) $\Rightarrow$ (4). Let us assume that (5) holds. Let F be a finite subset of V and let $N \in \mathbb{N}$. We want to show that

$$I(F,N) := \bigcap_{v \in F} \left\{ n \in \mathbb{N}_0 : \Big(\sum_{u \in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*} > N \right\} \in \mathcal{F}.$$

Set $g = \sum_{v \in F} e_v \in \ell^p(V,\mu)$. Let $U = \{ f \in \ell^p(V,\mu) : \|f - g\|_{p,\mu} < 1/(2N)\}$ and $W = \{ f \in \ell^p(V,\mu) : |\langle f - g, e_v \rangle| < 1/(2N), \ \forall v \in F\}$. By the hypothesis, we have $N(U,W) \in \mathcal{F}$ an so $N(U,W) \cap [n, +\infty[\in \mathcal{F}$ for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ big enough so

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that $\mathrm{Chi}^n(F) \cap F = \varnothing$, for all $n \geq n_0$. We will show that

$$N(U, W) \cap [n_0, +\infty[\subset I(F, N).$$

Let $n \in N(U, W) \cap [n_0, +\infty[$. Thus, $\mathrm{Chi}^n(F) \cap F = \varnothing$ and there exists $f \in \ell^p(V, \mu)$ such that

$$\|f - g\|_{p,\mu} < \frac{1}{2N} \quad \text{and} \quad |\langle B^n f - g, e_v \rangle| < \frac{1}{2N}, \quad \forall v \in F. \tag{4.3}$$

Fix now $v \in F$. Then the rightmost inequality in (4.3) gives

$$0 < \frac{1}{2} \leq 1 - \frac{1}{2N} < \sum_{u \in \mathrm{Chi}^n(v)} |f(u)|.$$

Since $\mathrm{Chi}^n(v) \cap F = \varnothing$, by using the leftmost inequality in (4.3), we obtain

$$\sum_{u \in \mathrm{Chi}^n(v)} |f(u)\mu_u|^p < \frac{1}{(2N)^p}.$$

Using Hölder's inequality (for $p > 1$) and the last two inequalities, we can conclude that:

$$\begin{aligned} \frac{1}{2N} &> \Big(\sum_{u \in \mathrm{Chi}^n(v)} |f(u)\mu_u|^p \Big)^{1/p} \\ &\geq \Big(\sum_{u \in \mathrm{Chi}^n(v)} |f(u)| \Big) \Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{-p^*} \Big)^{-1/p^*} \\ &> \frac{1}{2} \Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{-p^*} \Big)^{-1/p^*}, \end{aligned}$$

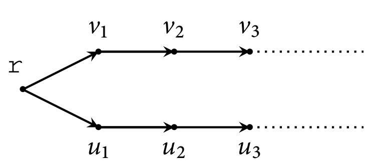
thus

$$\Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{-p^*} \Big)^{1/p^*} > N.$$

This holds for all $v \in F$. Therefore, $n \in I(F, N)$, hence $N(U, W) \cap [n_0, +\infty[\subset I(F, N)$. Consequently, $I(F, N) \in \mathcal{F}$. ∎

The following example shows that the equivalence between conditions (4) and (6) in Theorem 4.1 does not hold for Frustenberg families which are not filters.

Example 4.2 Let (V, E) be the following rooted directed tree.



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Let $(m_k)_{k\geq 1}$ be a strictly increasing sequence of nonnegative integers and $\mu = (\mu_v)_{v\in V}$ be the weight defined by $\mu_{\mathtt{r}} = 1$ and, for every $k \geq 1$, $\mu_{u_k} = 1/\mu_{v_k}$, where

$$(\mu_{v_k})_{k\geq 1} = \Big(\underbrace{\frac{1}{2},\dots,\frac{1}{2^{m_1}},\frac{2}{2^{m_1}},\dots,\frac{1}{2}}_{2m_1},1,\underbrace{2,\dots,2^{m_2},\frac{2^{m_2}}{2},\dots,2}_{2m_2},1,\dots\Big).$$

Let B be the backward shift on $\ell^p(V,\mu), 1 < p < +\infty$. By Proposition 2.2, B is bounded and $\|B\| = (2^{p^*} + 1/2^{p^*})^{1/p^*}$. Let $\mathcal{J}$ be the Furstenberg family of infinite subsets of $\mathbb{N}_0$. Note that $\mathcal{J}$ is not a filter. By the definition of the weight, for any $N \in \mathbb{N}$ and $k \in \mathbb{N}$, we have

$$I(\mathtt{r},N) := \left\{n \in \mathbb{N}_0 : \frac{1}{|\mu_{u_n}|^{p^*}} + \frac{1}{|\mu_{v_n}|^{p^*}} > N^{p^*}\right\} \in \mathcal{J},$$

$$I(v_k,N) := \left\{n \in \mathbb{N}_0 : \frac{1}{|\mu_{v_{k+n}}|} > N\right\} \in \mathcal{J} \quad \text{and} \quad I(u_k,N) := \left\{n \in \mathbb{N}_0 : \frac{1}{|\mu_{u_{k+n}}|} > N\right\} \in \mathcal{J}.$$

However, $I(u_k,N) \cap I(v_k,N) = \varnothing$, so $I(u_k,N) \cap I(v_k,N) \notin \mathcal{J}$.

A bounded linear operator T defined on a Banach space X is said to be weakly hypercyclic if there exists a vector $x \in X$ such that its orbit $\mathrm{Orb}(x,T)$ is dense in X with respect to the weak topology.

When the Furstenberg family $\mathcal{F}$ in the previous theorem is chosen as the collection of syndetic subsets of $\mathbb{N}_0$, we can derive a characterization of topologically ergodic backward shift operators on rooted directed trees. Furthermore, when $\mathcal{F}$ represents the family of infinite subsets, $\mathcal{F}$-transitivity coincides with hypercyclicity. This allows us to deduce the following corollary.

Corollary 4.3 *Let (V,E) be a rooted directed tree and let $\mu = (\mu_v)_{v\in V}$ be a weight on V. Let $X = \ell^p(V,\mu)$, $1 \leq p < +\infty$, or $X = c_0(V,\mu)$ and suppose that the backward shift B is a bounded operator on X. Then the following assertions are equivalent:*

(1) *B is hypercyclic.*
(2) *B is weakly hypercyclic.*
(3) *There exists a bounded subset C of $X\backslash\{0\}$ such that* $\mathrm{Orb}(C,B)$ *is weakly dense in X.*
(4) *For every $g \in X$, for every open neighborhood U of g, for every weakly open neighborhood W of g, there exists an integer $n \geq 1$ such that*

$$B^n(U) \cap W \neq \varnothing.$$

Note that the implication $(2) \Rightarrow (4)$ in the previous corollary holds for any operator (see [17, p. 42]). Furthermore, it is worth mentioning that if there exists an integer n satisfying condition (4), then the return set from U to W is infinite.

Remark 4.4 Let X be a Banach space and $T \in \mathcal{L}(X)$ be a linear operator. If for any nonempty weakly open subset U of X, there exists an integer $n \geq 1$ such that

$$T^n(U) \cap U \neq \varnothing,$$

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then there are infinitely many integers that satisfy this property. To see this, note that the range of T is weakly dense in X since it meets any nonempty weakly open subset of X. By Mazur's theorem, the weak and norm closures of the range of T coincide, which implies that T has dense range.

Now, let U be a nonempty weakly open subset of X, $x \in U$, and let $n \geq 1$ be an integer such that $T^n x \in U$. Then, the set $W = U \cap T^{-n}(U)$ is a nonempty weakly open subset of X. By the hypothesis, there exists an integer $m \geq 1$ such that $T^m(W) \cap W \neq \emptyset$, and so we have

$$T^{n+m}(U) \cap U \neq \emptyset.$$

Repeating this argument, we conclude that there exist infinitely many integers $k \geq 1$ such that $T^k(U) \cap U \neq \emptyset$.

In the case of rooted trees, by the previous corollary, recurrence and hypercyclicity coincide for backward shifts. This equivalence, in fact, holds for any bounded linear operator $T \in \mathcal{L}(X)$ whose generalized kernel (or, more generally, the set of vectors $x \in X$ such that $T^n x \longrightarrow 0$ as $n \longrightarrow +\infty$) is dense in the underlying space (see [12, Theorem 2.12]). Now, we will establish a similar result for $\mathcal{F}$-transitivity.

Proposition 4.5 *Let X be a Banach space, $T \in \mathcal{L}(X)$ and let $\mathcal{F}$ be a Furstenberg family on $\mathbb{N}_0$ such that $A \cap [n, +\infty[\in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $n \in \mathbb{N}$. Suppose that the set*

$$X_0 := \{x \in X : \lim_{n \to +\infty} T^n x = 0\},$$

is dense in X. Then the following statements are equivalent:

(1) *T is $\mathcal{F}$-transitive.*
(2) *T is topologically $\mathcal{F}$-recurrent.*

Proof It is clear that (1) $\Rightarrow$ (2). Assume that (2) holds. Let U, V be nonempty open subsets of X. There exist a nonempty open subset V' of X and a 0-neighborhood W such that $W + V' \subset V$. Assume that $U \neq V'$, otherwise there is nothing to prove. Let $y \in (U - V') \cap X_0$. Then there exists a nonnegative integer N such that $T^n y \in W$, for all $n \geq N$. By the hypothesis, we have $N(V', V') \in \mathcal{F}$ and so $N(V', V') \cap [N, +\infty[\in \mathcal{F}$. Now, we will show that

$$N(V', V') \cap [N, +\infty[\subset N(U, V).$$

Let $n \in N(V', V') \cap [N, +\infty[$. Thus, $T^n V' \cap V' \neq \emptyset$ and $T^n y \in W$. Then there exists $z \in V'$ such that $T^n z \in V'$. Hence $x = y + z \in U$ and

$$T^n x = T^n y + T^n z \in W + V' \subset V,$$

therefore, $n \in N(U, V)$. Consequently, $N(V', V') \cap [N, +\infty[\subset N(U, V)$ and so $N(U, V) \in \mathcal{F}$. This finishes the proof. ∎

5 $\mathcal{F}$-transitivity - unrooted case

We will now present our second main result, which characterizes $\mathcal{F}$-transitivity for backward shifts on unrooted directed trees and establishes its equivalence with topological $\mathcal{F}$-recurrence.

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Theorem 5.1 *Let (V,E) be an unrooted directed tree, let $\mu = (\mu_v)_{v\in V}$ be a weight on V, and let $\mathcal{F}$ be a Furstenberg family on $\mathbb{N}_0$. Let $X = \ell^p(V,\mu)$, $1 \le p < +\infty$, or $X = c_0(V,\mu)$ and suppose that the backward shift B is a bounded operator on X. The following assertions are equivalent:*

(1) *B is $\widetilde{\mathcal{F}}$-transitive.*
(2) *B is $\mathcal{F}$-transitive.*
(3) *For every $N \in \mathbb{N}$ and every finite subset $F \subset V$, we have $I(F,N) \cap J(F,N) \in \mathcal{F}$, where*

$$I(F,N) := \begin{cases} \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \sup_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} & \text{if } X = \ell^1(V,\mu); \\ \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \left(\sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|^{p^*}} \right)^{1/p^*} > N \right\} & \text{if } X = \ell^p(V,\mu),\ 1 < p < +\infty; \\ \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \sum_{u\in \mathrm{Chi}^n(v)} \frac{1}{|\mu_u|} > N \right\} & \text{if } X = c_0(V,\mu); \end{cases}$$

and

$$J(F,N) := \begin{cases} \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \min\left(|\mu_{\mathrm{par}^n(v)}|, \inf_{u\in \mathrm{Chi}^n(\mathrm{par}^n(v))} |\mu_u| \right) < \frac{1}{N} \right\} & \text{if } X = \ell^1(V,\mu); \\ \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \frac{1}{|\mu_{\mathrm{par}^n(v)}|^{p^*}} + \sum_{u\in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|^{p^*}} > N^{p^*} \right\} & \text{if } X = \ell^p(V,\mu),\ 1 < p < +\infty; \\ \bigcap_{v\in F} \left\{ n \in \mathbb{N}_0 : \frac{1}{|\mu_{\mathrm{par}^n(v)}|} + \sum_{u\in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|} > N \right\} & \text{if } X = c_0(V,\mu). \end{cases}$$

If $\mathcal{F}$ satisfies $A \cap [n,+\infty[\in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $n \in \mathbb{N}$, then the above conditions are equivalent to

(4) *B is topologically $\mathcal{F}$-recurrent.*

Proof We will only prove the equivalences in the case where $X = \ell^p(V,\mu)$ with $1 < p < +\infty$. A similar argument can be made to deduce the cases where $X = \ell^1(V,\mu)$ and $X = c_0(V,\mu)$. As in the proof of Theorem 4.1 and by using [22, Theorem 5.2], we obtain that $(1) \Longleftrightarrow (2)$. Note that it is clear that $(2) \Rightarrow (4)$.

Let us show that $(4) \Rightarrow (3)$. Let F be a finite subset of V and let $N \in \mathbb{N}$. Set $g = \sum_{v\in F} e_v \in \ell^p(V,\mu)$. Let $U = \{ f \in \ell^p(V,\mu) : \|f - g\|_{p,\mu} < \frac{1}{2N} \}$. Let $n_0 \in \mathbb{N}$ be big enough so that $\mathrm{Chi}^n(F) \cap F = \varnothing$ and $\mathrm{par}^n(F) \cap F = \varnothing$, for all $n \ge n_0$. By the hypothesis, $N(U,U) \in \mathcal{F}$ and so $N(U,U) \cap [n_0,+\infty[\in \mathcal{F}$. Since $\mathcal{F}$ is a Furstenberg family, it is enough to prove that

$$N(U,U) \cap [n_0,+\infty[\subset I(F,N) \cap J(F,N).$$

We can assume that N is sufficiently large such that $1 < |\mu_v| N$, $\forall v \in F$. Let $n \in N(U,U) \cap [n_0,+\infty[$. Then there then a function f in $\ell^p(V,\mu)$ such that

$$\|f - g\|_{p,\mu} < \frac{1}{2N} \quad \text{and} \quad \|B^n f - g\|_{p,\mu} < \frac{1}{2N}. \tag{5.1}$$

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Fix now $v \in F$. By the rightmost inequality above, we have

$$0 < \frac{1}{2} < 1 - \frac{1}{2N|\mu_v|} < \sum_{u \in \mathrm{Chi}^n(v)} |f(u)|. \tag{5.2}$$

Since $\mathrm{Chi}^n(v) \cap F = \varnothing$ and $F = \mathrm{supp}(g)$, by the leftmost inequality in (5.1), we obtain

$$\sum_{u \in \mathrm{Chi}^n(v)} |f(u)\,\mu_u|^p < \frac{1}{(2N)^p}. \tag{5.3}$$

By Hölder's inequality ($p > 1$) and (5.2), we get

$$\begin{aligned}\sum_{u \in \mathrm{Chi}^n(v)} |f(u)\mu_u|^p &\geqslant \Big(\sum_{u \in \mathrm{Chi}^n(v)} |f(u)|\Big)^p \Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{\frac{p}{1-p}}\Big)^{1-p} \\ &\geqslant \frac{1}{2^p}\Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{\frac{p}{1-p}}\Big)^{1-p},\end{aligned}$$

combining this with (5.3), we get

$$\Big(\sum_{u \in \mathrm{Chi}^n(v)} |\mu_u|^{-p^*}\Big)^{1/p^*} > N;$$

this holds for any fixed $v \in F$, hence $n \in I(F, N)$. Let us also show that $n \in J(F, N)$. By contradiction, suppose that there is a $v \in F$ for which it holds

$$\frac{1}{|\mu_{\mathrm{par}^n(v)}|^{p^*}} + \sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|^{p^*}} \leq N^{p^*}. \tag{5.4}$$

Thus, $\frac{1}{N} \leqslant |\mu_{\mathrm{par}^n(v)}|$. Set

$$h = (f - g)\chi_{\mathrm{Chi}^n(\mathrm{par}^n(v))}.$$

One has

$$\|h\|_{p,\mu} \leqslant \|f - g\|_{p,\mu} < \frac{1}{2N}. \tag{5.5}$$

Note now that we have

$$(B^n f)(\mathrm{par}^n(v)) = d_n + \sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} h(u),$$

where $d_n := \left|\mathrm{Chi}^n(\mathrm{par}^n(v)) \cap F\right| \geq 1$, where $|\cdot|$ stands for the cardinality. Since $\mathrm{par}^n(v) \in V \backslash F$, by using the rightmost inequality in (5.1), we get

$$\begin{aligned}\frac{1}{2N} &> |(B^n f)(\mathrm{par}^n(v))\,\mu_{\mathrm{par}^n(v)}| \\ &\geq \Big(d_n - \Big|\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} h(u)\Big|\Big)|\mu_{\mathrm{par}^n(v)}| \\ &\geq \frac{1}{N}\Big(1 - \Big|\sum_{u \in \mathrm{Chi}^{n_k}(\mathrm{par}^{n_k}(v))} h_k(u)\Big|\Big),\end{aligned}$$

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hence

$$\frac{1}{2} < \sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} |h(u)|.$$

By Hölder's inequality and (5.5), we have

$$\begin{aligned}\frac{1}{2N} &> \Big(\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} |h(u)|\Big)\Big(\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} |\mu_u|^{-p^*}\Big)^{-1/p^*} \\ &> \frac{1}{2}\Big(\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} |\mu_u|^{-p^*}\Big)^{-1/p^*},\end{aligned}$$

then

$$\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|^{p^*}} > N^{p^*}.$$

This contradicts (5.4). Therefore, $n \in J(F,N)$. Consequently, $N(U,U) \cap [n_0, +\infty[\subset I(F,N) \cap J(F,N)$, and so $I(F,N) \cap J(F,N) \in \mathcal{F}$.

Note that a minor adjustment to the proof of the implication $(4) \Rightarrow (3)$ allows us to deduce $(2) \Rightarrow (3)$. Let us show now that $(3) \Rightarrow (2)$. Let $\mathcal{B}$ be the filter base consisting of all subsets $I(F,N) \cap J(F,N)$ of $\mathbb{N}_0$, where $N > 0$ and $F \subset V$ is finite. Let $\mathcal{F}_{\mathcal{B}}$ be the filter generated by $\mathcal{B}$, that is,

$$\mathcal{F}_{\mathcal{B}} = \{A \subset \mathbb{N}_0 : B \subset A \text{ for some } B \in \mathcal{B}\}.$$

We will show that B satisfies the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion, which implies that B is $\mathcal{F}$-transitive since $\mathcal{F}_{\mathcal{B}} \subset \mathcal{F}$. Set $\mathcal{D} = \mathrm{span}\{e_v : v \in V\}$, which is dense in $\ell^p(V,\mu)$. For every $v \in V$ and $n \in \mathbb{N}$, by the continuity of B^n, we have

$$M_{v,n} := \frac{1}{|\mu_{\mathrm{par}^n(v)}|^{p^*}} + \sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|^{p^*}} < +\infty,$$

then either

$$\frac{1}{|\mu_{\mathrm{par}^n(v)}|^{p^*}} \geq \frac{M_{v,n}}{2}, \tag{5.6}$$

or

$$\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} \frac{1}{|\mu_u|^{p^*}} > \frac{M_{v,n}}{2}. \tag{5.7}$$

In the case where (5.6) holds, we set

$$I_n e_v = e_v,$$

and then

$$\|I_n e_v - e_v\|_{p,\mu} = 0 \quad \text{and} \quad \|B^n I_n e_v\|_{p,\mu} = \|e_{\mathrm{par}^n(v)}\|_{p,\mu} = |\mu_{\mathrm{par}^n(v)}| \leq \Big(\frac{2}{M_{v,n}}\Big)^{1/p^*}. \tag{5.8}$$

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In the case where (5.7) holds, by Lemma 2.4, there exists $h_{v,n} \in \mathbb{K}^V$ nonnegative, of support in $\mathrm{Chi}^n(\mathrm{par}^n(v))$ such that

(5.9)

$$\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} h_{v,n}(u) = 1 \quad \text{and} \quad \Big(\sum_{u \in \mathrm{Chi}^n(\mathrm{par}^n(v))} |h_{v,n}(u)\, \mu_u|^p \Big)^{1/p} \le \Big(\frac{2}{M_{v,n}} \Big)^{1/p^*}.$$

Hence, $h_{v,n} \in \ell^p(V,\mu)$, and by setting

$$I_n e_v = e_v - h_{v,n},$$

we obtain

$$\|I_n e_v - e_v\|_{p,\mu} \le \Big(\frac{2}{M_{v,n}} \Big)^{1/p^*} \quad \text{and} \quad B^n I_n e_v = 0. \tag{5.10}$$

In both cases, we extend linearly on $\mathcal{D}$ the maps I_n. Let $g = \sum_{a \in F} g(a) e_a \in \mathcal{D}$, where $F \subset V$ is its support. We will show that $\mathcal{F}_{\mathcal{B}}\text{-}\lim_n (I_n g, B^n I_n g) = (g, 0)$. Let $\varepsilon > 0$, $\mathcal{U} := \{f \in \ell^p(V,\mu) : \|f - g\|_{p,\mu} < \varepsilon\}$ and $\mathcal{V} := \{f \in \ell^p(V,\mu) : \|f\|_{p,\mu} < \varepsilon\}$. Let us check that

$$J(F,N) \subset \{n \in \mathbb{N}_0 : (I_n g, B^n I_n g) \in \mathcal{U} \times \mathcal{V}\}, \tag{5.11}$$

where

$$N = \max \Big\{ \frac{|g(a)|\,|F| 2^{1/p^*}}{\varepsilon} : a \in F \Big\}.$$

Let $n \in J(F,N)$. We have then $M_{a,n} > N^{p^*}$, for all $a \in F$. By the definition of N, (5.8) and (5.10), we obtain

$$\begin{aligned} \max \Big\{ \|I_n g - g\|, \|B^n I_n g\|_{p,\mu} \Big\} &\le \sum_{a \in F} |g(a)| \Big(\frac{2}{M_{a,n}} \Big)^{1/p^*} \\ &< \sum_{a \in F} |g(a)| \frac{2^{1/p^*}}{N} \\ &< \varepsilon, \end{aligned}$$

hence $I_n g \in \mathcal{U}$ and $B^n I_n g \in \mathcal{V}$. Therefore, (5.11) holds and so

$$\{n \in \mathbb{N}_0 : (I_n g, B^n I_n g) \in \mathcal{U} \times \mathcal{V}\} \in \mathcal{F}_{\mathcal{B}}.$$

Consequently, the first statement of the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion holds. As for the second statement, we can define the maps S_n as in the proof of Theorem 4.1. By employing the same arguments, we can deduce that

$$\{n \in \mathbb{N}_0 : (B^n S_n g, S_n g) \in \mathcal{U} \times \mathcal{V}\} \in \mathcal{F}_{\mathcal{B}}.$$

This shows that the second statement of the $\mathcal{F}_{\mathcal{B}}$-transitivity criterion holds. Hence, B is $\mathcal{F}$-transitive and statement (2) holds. ∎

Once again, if the Furstenberg family $\mathcal{F}$ in the previous theorem represents the collection of syndetic subsets, we can derive a characterization of topologically ergodic

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backward shift operators on unrooted directed trees. Additionally, in the case where $\mathcal{F}$ represents the family of infinite subsets, we deduce the following corollary.

Corollary 5.2 *Let (V, E) be an unrooted directed tree and $\mu = (\mu_v)_{v\in V}$ a weight on V. Let $X = \ell^p(V, \mu)$, $1 \le p < +\infty$, or $X = c_0(V, \mu)$ and suppose that the backward shift B is a bounded operator on X. Then B is hypercyclic if and only if B is recurrent.*

In the upcoming section, we will generalize this corollary by using the concept of Γ-supercyclicity.

6 Γ-supercyclicity

Supercyclicity is a weaker notion than hypercyclicity, which requires the density of a projective orbit instead of an orbit, see [8]. We can study both these notions by considering the notion of Γ-supercyclicity, see [14]. Let X be a complex Banach space and Γ be a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(X)$ is called Γ-*supercyclic* if there exists some vector $x \in X$ such that the set

$$\mathrm{Orb}(\Gamma x, T) := \{\lambda T^n x : \lambda \in \Gamma,\ n \in \mathbb{N}_0\},$$

is dense in X. In particular, $\mathbb{C}$-supercyclicity is simply called supercyclicity, and $\{1\}$-supercyclicity coincides with hypercyclicity. This notion has been studied for bilateral backward shifts in [3], and more generally for a family of translation operators on weighted L^p-spaces on locally compact groups in [2].

In the same way, we can extend the notion of recurrence as follows: a vector $x \in X$ is said to be Γ-*recurrent vector* for T if there exist a strictly increasing sequence $(n_k)_{k\ge1}$ of positive integers and a sequence $(\lambda_k)_{k\ge1}$ in Γ such that

$$\lambda_k T^{n_k} x \underset{k\to+\infty}{\longrightarrow} x.$$

The operator T is called Γ-recurrent if its set of Γ-recurrent vectors, denoted by Γ-Rec(T), is dense in X. This notion coincides with the so-called super-recurrence when Γ is equal to the complex plane [5].

In the following theorem, we provide a characterization of Γ-supercyclicity for backward shift operators on unrooted directed trees and give its equivalence with Γ-recurrence. The proof of this theorem is omitted, as it follows a similar approach to that of Theorem 5.1, specifically when considering the Furstenberg family $\mathcal{F}$ as the collection of infinite subsets of $\mathbb{N}_0$. Furthermore, instead of using the $\mathcal{F}$-transitivity criterion, we employ the Γ-supercyclicity criterion described below.

Theorem 6.1 *Let (V, E) be an unrooted directed tree, $\mu = (\mu_v)_{v\in V}$ a weight on V and let $\Gamma \subset \mathbb{C}$ be such that $\Gamma\backslash\{0\}$ is nonempty. Let $X = \ell^p(V, \mu)$, $1 \le p < +\infty$, or $X = c_0(V, \mu)$ and suppose that the backward shift B is a bounded operator on X. Then the following conditions are equivalent:*

(1) *B is Γ-supercyclic.*
(2) *B is Γ-recurrent.*

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(3) *There are an increasing sequence of positive integers* $(n_k)_{k\geq 1}$ *and a sequence* $(\lambda_k)_{\geq 1}$ *in* $\Gamma\backslash\{0\}$ *such that, for every* $v \in V$, *we have*

$$\begin{cases} \displaystyle\sum_{u\in \mathrm{Chi}^{n_k}(v)} \frac{|\lambda_k|^{p^*}}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty; \\ \displaystyle\inf_{u\in \mathrm{Chi}^{n_k}(v)} \frac{|\mu_u|}{|\lambda_k|} \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu); \\ \displaystyle\sum_{u\in \mathrm{Chi}^{n_k}(v)} \frac{|\lambda_k|}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu), \end{cases}$$

and

$$\begin{cases} \displaystyle\frac{1}{|\lambda_k\, \mu_{\mathrm{par}^{n_k}(v)}|^{p^*}} + \sum_{u\in \mathrm{Chi}^{n_k}(\mathrm{par}^{n_k}(v))} \frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty; \\ \displaystyle\min\Big(|\lambda_k\, \mu_{\mathrm{par}^{n_k}(v)}|, \inf_{u\in \mathrm{Chi}^{n_k}(\mathrm{par}^{n_k}(v))} |\mu_u|\Big) \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu); \\ \displaystyle\frac{1}{|\lambda_k\, \mu_{\mathrm{par}^{n_k}(v)}|} + \sum_{u\in \mathrm{Chi}^{n_k}(\mathrm{par}^{n_k}(v))} \frac{1}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu) \end{cases}$$

The proof of the Γ-supercyclicity criterion presented below is a straightforward adaptation of the proof of the supercyclicity criterion given in [8, Theorem 1.14].

Theorem 6.2 (Γ-supercyclicity Criterion) *Let* X *be an infinite-dimensional Banach space,* $T \in \mathcal{L}(X)$ *and let* $\Gamma \subset \mathbb{C}$ *be such that* $\Gamma\backslash\{0\}$ *is nonempty. Assume that there exists an increasing sequence of positive integers* $(n_k)_{k\geq 1}$, *a sequence* $(\lambda_k)_{k\geq 1}$ *in* $\Gamma\backslash\{0\}$, *two dense subsets* X_0 *and* Y_0 *in* X, *and two families of applications* $I_{n_k} : X_0 \to X$ *and* $S_{n_k} : Y_0 \to X$ *such that, for any* $x \in X_0$ *and any* $y \in Y_0$, *the following conditions hold:*

(a) $I_{n_k}x \underset{k\to+\infty}{\longrightarrow} x$;
(b) $\lambda_k T^{n_k} I_{n_k} x \underset{k\to+\infty}{\longrightarrow} 0$;
(c) $\dfrac{1}{\lambda_k} S_{n_k} y \underset{k\to+\infty}{\longrightarrow} 0$;
(d) $T^{n_k} S_{n_k} y \underset{k\to+\infty}{\longrightarrow} y$.

Then $T \oplus T$ *is* Γ*-supercyclic.*

In the case of rooted trees, we can deduce from Corollary 4.3 that for any bounded subset Γ of $\mathbb{C}\backslash\{0\}$, a backward shift operator B is hypercyclic if and only if it is Γ-supercyclic. In addition, in the case where Γ is an unbounded subset of $\mathbb{C}$, B is always Γ-supercyclic, because in this case B has dense generalized kernel and dense range, see Proposition 6.3 below.

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The generalized kernel of a bounded linear operator T defined on some Banach space X is the subspace

$$\ker^*(T) := \bigcup_{n=1}^{+\infty} \ker(T^n).$$

In [10, Corollary 3.3], it was proven that an operator with dense generalized kernel is supercyclic if and only if it has dense range.

Proposition 6.3 *Let X be a separable infinite-dimensional Banach space, and let $T \in \mathcal{L}(X)$ have a dense generalized kernel. Then the following assertions are equivalent:*

(1) *For any unbounded subset Γ of $\mathbb{C}$, T is Γ-supercyclic.*
(2) *There exists an unbounded subset Γ of $\mathbb{C}$ such that T is Γ-supercyclic.*
(3) *T has dense range.*

Proof It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Let us show that (3) $\Rightarrow$ (1). Assume that T has dense range. Let Γ be an unbounded subset of $\mathbb{C}$. We will show that $T \oplus T$ is Γ-supercyclic. Let U_1, U_2, V_1, V_2 be nonempty open subsets of X. Since $\ker^*(T)$ is dense in X, there exists $(x_1, x_2) \in U_1 \times U_2$ such that $T^n x_1 = T^n x_2 = 0$ for some $n \in \mathbb{N}$. Since T has dense range, T^n also does; therefore, there exist $y_1, y_2 \in X$ such that $(T^n y_1, T^n y_2) \in V_1 \times V_2$. By the unboundedness of Γ, there exists $\lambda \in \Gamma$ such that

$$\frac{1}{\lambda}(y_1, y_2) + (x_1, x_2) \in U_1 \times U_2,$$

and

$$\lambda(T \oplus T)^n\left(\frac{1}{\lambda}(y_1, y_2) + (x_1, x_2)\right) = (T^n y_1, T^n y_2) + \lambda(T^n x_1, T^n x_2) \in V_1 \times V_2.$$

Hence, $T \oplus T$ is Γ-supercyclic. ∎

7 Zero-one law limit point

In [18], Chan and Seceleanu showed that any weighted unilateral/bilateral backward shift that has an orbit with a nonzero limit point is hypercyclic. However, comparing the characterization of hypercyclicity for backward shift operators on rooted directed trees with the following theorem, we deduce that Chan and Seceleanu's result does not hold for rooted directed trees.

Theorem 7.1 *Let (V, E) be a rooted directed tree and let $\mu = (\mu_v)_{v \in V}$ be a weight on V. Let $X = \ell^p(V, \mu)$, $1 \le p < +\infty$, or $X = c_0(V, \mu)$ and suppose that the backward shift B is a bounded operator on X. The following conditions are equivalent:*

(1) *B has an orbit with a nonzero limit point.*
(2) *B has an orbit with a nonzero weak limit point.*
(3) *B has an orbit with $e_{\mathtt{r}}$ as a limit point.*
(4) *B has an orbit with $e_{\mathtt{r}}$ as a weak limit point.*

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(5) *There are an increasing sequence* $(n_k)_k$ *of positive integers and a vertex* $v \in V$ *such that*

$$\begin{cases} \sum_{u\in \mathrm{Chi}^{n_k}(v)} \frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty; \\ \inf_{u\in \mathrm{Chi}^{n_k}(v)} |\mu_u| \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu); \\ \sum_{u\in \mathrm{Chi}^{n_k}(v)} \frac{1}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu). \end{cases}$$

(6) *There is an increasing sequence* $(n_k)_k$ *of positive integers such that*

$$\begin{cases} \sum_{u\in \mathrm{Chi}^{n_k}(\mathtt{r})} \frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty; \\ \inf_{u\in \mathrm{Chi}^{n_k}(\mathtt{r})} |\mu_u| \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu); \\ \sum_{u\in \mathrm{Chi}^{n_k}(\mathtt{r})} \frac{1}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu), \end{cases}$$

(7) *There exist an increasing sequence* $(n_k)_{k\in\mathbb{N}}$ *of positive integers and a vertex* $v \in V$ *such that, for any* $l \in \mathbb{N}_0$,

$$\begin{cases} \sum_{u\in \mathrm{Chi}^{n_k+l}(v)} \frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = \ell^p(V,\mu),\ 1<p<+\infty; \\ \inf_{u\in \mathrm{Chi}^{n_k+l}(v)} |\mu_u| \underset{k\to+\infty}{\longrightarrow} 0 & \text{if } X = \ell^1(V,\mu); \\ \sum_{u\in \mathrm{Chi}^{n_k+l}(v)} \frac{1}{|\mu_u|} \underset{k\to+\infty}{\longrightarrow} +\infty & \text{if } X = c_0(V,\mu). \end{cases}$$

Proof It is enough to prove that (2) ⇒ (5) ⇒ (6) ⇒ (3) and (5) ⇒ (7). We will prove these implications when $X = \ell^p(V,\mu)$, with $1 < p < +\infty$, the same reasoning works for the other cases. Let us show that (2) implies (5). Let $f, g \in \ell^p(V,\mu)$ be nonzero vectors. Let $v_0 \in V$ such that $g(v_0) \neq 0$. Let $(\delta_k)_k$ be a decreasing sequence of positive numbers such that $2\delta_k < |g(v_0)|$. Let $(n_k)_k$ be an increasing sequence of nonnegative integers such that

$$|\langle B^{n_k} f - g, e_{v_0}\rangle| < \delta_k,$$

that is

$$|(B^{n_k} f)(v_0) - g(v_0)| < \delta_k,$$

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thus

$$0 < \frac{|g(v_0)|}{2} < |g(v_0)| - \delta_k < \sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)|. \tag{7.1}$$

Since $\|f\|_{p,\mu} < +\infty$, we obtain

$$\sum_{k \geq 0} \sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)\mu_u|^p < +\infty,$$

hence

$$\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)\mu_u|^p \underset{k \to +\infty}{\longrightarrow} 0. \tag{7.2}$$

By Hölder's inequality ($p > 1$), we get

$$\begin{aligned} \sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)\mu_u|^p &\geqslant \Big(\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)| \Big)^p \Big(\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |\mu_u|^{\frac{p}{1-p}} \Big)^{1-p} \\ &\overset{(7.1)}{\geqslant} \frac{|g(v_0)|^p}{2^p} \Big(\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |\mu_u|^{\frac{p}{1-p}} \Big)^{1-p} \end{aligned}$$

hence

$$\Big(\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |\mu_u|^{-p^*} \Big)^{-1/p^*} \leqslant \frac{2}{|g(v_0)|} \Big(\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} |f(u)\mu_u|^p \Big)^{1/p},$$

combining this with (7.2), we get

$$\sum_{u \in \mathrm{Chi}^{n_k}(v_0)} \frac{1}{|\mu_u|^{p^*}} \underset{k \to +\infty}{\longrightarrow} +\infty.$$

Let us show now that (5) implies (6). Suppose that (5) holds for some sequence $(n_k)_k$ and $v \in V$. Let $m \in \mathbb{N}$ be such that $v \in \mathrm{Chi}^m(\mathtt{r})$. Set $m_k = n_k + m$. Thus, $\mathrm{Chi}^{n_k}(v) \subset \mathrm{Chi}^{m_k}(\mathtt{r})$ and hence

$$\sum_{u \in \mathrm{Chi}^{n_k}(v)} \frac{1}{|\mu_u|^{p^*}} \leqslant \sum_{u \in \mathrm{Chi}^{m_k}(\mathtt{r})} \frac{1}{|\mu_u|^{p^*}},$$

since the term at the left goes to infinity, as $k \to +\infty$, we get

$$\sum_{u \in \mathrm{Chi}^{m_k}(\mathtt{r})} \frac{1}{|\mu_u|^{p^*}} \underset{k \to +\infty}{\longrightarrow} +\infty.$$

Let us show that (6) implies (3). Let $(n_k)_{k \geq 0}$ be an increasing sequence of positive integers such that

$$\sum_{u \in \mathrm{Chi}^{n_k}(\mathtt{r})} \frac{1}{|\mu_u|^{p^*}} \underset{k \to +\infty}{\longrightarrow} +\infty.$$

Set $J_k = \mathrm{Chi}^{n_k}(\mathtt{r})$, for $k \in \mathbb{N}$. By Lemma 2.4, we obtain

$$\inf_{\|x\|_1 = 1} \Big(\sum_{u \in J_k} |x_u \mu_u|^p \Big)^{1/p} \underset{k \to +\infty}{\longrightarrow} 0$$

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There exists then an increasing sequence $(k_j)_{j\geq 0}$ of positive integers such that, for each $j \in \mathbb{N}_0$, we have

$$\inf_{\|x\|_1=1} \Big(\sum_{u\in J_{k_j}} |x_u \mu_u|^p \Big)^{1/p} \leqslant \frac{1}{2^{j+1} c_j}, \quad \text{where} \quad c_j = \max\{1; \|B^{n_{k_l}}\|, 0 \leqslant l \leqslant j-1\},$$

hence, there exists $f_{k_j} \in \mathbb{K}^V$, of support in $J_{k_j} = \mathrm{Chi}^{n_{k_j}}(\mathtt{r})$ such that

$$\|f_{k_j}\|_1 = \sum_{u\in \mathrm{Chi}^{n_{k_j}}(\mathtt{r})} |f_{k_j}(u)| = 1, \tag{7.3}$$

and

$$\Big(\sum_{u\in J_{k_j}} |f_{k_j}(u)\mu_u|^p \Big)^{1/p} \leqslant \frac{1}{2^j c_j}. \tag{7.4}$$

For each $u \in V$, set

$$f(u) = \begin{cases} g_j(u) & \text{if } u \in \mathrm{Chi}^{n_{k_j}}(\mathtt{r}) \text{ for some } j \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } g_j = |f_{k_j}|.$$

By (7.4), we obtain

$$\|f\|_{p,\mu}^p = \sum_{u\in V} |f(u)\mu_u|^p = \sum_{j\in\mathbb{N}_0} \sum_{u\in \mathrm{Chi}^{n_{k_j}}(\mathtt{r})} |f_{k_j}(u)\mu_u|^p \leqslant \sum_{j\in\mathbb{N}_0} \frac{1}{2^{pj}} < +\infty,$$

thus $f \in \ell^p(V,\mu)$. Moreover, we have

$$\begin{aligned} \|B^{n_{k_j}} f - e_{\mathtt{r}}\|_{p,\mu} &\leq \|B^{n_{k_j}} g_j - e_{\mathtt{r}}\|_{p,\mu} + \sum_{l=j+1} \|B^{n_{k_j}} g_l\|_{p,\mu} \\ &= \sum_{l=j+1} \|B^{n_{k_j}} g_l\|_{p,\mu} \qquad (\text{by } (7.3)) \\ &\leqslant \sum_{l=j+1} \|B^{n_{k_j}}\| \|f_{k_l}\|_{p,\mu} \\ &< \sum_{l=j+1} \frac{\|B^{n_{k_j}}\|}{2^l c_l} \qquad (\text{by } (7.4)) \\ &\leqslant \sum_{l=j+1} \frac{1}{2^l} \underset{j\to+\infty}{\longrightarrow} 0, \end{aligned}$$

hence condition (3) holds.

Let us show that (5) implies (7). Set $C = \max\{2, \|B\|\}$. By assuming that (5) holds, there exist an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a vertex $v \in V$ such that

$$\Big(\sum_{u\in \mathrm{Chi}^{n_k+k}(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*} < C^{-2k}. \tag{7.5}$$

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For every $u \in V\backslash\{\mathtt{r}\}$, by Proposition 2.2, we obtain

$$C \geqslant \Big(\sum_{v \in \mathrm{Chi}(\mathrm{par}(u))} \left| \frac{\mu_{\mathrm{par}(u)}}{\mu_v} \right|^{p^*} \Big)^{1/p^*} \geqslant \left| \frac{\mu_{\mathrm{par}(u)}}{\mu_u} \right|,$$

that is, for every $u \in V\backslash\{\mathtt{r}\}$,

$$\left| \frac{\mu_u}{\mu_{\mathrm{par}(u)}} \right| \geqslant C^{-1}. \tag{7.6}$$

Fix now $l \in \mathbb{N}_0$. Let $k \in \mathbb{N}$ be such that $k > l$. Note that

$$u \in \mathrm{Chi}^{n_k+k}(v) \Longleftrightarrow \mathrm{par}^{k-l}(u) \in \mathrm{Chi}^{n_k+l}(v), \tag{7.7}$$

and for every $u \in \mathrm{Chi}^{n_k+k}(v)$, we have

$$\mu_u = \frac{\mu_u}{\mu_{\mathrm{par}(u)}} \times \frac{\mu_{\mathrm{par}(u)}}{\mu_{\mathrm{par}^2(u)}} \times \cdots \times \frac{\mu_{\mathrm{par}^{k-l-1}(u)}}{\mu_{\mathrm{par}^{k-l}(u)}} \times \mu_{\mathrm{par}^{k-l}(u)}.$$

By (7.6), we obtain

$$\frac{1}{|\mu_{\mathrm{par}^{k-l}(u)}|^{p^*}} \geqslant C^{(l-k)p^*} \frac{1}{|\mu_u|^{p^*}}.$$

Combining this with (7.7), we obtain

$$\Big(\sum_{u \in \mathrm{Chi}^{n_k+l}(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*} \geqslant C^{l-k} \Big(\sum_{u \in \mathrm{Chi}^{n_k+k}(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*}.$$

Thus,

$$\Big(\sum_{u \in \mathrm{Chi}^{n_k+l}(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*} \leqslant C^{k-l} \Big(\sum_{u \in \mathrm{Chi}^{n_k+k}(v)} \frac{1}{|\mu_u|^{p^*}} \Big)^{-1/p^*}$$
$$\overset{(7.5)}{\leqslant} C^{k-l} C^{-2k} \leqslant C^{-k} \underset{k \to +\infty}{\longrightarrow} 0.$$

■

We now provide an example of a backward shift operator on a rooted directed tree that is not hypercyclic, but it has an orbit with a nonzero limit point.

Example 7.2 Let B be the backward shift operator given in Example 4.2. It is clear that, by using Theorem 4.1, B is not hypercyclic. For every $N \in \mathbb{N}$, we have

$$I(\mathtt{r}, N) := \Big\{ n \in \mathbb{N}_0 : \Big(\sum_{u \in \mathrm{Chi}^n(\mathtt{r})} \frac{1}{|\mu_u|^{p^*}} \Big)^{1/p^*} > N \Big\} \in \mathcal{J},$$

hence, by using Theorem 7.1, we deduce that B has an orbit with a nonzero limit point.

The following proposition provides a characterization of backward shift operators that have an orbit of a non-negative function with a nonzero limit point, specifically in the context of unrooted directed trees. We omit the proof as it can be derived through straightforward modifications of the arguments presented in the proofs of the preceding theorems.

Proposition 7.3 *Let (V,E) be an unrooted directed tree and let $\mu=(\mu_v)_{v\in V}$ be a weight on V. Let $X=\ell^p(V,\mu)$, $1\le p<+\infty$, or $X=c_0(V,\mu)$ and suppose that the backward shift B is a bounded operator on X. The following conditions are equivalent:*

(1) *There exists a nonnegative vector in X whose orbit under B has a nonzero limit point.*
(2) *There exist a vertex $v\in V$ and a nonnegative vector in X whose orbit under B has e_v as a limit point.*
(3) *There exist an increasing sequence $(n_k)_k$ of positive integers and a vertex $v\in V$ such that*

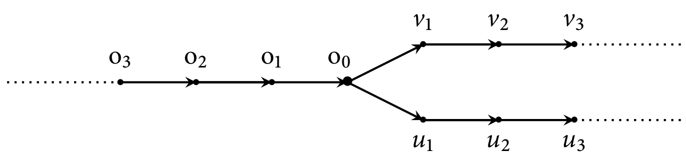
$$\begin{cases}\displaystyle\sum_{u\in\mathrm{Chi}^{n_k}(v)}\frac{1}{|\mu_u|^{p^*}}\underset{k\to+\infty}{\longrightarrow}+\infty & \text{if } X=\ell^p(V,\mu),\ 1<p<+\infty;\\ \displaystyle\inf_{u\in\mathrm{Chi}^{n_k}(v)}|\mu_u|\underset{k\to+\infty}{\longrightarrow}0 & \text{if } X=\ell^1(V,\mu);\\ \displaystyle\sum_{u\in\mathrm{Chi}^{n_k}(v)}\frac{1}{|\mu_u|}\underset{k\to+\infty}{\longrightarrow}+\infty & \text{if } X=c_0(V,\mu),\end{cases}$$

and, for every $i\in\mathbb{N}$, we have

$$\mu_{\mathrm{par}^{n_k-n_i}(v)}\underset{k\to+\infty}{\longrightarrow}0.$$

In the following example, we provide a nonhypercyclic bounded backward shift on an unrooted directed tree that has an orbit with a nonzero limit point, but it does not possess an orbit of a nonnegative function with a nonzero limit point.

Example 7.4 Let (V,E) be the following unrooted directed tree:



that is

$$V:=\{u_k:k\in\mathbb{N}\}\cup\{v_k:k\in\mathbb{N}\}\cup\{o_k:k\in\mathbb{N}_0\}.$$

Let $\mu=(\mu_v)_{v\in V}$ be the weight on V defined by

$$\mu_{o_k}=1,\quad \mu_{u_k}=\frac{1}{2^k},\quad\text{and}\quad \mu_{v_k}=\frac{1}{2^k}.$$

Let B be the backward shift on $\ell^p(V,\mu)$, $1<p<+\infty$. By Proposition 2.2, it is clear that B is bounded and $\|B\|=2^{2-\frac{1}{p}}$. Now, consider any vertex $v\in V$, any increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, and $i\in\mathbb{N}$, we have

$$\mu_{\mathrm{par}^{n_k-n_i}(v)}\underset{k\to+\infty}{\longrightarrow}1\neq 0,$$

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thus, the condition (3) of Proposition 7.3 does not hold. Consequently, B does not possess an orbit of a nonnegative function with a nonzero limit point. Moreover, for any vertex $v \in V$ and any increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, we have

$$\frac{1}{|\mu_{\text{par}^{n_k}(v)}|^{p^*}} + \sum_{u\in\text{Chi}^{n_k}(\text{par}^{n_k}(v))} \frac{1}{|\mu_u|^{p^*}} \underset{k\to+\infty}{\not\longrightarrow} +\infty,$$

thus, by using Theorem 5.1 (when $\mathcal{F}$ is the Furstenberg family of infinite subsets of $\mathbb{N}_0$), we deduce that B is not hypercyclic.

Let $f : V \longrightarrow \mathbb{C}$ be the function defined by

$$f(o_k) = 0,\ f(u_{2^k}) = 1,\ f(v_{2^k}) = -1, \text{ and } f(u_j) = f(v_j) = 0, \text{ if } j \neq 2^k.$$

We have $f \in \ell^p(V,\mu)$, since

$$\|f\|^p_{p,\mu} = \sum_{v\in V} |f(v)|^p \mu^p_v = 2\sum_{k\geq 0} \frac{1}{2^{p2^k}} < +\infty.$$

Let us show that $B^{2^k-1}f \underset{k\to+\infty}{\longrightarrow} e_{u_1} - e_{v_1}$. One has

$$\begin{aligned}\|B^{2^k-1}f - e_{u_1} + e_{v_1}\|^p_{p,\mu} &= \sum_{j\geq 0} |(B^{2^k-1}f)(o_j)|^p \mu^p_{o_j} \\ &+ |(B^{2^k-1}f)(u_1) - 1|^p \mu^p_{u_1} + \sum_{j\geq 2} |(B^{2^k-1}f)(u_j)|^p \mu^p_{u_j} \\ &+ |(B^{2^k-1}f)(v_1) + 1|^p \mu^p_{v_1} + \sum_{j\geq 2} |(B^{2^k-1}f)(v_j)|^p \mu^p_{v_j}.\end{aligned}$$

We will show now that all these terms converge to 0, as k goes to $+\infty$.

- Let $k, j \in \mathbb{N}$. If $2^k - 1 \leq j$, then $\text{Chi}^{2^k-1}(o_j) = \{o_{j-2^k+1}\}$, hence.

  $$(B^{2^k-1}f)(o_j) = f(o_{j-2^k+1}) = 0.$$

  If $2^k - 1 > j$, then $\text{Chi}^{2^k-1}(o_j) = \{u_{2^k-1-j}, v_{2^k-1-j}\}$, hence

  $$(B^{2^k-1}f)(o_j) = f(u_{2^k-1-j}) + f(v_{2^k-1-j}) = 0.$$

  Thus

  $$\sum_{j\geq 0} |(B^{2^k-1}f)(o_j)|^p \mu^p_{o_j} = 0.$$

- Let $k \in \mathbb{N}$. We have

  $$(B^{2^k-1}f)(u_1) = \sum_{v\in\text{Chi}^{2^k-1}(u_1)} f(v) = f(u_{2^k}) = 1,$$

  and

  $$(B^{2^k-1}f)(v_1) = \sum_{v\in\text{Chi}^{2^k-1}(v_1)} f(v) = f(v_{2^k}) = -1.$$

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- Let $k \in \mathbb{N}$. We have

$$\begin{aligned}\sum_{j\geq 2}|(B^{2^k-1}f)(u_j)|^p\mu_{u_j}^p &= \sum_{j\geq 2}|f(u_{2^k-1+j})|^p\mu_{u_j}^p\\ &= \sum_{l\geq k+1}|f(u_{2^l})|^p\mu_{u_{2^l-2^k+1}}^p\\ &= \frac{1}{2^p}\sum_{l\geq k+1}\frac{1}{2^{p(2^l-2^k)}},\end{aligned}$$

since, for every $l \geq k+1$, $2^l - 2^k \geq l$, we get

$$\sum_{j\geq 2}|(B^{2^k-1}f)(u_j)|^p\mu_{u_j}^p \leqslant \frac{1}{2}\sum_{l\geq k+1}\frac{1}{2^l}\underset{k\to+\infty}{\longrightarrow} 0.$$

- Similarly, for each $k \in \mathbb{N}$, we have

$$\sum_{j\geq 2}|(B^{2^k-1}f)(v_j)|^p\mu_{v_j}^p = \frac{1}{2^p}\sum_{l\geq k+1}\frac{1}{2^{p(2^l-2^k)}}\underset{k\to+\infty}{\longrightarrow} 0.$$

Hence,

$$\|B^{2^k-1}f - e_{u_1} + e_{v_1}\|_{p,\mu}\underset{k\to+\infty}{\longrightarrow} 0.$$

Therefore, B has an orbit with a nonzero limit point. Note that the above computations also work when considering the backward shift operator B as an operator acting on $\ell^1(V,\mu)$ or $c_0(V,\mu)$.

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