SPREADS WHICH ARE NOT DUAL SPREADS

A. Bruen and J. C. Fisher

(received April 25, 1969)

In this note we show the existence of a spread which is not a dual spread, thus answering a question in [1]. We also obtain some related results on spreads and partial spreads.

Let $\Sigma = PG(2t-1, F)$ be a projective space of odd dimension $(2t-1, t \geq 2)$ over the field F. In accordance with [1] we make the following definitions. A *partial spread* S of Σ is a collection of (t-1)-dimensional projective subspaces of Σ which are pairwise disjoint (skew). S is *maximal* if it is not properly contained in any other partial spread; in particular, if every point of Σ is contained in some member of S then S is a *spread*. If each (2t-2)-dimensional projective subspace of Σ contains exactly one member of S then S is called a *dual spread*. $|S|$ will denote the number of subspaces in S.

THEOREM 1. *If* F *is finite then* S *is a spread if and only if* S *is a dual spread.*

Proof. Suppose S is a spread which is not a dual spread of Σ. Let δ be any correlation of Σ (for the existence of such a δ see [3, page 41]). Then S^{δ}, the image of S under δ, is a partial spread which is not a spread. But $|S^{\delta}| = |S|$ and F is finite so we obtain a contradiction. Similarly every dual spread is a spread.

For simplicity we now specialize to the case t=2 and we assume that F is commutative to facilitate the notion of regulus. We say a spread S is *regular* provided that, for every line ℓ of Σ which is not in S, the lines of S meeting ℓ form a regulus R of Σ. Not all spreads are regular: we can obtain a new non-regular spread S' from S by the process of replacing some regulus R by its opposite regulus R'. If S' can be obtained from a regular spread S by finitely many iterations of such a process, S is called *subregular*.

Canad. Math. Bull. vol. 12, no. 6, 1969

https://doi.org/10.4153/CMB-1969-103-7 Published online by Cambridge University Press

THEOREM 2. Every regular spread S of Σ is a dual spread.

Proof. Let π be any plane of Σ; π contains at most one line of S. To show that there must be one let ℓ be any line of π which is not in S. The lines of S meeting ℓ form a regulus R. Let p and q be any two lines of the opposite regulus R' different from ℓ. p and q meet π in distinct points P and Q not on ℓ. The line PQ of π meets ℓ and hence meets three lines of R'. Thus PQ is a line of R, that is, of S.

A straightforward extension of this argument yields the following result.

THEOREM 3. Let S be a spread which is a dual spread. Suppose S contains a regulus R. Then the spread S' obtained from S by replacing the regulus R by its opposite regulus R' is also a dual spread.

COROLLARY. Every subregular spread is a dual spread.

THEOREM 4. There exists a spread S of Σ such that

(1) S is not a dual spread;

(2) no four lines of S are contained in a regulus.

Proof. Let F be infinite and countable. Choose any plane π and list the points in π $(P_1, P_2, P_3, \ldots)$ and the points not in π $(Q_1, Q_2, Q_3, \ldots)$. Through P_1 construct the line $\ell_1 = P_1 Q_1$. Suppose $\ell_1, \ldots, \ell_n$ have been constructed such that (i) no ℓ_i is in π, (ii) no two ℓ_i intersect, and (iii) no four ℓ_i are in a regulus. We now show that ℓ_{n+1} can be constructed in such a way that (i) - (iii) are satisfied also by $\{\ell_1, \ldots, \ell_{n+1}\}$.

If n is odd, let $X = P_j$ be the first point in π which is on none of the lines $\ell_1, \ldots, \ell_n$ and $Y = Q_k$ the first point not in π such that (a) Y is on none of the n planes $X\ell_i$ $(i = 1, \ldots, n)$ and (b) XY does not belong to any one of the $\binom{n}{3}$ reguli determined by $\ell_1, \ldots, \ell_n$. Then put $\ell_{n+1} = XY = P_j Q_k$.

If n is even, let $X = Q_s$ be the first point not in π which is on none of the ℓ_i, $i = 1, \ldots, n$ and $Y = P_t$ the first point in π such that (a) and (b) are satisfied. Then put $\ell_{n+1} = XY = Q_s P_t$.

Clearly $\ell_1, \ldots, \ell_{n+1}$ satisfy conditions (i) - (iii). Furthermore, our construction guarantees that each point of Σ is on a line of S. Thus the theorem is proved.

There is an interesting consequence of Theorem 4.

COROLLARY. Maximal partial spreads W, which are not spreads, exist in Σ.

Proof. Consider the image W of S under any correlation of Σ.

Remark. The above corollary is also true if F is finite (for an example in PG(3,4) see [4]). One of the authors [2] has constructed such maximal partial spreads W, with

$q^2 - q + 1 \leq |W| \leq q^2 - q + 2$ in PG(3,q), for any q.

REFERENCES

1. R.H. Bruck and R.C. Bose, The construction of translation planes from projective spaces. J. Algebra 1 (1964) 85-102.

2. A. Bruen, Blocking sets in finite projective planes. (unpublished).

3. P. Dembowski, Finite eometries (Springer-Verlag, 1968).

4. D.M. Mesner, Sets of disjoint lines in PG(3,q). Canad. J. Math. 19 (1967) 273-280.

University of Toronto