Proceedings of the Royal Society of Edinburgh, **148A**, 659–668, *2018*
DOI:10.1017/S0308210517000269

Evolution of non-simple closed curves in the area-preserving curvature flow

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(MS received 12 September 2015; accepted 4 July 2016)

The convergence and blow-up results are established for the evolution of non-simple closed curves in an area-preserving curvature flow. It is shown that the global solution starting from a locally convex curve converges to an m-fold circle if the enclosed algebraic area A_0 is positive, and evolves into a point if $A_0 = 0$.

Keywords: curvature flow; non-local; blow-up; convergence

2010 *Mathematics subject classification:* Primary 53C44; 35B40; 35K59

1. Introduction

In this paper we investigate the evolution of non-simple closed curves $X(p,t)$ parametrized by p and driven by the inner normal speed

$$V(p,t) = \left(k(p,t) - \frac{2m\pi}{L(t)}\right)\boldsymbol{n}(p,t), \tag{1.1}$$

where $k(p,t)$ and $L(t)$ denote the curvature (with respect to the inner normal $\boldsymbol{n}(p,t)$) and the length of $X(p,t)$, respectively, and $2m\pi$ $(m > 0)$ denotes its total curvature. This flow was studied by Gage [10] when the initial curve X_0 is a simple convex closed curve. It was shown that the flow preserves the evolving curve's convexity and enclosed area while decreasing its length, finally converging to a round circle under the C^∞ metric.

When the initial curve X_0 is non-simple closed convex, it can still be shown that the enclosed algebraic area is preserved under the flow, but some things become different. Two relevant studies are [9, 23]. In [9] it was shown, among other things, that the singularity happens at a finite time in the flow (1.1) (that is, the curvature blows up) when the initial curve X_0 satisfies $A_0 < 0$ or $L_0^2 < 4m\pi A_0$, where L_0 and A_0 are the length and the enclosed algebraic area of X_0, respectively. In [23] two classes of rotationally symmetric, locally convex closed initial curves, which both enclose a positive algebraic area, were found to guarantee the convergence of the flow (1.1) to m-fold circles. One class is that of highly symmetric convex curves, which are defined by the locally convex closed curves having n-fold rotational symmetry and total curvature of $2m\pi$ with $n > 2m$. The other is that of Abresch–Langer-type convex curves, which still have n-fold rotational symmetry and a total curvature of $2m\pi$ but with $n < 2m$. See its detailed definition in [23]. We note that the Abresch–Langer curves in [1] belong to this class.

From the works mentioned above, one can see that the enclosed algebraic area of the initial curve plays an important role in the behaviour of the flow. In [9] an open problem was proposed, namely, what is the evolution of the flow when $A_0 = 0$? Another natural question is, what happens for a general initial curve with $A_0 > 0$? On the other hand, if the singularity appears, what about the blow-up property of the curvature? We will partially answer these questions in this short note. The following is our main theorem.

THEOREM 1.1. *Let $X(\cdot, t)$ be a solution of (1.1) in a maximal interval $[0, T)$ ($T \leqslant \infty$), where the initial curve X_0 is a closed and smooth and may be non-simple.*

(1) *Assume that the time span is infinite ($T = \infty$) and that X_0 is locally convex with total curvature $2m\pi$ enclosing an algebraic area A_0. If $A_0 > 0$, $X(\cdot, t)$ converges smoothly to an m-fold circle enclosing the same algebraic area as X_0. If $A_0 = 0$, $X(\cdot, t)$ evolves into a point finally.*

(2) *Assume that the time span is finite ($T < \infty$). If the length of the evolving curve satisfies $\lim_{t\to T} L(t) > 0$, then*

$$\int_{X(\cdot,t)} k^2(s, t)\,\mathrm{d}s \geqslant C(T - t)^{-1} \tag{1.2}$$

for some constant C, as t is close to T.

We note that the proof of theorem 1.1 depends crucially on using the interpolation inequalities. The idea is from Dziuk–Kuwert–Schätzle [8] and Chou [6], in both of which another kind of area-preserving flow, namely, the curve diffusion flow, was investigated.

Before concluding this section, we say more about non-local flow. As an interesting variant of the popular curve shortening flow [2, 3, 7, 11], the non-local curvature flow, arising in many applications [4, 21, 24] such as phase transitions, image processing, etc., has received much attention in recent years. In addition to the area-preserving flow considered here, there are other non-local flows. For example, Ma and Zhu [16] studied a length-preserving flow, and Jiang and Pan [13] studied a non-local flow that increased the enclosed area of the evolving curves and decreased their length. The curves in their papers and [10] are all simple closed and convex, and the convergence to a round circle is proved by way of modifying the argument in [11]. Their proof depends crucially on the Bonnesen inequality. Since it is unknown whether the Bonnesen inequality holds for a general non-simple closed curve, we are compelled to use a different method in this paper, that is, an energy method. For other types of planar non-local flow, the reader is referred to [14, 15, 19, 20, 22]. In the higher dimensional case, people also consider non-local flows. For example, there are Huisken's volume-preserving mean-curvature flow [12] and McCoy's surface-area-preserving mean-curvature flow [17]. Recently, the study of non-local flow was extended to the case of the Riemannian manifold; see [25].

We organize this paper in the following way. Some basic formulae are given in § 2. We then prove theorem 1.1 in § 3.

2. Reformulation

We use the following notation:

- $s(t)$ denotes the arc length of $X(\cdot, t)$,
- $\mathrm{d}s$ denotes the differential element of arc-length,
- $\boldsymbol{t}$ and $\boldsymbol{n}$ respectively denote the unit tangent and inner normal of $X(\cdot, t)$,
- θ denotes the tangential angle of $X(\cdot, t)$,
- $L(t)$ denotes the length of $X(\cdot, t)$,
- $A(t)$ denotes the algebraic area of $X(\cdot, t)$ defined by $-\frac{1}{2}\int_X \langle X, \boldsymbol{n}\rangle \,\mathrm{d}s$,
- $k(\cdot, t)$ denotes the curvature of $X(\cdot, t)$ with respect to $\boldsymbol{n}$.

Here, we always take the orientation of $X(\cdot, t)$ to be counterclockwise.

We write down the evolution of various geometric quantities along the flow (1.1), which can be deduced from the general formulae in [7], as follows:

$$\bar{k} = \frac{1}{L(t)}\int_X k \,\mathrm{d}s = \frac{2m\pi}{L(t)}, \qquad \frac{\partial(\mathrm{d}s)}{\partial t} = -k(k-\bar{k})\,\mathrm{d}s,$$
$$\frac{\partial k}{\partial t} = k_{ss} + k^2(k-\bar{k}), \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\int_X k(k-\bar{k})\,\mathrm{d}s \leqslant 0,$$

and

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_X (k-\bar{k})\,\mathrm{d}s = 0.$$

When the locally convex solution $X(\cdot, t)$ is considered, each point on it has a unique tangent and one can use the tangent angle $\theta \in S^1_m := \mathbb{R}/2m\pi\mathbb{Z}$ to parametrize it. Generally speaking, θ is a function depending on t. In order to make θ independent of the time t, one can add a tangential component to the velocity vector $\partial X/\partial t$, which does not affect the geometric shape of the evolving curve (see, for instance, [10]). Then the evolution equations can be expressed in the coordinates θ and t. If we denote by $k(\theta, t)$ the curvature function of $X(\theta, t)$, problem (1.1) can be reformulated as

$$\left.\begin{aligned} k_t &= k^2[k_{\theta\theta} + k - \bar{k}], \quad && (\theta, t) \in I \times (0, T),\\ k(\theta, 0) &= k_0, && \theta \in I, \end{aligned}\right\} \tag{2.1}$$

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where k_0 denotes the curvature of X_0. Hereafter, the notation I is always used to denote the circle S^1_m.

By the maximum principle, we can show that the local convexity of the initial curve is preserved along the flow (1.1).

LEMMA 2.1. *If the initial curve X_0 is locally convex, then $X(\cdot,t)$ remains locally convex as long as the flow exists.*

Proof. By continuity, $\min_{\theta\in I} k(\theta,t)$ remains positive on small time intervals. Assume that the time span of the flow is T. Suppose that the conclusion is not true. Then there must be a first time, say $t_1 < T$, such that $\min_{\theta\in I} k(\theta,t_1) = 0$. We will deduce a contradiction. Consider the quantity

$$\Phi(\theta,t) = \frac{1}{k(\theta,t)} - \frac{L(t)}{2m\pi} - \frac{1}{2m\pi}\int_0^t\int_0^{2m\pi} k(\theta,\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau$$

with $(\theta,t)\in I\times[0,t_1)$. By (2.1), we have

$$\Phi_t(\theta,t) = -k_{\theta\theta} - k \leqslant k^2(\theta,t)\Phi_{\theta\theta}(\theta,t).$$

Hence, by the maximum principle,

$$\frac{1}{k(\theta,t)} \leqslant \max_{\theta\in I}\left(\frac{1}{k_0(\theta)}\right) + \frac{L(t)-L(0)}{2m\pi} + \frac{1}{2m\pi}\int_0^t\int_0^{2m\pi} k(\theta,\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau$$

for all $(\theta,t)\in I\times[0,t_1)$, where we note that

$$\max_{\theta\in I}\left(\frac{1}{k_0(\theta)}\right) - \frac{L(0)}{2m\pi} = \max_{\theta\in I}\left(\frac{1}{k_0(\theta)}\right) - \frac{1}{2m\pi}\int_0^{2m\pi}\frac{1}{k_0(\theta)}\,\mathrm{d}\theta \geqslant 0$$

and

$$\max_{(\theta,t)\in I\times[0,t_1)} k(\theta,t) \leqslant C_1(t_1) < \infty$$

for some constant $C_1(t_1)$. Therefore,

$$\min_{\theta\in I} k(\theta,t) \geqslant C_2(t_1) > 0 \quad \forall t\in[0,t_1),$$

for some constant $C_2(t_1)$. This is a contradiction! The proof is done. □

3. Proof of theorem 1.1

The crucial point in the proof of theorem 1.1 is to obtain an estimate for the evolving curve's energy

$$E(t) = \int_{X(\cdot,t)} (k-\bar{k})^2\,\mathrm{d}s.$$

LEMMA 3.1. *For the flow (1.1), if*

$$\lim_{t\to T} L(t) := L(T) > 0,$$

then the energy satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leqslant C(E+E^3),$$

where C only depends on the initial curve and $L(T)$.

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Proof. First, by using the evolution formulae for $\mathrm{d}s$ and k, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_X k^2\,\mathrm{d}s &= 2\int_X kk_t\,\mathrm{d}s - \int_X k^3(k-\bar{k})\,\mathrm{d}s\\ &= 2\int_X k[k_{ss}+k^2(k-\bar{k})]\,\mathrm{d}s - \int_X k^3(k-\bar{k})\,\mathrm{d}s\\ &= -2\int_X k_s^2\,\mathrm{d}s + \int_X k^3(k-\bar{k})\,\mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_X (k-\bar{k})^2\,\mathrm{d}s &= \frac{\mathrm{d}}{\mathrm{d}t}\biggl[\int_X k^2\,\mathrm{d}s - \frac{4m^2\pi^2}{L}\biggr]\\ &= -2\int_X k_s^2\,\mathrm{d}s + \int_X k^3(k-\bar{k})\,\mathrm{d}s - \bar{k}^2\int_X k(k-\bar{k})\,\mathrm{d}s\\ &= -2\int_X k_s^2\,\mathrm{d}s + \int_X (k-\bar{k})^4\,\mathrm{d}s + 3\bar{k}\int_X (k-\bar{k})^3\,\mathrm{d}s\\ &\quad + 2\bar{k}^2\int_X (k-\bar{k})^2\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

We shall use the interpolation inequalities. For periodic functions with zero mean,

$$\|u^{(j)}\|_{L^r} \leqslant C\|u\|_{L^p}^{1-\theta}\|u^{(k)}\|_{L^q}^{\theta}, \quad \theta\in(0,1),$$

where r, p, q, j, k and θ satisfy $p,q,r>1$, $j\geqslant 0$,

$$\frac{1}{r} = j + \theta\biggl(\frac{1}{q}-k\biggr) + (1-\theta)\frac{1}{p}$$

and

$$\frac{j}{k}\leqslant\theta\leqslant 1.$$

Here the constant C depends on r, p, q, j and k only. Using the interpolation inequality, we have

$$\int_X (k-\bar{k})^3\,\mathrm{d}s \leqslant C\biggl(\int_X (k-\bar{k})^{4/3}\,\mathrm{d}s\biggr)^{3/2}\biggl(\int_X k_s^2\,\mathrm{d}s\biggr)^{1/2}$$

and

$$\int_X (k-\bar{k})^4\,\mathrm{d}s \leqslant C\biggl(\int_X (k-\bar{k})^{2}\,\mathrm{d}s\biggr)^{3/2}\biggl(\int_X k_s^2\,\mathrm{d}s\biggr)^{1/2}.$$

Hence,

$$\int_X (k-\bar{k})^3\,\mathrm{d}s \leqslant \varepsilon_1\int_X k_s^2\,\mathrm{d}s + \frac{C^2L(t)}{4\varepsilon_1}\biggl(\int_X (k-\bar{k})^2\,\mathrm{d}s\biggr)^2$$

and

$$\int_X (k-\bar{k})^4\,\mathrm{d}s \leqslant \varepsilon_2\int_X k_s^2\,\mathrm{d}s + \frac{C^2}{4\varepsilon_2}\biggl(\int_X (k-\bar{k})^2\,\mathrm{d}s\biggr)^3.$$

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Putting these estimates into (3.1) and noticing that $L(t) \leqslant L_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leqslant -2\int_X k_s^2\,\mathrm{d}s + 3\bar{k}\left(\varepsilon_1\int_X k_s^2\,\mathrm{d}s + \frac{C^2L_0}{4\varepsilon_1}E^2\right) + \varepsilon_2\int_X k_s^2\,\mathrm{d}s + \frac{C^2}{4\varepsilon_2}E^3 + 2\bar{k}^2E.$$

Since $L(t) \geqslant L(T)$, we have $\bar{k} \leqslant 2m\pi/L(T)$. By choosing ε_1, ε_2 such that

$$6m\pi\varepsilon_1/L(T) + \varepsilon_2 = 2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leqslant C_1(E + E^2 + E^3),$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leqslant C_2(E + E^3),$$

where the constants C_1, C_2 depend only on the initial data, $L(T)$ and the best constants in these interpolation inequalities. □

If the time span is infinite ($T = \infty$), we have the following lemma.

Lemma 3.2. *For the flow (1.1), if the flow exists for all time and* $\lim_{t\to\infty} L(t) > 0$*, then*

$$E(t) \leqslant C \quad \forall t \geqslant 0,$$

for some constant C*. Furthermore, we have*

$$\int_{X(\cdot,t)} k^2\,\mathrm{d}s \leqslant C \quad \forall t \geqslant 0,$$

for some constant C*.*

Proof. By the evolution equation of $L(t)$, we have

$$-\int_0^t\int_{X(\cdot,\tau)} k(k-\bar{k})\,\mathrm{d}s\,\mathrm{d}\tau = L(t) - L_0.$$

If the flow exists for all time, then

$$\int_0^\infty\int_{X(\cdot,\tau)} k(k-\bar{k})\,\mathrm{d}s\,\mathrm{d}\tau \leqslant L_0.$$

That is,

$$\int_0^\infty E(\tau)\,\mathrm{d}\tau \leqslant L_0.$$

Hence, for any $\varepsilon > 0$, there exists a j_0 such that we can find, by the mean-value theorem, $t_j \in [j, j+1]$ satisfying $E(t_j) \leqslant \varepsilon$ for all $j \geqslant j_0$. From lemma 3.1 it is clear that we can find a sufficiently small ε such that $E(t)$ is less than 1 for all $t \in [t_j, t_j + 2]$. This means that $E(t)$ is uniformly bounded in $[j_0 + 1, \infty)$. Hence, it is uniformly bounded in $[0, \infty)$. □

Now, we can study the convergence of evolving curves when the flow exists for all time.

LEMMA 3.3. *Assume that the initial curve X_0 is locally convex. If the flow (1.1) exists for all time and $\lim_{t\to\infty} L(t) = L(\infty) > 0$, then $X(\cdot, t)$ converges smoothly to an m-fold circle.*

Proof. Since X_0 is locally convex, lemma 2.1 tells us that the evolving curves are also always locally convex. This means that we can use the tangential angle θ as a parameter, as explained in § 2. First, we claim that

$$\int_I (k_\theta)^2 \, \mathrm{d}\theta \leqslant \int_I k^2 \, \mathrm{d}\theta + C \tag{3.2}$$

holds, where C is a constant depending on X_0 and $\sup_{[0,\infty)} E(t)$ only. Indeed, by using the equation for k, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_I \left[(k_\theta)^2 - k^2 + \frac{4m\pi}{L} k\right] \mathrm{d}\theta &= -2 \int_I k^2 \left(k_{\theta\theta} + k - \frac{2m\pi}{L}\right)^2 \mathrm{d}\theta - \frac{4m\pi}{L^2} \frac{\mathrm{d}L}{\mathrm{d}t} \int_I k \, \mathrm{d}\theta \\ &\leqslant -\frac{4m\pi}{L^2} \frac{\mathrm{d}L}{\mathrm{d}t} \int_I k \, \mathrm{d}\theta.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I (k_\theta)^2 \, \mathrm{d}\theta \leqslant \frac{\mathrm{d}}{\mathrm{d}t} \int_I k^2 \, \mathrm{d}\theta - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{4m\pi}{L} \int_I k \, \mathrm{d}\theta\right) - \frac{4m\pi}{L^2} \frac{\mathrm{d}L}{\mathrm{d}t} \int_I k \, \mathrm{d}\theta.$$

Integration yields

$$\int_I (k_\theta)^2 \, \mathrm{d}\theta \leqslant \int_I k^2 \, \mathrm{d}\theta - \int_0^t \frac{4m\pi}{L^2} \frac{\mathrm{d}L}{\mathrm{d}t} \int_I k \, \mathrm{d}\theta \, \mathrm{d}t + C_3,$$

where

$$C_3 = \int_I (k_\theta(\theta, 0))^2 \, \mathrm{d}\theta + \frac{4m\pi}{L(0)} \int_I k(\theta, 0) \, \mathrm{d}\theta.$$

By lemma 3.2, we have

$$\begin{aligned}\int_I (k_\theta)^2 \, \mathrm{d}\theta &\leqslant \int_I k^2 \, \mathrm{d}\theta - C \int_0^t \frac{4m\pi}{L^2} \frac{\mathrm{d}L}{\mathrm{d}\tau} \, \mathrm{d}\tau + C_3 \\ &= \int_I k^2 \, \mathrm{d}\theta + 4m\pi C \left(\frac{1}{L(t)} - \frac{1}{L(0)}\right) + C_3.\end{aligned}$$

Since $L(t) \geqslant L(\infty) > 0$, we have proved the claim that (3.2) holds.

Let $k_{\max}(t) = \max_I k(\theta, t)$. Assume that $k_{\max}(t) = k(\theta_t, t)$ for some $\theta_t \in I$. We make the following claim: for any small $\varepsilon > 0$ there exists a number $\delta > 0$, depending only on ε, such that

$$(1 - \varepsilon) k_{\max}(t) \leqslant k(\theta, t) + \sqrt{2m\pi C} \tag{3.3}$$

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for all $\theta \in (\theta_t - \delta^2, \theta_t + \delta^2)$ and all $t \in (0, T)$, where C is the constant appearing in (3.2). Indeed, we have

$$
\begin{aligned}
k_{\max}(t) &= k(\theta, t) + \int_{\theta}^{\theta_t} k_\theta(\theta, t)\, \mathrm{d}\theta \\
&\leqslant k(\theta, t) + |\theta_t - \theta|^{1/2} \left(\int_{\theta}^{\theta_t} k_\theta^2 \, \mathrm{d}\theta \right)^{1/2} \\
&\leqslant k(\theta, t) + |\theta_t - \theta|^{1/2} \left(\int_I k^2 \, \mathrm{d}\theta + C \right)^{1/2} \\
&\leqslant k(\theta, t) + |\theta_t - \theta|^{1/2} (2m\pi k_{\max}^2(t) + C)^{1/2} \\
&\leqslant k(\theta, t) + |\theta_t - \theta|^{1/2} \sqrt{2m\pi} k_{\max}(t) + |\theta_t - \theta|^{1/2} C^{1/2} \\
&\leqslant k(\theta, t) + |\theta_t - \theta|^{1/2} \sqrt{2m\pi} k_{\max}(t) + \sqrt{2m\pi C}.
\end{aligned}
$$

Take δ such that $|\theta_t - \theta|^{1/2} \leqslant \delta := \varepsilon/\sqrt{2m\pi}$. Thus the claim is proved.

In order to show that our lemma holds, it is sufficient to show that the curvature k has a uniform upper bound independent of time. By contradiction, if $\lim_{t\to T} \sup k_{\max}(t) = \infty$, we can find a sequence $\{t_j\}_{j=1}^{\infty} \to \infty$ and a sequence $\{\theta_j\}_{j=1}^{\infty} \subset I$ such that $k(\theta_j, t_j) \to \infty$. By the above claim, we have that

$$
\int_I k(\theta, t_j)\, \mathrm{d}\theta \to \infty,
$$

contradicting

$$
\int_I k \, \mathrm{d}\theta < C,
$$

which was shown in lemma 3.2! Thus we must have $k_{\max}(t) \leqslant C$ for a constant C independent of time. By following a routine step as in [23] or [5], we can show that the curvature converges smoothly to the curvature of an m-fold circle. Then using the technique in [11, 13] or [16] to clarify the decay of the derivatives of the curvature exponentially in time, we finally obtain the convergence of the flow to a unique m-fold circle. □

Proof of theorem 1.1(1). An isoperimetric inequality of Rado [18] tells that for any closed immersed curve,

$$
L^2 \geqslant 4\pi \sum_j |m_j| A_j,
$$

where m_j and A_j are the winding number and the area (in the usual sense) of the jth component of the curve, respectively. Since the flow preserves the (algebraic) area of the curve, we have

$$
L^2(t) \geqslant 4\pi \left| \sum_j m_j A_j \right| = 4\pi |A_0|.
$$

If $A_0 > 0$ and $T = +\infty$, then $\lim_{t\to\infty} L(t) > 0$. From lemma 3.3, we obtain the convergence of the flow. If $A_0 = 0$ and the flow does not go to a point, it means

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that $\lim_{t\to\infty} L(t) > 0$, which from lemma 3.3 also yields the convergence of the flow to an m-fold circle. However, this will contradict the fact that the flow preserves the enclosed area. Thus the proof is complete. □

Proof of theorem 1.1(2). Since the lifespan of the flow is finite, the curvature k of the evolving curve satisfies $\limsup_{t\to T} k_{\max}(t) = \infty$. By a careful choice, we can find the sequences $\{\theta_j\}_{j=1}^{\infty} \subset I$ and $\{t_j\}_{j=1}^{\infty} \to T$ such that

$$k(\theta_j, t_j) = \max_{I\times[0,t_j]} k(\theta, t)$$

and

$$k(\theta_j, t_j) \to \infty \quad \text{as } j \to \infty.$$

We then claim that $E(t_j) \to \infty$ as $j \to \infty$. A similar proof to that of lemma I1.12 in [2] shows that the following estimate holds for any $t \in (0, T)$:

$$\sup_{I\times[0,t]} (k_\theta^2 + k^2) \leqslant \max\Big\{ \sup_{I\times[0,t]} k^2, \sup_{I\times\{0\}} (k_\theta^2 + k^2) \Big\}.$$

According to the definition of the sequence $\{k(\theta_j, t_j)\}_{j=1}^{\infty}$, we have

$$\sup_I k_\theta^2(\theta, t_j) \leqslant k^2(\theta_j, t_j) + C,$$

and C only depends on initial data. Then, using the same proof as that of lemma 3.3, we obtain the estimate (3.3) along the time sequence $\{t_j\}_{j=1}^{\infty}$. Since

$$E(t) = \int_I k(\theta, t)\,\mathrm{d}\theta - (2m\pi)^2 L^{-1}$$

and $L(t) \geqslant L(\infty) > 0$, we immediately have that $E(t_j) \to \infty$ as $j \to \infty$.

By integrating the differential inequality obtained in lemma 3.1 on $[t, t_j]$ and then taking the limit of $t_j \to T$, we have $E(t) \geqslant [\mathrm{e}^{2C(T-t)} - 1]^{-1}$, and thus $E(t) \geqslant [4C(T-t)]^{-1}$ as t is close to T. □

Acknowledgements

We thank the referees for their careful reading and useful comments, which have helped improve the presentation of this paper. Our work is partly supported by the Fundamental Research Funds for the Central Universities (Grant no. 2242015R 30012), the NSF of China (Grant no. 11201249) and the Natural Science Foundation of Jiangsu Province (Grant no. BK20130596).

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