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COMPARING ℂ AND ZILBER'S EXPONENTIAL FIELDS: ZERO SETS OF EXPONENTIAL POLYNOMIALS

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To the memory of Professor Steve Schanuel.

Abstract We continue the research programme of comparing the complex exponential with Zilberś exponential. For the latter, we prove, using diophantine geometry, various properties about zero sets of exponential functions, proved for ℂ using analytic function theory, for example, the Identity Theorem.

Keywords: exponential fields; Zilber fields; Schanuel's Conjecture; Identity Theorem

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1. Introduction

In [19], Zilber introduced, and studied deeply, a class of exponential fields now known as Zilber fields. There are many novelties in his analysis, including a reinterpretation of Schanuel's Conjecture in terms of Hrushovski's very general theory of predimension and strong extensions. By now there is no need to spell out yet again all his ingredients and results (see [6, 19]). The most dramatic aspect is that his fields satisfy Schanuel's Conjecture and a beautiful *Nullstellensatz* for exponential equations. Moreover, in each uncountable cardinal there is a privileged such field, satisfying a *countable closure condition* and a strengthened Nullstellensatz. *Privileged* means that the structure in each uncountable cardinal is unique up to isomorphism. The one in cardinal $2^{\aleph_0}$ is called $\mathbb{B}$ by us. Zilber conjectured that $\mathbb{B} \cong \mathbb{C}$ as exponential fields. This would, of course, imply that ℂ satisfies Schanuel's Conjecture, and Zilber's Nullstellensatz, which seem far out of reach of current analysis of several complex variables.

Zilber fields are constructed model theoretically, and they have no visible topology except an obvious exponential Zariski topology. Zilber's countable closure condition is somehow an analogue of the separability of the complex field, and the fact that any finite system of exponential equations has only countably many isolated points. Isolation in ℂ relates, via the Jacobian criterion, to definability and dimension in $\mathbb{B}$; see [19].

We have undertaken a research programme of taking results from $\mathbb{C}$, proved using analysis and/or topology, and seeking exponential–algebraic proofs in $\mathbb{B}$. An early success was a proof in $\mathbb{B}$ of the Schanuel's Nullstellensatz [6], proved in $\mathbb{C}$ [13] using Nevanlinna theory. More recently, in [7], we derived, in an exponential–algebraic way, Shapiro's Conjecture from Schanuel's Conjecture, thereby getting Shapiro's Conjecture in $\mathbb{B}$. In the present paper, we put the ideas of [7] to work on some problems connected to the *Identity Theorem* of complex analysis [5, Theorem 3.7, p. 78]. That fundamental theorem says in particular that, if the zero set of an entire function f has an accumulation point, then $f \equiv 0$. We specialize to exponential functions, and face the obvious difficulty that the concept of accumulation point has no general meaning in Zilber fields.

We prove, not only for Zilber fields, but for the much more general classes of ***LEC***-fields and ***LECS***-fields (see § 2) various results proved for $\mathbb{C}$ using the Identity Theorem. Our replacement techniques come from diophantine geometry. Our results are not confined to ones proved in $\mathbb{C}$ using the Identity Theorem; see for example Theorem 6.2. This theorem is not true for all entire functions; see § 4.1. An analysis of the location of the zeros of exponential polynomials in the complex plane was obtained by Pólya *et al.* (see, e.g., [17]) in terms of lines in the plane determined by the polynomial itself. For more recent work on this, see [9]. Notice that this analysis makes no sense in Zilber fields, as the very notion of a line (defined over $\mathbb{R}$) makes no sense.

2. Exponential fields and exponential polynomials

We will be working over an algebraically closed field K of characteristic 0, with a surjective exponential map to $K^\times$ whose kernel is an infinite cyclic group written as $2\pi i\mathbb{Z}$, as if we were in $\mathbb{C}$. In fact, here π has a well-defined meaning; see [14]. We will call these fields ***LEC***-fields. In some cases, we will assume that the field K satisfies the following transcription of Schanuel's Conjecture with transcendental number theory.

(SC) If $\alpha_1, \ldots, \alpha_n \in K$, then

$$tr.d._{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geqslant l.d._{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n),$$

where $tr.d._{\mathbb{Q}}$ stands for transcendence degree over $\mathbb{Q}$ and $l.d._{\mathbb{Q}}$ stands for $\mathbb{Q}$-linear dimension.

We will refer to these fields as ***LECS***-fields. The class of ***LECS***-fields includes the exponential fields introduced by Zilber in [19].

We will consider exponential polynomial functions over K of the following form:

$$f(z) = \lambda_1 e^{\mu_1 z} + \cdots + \lambda_N e^{\mu_N z}, \qquad \text{where } \lambda_i, \mu_i \in K. \tag{1}$$

The set of these polynomials forms a ring $\mathcal{E}$ under the usual addition and multiplication operations. We will study subsets of the zero set of polynomials in $\mathcal{E}$. We will denote the zero set of f by $Z(f)$.

We recall some basic definitions and results for the exponential polynomials in the ring $\mathcal{E}$. The units in $\mathcal{E}$ are of the form $\lambda e^{\mu z}$, where $\lambda, \mu \in K$, and $\lambda \neq 0$.

Definition 2.1. An element f in $\mathcal{E}$ is irreducible if there are no non-units g and h such that $f = gh$.

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Definition 2.2. Let $f = \sum_{i=1}^{N} \lambda_i e^{\mu_i z}$ be an exponential polynomial. The support of f, denoted by $\mathrm{supp}(f)$, is the $\mathbb{Q}$-space generated by $\mu_1, \ldots, \mu_N$.

Definition 2.3. An exponential polynomial $f(z)$ in $\mathcal{E}$ is simple if $\dim(\mathrm{supp}(f)) = 1$.

It is easily seen that, up to a unit, a simple exponential polynomial is a polynomial in $e^{\mu z}$, for some $\mu \in K$. An important example of a simple exponential polynomial is $\sin(\pi z)$. A simple polynomial f can be factored, up to a unit in $\mathcal{E}$, as $f = \prod(1 - ae^{\mu z})$, where $a, \mu \in K$. If a simple polynomial f has infinitely many roots, then by the Pigeon-hole Principle one factor of f, say $1 - ae^{\mu z}$, has infinitely many zeros, and these are of the form $z = (2k\pi i - \log a)/\mu$ with $k \in \mathbb{Z}$, for a fixed value of $\log a$.

We will refer to $\lambda_1 e^{\mu_1 z} + \lambda_2 e^{\mu_2 z}$ (or to the equivalent form $1 - ae^{\mu z}$) as a simple polynomial of length 2. We have a complete description of the zero set of these polynomials.

Lemma 2.4. *If $f(z) = \lambda_1 e^{\mu_1 z} + \lambda_2 e^{\mu_2 z}$, then $Z(f)$ has dimension less than or equal to 2 over $\mathbb{Q}$. Moreover, $Z(f) = A_f + \mathbb{Z}B_f$, where $A_f = \frac{\log(-\lambda_1^{-1}\lambda_2)}{\mu_1 - \mu_2}$ and $B_f = \frac{2i\pi}{\mu_1 - \mu_2}$. So, the zero set of f is a translate of a rank-1 free abelian group.*

It seems that the first to consider a factorization theory for exponential polynomials over $\mathbb{C}$ was Ritt, in [18]. His original idea was to reduce the factorization of an exponential polynomial to that of a classical polynomial in many variables by replacing the variables with their powers. Let $f(z) = \lambda_1 e^{\mu_1 z} + \cdots + \lambda_N e^{\mu_N z}$ be an exponential polynomial where $\lambda_i, \mu_i \in \mathbb{C}$, and let $b_1, \ldots, b_D$ be a basis of the $\mathbb{Z}$-module spanned by $\mu_1, \ldots, \mu_N$. Let $Y_i = e^{b_i z}$, with $i = 1, \ldots, D$. If each μ_i is expressed in terms of the b_i, then $f(z)$ is transformed into a classical Laurent polynomial $F(Y_1, \ldots, Y_D) \in \mathbb{Q}(\overline{\lambda})[Y_1, \ldots, Y_D]$. Clearly any factorization of f produces a factorization of $F(Y_1, \ldots, Y_D)$.

In general, an irreducible classical polynomial $F(Y_1, \ldots, Y_D)$ can become reducible after a substitution of the variables by powers.

Definition 2.5. A polynomial $F(Y_1, \ldots, Y_D)$ is power irreducible if, for each $n_1, \ldots, n_D$ in $\mathbb{N} - \{0\}$, $F(Y_1^{n_1}, \ldots, Y_D^{n_D})$ is irreducible.

Notice that if f is irreducible in $\mathcal{E}$ then the associated polynomial $F(Y_1, \ldots, Y_D)$ is power irreducible.

Ritt saw the importance of understanding the ways in which an irreducible classical polynomial $F(Y_1, \ldots, Y_D)$ can become reducible when the variables are replaced by their powers. His analysis gave the following result.

Theorem 2.6. *Let $f(z) = \lambda_1 e^{\mu_1 z} + \cdots + \lambda_N e^{\mu_N z}$, where $\lambda_i, \mu_i \in \mathbb{C}$. Then $f(z)$ can be written uniquely up to order and multiplication by units as*

$$f(z) = S_1 \cdot \ldots \cdot S_k \cdot I_1 \cdot \ldots \cdot I_m,$$

where the S_j are simple polynomials with different supports and the I_h are irreducible in $\mathcal{E}$.

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In our setting, we use an analogous result for exponential polynomials as in (1) over any algebraically closed field of characteristic 0 carrying an exponentiation (see also [8]). Notice that the factorization theorem of Ritt is a result on the free E-ring over exponential fields, and need not involve any analysis of the zero set of an exponential polynomial.

3. Zero sets in $\mathbb{C}$

We are interested in putting restrictions on infinite subsets of the zeros sets of certain totally defined functions, such as exponential functions. These satisfy special properties such as Schanuel's Nullstellensatz (see [6] and [13]). For such functions in $\mathbb{C}$, one may use the Identity Theorem to get information about the zero set. We show below that, in some cases that we list, and related ones, the topology on $\mathbb{C}$ is not needed, and the use of Identity Theorem can be replaced by diophantine geometry. Here are the examples we will consider.

(1) Let $X \subseteq \mathbb{Q}$. There is a unique copy of the field of rationals in K, since it is a characteristic 0 field. We will say that a subset X of the rationals *accumulates* if there exists a Cauchy sequence of distinct elements of X. Note that this makes sense, since the definition is given inside $\mathbb{Q}$, and in $\mathbb{Q}$ there is a perfectly good notion of a Cauchy sequence. $\mathbb{Q}$ clearly accumulates in this sense, while $\mathbb{Z}$ and $\frac{1}{N}\mathbb{Z}$, where $N \in \mathbb{N} - \{0\}$, do not. Note that we are not explicitly considering the question whether a subset X of $\mathbb{Q}$ that accumulates has an accumulation point in a Zilber field, although it has an accumulation point in $\mathbb{C}$.

(2) Let $\mathbb{U}$ be the multiplicative group of roots of unity. This has an invariant meaning in any algebraically closed field. In $\mathbb{C}$, any infinite subset of $\mathbb{U}$ has an accumulation point in $\mathbb{C}$, since it is a subset of the unit disc, which is compact. In our more abstract situation the subset accumulates if it is infinite.

(3) Let X be an infinite subset of a cyclic group $\langle\alpha\rangle$ (under multiplication). In $\mathbb{C}$, we have the following three cases.
Case 1. If $\|\alpha\| = 1$, then X has an accumulation point on the unit circle, by compactness. If an entire function f vanishes on X, then $f \equiv 0$.
Case 2. If $\|\alpha\| < 1$, then $\{\alpha^n : n \geqslant 0\}$ has 0 as an accumulation point, and, if $X \cap \{\alpha^n : n \geqslant 0\}$ is infinite, then X has 0 as an accumulation point. Again we conclude that, if f is an entire function and vanishes on X, then $f \equiv 0$. If, however, $X \cap \{\alpha^n : n \geqslant 0\}$ is finite, there is nothing we can say for a general entire function f, since X need not have an accumulation point in $\mathbb{C}$ (so Weierstrass' work [5, Theorem 5.14, p. 170] allows X to be the zero set of a non-identically zero entire function). Nevertheless, if we restrict f to be of type (1), α is subject to severe constraints, relating to work of Győry and Schinzel on trinomials; see [12]. This is proved in § 6, assuming Schanuel's Conjecture. The proof works uniformly for $\mathbb{C}$, and $\mathbb{B}$ (assuming (SC) for $\mathbb{C}$), and indeed for a much wider class of E-fields (the *LECS*-fields).
Case 3. $\|\alpha\| > 1$. This is dual to Case 2, replacing α by $\frac{1}{\alpha}$, and is treated accordingly in § 6.

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(4) Let X be a finite-dimensional $\mathbb{Q}$-vector space. In $\mathbb{C}$, we fix a basis, and identify X with $\mathbb{Q}^n$, where n is the dimension of the space. Then we extend the definition given in (1) of a Cauchy sequence to the case $n > 1$ using the supremum metric. This gives us a notion of a subset of X accumulating. Notice that the notion does not depend on the choice of the basis, by elementary matrix theory.

4. Results about general infinite set of zeros

We are going to employ some of the arguments used in [7] for the proof of Shapiro's Conjecture from Schanuel's Conjecture. Here, we are principally interested in conditions on an infinite set X in order for it to be contained in the zero set of some exponential polynomial. We will use these results in § 6 to deal with the case of an infinite set of roots X contained in an infinite cyclic group.

The assumptions we have through this section are as follows.

1. K is an *LECS*-field.
2. X is an infinite subset of K.
3. $tr.d._{\mathbb{Q}}(X) = M < \infty$.
4. X is contained in the zero set of some exponential polynomial f in $\mathcal{E}$.

Our objective is to understand the strength of these assumptions. In order to do this, we proceed through various reductions, roughly following [7].

Reduction 1. We apply Ritt Factorization (Theorem 2.6) to f in $\mathcal{E}$, and we use the Pigeon-hole Principle to go to the case of X infinite and f either irreducible or simple of length 2. We will consider the case of f simple at the end of the section, and for now we assume f to be irreducible until further notice.

Reduction 2. This involves the very first step in our proof of Shapiro's Conjecture from Schanuel's Conjecture (see § 5 of [7]). Here, we have the hypothesis that the transcendence degree of X is finite, in contrast to our previous paper on Shapiro's Conjecture, where we had to prove that the set of common zeros of two exponential polynomials have finite transcendence degree. Both in [7] and in this paper, Schanuel's Conjecture is crucial.

We also use the same notation, and for convenience we recall the following.

(i) $D = l.d._{\mathbb{Q}}(\mu_1, \ldots, \mu_N)$.

(ii) $\delta_1 = tr.d._{\mathbb{Q}}(\lambda_1, \ldots, \lambda_N)$.

(iii) $\delta_2 = tr.d._{\mathbb{Q}}(\mu_1, \ldots, \mu_N)$.

(iv) L is the algebraic closure of $\mathbb{Q}(\overline{\lambda})$.

(v) G_m^D is the multiplicative group variety.

Let $\alpha_1, \ldots, \alpha_k \in X$ be solutions of $f(z) = 0$. An upper bound for the linear dimension of the set $\{\alpha_j \mu_i : 1 \leqslant j \leqslant k, 1 \leqslant i \leqslant N\}$ over $\mathbb{Q}$ is Dk. This is the actual dimension D when $k = 1$. We exploit the fact that Schanuel's Conjecture puts restrictions on k for the above linear dimension to be Dk. Let $F(Y_1, \ldots, Y_D)$ be the Laurent polynomial over L associated to f. The condition $F(Y_1, \ldots, Y_D) = 0$ defines an irreducible subvariety V of G_m^D of dimension $D-1$ over L (here we use that f is irreducible in $\mathcal{E}$, and hence

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$F(Y_1, \ldots, Y_D)$ is power irreducible). We get the following easy estimates:

$$tr.d._{\mathbb{Q}}(\overline{\mu}\alpha_1, \ldots, \overline{\mu}\alpha_k, e^{\overline{\mu}\alpha_1}, \ldots, e^{\overline{\mu}\alpha_k}) \leqslant M + \delta_2 + \delta_1 + k(D-1).$$

By Schanuel's Conjecture, the above transcendence degree is greater than or equal to $l.d._{\mathbb{Q}}(\overline{\mu}\alpha_1, \ldots, \overline{\mu}\alpha_k)$, so if this dimension is kD we get

$$kD \leqslant M + \delta_2 + \delta_1 + k(D-1).$$

Hence, $k \leqslant M + \delta_2 + \delta_1$.

Reduction 3. We now appeal, as in [7], to work of Bombieri *et al.* [2] on anomalous subvarieties.

If $\alpha \in X$, we write $e^{\overline{b}\alpha}$ for the tuple $(e^{b_1\alpha}, \ldots, e^{b_D\alpha})$. The variety V contains all points of the form $(e^{b_1\alpha}, \ldots, e^{b_D\alpha})$ for $\alpha \in X$. We consider points $\alpha_1, \ldots, \alpha_k \in X$ for $k > \delta_1 + \delta_2 + M$, so among $\overline{b}\alpha_1, \ldots, \overline{b}\alpha_k$ there are non-trivial linear relations over $\mathbb{Q}$, and hence the linear dimension over $\mathbb{Q}$ of $\overline{b}\alpha_1, \ldots, \overline{b}\alpha_k$ is $< Dk$. The point $(e^{\overline{b}\alpha_1}, \ldots, e^{\overline{b}\alpha_k})$ in the Dk-space lies on V^k.

The non-trivial $\mathbb{Q}$-linear relations on the $b_j\alpha_r$ induce algebraic relations between the $e^{b_j\alpha_r}$. These latter relations define an algebraic subgroup $\Gamma_{\overline{\alpha}}$ of $(G_m^D)^k$ of dimension $d(\overline{\alpha})$ over $\mathbb{Q}$ and codimension $Dk - d(\overline{\alpha})$. The dimension $d(\overline{\alpha})$ is strictly connected to the linear dimension of the $b_j\alpha_r$.

The point $(e^{\overline{b}\alpha_1}, \ldots, e^{\overline{b}\alpha_k})$ lies on $\Gamma_{\overline{\alpha}} \cap V$. Let $W_{\overline{\alpha}}$ be the variety of the point $(e^{\overline{b}\alpha_1}, \ldots, e^{\overline{b}\alpha_k})$ over L. Now, we examine the issue when $W_{\overline{\alpha}}$ is anomalous in V^k. See [2] for definitions and properties of anomalous subvarieties.

Suppose that $W_{\overline{\alpha}}$ is neither anomalous nor of dimension 0. Then, in particular, $\dim W_{\overline{\alpha}} \leqslant \dim(V^k) -$ codim $(\Gamma_{\overline{\alpha}})$, i.e., $\dim W_{\overline{\alpha}} \leqslant k(D-1) - (Dk - d(\overline{\alpha})) = d(\overline{\alpha}) - k$. Schanuel's Conjecture implies that $d(\overline{\alpha}) \leqslant d(\overline{\alpha}) - k + \delta_1 + M + \delta_2$; hence $k \leqslant \delta_1 + M + \delta_2$. So, if $k > \delta_1 + M + \delta_2$, then $W_{\overline{\alpha}}$ is either anomalous or of dimension 0.

Reduction 4. Suppose that such a $W_{\overline{\alpha}}$ has dimension 0. Then all the $e^{b_j\alpha_r}$ are algebraic over L. So, $tr.d._{\mathbb{Q}}(e^{\overline{b}\alpha_1}, \ldots, e^{\overline{b}\alpha_k}) \leqslant \delta_1$. Moreover, $tr.d._{\mathbb{Q}}(\overline{b}\alpha_1, \ldots, \overline{b}\alpha_k) \leqslant \delta_2 + M$. So, by Schanuel's Conjecture,

$$d(\overline{\alpha}) \leqslant \delta_1 + \delta_2 + M. \tag{2}$$

Reduction 5. We worked locally with $\overline{\alpha}$, and now we have to work independently of $\overline{\alpha}$. Notice that, under permutations of the $\overline{\alpha}$ the two properties, namely that $W_{\overline{\alpha}}$ has dimension 0, or $W_{\overline{\alpha}}$ is anomalous, are invariant. Suppose that $l.d._{\mathbb{Q}}(X)$ is infinite. Choose an infinite independent subset X_1 of X. Let $k > \delta_1 + \delta_2 + M$, and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, with $\alpha_1, \ldots, \alpha_k$ distinct elements of X_1, hence linearly independent over $\mathbb{Q}$. Then $W_{\overline{\alpha}}$ cannot have dimension 0, since $d(\overline{\alpha}) \geqslant k$ and (2) holds. Thus, for any k-element subset $\{\alpha_1, \ldots, \alpha_k\}$ of X_1, $W_{\overline{\beta}}$ is anomalous in V^k for any permutation $\overline{\beta}$ of $(\alpha_1, \ldots, \alpha_k)$. We thin X to X_1, and we work with X_1.

Reduction 6. We work with X_1, which we still call X. Let k be minimal such that, for any $k+1$ elements $\eta_1, \ldots, \eta_{k+1}$ of X, the variety of the point $(e^{\overline{b}\eta_1}, \ldots, e^{\overline{b}\eta_{k+1}})$ is anomalous in V^{k+1}. From [2], it follows that there is a finite collection Φ of proper tori $H_1, \ldots, H_t$

in $G_m^{D(k+1)}$ such that each maximal anomalous subvariety of V^{k+1} is a component of the intersection of V^{k+1} with a coset of one of the H_i. We now discard much information. For each H_i, we pick one of the multiplicative conditions defining it. These define a finite set $\{J_1, \dots, J_t\}$ of codimension 1 subgroups so each anomalous subvariety is contained in one of them. Now, exactly as in [7], by using Ramsey's Theorem and Schanuel's Conjecture yet again, we get an infinite $X_2 \subseteq X$ so the $\mathbb{Q}$-linear dimension of X_2 is finite.

Reduction 7. Without loss of generality, we can assume that the set X of solutions of the exponential polynomial f is infinite and of finite linear dimension over $\mathbb{Q}$. Each α in X gives rise to a solution $(e^{\mu_1\alpha}, \dots, e^{\mu_N\alpha})$ of the linear equation $\lambda_1 Y_1 + \cdots + \lambda_N Y_N = 0$. We change the latter to the equation

$$\left(\frac{-\lambda_1}{\lambda_N}\right) Z_1 + \cdots + \left(\frac{-\lambda_{N-1}}{\lambda_N}\right) Z_{N-1} = 1, \tag{3}$$

where Z_j stands for $\frac{Y_j}{Y_N}$. Note that $(e^{(\mu_1-\mu_N)\alpha}, \dots, e^{(\mu_{N-1}-\mu_N)\alpha})$ is a solution of (3). In [7, Lemma 5.6], we observed that distinct α give distinct roots of (3), unless f is a simple polynomial. The finite dimensionality of X implies that the multiplicative group generated by the $e^{(\mu_j-\mu_N)\alpha}$ for $\alpha \in X$ and $j = 1, \dots, N-1$ has finite rank. We can then apply a basic result on solving linear equations over a finite rank multiplicative group, due to Evertse, Schlickewei, and Schmidt in [10]. From this result it follows that only finitely many solution of (3) of the form $(e^{(\mu_1-\mu_N)\alpha}, \dots, e^{(\mu_{N-1}-\mu_N)\alpha})$, for $\alpha \in X$, are non-degenerate. We then thin X again to an infinite set, which we still call X, generating infinitely many degenerate solutions of (3). For any proper subset I of $\{1, \dots, N-1\}$ with $|I| > 1$, let

$$f_I = \sum_{j\in I} \left(\frac{-\lambda_j}{\lambda_N}\right) e^{(\mu_j-\mu_N)z}.$$

By repeated applications of the Pigeon-hole Principle and the Evertse, Schlickewei, and Schmidt result in [10], we construct a finite chain of subsets I_j of $\{1, \dots, N-1\}$ so f_{I_j} has infinitely many common solutions with f. For cardinality reasons, we have to reach an I_{j_0} of cardinality 2 whose corresponding polynomial is simple of length 2. By our result [7, Theorem 5.7], we get that f divides a simple polynomial, and so f is necessarily simple.

Recall that in Reduction 1 we postponed the discussion for simple polynomials, and we did the reductions for irreducible polynomials in $\mathcal{E}$. Now we have reached the conclusion that only a simple polynomial can satisfy assumptions 1–4 of this section. We have then proved the following.

Theorem 4.1 (SC). *Let f be an exponential polynomial in $\mathcal{E}$. If $Z(f)$ contains an infinite set X of finite transcendence degree, then f is divisible by a simple polynomial. Every infinite subset of the zero set of an irreducible polynomial has infinite transcendence degree (and hence infinite linear dimension).*

Remark 4.2. Notice that the above theorem and Lemma 2.4 give the following conclusions.

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1. (SC) If an exponential polynomial f vanishes on an infinite set X of finite transcendence degree, then in fact X has finite linear dimension.
2. (Unconditionally) If f vanishes on an infinite set X of finite dimension, that dimension has a bound depending only on the number of simple factors of length 2 dividing f.

The above results also imply that the zero set of an irreducible exponential polynomial cannot contain an infinite set of finite linear dimension over $\mathbb{Q}$. This does not depend on Schanuel's Conjecture. So, there is no irreducible exponential polynomial f such that $Z(f)$ contains an infinite set of algebraic numbers. If $Z(g)$ contains an infinite set of finite dimension and g is simple of length 2, then this dimension has to be at most 2.

4.1. The case of subsets of $\mathbb{Q}$

An immediate consequence of the above results is that, if f is an exponential polynomial whose zero set contains an infinite subset X of rationals, then f is divisible by a simple polynomial. We can then assume without loss of generality that f is simple. We now show that the elements in X have bounded denominators. Indeed, factor f into simple polynomials $f_1, \ldots, f_k$ of length 2, and let $X_i = X \cap Z(f_i)$. For any i, $X_i \subseteq Z(f_i) \cap \mathbb{Q}$, and, moreover, $X_i \subseteq A_{f_i} + \mathbb{Z}B_{f_i}$ in the notation of Lemma 2.4. If $q_1, q_2 \in X_i$, then $q_1 - q_2 = (k-h)B_{f_i}$ for some $k, h \in \mathbb{Z}$. Hence, $B_{f_i} \in \mathbb{Q}$, and so also $A_{f_i} \in \mathbb{Q}$. This implies that there is $N_i \in \mathbb{N}$ such that $X_i \subseteq (1/N_i)\mathbb{Z}$. So, X has bounded denominators, since $X = X_1 \cup \cdots \cup X_k$.

There is a recently published paper by Gunaydin [11] about solving in the rationals exponential polynomials in many variables $\overline{X} = (X_1, \ldots, X_t)$ over $\mathbb{C}$ of the form

$$\sum_{i=1}^{s} P_i(\overline{X}) e^{((\overline{X} \cdot \overline{\alpha_i}))}, \tag{4}$$

where $P_i(\overline{X}) \in \mathbb{C}[\overline{X}]$ and $\overline{\alpha_i} \in \mathbb{C}^t$. His result is the following.

Theorem 4.3 ([11]). *Given $P_1, \ldots, P_s \in \mathbb{C}[\overline{X}]$ and $\overline{\alpha_1}, \ldots, \overline{\alpha_s} \in \mathbb{C}^t$, there is $N \in \mathbb{N}^{>0}$ such that, if $\overline{q} \in \mathbb{Q}^t$ is a non-degenerate solution of*

$$\sum_{i=1}^{s} P_i(\overline{X}) e^{((\overline{X} \cdot \overline{\alpha_i}))} = 0,$$

then $\overline{q} \in (\frac{1}{N}\mathbb{Z})^t$.

He makes no reference to any other exponential field, but we have verified that his results hold for exponential polynomials as in (4) over an *LEC*-field.

His conclusion for $f(z)$ in a single variable implies that the rationals in the zero set of the function do not *accumulate*. So we have an algebraic proof of a result proved for $\mathbb{C}$ via analytic methods.

Open problem. It is a natural question to ask if the result of Theorem 4.3 can be extended to an arbitrary exponential polynomial with iterations of exponentiation over an *LEC*-field or more generally for a Zilber field.

If Zilber's Conjecture is true, there is no exponential polynomial in this general sense vanishing on a set of rationals which has an accumulation point. Note that there are infinite subsets of $\mathbb{Q}$ with unbounded denominators, without accumulation points, e.g.,

$$X = \left\{ \frac{m_n}{p_n} : p_n \text{ is } n\text{th prime}, \, m_n \in \mathbb{N}, \text{ and } n < \frac{m_n}{p_n} < n+1 \right\}.$$

It seems inconceivable to us that there is a non-trivial exponential polynomial vanishing on X. The classic Weierstrass Theorem (see [5]) provides a non-trivial entire function vanishing on X, but the usual proof gives a complicated infinitary definition.

In the special case of exponential polynomials over $\mathbb{C}$ of the form

$$\sum_{i=1}^{s} \lambda_i e^{\mu_i e^{2\pi i z}},$$

Theorem 4.3 is true, because otherwise the polynomial

$$\sum_{i=1}^{s} \lambda_i e^{\mu_i w}$$

would have infinitely many roots of unity as solutions. We prove in the next section that this cannot happen unless the polynomial is identically zero.

5. Case of roots of unity

Let $\mathbb{U}$ denote the set of roots of unity. We know that an entire function over $\mathbb{C}$ cannot have infinitely many roots of unity as zeros unless it is identically zero. Let K be an *LECS*-field.

Theorem 5.1. *If $f(z) \in \mathcal{E}$ over K vanishes on an infinite subset X of $\mathbb{U}$, then $f(z)$ is the zero polynomial in $\mathcal{E}$.*

Proof. From Theorem 4.1, f is necessarily simple, and so X is of finite linear dimension over $\mathbb{Q}$. Hence, X lies in a finite extension of $\mathbb{Q}$, and it is a very well-known fact that any finitely generated field F of characteristic 0 cannot contain infinitely many roots of unity; see [3]. So we get a contradiction. □

Notice that the above result is unconditional; Schanuel's Conjecture has not been used.

5.1. Torsion points of elliptic curves

We generalize the above argument to the case of coordinates of torsion points of an elliptic curve E over any fixed number field F.

We consider the affine part of E given as usual by an equation that is cubic in x and quadratic in y. Let $\mathrm{tor}(E)$ be the set of torsion points of E. Notice that not all torsion points of E are in F. Define π_1 as the projection to the x-coordinate, and π_2 as the projection to the y-coordinate. Let X be an infinite subset of $\pi_1(\mathrm{tor}(E)) \cap Z(f)$. Then X is contained in F^{alg} (see [15]), and also in $\mathbb{Q}^{\mathrm{alg}}$. From the reductions of § 4, assuming (SC), the set X is finite dimensional over $\mathbb{Q}$, and hence there is a finite extension L of

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F which contains X. L is of finite dimension over $\mathbb{Q}$, say d. Now, each y in $\pi_2(\text{tor}(E))$ associated to an element of X is quadratic over L, and so each torsion point of E belongs to an extension of $\mathbb{Q}$ of degree at most $2d$. Recall that for each n the group of n-torsion points of an elliptic curve E is isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

In Merel's paper [16], the following major result on a uniform bound on the order of torsion points of an elliptic curve over a number field is proved.

Theorem 5.2. *For all $d \in \mathbb{Z}$, $d \geqslant 1$, there exists a constant $n(d) \geqslant 0$ such that, for all elliptic curves E over a number field F with $[F : \mathbb{Q}] = d$, any torsion point of $E(F)$ is of order less than $n(d)$.*

The bound in Merel's result depends only on the dimension of the number field, and not on the particular number field we are considering. We apply Merel's result to number fields of dimension greater than d and at most $2d$, and we get that there can be only finitely many torsion points. We have then proved the following result.

Theorem 5.3 (SC)**.** *Let f be an exponential polynomial in $\mathcal{E}$. If $X \cap Z(f)$ has infinite cardinality, then f is identically zero.*

A natural generalization of Theorem 5.3 is stated in the following.

Open Problem: Is the intersection of an infinite set of solutions of a non-zero exponential polynomial with the x-coordinates of the torsion points of an abelian variety over $\mathbb{Q}$ finite?

The crucial obstruction is that as far as we know there is not an analogous result to Merel's for abelian varieties. Using simply the reduction of § 4, we obtain that, if X is the set of x-coordinates and y-coordinates of torsion points of an abelian variety over $\mathbb{Q}$, then the intersection of X with the zero set of an exponential polynomial f is finite unless f is identically zero (no use of Merel's result is needed here).

6. Case of an infinite cyclic group

In this section, we examine the case of an infinite $X \subseteq Z(f)$ such that $X \subseteq \langle \alpha \rangle$, where $\alpha \in K$. Clearly, the transcendence degree of X is finite, and by Theorem 4.1, without loss of generality, we can assume that f is simple (modulo Schanuel's Conjecture). We are going to show that some power of α is in $\mathbb{Z}$. Without loss of generality, by the classical factorization of 1-variable polynomials, f can be chosen to be of length 2.

In this section, we will often use the following basic thinning and reduction argument for the set X of solutions using the Euclidean reduction.

Euclidean reduction. Suppose that $\alpha^r \in X$ for some $r \in \mathbb{N}$ (when r is negative, we work with $\frac{1}{\alpha}$). Let $s_0 < r$ such that there are infinitely many m with $\alpha^{rm+s_0} \in X$. Via a change of variable, we work with the polynomial $g(z) = f(\alpha^{s_0} z)$, which vanishes on an infinite subset of $\langle \alpha^r \rangle$. If f is simple, of length 2, then also g is simple, of length 2. Hence, by Lemma 2.4, $Z(g)$ is the translate $A_g + \mathbb{Z}B_g$, where A_g and B_g are in K, and in general different from A_f and B_f, respectively. In what follows, it will not make any difference if we work with either f or g. The infinite set X of solutions of f contains a translate X'

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of an infinite subset of $Z(g)$. We will not make any distinction between the two, and we will still use the notation X for X'.

Lemma 6.1. *Let $\alpha \in K^\times$ and $\alpha \notin \mathbb{U}$. Suppose that the set $X \cap \langle \alpha \rangle$ is infinite. If, for some $r \neq 0$ in $\mathbb{Z}$, $\alpha^r \in \mathbb{Q}$, then $\alpha^r \in \mathbb{Z}$ or $\frac{1}{\alpha^r} \in \mathbb{Z}$.*

Proof. Using the Euclidean reduction, we can assume that there are infinitely many m with α^{rm} rational solutions of g. Choose two of them, say α^{rm_1} and α^{rm_2}, so $\alpha^{rm_1} = A_g + kB_g$ and $\alpha^{rm_2} = A_g + hB_g$, for some $k, h \in \mathbb{Z}$. Moreover, $\alpha^{rm_1} \neq \alpha^{rm_2}$, since $\alpha \notin \mathbb{U}$. Clearly, A_g and B_g are rationals. We now use the result in § 4.1, and we get a contradiction unless either $\alpha^r \in \mathbb{Z}$ or $\frac{1}{\alpha^r} \in \mathbb{Z}$. □

Theorem 6.2. *Let $f \in \mathcal{E}$. Suppose that $X \subseteq Z(f)$, X is contained in the infinite cyclic group generated by α, and that X is infinite. Then f is identically zero unless $\alpha^r \in \mathbb{Z}$ for some $r \in \mathbb{Z}$.*

Proof. As already observed, f is simple, of length 2, and so $Z(f) = A_f + \mathbb{Z}B_f$. Hence X has linear dimension over $\mathbb{Q}$ less than or equal to 2. In particular, α is a root of infinitely many trinomials over $\mathbb{Q}$, and so it is algebraic, with minimum polynomial $p(x)$ dividing infinitely many trinomials. By work of Györy and Schinzel in [12], there exists a polynomial $q(x) \in \mathbb{Q}[x]$ of degree $\leqslant 2$ such that $p(x)$ divides $q(x^r)$ for some r.
If $q(x)$ is linear, then α^r is rational, and we have finished, thanks to Lemma 6.1.
If $q(x)$ is quadratic, then $\alpha^r \in \mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}$. By Euclidean reduction, we reduce to the case of a simple polynomial g of length 2 for which the corresponding A_g and B_g are in $\mathbb{Q}(\sqrt{d})$. Using again Euclidean reduction, we can assume without loss of generality that $\alpha \in \mathbb{Q}(\sqrt{d})$. The polynomial g may have changed, but we will continue to refer to it as g. Note that, if $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}$, then $\alpha \in \mathbb{Q}$, and so by Lemma 6.1 either $\alpha \in \mathbb{Z}$ or $\frac{1}{\alpha} \in \mathbb{Z}$. So, we can assume that $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}$.

We will also drop the subscript g from A_g and B_g in the rest of the proof, since no confusion can arise.

Claim 1. Either α or $1/\alpha$ is an algebraic integer in $\mathbb{Q}(\sqrt{d})$.

Proof of the Claim 1: Suppose that α is not an algebraic integer in $\mathbb{Q}(\sqrt{d})$. By a basic result in algebraic number theory (e.g., see [4, Chapter 1]), there is a valuation v on $\mathbb{Q}(\sqrt{d})$ such that $v(\alpha) < 0$. If, for infinitely many positive $m \in \mathbb{Z}$, $\alpha^m \in X$, then there is no lower bound on the valuations of elements of X. We get a contradiction, since, for all m, $\alpha^m = A + k_m B$ for some $k_m \in \mathbb{Z}$, and $v(A + k_m B) \geqslant \min\{v(A), v(B)\}$. So the v of elements of X have to be bounded below. If there exist infinitely many integers m such that $\alpha^m \in X$ are negative then apply the same argument to $\frac{1}{\alpha}$.

Let $\alpha^m \in X$ so $\alpha^m = A + k_m B$ for some $k_m \in \mathbb{Z}$, and let σ be a generator of the Galois group of $\mathbb{Q}(\sqrt{d})$ over $\mathbb{Q}$. Then $\sigma(\alpha)^m = \sigma(A) + k_m \cdot \sigma(B)$. The norm function is defined as $Nm(\alpha) = \alpha \cdot \sigma(\alpha)$. So,

$$Nm(\alpha)^m = Nm(A) + Tr(A \cdot \sigma(B))k_m + Nm(B)k_m^2,$$

where Tr denotes the trace function. The polynomial $Nm(A) + Tr(A \cdot \sigma(B))x + Nm(B)x^2$ is over $\mathbb{Q}$. By Euclidean reduction with $r = 3$ and some s_0 with $0 \leqslant s_0 < 3$, the equation

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$$Nm(\alpha^{s_0})y^3 = Nm(A) + Tr(A\cdot\sigma(B))x + Nm(B)x^2 \tag{5}$$

has infinitely many integer solutions of the form $(Nm(\alpha^j), k_{m_j})$, for $m_j = 3j + s_0$ and $j \in \mathbb{Z}$. Notice that, according to the sign of the infinitely many m, we work either with α or with $\frac{1}{\alpha}$. If the polynomial

$$P(x) = Nm(A) + Tr(A\cdot\sigma(B))x + Nm(B)x^2$$

has non-zero discriminant, then (5) defines (the affine part of) an elliptic curve over $\mathbb{Q}$, and by Siegel's Theorem (see [1]) we get a contradiction.

It remains to consider the case when the discriminant of $P(x)$ is zero, i.e., $(Tr(A\cdot\sigma(B)))^2 - 4Nm(A\cdot B) = 0$. In this case, we have

$$P(x) = Nm(B)\left(x + \frac{Tr(A\cdot\sigma(B))}{2Nm(B)}\right)^2,$$

where $-\frac{Tr(A\cdot\sigma(B))}{2Nm(B)}$ is the multiple root, and equation (5) becomes

$$Nm(\alpha^{s_0})y^3 = Nm(B)\left(x + \frac{Tr(A\cdot\sigma(B))}{2Nm(B)}\right)^2. \tag{6}$$

Equation (6) has infinitely many rational solutions of the form

$$\left(Nm(\alpha^j), k_{m_j} + \frac{Tr(A\cdot\sigma(B))}{2Nm(B)}\right), \tag{7}$$

where j varies in $\mathbb{Z}$. By the change of variable $x \mapsto x + \frac{Tr(A\cdot\sigma(B))}{2Nm(B)}$, and dividing by $Nm(\alpha^{s_0})$, we transform equation (6) to one of the form $y^3 = cx^2$, where $c = \frac{Nm(B)}{Nm(\alpha^{s_0})} \in \mathbb{Q}^\times$. The equation defines a rational curve which we now parameterize.

Claim 2. The rational solutions of $y^3 = cx^2$ are of the following form:

$$\begin{cases} x = \theta^3 c^{-5} \\ y = \theta^2 c^{-3} \end{cases} \tag{8}$$

with $\theta \in \mathbb{Q}$.

Proof of the Claim 2: Consider the p-adic valuation v_p for some p. Suppose that $y^3 = cx^2$ with $x, y \in \mathbb{Q}$. Then we have

(1) $3v_p(y) = 2v_p(x) + v_p(c)$, and so $v_p(y) = v_p\left(\left(\frac{x}{y}\right)^2 c\right)$

(2) $15v_p(y) = 10v_p(x) + v_p(c)$, and so $v_p(x) = v_p\left(\left(\frac{y^5}{x^2}\right)^3 c^{-5}\right)$.

Hence,
$x = \theta^3 c^{-5}$ and $y = \xi^2 c$, for some θ, ξ, and so $\xi^6 c^3 = \theta^6 c^{-9}$. Therefore, $(\xi/\theta)^6 = c^{-12}$, and this implies that $\xi/\theta = \pm c^{-2}$, i.e., $\xi = \pm\theta c^{-2}$. Notice that in the equation $y^3 = cx^2$ the variable x occurs in even power while the variable y occurs in odd power, so we get the following parameterization of the curve $y^3 = cx^2$:

$$\begin{cases} x = \pm\,\theta^3 c^{-5} \\ y = \theta^2 c^{-3}. \end{cases} \tag{9}$$

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We have then obtained that all the rational solutions of $y^3 = cx^2$ are of the form in (9) for θ rational. From (7) and (9), it follows that, for infinitely many $j \in \mathbb{N}$,

$$\left(Nm(\alpha^j), k_{m_j} + \frac{Tr(A \cdot \sigma(B))}{2Nm(B)}\right) = (\theta_{m_j}^2 c^{-3}, \theta_{m_j}^3 c^{-5}), \tag{10}$$

for $\theta_{m_j} \in \mathbb{Q}$. Hence $k_{m_j} = \theta_{m_j}^3 c^{-5} - \frac{Tr(A \cdot \sigma(B))}{2Nm(B)}$.

We have then that there are infinitely many $j \in \mathbb{N}$ such that $(\alpha^j)^3 \alpha^{s_0} = A + k_{m_j} B$, where as before $m_j = 3j + s_0$. Now we consider the new curve in y and θ,

$$y^3 \alpha^{s_0} = \theta^3 c^{-5} B + \left(A - \frac{Tr(A \cdot \sigma(B))}{2Nm(B)} B\right). \tag{11}$$

We now distinguish two cases.

Case 1: $A - \frac{Tr(A \cdot \sigma(B))}{2Nm(B)} B = 0$. Let $m, m' \in \mathbb{Z}$, $m \neq m'$, and let both belong to X. Then $\alpha^m \neq \alpha^{m'}$ and

$$\alpha^{m-m'} = \left(\frac{\theta_m}{\theta_{m'}}\right)^3,$$

and, by Lemma 6.1, we complete the proof in this case.

Case 2: $A - \frac{Tr(A \cdot \sigma(B))}{2Nm(B)} B \neq 0$. Dividing both sides of equation (11) by α^{s_0}, we obtain a new elliptic curve defined on the number field $\mathbb{Q}(\sqrt{d})$. In this case, we apply a generalization of Siegel's Theorem (see [1]) to the new elliptic curve defined on $\mathbb{Q}(\sqrt{d})$. Fix a prime ideal $\mathcal{P}$ in $\mathbb{Q}(\sqrt{d})$. Then, by (10), we have that

$$v_{\mathcal{P}}(\theta_{m_j}) \geqslant \frac{1}{3}\left(5v_{\mathcal{P}}(c) + \min\left\{0, v_{\mathcal{P}}\left(\frac{Tr(A \cdot \sigma(B))}{2Nm(B)}\right)\right\}\right).$$

For all but finitely many prime ideals $\mathcal{P}$, we have both $v_{\mathcal{P}}(c)$ and $v_{\mathcal{P}}\left(\frac{Tr(A \cdot \sigma(B))}{2Nm(B)}\right) \geqslant 0$. So there is a finite set $\mathcal{S}$ of prime ideals $\mathcal{P}$ in $\mathbb{Q}(\sqrt{d})$ so all $v_{\mathcal{P}}(\theta_{m_j})$ are $\mathcal{S}$-integers. Recall also that α^{s_0} is an integer in $\mathbb{Q}(\sqrt{d})$, so we get a contradiction with Siegel's Theorem.

We are now able to characterize those infinite subsets of an infinite cyclic group which may occur as zero set of an exponential polynomial in $\mathcal{E}$. In particular, this polynomial is divisible by a simple polynomial. □

Corollary 6.3. *The following characterization holds:*

$$\{\alpha \in K : |\langle\alpha\rangle \cap Z(f)| = \infty \text{ for some } f \in \mathcal{E}\} = \{\alpha \in K : \alpha^r \in \mathbb{Z} \text{ for some } r \in \mathbb{Z} - \{0\}\}.$$

Proof. ($\subseteq$) This inclusion follows from previous theorem. □

($\supseteq$) Clearly α is algebraic over $\mathbb{Q}$. Suppose that $r \in \mathbb{Z}$ and $r > 0$. Then, for any $s \geqslant r$, $\alpha^s \in \mathbb{Z}$. Let $j_0 < r$, and consider the infinitely many integers of the form $s = rq + j_0$ as q varies in $\mathbb{N}$. Then the polynomial $f(z) = 1 - e^{\mu z}$, where $\mu = \alpha^{j_0} 2i\pi$, has infinitely many roots in $\langle\alpha\rangle$. If $r < 0$, then use the same argument for $\frac{1}{\alpha}$. □

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