Probability in the Engineering and Informational Sciences, **26**, 2012, 375–391.
doi:10.1017/S026996481200006X

LIKELIHOOD RATIO AND HAZARD RATE ORDERINGS OF THE MAXIMA IN TWO MULTIPLE-OUTLIER GEOMETRIC SAMPLES

BAOJUN DU
School of Mathematics and Statistics
Lanzhou University, Lanzhou 730000, China

PENG ZHAO
School of Mathematical Sciences
Jiangsu Normal University, Xuzhou 221116, China

N. BALAKRISHNAN
Department of Mathematics and Statistics
McMaster University, Hamilton, Ontario, Canada L8S 4K1
E-mail: bala@mcmaster.ca

In this paper, we study some stochastic comparisons of the maxima in two multiple-outlier geometric samples based on the likelihood ratio order, hazard rate order, and usual stochastic order. We establish a sufficient condition on parameter vectors for the likelihood ratio ordering to hold. For the special case when $n = 2$, it is proved that the p-larger order between the two parameter vectors is equivalent to the hazard rate order as well as usual stochastic order between the two maxima. Some numerical examples are presented for illustrating the established results.

1. INTRODUCTION

Order statistics have received considerable attention in the literature since they are very useful in statistical inference, goodness-of-fit tests, reliability theory, operations research, applied probability, and many other areas. Let $X_1, \ldots, X_n$ be independent random variables having possibly different probability distributions. Denote by $X_{i:n}$

 0269-9648/12 $25.00

the ith order statistic among $X_1, \ldots, X_n$. Then, the lifetime of a k-out-of-n system is given by $X_{n-k+1:n}$, with $X_{n:n}$ and $X_{1:n}$ corresponding to the lifetimes of parallel and series systems, respectively. Majority of the work on order statistics have been on the case when the underlying variables are independent and identically distributed (i.i.d.). The case when samples are non-i.i.d., however, often arises in a natural way in many practical situations; see, for example, Andrews et al. [1], Tiku, Tan, and Balakrishnan [25], and Barnett and Lewis [5]. Due to the complexity of the distribution theory in this case, only limited results are available in this direction; see, for example, Balakrishnan and Rao [3,4] and David and Nagaraja [9] for a comprehensive discussion on this aspect, and Balakrishnan [2] for an elaborate review of recent developments on order statistics arising from independent and non-identically distributed (i.ni.d.) random variables.

In life-testing and reliability analysis, the exponential distribution has been widely applied since it possesses the unique memoryless property and has the constant failure rate. Stochastic comparison of order statistics from heterogeneous exponential variables was first discussed by Pledger and Proschan [22]. Since then, many researchers have worked on this topic, including Proschan and Sethuraman [23], Kochar and Rojo [16], Dykstra [10], Khaledi and Kochar [14], Bon and Păltănea [8], Kochar and Xu [17], Păltănea [21], Zhao and Balakrishnan [28–30], Zhao, Li, and Balakrishnan [31,32], Joo and Mi [13], and Mao and Hu [19]. More recently, stochastic comparison results for order statistics and spacings from single- and multiple-outlier exponential models; see, for example, Khaledi and Kochar [15], Hu, Wang, and Zhu [12], Wen, Lu, and Hu [26], Xu et al. [27], Hu, Lu, and Wen [11], Kochar and Xu [18], and the references therein.

Suppose $X_1, \ldots, X_n$ are independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$, respectively. Similarly, let $X_1^*, \ldots, X_n^*$ be another set of independent exponential random variables with respective hazard rates $\lambda_1^*, \ldots, \lambda_n^*$. Then, Pledger and Proschan [22] showed, for $1 \le k \le n$, that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{k:n} \ge_{\text{st}} X_{k:n}^*. \tag{1.1}$$

Formal definitions of this and other stochastic orderings pertinent to the developments here will all be presented in the next section. Proschan and Sethuraman [23] strengthened this result from componentwise stochastic order to multivariate stochastic order, while Khaledi and Kochar [14] partially improved (1.1) as

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \ge_{\text{st}} X_{n:n}^*. \tag{1.2}$$

Boland, EL-Neweihi, and Proschan [6] showed by a counterexample that (1.1) cannot be strengthened from stochastic order to hazard rate order even for parallel systems with three independent exponential components; they established, however, for the case when $n = 2$ that

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{hr}} X_{2:2}^*. \tag{1.3}$$

Dykstra et al. [10] further improved (1.3) from the hazard rate order to likelihood ratio order as

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{lr}} X_{2:2}^*. \quad \textbf{(1.4)}$$

Joo and Mi [13] gave some conditions under which the hazard rate order in (1.3) holds. Specially, they proved, under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, that

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{hr}} X_{2:2}^*. \quad \textbf{(1.5)}$$

Recently, Zhao and Balakrishnan [29] strengthened the result in (1.5) and established the following two equivalent characterizations, under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$:

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\text{lr}} [\geq_{\text{rh}}] X_{2:2}^* \quad \textbf{(1.6)}$$

and

$$(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\text{hr}} [\geq_{\text{st}}] X_{2:2}^*. \quad \textbf{(1.7)}$$

In the robustness literature, several robust estimators have been proposed including trimmed estimators, winsorized estimators, linearly weighted means, and Gastwirth means. Though the robustness feature of these estimators are often intuitively clear, its demonstration in the presence of outliers is often difficult due to the complicated form of density functions of order statistics arising from a sample containing outliers. For this reason, this has been done in the outlier literature mostly only for the case when there is only one outlier in the sample; see, for example, the book by Barnett and Lewis [5]. However, the necessity for the consideration of a multiple-outlier model (i.e., a sample containing $n-p$ i.i.d. observations from a distribution F and the remaining p i.i.d. observations from yet another distribution G) is clear in order to study the performance of the robust estimators when there are multiple outliers present in the sample. But, due to the extremely complicated form of the distribution and joint distribution of order statistics arising from such a multiple-outlier model, limited studies have been done in this regard.

Zhao and Balakrishnan [30] further extended the results in (1.6) and (1.7) to the case of multiple-outlier exponential samples. To be specific, let $X_1, \ldots, X_n$ be independent exponential variables with parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_1, \ldots, \lambda_1}_{q}),$$

where $p + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent exponential variables with parameters

$$(\underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}).$$

Then, under the condition $\lambda_1^* \le \lambda_2^* \le \lambda_2 \le \lambda_1$, Zhao and Balakrishnan [30] established that

$$(\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_1,\ldots,\lambda_1}_{q}) \overset{\text{w}}{\succeq} (\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \Longleftrightarrow X_{n:n} \ge_{\text{lr}} [\ge_{\text{rh}}] Y_{n:n} \tag{1.8}$$

and

$$(\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_1,\ldots,\lambda_1}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \Longleftrightarrow X_{n:n} \ge_{\text{hr}} [\ge_{\text{st}}] Y_{n:n}. \tag{1.9}$$

It is well known that the geometric distribution can be regarded as the discrete counterpart of the exponential distribution as they both possess lack of memory property and constant hazard rates. The geometric distribution is one of the fundamental distributions in statistics, and has wide applications in reliability theory, engineering, game theory, quality control, and communication theory. For a geometric random variable X with parameter $p \in (0,1)$, we have the probability mass function as

$$\mathsf{P}(X=k) = p(1-p)^k, \qquad k \in \mathbb{N}_0 = \{0,1,2,\ldots\}.$$

Let $X_1,\ldots,X_n$ be independent geometric random variables with parameters $p_1,\ldots,p_n$, respectively, and $X_1^*,\ldots,X_n^*$ be another set of independent geometric random variables with respective parameters $p_1^*,\ldots,p_n^*$. Then, Mao and Hu [19] showed that

$$(p_1,\ldots,p_n) \overset{\text{p}}{\succeq} (p_1^*,\ldots,p_n^*) \Longrightarrow X_{n:n} \ge_{\text{st}} X_{n:n}^*, \tag{1.10}$$

which can be seen to be an analog of (1.2) for the geometric case. Let $Y_1,\ldots,Y_n$ be i.i.d. geometric random variables with common parameter p. Denote by $p_{\text{cg}} = 1-\{\prod_{i=1}^n(1-p_i)\}^{1/n}$ the complementary geometric mean of the p_i's and by $\bar{p} = \frac{1}{n}\sum_{i=1}^n p_i$ the arithmetic mean of the p_i's. Mao and Hu [19] then showed that

$$p \ge p_{\text{cg}} \Longrightarrow X_{n:n} \ge_{\text{lr}} Y_{n:n}, \tag{1.11}$$

and they also pointed out that the reversed hazard rate order (and hence the likelihood ratio order) does not hold between $X_{n:n}$ and $Y_{n:n}$ under the condition $p \ge \bar{p}$ even though it does hold for the corresponding exponential case; see Kochar and Xu [17]. Moreover, they left the question whether the hazard rate order holds between $X_{n:n}$ and $Y_{n:n}$ under the condition $p \ge \bar{p}$ as an open problem.

In this paper, we compare stochastically the maxima from two multiple-outlier geometric samples through the likelihood ratio order and hazard rate order (usual stochastic order). In this regard, let $X_1,\ldots,X_n$ be independent geometric variables with parameters

$$(\underbrace{p_1^*, \ldots, p_1^*}_{r}, \underbrace{p_1, \ldots, p_1}_{q}),$$

where $r + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent geometric variables with parameters

$$(\underbrace{p_2^*, \ldots, p_2^*}_{r}, \underbrace{p_2, \ldots, p_2}_{q}),$$

respectively. Then, we establish that, if $p_1^* \le p_2^* \le p_2 \le p_1$ and

$$(\underbrace{-\log(1-p_1^*), \ldots, -\log(1-p_1^*)}_{r}, \underbrace{-\log(1-p_1), \ldots, -\log(1-p_1)}_{q})$$

$$\overset{\mathrm{w}}{\succeq} (\underbrace{-\log(1-p_2^*), \ldots, -\log(1-p_2^*)}_{r}, \underbrace{-\log(1-p_2), \ldots, -\log(1-p_2)}_{q}),$$

we have

$$X_{n:n} \ge_{\mathrm{lr}} Y_{n:n}.$$

For the special case when $n = 2$, we have

$$(p_1, p_2) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*) \Longleftrightarrow X_{2:2} \ge_{\mathrm{hr}} [\ge_{\mathrm{st}}]\, Y_{2:2}, \tag{1.12}$$

which is an analog of the result in (1.7) for the geometric case. Incidentally, it provides a partial answer to the open problem posed by Mao and Hu [19]. We also show with the help of a counterexample that an analog of (1.6) does not hold. Finally, we provide some numerical examples for illustrating the established results.

2. DEFINITIONS AND NOTATION

In this section, we recall some notions of stochastic orders, and majorization and related orders which are most pertinent to the developments in the subsequent sections. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

2.1. Stochastic Orders

Here, we are concerned with non-negative random variables that are either absolutely continuous or discrete with support on integers $\mathbb{N}_0$.

DEFINITION 2.1: *For two absolutely continuous [discrete] random variables V_1 and V_2 with their density [probability mass] functions $g_1[p_1]$ and $g_2[p_2]$, distribution*

https://doi.org/10.1017/S026996481200006X Published online by Cambridge University Press

functions G_1 and G_2, survival functions $\overline{G}_1$ and $\overline{G}_2$, as the ratios in the statements below are well-defined, V_1 is said to be smaller than V_2 in the

(i) likelihood ratio order *(denoted by $V_1 \leq_{\rm lr} V_2$) if $g_2(t)/g_1(t)$ $[p_2(k)/p_1(k)]$ is increasing in t $[k \in \mathbb{N}_0]$;*

(ii) hazard rate order *(denoted by $V_1 \leq_{\rm hr} V_2$) if $\overline{G}_2(t)/\overline{G}_1(t)$ $[\overline{G}_2(k)/\overline{G}_1(k)]$ is increasing in t $[k \in \mathbb{N}_0]$;*

(iii) reversed hazard rate order *(denoted by $V_1 \leq_{\rm rh} V_2$) if $G_2(t)/G_1(t)$ $[G_2(k)/G_1(k)]$ is increasing in t $[k \in \mathbb{N}_0]$;*

(iv) stochastic order *(denoted by $V_1 \leq_{\rm st} V_2$) if $\overline{G}_2(t)$ $[\overline{G}_2(k)] \geq \overline{G}_1(t)$ $[\overline{G}_1(k)]$ for all x* $[k \in \mathbb{N}_0]$.

The following implications among these stochastic orders is well known (see Shaked and Shanthikumar, [24]):

$$V_1 \leq_{\rm lr} V_2 \Longrightarrow V_1 \leq_{\rm hr} [\leq_{\rm rh}] V_2 \Longrightarrow V_1 \leq_{\rm st} V_2.$$

2.2. Majorization and Related Orders

We will use the notion of majorization extensively in this paper as it is quite useful in establishing various inequalities. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2: *(i) A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n$ is said to* majorize *another vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re^n$ (written as $\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

(ii) A vector $\mathbf{x} \in \Re^n$ is said to weakly supmajorize *another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \overset{\rm w}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

(iii) A vector $\mathbf{x} \in \Re^n$ is said to weakly submajorize *another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \succeq_{\rm w} \mathbf{y}$) if*

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{m} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

*(iv) A vector $\mathbf{x} \in \Re^n_+$ is said to be p-*larger *than another vector $\mathbf{y} \in \Re^n_+$ (written as $\mathbf{x} \overset{\rm p}{\succeq} \mathbf{y}$) if*

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

It is clear that $\mathbf{x} \stackrel{\mathrm{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \stackrel{\mathrm{w}}{\succeq} \mathbf{y}$, and $\mathbf{x} \stackrel{\mathrm{p}}{\succeq} \mathbf{y}$ is equivalent to $\log(\mathbf{x}) \stackrel{\mathrm{w}}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ is the vector of componentwise logarithms of $\mathbf{x}$. It is known that $\mathbf{x} \stackrel{\mathrm{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \stackrel{\mathrm{p}}{\succeq} \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \Re_+^n$. The converse is, however, not true. For more details on majorization and p-larger orders and their applications, one may refer to Marshall and Olkin [20] and Bon and Păltănea [7].

3. LIKELIHOOD RATIO ORDERING OF THE MAXIMA IN TWO MULTIPLE-OUTLIER GEOMETRIC SAMPLES

In this section, we examine the likelihood ratio order of the maxima in two multiple-outlier geometric samples.

THEOREM 3.1: *Let $X_1, \ldots, X_n$ be independent geometric variables with parameters*

$$(\underbrace{p_1^*, \ldots, p_1^*}_{r}, \underbrace{p_1, \ldots, p_1}_{q}),$$

where $r + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent geometric variables with parameters

$$(\underbrace{p_2^*, \ldots, p_2^*}_{r}, \underbrace{p_2, \ldots, p_2}_{q}),$$

respectively. Then, if $p_1^* \le p_2^* \le p_2 \le p_1$ and

$$(\underbrace{-\log(1-p_1^*), \ldots, -\log(1-p_1^*)}_{r}, \underbrace{-\log(1-p_1), \ldots, -\log(1-p_1)}_{q})$$
$$\stackrel{\mathrm{w}}{\succeq} (\underbrace{-\log(1-p_2^*), \ldots, -\log(1-p_2^*)}_{r}, \underbrace{-\log(1-p_2), \ldots, -\log(1-p_2)}_{q}), \quad \textbf{(3.1)}$$

we have

$$X_{n:n} \ge_{\mathrm{lr}} Y_{n:n}.$$

PROOF: Let $S_1, \ldots, S_n$ be independent exponential variables with respective parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{r}, \underbrace{\lambda_1, \ldots, \lambda_1}_{q}),$$

where $r + q = n$, and $T_1, \ldots, T_n$ be another set of independent exponential variables with respective parameters

$$(\underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{r}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

https://doi.org/10.1017/S026996481200006X Published online by Cambridge University Press

where $\lambda_i = -\log(1-p_i)$ and $\lambda_i^* = -\log(1-p_i^*)$, $i=1,2$. We then have

$$(\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{r},\underbrace{\lambda_1,\ldots,\lambda_1}_{q}) \overset{\mathrm{w}}{\succeq} (\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{r},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}).$$

Let us denote by $f_{S_{n:n}}$ and $f_{T_{n:n}}$ the density functions of $S_{n:n}$ and $T_{n:n}$, respectively. Then, for $t \in \Re_+$, we have

$$\begin{aligned} f_{S_{n:n}}(t) &= \left\{\frac{r\lambda_1^* e^{-\lambda_1^* t}}{1-e^{-\lambda_1^* t}} + \frac{q\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}}\right\}(1-e^{-\lambda_1^* t})^r(1-e^{-\lambda_1 t})^q \\ &= -\left\{\frac{r(1-p_1^*)^t\log(1-p_1^*)}{1-(1-p_1^*)^t} + \frac{q(1-p_1)^t\log(1-p_1)}{1-(1-p_1)^t}\right\} \\ &\qquad \times [1-(1-p_1^*)^t]^r[1-(1-p_1)^t]^q \\ &= \frac{d}{dt}\{[1-(1-p_1^*)^t]^r[1-(1-p_1)^t]^q\}. \end{aligned}$$

Similarly, we have

$$f_{T_{n:n}}(t) = \frac{d}{dt}\{[1-(1-p_2^*)^t]^r[1-(1-p_2)^t]^q\},$$

which means, for $k \in \mathbb{N}_0$, that

$$\begin{aligned} &\frac{\mathsf{P}(X_{n:n}=k)}{\mathsf{P}(Y_{n:n}=k)} \\ &\quad = \frac{[1-(1-p_1^*)^{k+1}]^r[1-(1-p_1)^{k+1}]^q - [1-(1-p_1^*)^k]^r[1-(1-p_1)^k]^q}{[1-(1-p_2^*)^{k+1}]^r[1-(1-p_2)^{k+1}]^q - [1-(1-p_2^*)^k]^r[1-(1-p_2)^k]^q} \\ &\quad = \frac{\displaystyle\int_k^{k+1} f_{S_{n:n}}(u)\,du}{\displaystyle\int_k^{k+1} f_{T_{n:n}}(u)\,du}. \end{aligned}$$

From Theorem 3.6 in Zhao and Balakrishnan [30], we then have $S_{n:n} \geq_{\mathrm{lr}} T_{n:n}$; that is,

$$f_{S_{n:n}}(u) f_{T_{n:n}}(v) \leq f_{S_{n:n}}(v) f_{T_{n:n}}(u), \qquad 0 < u \leq v.$$

Upon integrating both sides with respect to u and v over $(u,v) \in [k,k+1]\times[k+1,k+2]$, we obtain

$$\int_k^{k+1} f_{S_{n:n}}(u)\,du \int_{k+1}^{k+2} f_{T_{n:n}}(v)\,dv \leq \int_k^{k+1} f_{T_{n:n}}(u)\,du \int_{k+1}^{k+2} f_{S_{n:n}}(v)\,dv, \qquad k \in \mathbb{N}_0.$$

From here, we can conclude that $\mathsf{P}(X_{n:n}=k)/\mathsf{P}(Y_{n:n}=k)$ is increasing in $k \in \mathbb{N}_0$, and the required result then follows immediately. ■

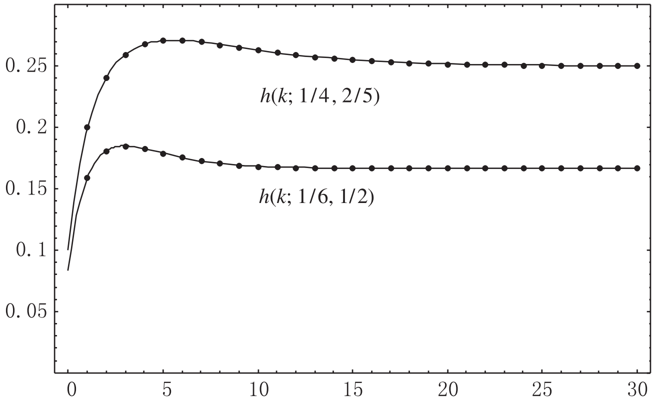

FIGURE 1. Plots of the hazard rate functions of the maxima of geometric variables with parameter vector $\left(\frac{1}{6}, \frac{1}{2}\right)$ and $\left(\frac{1}{4}, \frac{2}{5}\right)$.

Since the likelihood ratio order implies the hazard rate order, the result in Theorem 3.1 can be used to compare the hazard rate functions of the maxima from two multiple-outlier geometric samples. To illustrate this fact, we present the following numerical example.

Example 3.2: Let (X_1, X_2, X_3) be a vector of independent geometric random variables with parameter vector $(p_1^*, p_1^*, p_1) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{7}\right)$, and let $h\left(k; \frac{1}{6}, \frac{1}{6}, \frac{2}{7}\right)$ be the corresponding hazard rate function of $X_{3:3}$. Let (Y_1, Y_2, Y_3) be another vector of independent geometric random variables with parameter vector $(p_2^*, p_2^*, p_2) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{4}\right)$, and let $h\left(k; \frac{1}{5}, \frac{1}{5}, \frac{1}{4}\right)$ be the corresponding hazard rate function of $Y_{3:3}$. It can be readily verified that condition (3.1) in Theorem 3.1 is satisfied in this case. Figure 1 presents plots of the hazard rate functions of these two maxima which are readily seen to be in accordance with the result of Theorem 3.1.

As immediate consequences of Theorem 3.1, we obtain the following corollaries.

COROLLARY 3.3: *Let $X_1, \ldots, X_n$ be independent geometric variables with parameters*

$$(\underbrace{p_1, \ldots, p_1}_{r}, \underbrace{p_2, \ldots, p_2}_{q}),$$

where $r + q = n$, and $Y_1, \ldots, Y_n$ be another set of i.i.d. geometric variables with a common parameter p. Then, the necessary and sufficient condition for $X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}$ is

$$p \geq 1 - \{(1 - p_1)^r (1 - p_2)^q\}^{1/n}.$$

COROLLARY 3.4: *Under the same setup as in Theorem* 3.1*, if*

$$(\underbrace{\log(1-p_1^*),\ldots,\log(1-p_1^*)}_{r},\underbrace{\log(1-p_1),\ldots,\log(1-p_1)}_{q})$$
$$\succeq_{\mathrm{w}} (\underbrace{\log(1-p_2^*),\ldots,\log(1-p_2^*)}_{r},\underbrace{\log(1-p_2),\ldots,\log(1-p_2)}_{q}),$$

then

$$X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}.$$

COROLLARY 3.5: *Let $X_1,\ldots,X_n$ be independent geometric variables with parameters*

$$(\underbrace{p_1,\ldots,p_1}_{r},\underbrace{p,\ldots,p}_{q}),$$

where $r+q=n$, and $Y_1,\ldots,Y_n$ be another set of independent geometric variables with parameters

$$(\underbrace{p_2,\ldots,p_2}_{r},\underbrace{p,\ldots,p}_{q}),$$

respectively. Further, suppose $p \geq \max\{p_1,p_2\}$. Then, the necessary and sufficient condition for $X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}$ is $p_1 \leq p_2$.

PROOF: The *sufficiency* can be directly obtained from Theorem 3.1, and so we only need to show the *necessity*. Suppose $X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}$, we then have $X_{n:n} \geq_{\mathrm{st}} Y_{n:n}$; that is,

$$\{1-(1-p_1)^k\}^r\{1-(1-p)^k\}^q \leq \{1-(1-p_2)^k\}^r\{1-(1-p)^k\}^q, \qquad k \in \mathbb{N}_0,$$

which in turn implies $p_1 \leq p_2$, as required. ∎

4. HAZARD RATE ORDERING OF THE MAXIMA IN THE TWO-DIMENSIONAL CASE

In this section, we discuss the hazard rate ordering of the maxima for the special case when $n=2$.

LEMMA 4.1 (Marshall and Olkin [20] p. 57): *Let $I \subset \Re$ be an open interval, and let $\phi : I^n \to \Re$ be continuously differentiable. Then, ϕ is Schur-convex (Schur-concave) on I^n if and only if ϕ is symmetric on I^n and for all $i \neq j$,*

$$(z_i - z_j)\left[\frac{\partial}{\partial z_i}\phi(z) - \frac{\partial}{\partial z_j}\phi(z)\right] \geq [\leq] 0 \quad \textit{for all } z \in I^n,$$

where $(\partial/\partial z_i)\phi(z)$ denotes the partial derivative of $\phi(z)$ with respect to its i-th argument.

THEOREM 4.2: *Let X_1, X_2 be independent geometric variables with respective parameters p_1^*, p_1, and Y_1, Y_2 be another set of independent geometric variables with respective parameters p_2^*, p_2. Then, if $p_1^* \le p_2^* \le p_2 \le p_1$ and $p_1^* p_1 = p_2^* p_2$, we have*

$$X_{2:2} \ge_{\text{hr}} Y_{2:2}.$$

PROOF: The hazard rate function of $X_{2:2}$ is given by

$$\begin{aligned} h_{X_{2:2}}(k) &= \frac{[1-(1-p_1^*)^{k+1}][1-(1-p_1)^{k+1}] - [1-(1-p_1^*)^k][1-(1-p_1)^k]}{1-[1-(1-p_1^*)^k][1-(1-p_1)^k]} \\ &= \frac{\begin{array}{c} p_1^*(1-p_1^*)^k[1-(1-p_1)^k] + p_1(1-p_1)^k[1-(1-p_1^*)^k] \\ +p_1^* p_1 (1-p_1^*)^k (1-p_1)^k \end{array}}{1-[1-(1-p_1^*)^k][1-(1-p_1)^k]} \end{aligned}$$

for $k \in \mathbb{N}_0$. Similarly, the hazard rate function of $Y_{2:2}$ is given by

$$h_{Y_{2:2}}(k) = \frac{\begin{array}{c} p_2^*(1-p_2^*)^k[1-(1-p_2)^k] + p_2(1-p_2)^k[1-(1-p_2^*)^k] \\ +p_2^* p_2 (1-p_2^*)^k (1-p_2)^k \end{array}}{1-[1-(1-p_2^*)^k][1-(1-p_2)^k]}.$$

To establish that $X_{2:2} \ge_{\text{hr}} Y_{2:2}$, it suffices to show that

$$h_{X_{2:2}}(k) \le h_{Y_{2:2}}(k), \qquad k \in \mathbb{N}_0. \tag{4.1}$$

The inequality in (4.1) is trivially true for the case $k = 0$ upon noting that

$$h_{X_{2:2}}(0) = p_1^* p_1 = p_2^* p_2 = h_{Y_{2:2}}(0),$$

and so in what follows we shall assume $k \ge 1$.

For simplicity, let us denote $x_1^* = \log p_1^*, x_1 = \log p_1, x_2^* = \log p_2^*$ and $x_2 = \log p_2$. We then observe that

$$(x_1^*, x_1) \overset{\text{m}}{\succeq} (x_2^*, x_2).$$

It is then easy to see that it suffices to show that the symmetric differentiable function $H : (-\infty, 0)^2 \to (0, \infty)$ given by

$$\begin{aligned} H(x_1^*, x_1) &= \frac{\begin{array}{c} e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k] + e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] \\ +e^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k \end{array}}{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]} \\ &= H_1(x_1^*, x_1) + H_2(x_1^*, x_1) \end{aligned}$$

is Schur-concave, where

$$H_1(x_1^*, x_1) = \frac{e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k] + e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k]}{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]}$$

and

$$H_2(x_1^*, x_1) = \frac{e^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k}{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]}.$$

We shall now show that $H_1(x_1^*, x_1)$ and $H_2(x_1^*, x_1)$ are both Schur-concave. For $H_1(x_1^*, x_1)$, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}H_1(x_1^*, x_1)\{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}^2 \\
&\quad = \{ke^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^{k-1} - ke^{2x_1}(1-e^{x_1^*})^k(1-e^{x_1})^{k-1}[1-(1-e^{x_1^*})^k]\} \\
&\qquad + e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k]\{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\} \\
&\quad = ke^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k\left\{1-\frac{e^{x_1}}{1-e^{x_1}}\sum_{i=1}^{k-1}(1-e^{x_1^*})^i\right\} \\
&\qquad + e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k]\{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}.
\end{aligned}$$

We then have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}H_1(x_1^*, x_1) - \frac{\partial}{\partial x_1^*}H_1(x_1^*, x_1) \\
&\quad \overset{\text{sgn}}{=} ke^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k\left\{\frac{e^{x_1^*}}{1-e^{x_1^*}}\sum_{i=1}^{k-1}(1-e^{x_1})^i - \frac{e^{x_1}}{1-e^{x_1}}\sum_{i=1}^{k-1}(1-e^{x_1^*})^i\right\} \\
&\qquad + \{e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k]\} \\
&\qquad \times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\} \\
&\quad = \alpha + \beta, \quad \text{say},
\end{aligned}$$

where

$$\alpha = ke^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k\left\{\frac{e^{x_1^*}}{1-e^{x_1^*}}\sum_{i=1}^{k-1}(1-e^{x_1})^i - \frac{e^{x_1}}{1-e^{x_1}}\sum_{i=1}^{k-1}(1-e^{x_1^*})^i\right\}$$

and

$$\begin{aligned}
\beta = &\{e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k]\} \\
&\times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\alpha &\overset{\text{sgn}}{=} \frac{e^{x_1^*}}{1-e^{x_1^*}}\sum_{i=1}^{k-1}(1-e^{x_1})^i - \frac{e^{x_1}}{1-e^{x_1}}\sum_{i=1}^{k-1}(1-e^{x_1^*})^i \\
&\overset{\text{sgn}}{=} x_1^* - x_1
\end{aligned}$$

and

$$\begin{aligned}\beta &\overset{\text{sgn}}{=} e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k] \\ &\overset{\text{sgn}}{=} \frac{1}{\sum_{i=1}^k [1/(1-e^{x_1})^i]} - \frac{1}{\sum_{i=1}^k [1/(1-e^{x_1^*})^i]} \\ &\overset{\text{sgn}}{=} x_1^* - x_1.\end{aligned}$$

Thus, we have

$$(x_1 - x_1^*)\left(\frac{\partial}{\partial x_1}H_1(x_1^*, x_1) - \frac{\partial}{\partial x_1^*}H_1(x_1^*, x_1)\right) \le 0,$$

which means, from Lemma 4.1, that $H_1(x_1^*, x_1)$ is Schur-concave.

For $H_2(x_1^*, x_1)$, we similarly have

$$\begin{aligned}&\frac{\partial}{\partial x_1}H_2(x_1^*, x_1)\{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}^2 \\ &= \{e^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k - ke^{2x_1}e^{x_1^*}(1-e^{x_1^*})^k(1-e^{x_1})^{k-1}[1-(1-e^{x_1^*})^k]\} \\ &\quad \times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\} \\ &\quad + ke^{x_1^*}e^{2x_1}(1-e^{x_1^*})^k(1-e^{x_1})^{2k-1}[1-(1-e^{x_1^*})^k].\end{aligned}$$

We then have

$$\begin{aligned}&\frac{\partial}{\partial x_1}H_2(x_1^*, x_1) - \frac{\partial}{\partial x_1^*}H_2(x_1^*, x_1) \\ &\overset{\text{sgn}}{=} k(1-e^{x_1^*})^{k-1}(1-e^{x_1})^{k-1}\{e^{x_1^*}(1-e^{x_1}) - e^{x_1}(1-e^{x_1^*})\} \\ &\quad \times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\} \\ &\quad + ke^{x_1^*}e^{x_1}(1-e^{x_1^*})^k(1-e^{x_1})^k \\ &\quad \times \{e^{x_1}(1-e^{x_1})^{k-1}[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^{k-1}[1-(1-e^{x_1})^k]\} \\ &= \gamma + \delta, \quad \text{say},\end{aligned}$$

where

$$\begin{aligned}\gamma &= k(1-e^{x_1^*})^{k-1}(1-e^{x_1})^{k-1}\{e^{x_1^*}(1-e^{x_1}) - e^{x_1}(1-e^{x_1^*})\} \\ &\quad \times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}\end{aligned}$$

and

$$\begin{aligned}\delta &= \{e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k]\} \\ &\quad \times \{1-[1-(1-e^{x_1^*})^k][1-(1-e^{x_1})^k]\}.\end{aligned}$$

https://doi.org/10.1017/S026996481200006X Published online by Cambridge University Press

Note that

$$\gamma \overset{\text{sgn}}{=} e^{x_1^*}(1-e^{x_1}) - e^{x_1}(1-e^{x_1^*})$$
$$\overset{\text{sgn}}{=} x_1^* - x_1$$

and

$$\delta \overset{\text{sgn}}{=} e^{x_1}(1-e^{x_1})^k[1-(1-e^{x_1^*})^k] - e^{x_1^*}(1-e^{x_1^*})^k[1-(1-e^{x_1})^k]$$
$$\overset{\text{sgn}}{=} \frac{1}{\sum_{i=0}^{k-1}[1/(1-e^{x_1})^i]} - \frac{1}{\sum_{i=0}^{k-1}[1/(1-e^{x_1^*})^i]}$$
$$\overset{\text{sgn}}{=} x_1^* - x_1;$$

that is,

$$(x_1 - x_1^*)\left(\frac{\partial}{\partial x_1}H_2(x_1^*, x_1) - \frac{\partial}{\partial x_1^*}H_2(x_1^*, x_1)\right) \leq 0.$$

Upon using Lemma 4.1 once again, we have $H_2(x_1^*, x_1)$ also to be Schur-concave, and that completes the proof of the theorem. ■

THEOREM 4.3: *Let X_1, X_2 be independent geometric variables with respective parameters p_1^*, p_1, and Y_1, Y_2 be another set of independent geometric variables with respective parameters p_2^*, p_2. Suppose $p_1^* \leq p_2^* \leq p_2 \leq p_1$. Then, the following three statements are equivalent:*

(a) $(p_1^*, p_1) \overset{\text{p}}{\succeq} (p_2^*, p_2)$;
(b) $X_{2:2} \geq_{\text{hr}} Y_{2:2}$;
(c) $X_{2:2} \geq_{\text{st}} Y_{2:2}$.

PROOF: Since the hazard rate order implies the usual stochastic order, it is sufficient to prove that (a) ⇒ (b) and (c) ⇒ (a).
(a) ⇒ *(b)* Suppose $(p_1^*, p_1) \overset{\text{p}}{\succeq} (p_2^*, p_2)$. Clearly, we have $p_1^* \leq p_2^* \leq p_2 \leq p_1$ and $p_1^* p_1 \leq p_2^* p_2$. The result for the case when $p_1^* p_1 = p_2^* p_2$ follows immediately from Theorem 4.2. Next, let us assume that $p_1^* p_1 < p_2^* p_2$, and let $p' = p_2^* p_2 / p_1$. We then have $p' p_1 = p_2^* p_2$ and $p_1^* < p' \leq p_2^*$. Let $Z_{2:2}$ be the maximum among two independent geometric variables with respective parameters p' and p_1. Then, from Theorem 4.2, it follows that $Z_{2:2} \geq_{\text{hr}} Y_{2:2}$. Also, we have $X_{2:2} \geq_{\text{hr}} Z_{2:2}$ from Corollary 3.5, and we thus obtain the desired result that $X_{2:2} \geq_{\text{hr}} Y_{2:2}$.
(c) ⇒ *(a)* Suppose $X_{2:2} \geq_{\text{st}} Y_{2:2}$, that is, $F_{X_{2:2}}(k) \leq F_{Y_{2:2}}(k)$ for all $k \in \mathbb{N}_0$. It then follows that

$$F_{X_{2:2}}(1) = p_1^* p_1 \leq p_2^* p_2 = F_{Y_{2:2}}(1),$$

from which the desired result follows. ■

Remark 4.4: The result in Theorem 4.3 is an analog of (1.7) for the hazard rate order [usual stochastic order]. It is, therefore, natural to ask the question whether an analog of (1.6) for the likelihood ratio order [reversed hazard rate order] also holds. Unfortunately, the answer is negative. To see this, let us take (X_1, X_2) to be a vector of independent geometric variables with parameter vector $(p_1^*, p_1) = \left(\frac{1}{6}, \frac{1}{2}\right)$, and (Y_1, Y_2) to be another vector of independent geometric variables with parameter vector $(p_2^*, p_2) = \left(\frac{1}{5}, \frac{5}{11}\right)$. Denote by $F_{X_{2:2}}$ and $F_{Y_{2:2}}$ the corresponding distribution functions of $X_{2:2}$ and $Y_{2:2}$, respectively. Clearly, we have $(p_1^*, p_1) \overset{\text{p}}{\succeq} (p_2^*, p_2)$. However,

$$\frac{F_{X_{2:2}}(2)}{F_{Y_{2:2}}(2)} \approx 0.906 \geq 0.902 \approx \frac{F_{X_{2:2}}(3)}{F_{Y_{2:2}}(3)},$$

which implies that $X_{2:2} \not\geq_{\text{rh}} Y_{2:2}$. This shows that some differences do exist between ordering properties of maxima from geometric variables and from exponential variables even though there are similarities in many cases.

Next, we present a numerical example to illustrate the results established in Theorem 4.3.

Example 4.5: Let (X_1, X_2) be a vector of independent geometric variables with parameter vector $(p_1^*, p_1) = \left(\frac{1}{6}, \frac{1}{2}\right)$, and let $h\left(k; \frac{1}{6}, \frac{1}{2}\right)$ be the corresponding hazard rate function of $X_{2:2}$. Let (Y_1, Y_2) be another vector of independent geometric variables with parameter vector $(p_2^*, p_2) = \left(\frac{1}{4}, \frac{1}{5}\right)$, and let $h\left(k; \frac{1}{4}, \frac{2}{5}\right)$ be the corresponding hazard rate function of $Y_{2:2}$. It can be readily seen that $(p_1^*, p_1) \overset{\text{p}}{\succeq} (p_2^*, p_2)$. Figure 2 presents plots of the hazard rate functions of these two maxima, which are in accordance with the result in Theorem 4.3.

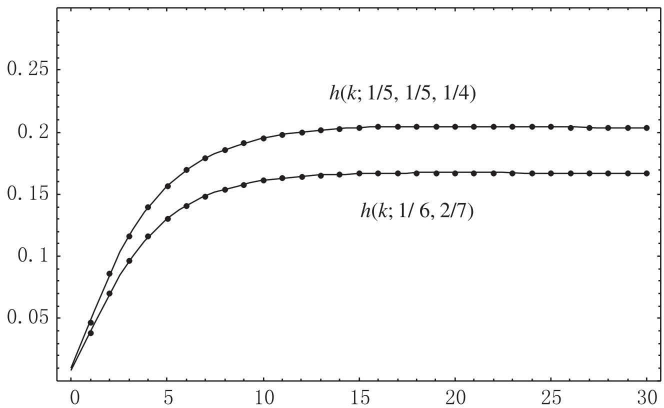

FIGURE 2. Plot of the hazard rate functions of maxima from two geometric samples parameter vectors as $\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{7}\right)$ and $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{4}\right)$.

https://doi.org/10.1017/S026996481200006X Published online by Cambridge University Press

The following corollary is a direct consequence of Theorem 4.3.

COROLLARY 4.6: *Let X_1, X_2 be independent geometric variables with respective parameters p_1, p_2, and Y_1, Y_2 be another pair of i.i.d. geometric variables with common parameter p. Let $p \leq \max(p_1, p_2)$. Then, the following three statements are equivalent:*

(a) $p \geq \sqrt{p_1 p_2}$;
(b) $X_{2:2} \geq_{\text{hr}} Y_{2:2}$;
(c) $X_{2:2} \geq_{\text{st}} Y_{2:2}$.

Remark 4.7: It is worth mentioning that the result in Corollary 4.6 is actually valid even without the condition $p \leq \max(p_1, p_2)$. Let Z_p $[Z_q]$ be the maximum from a random sample of size 2 from a geometric distribution with common hazard rate p $[q]$. Let $p < q$. We then have $Z_p \geq_{\text{lr}} Z_q$. Based on this fact, we can conclude that the result of Corollary 4.6 is also valid for the case when $p > \max(p_1, p_2)$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11001112), Research Fund for the Doctoral Program of Higher Education (20090211120019).

References

1. Andrews, D.F., Bickel, P.J., Hampel, F., Huber, P.J., Rogers, W., & Tukey, J.W. (1972). *Robust estimates of location: Survey and advances*. Princeton, NJ: Princeton University Press.
2. Balakrishnan, N. (2007). Permanents, order statistics, outliers, and robustness. *Revista Matemática Complutense* 20: 7–107.
3. Balakrishnan, N. & Rao, C.R. (eds.) (1998). *Handbook of statistics*, vol. 16: *Order statistics: Theory and methods*. Amsterdam: Elsevier.
4. Balakrishnan, N. & Rao, C.R. (eds.) (1998). *Handbook of statistics*, vol. 17: *Order statistics: Applications*. Amsterdam: Elsevier.
5. Barnett, V. & Lewis, T. (1993). *Outliers in statistical data*, 3rd ed. Chichester, England: John Wiley & Sons.
6. Boland, P.J., EL-Neweihi, E., & Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics. *Journal of Applied Probability* 31: 180–192.
7. Bon, J.L. & Păltănea, E. (1999). Ordering properties of convolutions of exponential random variables. *Lifetime Data Analysis* 5: 185–192.
8. Bon, J.L. & Păltănea, E. (2006). Comparisons of order statistics in a random sequence to the same statistics with i.i.d. variables. *ESAIM: Probability and Statistics* 10: 1–10.
9. David, H.A. & Nagaraja, H.N. (2003). *Order statistics*, 3rd ed. Hoboken, NJ: John Wiley & Sons.
10. Dykstra, R., Kochar, S.C., & Rojo, J. (1997). Stochastic comparisons of parallel systems of heterogeneous exponential components. *Journal of Statistical Planning and Inference* 65: 203–211.
11. Hu, T., Wang, F., & Zhu, Z. (2006). Stochastic comparisons and dependence of spacings from two samples of exponential random variables. *Communications in Statistics — Theory and Methods* 35: 979–988.
12. Hu, T., Lu, Q., & Wen, S. (2008). Some new results on likelihood ratio orderings for spacings of heterogeneous exponential random variables. *Communications in Statistics — Theory and Methods* 37: 2506–2515.

13. Joo, S. & Mi, J. (2010). Some properties of hazard rate functions of systems with two components. *Journal of Statistical Planning and Inference* 140: 444–453.
14. Khaledi, B. & Kochar, S.C. (2000). Some new results on stochastic comparisons of parallel systems. *Journal of Applied Probability* 37: 283–291.
15. Khaledi, B.E. & Kochar, S. (2001). Stochastic properties of spacings in a single-outlier exponential model. *Probability in the Engineering and Informational Sciences* 15: 401–408.
16. Kochar, S.C. & Rojo, J. (1996). Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions. *Journal of Multivariate Analysis* 59: 272–281.
17. Kochar, S.C. & Xu, M. (2007). Stochastic comparisons of parallel systems when components have proportional hazard rates. *Probability in the Engineering and Informational Science* 21: 597–609.
18. Kochar, S.C. & Xu, M. (2011). On the skewness of order statistics in the multiple-outlier models. *Journal of Applied Probability* 48: 271–284.
19. Mao, T. & Hu, T. (2010). Equivalent characterizations on orderings of order statistics and sample ranges. *Probability in the Engineering and Informational Sciences* 24: 245–262.
20. Marshall, A.W. & Olkin, I. (1979). *Inequalities: Theory of majorization and its applications*. New York: Academic Press.
21. Păltănea, E. (2008). On the comparison in hazard rate ordering of fail-safe systems. *Journal of Statistical Planning and Inference* 138: 1993–1997.
22. Pledger, P. & Proschan, F. (1971). Comparisons of order statistics and of spacings from heterogeneous distributions. In: (J.S. Rustagi) (ed.), *Optimizing methods in statistics*, New York: Academic Press, pp. 89–113.
23. Proschan, F. & Sethuraman, J. (1976). Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability. *Journal of Multivariate Analysis* 6: 608–616.
24. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer-Verlag.
25. Tiku, M.L., Tan, W.Y., & Balakrishnan, N. (1986). *Robust inference*. New York: Marcel Dekker.
26. Wen, S., Lu, Q., & Hu, T. (2007). Likelihood ratio order of spacings from two samples of exponential distributions. *Journal of Multivariate Analysis* 98: 743–756.
27. Xu, M., Li, X., Zhao, P., & Li, Z. (2007). Likelihood ratio order of m-spacings in multiple-outlier models. *Communications in Statistics — Theory and Methods* 36: 1507–1525.
28. Zhao, P. & Balakrishnan, N. (2009). Characterization of MRL order of fail-safe systems with heterogeneous exponentially distributed components. *Journal of Statistical and Planning Inference* 139: 3027–3037.
29. Zhao, P. & Balakrishnan, N. (2011). Some characterization results for parallel systems with two heterogeneous exponential components. *Statistics* 45(6): 593–604.
30. Zhao, P. & Balakrishnan, N. (2011). Stochastic comparisons of largest order statistics from multiple-outlier exponential models. *Probability in the Engineering and Informational Sciences*. (in press)
31. Zhao, P., Li, X., & Balakrishnan, N. (2008). Conditional ordering of k-out-of-n systems with independent but non-identical components. *Journal of Applied Probability* 45: 1113–1125.
32. Zhao, P., Li, X., & Balakrishnan, N. (2009). Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables. *Journal of Multivariate Analysis* 100: 952–962.