Ergod. Th. & Dynam. Sys. (2018), **38**, 3170–3187
doi:10.1017/etds.2017.2

Directed harmonic currents near hyperbolic singularities

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(*Received* 11 *August* 2015 *and accepted in revised form* 21 *December* 2016)

Abstract. Let $\mathscr{F}$ be a holomorphic foliation by curves defined in a neighborhood of 0 in $\mathbb{C}^2$ having 0 as a hyperbolic singularity. Let T be a harmonic current directed by $\mathscr{F}$ which does not give mass to any of the two separatrices. We show that the Lelong number of T at 0 vanishes. Then we apply this local result to investigate the global mass distribution for directed harmonic currents on singular holomorphic foliations living on compact complex surfaces. Finally, we apply this global result to study the recurrence phenomenon of a generic leaf.

1. *Introduction*

While investigating the unique ergodicity of harmonic currents on singular holomorphic foliations in $\mathbb{P}^2$, Fornæss and Sibony in [**7**, Corollary 2] established, among other things, the following remarkable result.

THEOREM 1.1. (Fornæss and Sibony [**7**]) *Let $(M, \mathscr{F}, E)$ be a hyperbolic foliation with the set of singularities E in a compact complex surface M. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Then for every harmonic current T directed by $\mathscr{F}$, its transverse measure is diffuse, that is, T gives no mass to each single leaf.*

In fact, the original version of the Fornæss–Sibony theorem was only formulated for the case $M = \mathbb{P}^2$. However, their argument still goes through (at least) in the above general context. On the other hand, a convenient way to quantify the density of harmonic currents is to use the notion of the Lelong number introduced by Skoda [**12**]. Indeed, Theorem 1.1 is equivalent to the assertion that the Lelong number of T vanishes everywhere *outside* E. Complementarily to this theorem, the main purpose of the present work is to investigate the mass-clustering phenomenon of T *near* the set of singularities E. Here is our main result, which is of local nature.

THEOREM 1.2. (Main theorem) *Let* $(\mathbb{D}^2, \mathscr{F}, \{0\})$ *be a holomorphic foliation on the unit bidisc* $\mathbb{D}^2$ *associated to the linear vector field* $\Phi(z, w) = \mu z\partial/(\partial z) + \lambda w\partial/(\partial w)$*, where* λ, μ *are non-zero complex numbers such that* $\lambda/\mu \notin \mathbb{R}$*. Then for every harmonic current* T *directed by* $\mathscr{F}$ *which does not give mass to any of the two separatrices* $(z = 0)$ *and* $(w = 0)$, *the Lelong number of* T *at* 0 *vanishes.*

Note that the hypothesis on the linear vector field means that 0 is an isolated hyperbolic singularity of the foliation (see, for example, the recent survey [**6**]). Our proof of Theorem 1.2 is inspired by the approach of Fornæss and Sibony in [**6**, **7**] which is based on integral formulas. Indeed, the nature of the holonomy maps associated to a hyperbolic singularity allows us to use the Poisson representation formula for harmonic functions on leaves associated to a given harmonic current near the singularity. Therefore, we are led to analyze the behavior of some singular integrals at infinity, that is, when the leaves get close to the separatrices. Using delicate Poisson kernel estimates, we are able to handle these singular integrals.

Combining Theorems 1.1 and 1.2, we obtain the following result which gives a rather complete picture of the mass distribution of directed harmonic currents in dimension 2.

THEOREM 1.3. *Let* $(M, \mathscr{F}, E)$ *be a hyperbolic foliation with the set of singularities* E *in a compact complex surface* M*. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Then for every harmonic current* T *directed by* $\mathscr{F}$*, the Lelong number of* T *vanishes everywhere in* M*.*

The above theorem and a result by Glutsyuk [**8**] and Lins Neto [**9**] gives us the following corollary. It can be applied to every generic foliation in $\mathbb{P}^2$ with given degree $d > 1$ (see [**10**]).

COROLLARY 1.4. *Let* $(\mathbb{P}^2, \mathscr{F}, E)$ *be a singular foliation by Riemann surfaces on the complex projective plane* $\mathbb{P}^2$*. Assume that all the singularities are hyperbolic and that* $\mathscr{F}$ *has no invariant algebraic curve. Then for every harmonic current* T *directed by* $\mathscr{F}$*, the Lelong number of* T *vanishes everywhere in* $\mathbb{P}^2$*.*

It is worth noting that under the hypothesis of Corollary 1.4 there is a unique harmonic current T of mass 1 directed by $\mathscr{F}$. Indeed, this is a consequence of the Fornæss–Sibony theorem on the unique ergodicity of harmonic currents (see [**7**, Theorem 4]).

As an application of our results we will study the problem of leaf recurrence. This problem asks how often the leaf L_a of a point a, which is generic with respect to a directed harmonic current T, visits the neighborhood of a given point x. Our approach to this question is to apply a geometric Birkhoff ergodic theorem which has recently been obtained in our joint work with Dinh and Sibony [**2**]. The theorem permits us to define, using the leafwise Poincaré metric, an indicator which measures the frequency of a generic leaf visiting a small ball near a given point in terms of the radius of the ball. This, combined Theorem 1.3, gives us an upper estimate on the frequency outside and near singularities (see Theorem 5.2 below).

This paper is organized as follows. In §2 we set up the background for the paper. Next, we develop our main estimates in §3, which are the core of the work. The proof of Theorem

1.2 and Theorem 1.3 is provided in §4. The recurrence phenomenon of a generic leaf is studied in §5. The paper concludes with some remarks and open questions.

Note added in proof. If, in Theorem 1.3, we assume in addition that M is a projective surface, then our recent work [**11**, integrability condition (1.1)] provides the following estimate:

$$\int_X \frac{T \wedge \omega(x)}{(\mathrm{dist}(x, E))^2 \log^* \mathrm{dist}(x, E)} < \infty.$$

Here ω is a Hermitian metric on X and dist is the distance on M induced by ω, and $\log^*(\cdot) := 1 + |\log(\cdot)|$ is a log-type function. This inequality is much more difficult to obtain than the vanishing of the Lelong number of T at singularities established in Theorem 1.3.

2. *Background*

Let M be a complex surface. A *holomorphic foliation (by Riemann surfaces)* $(M, \mathscr{F})$ on M is the data of a *foliation atlas* with charts

$$\Phi_p : \mathbb{U}_p \to \mathbb{B}_p \times \mathbb{T}_p.$$

Here, $\mathbb{T}_p$ and $\mathbb{B}_p$ are domains in $\mathbb{C}$, $\mathbb{U}_p$ is a domain in M, and Φ_p is biholomorphic, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t).$$

The open set $\mathbb{U}_p$ is called a *flow box* and the Riemann surface $\Phi_p^{-1}\{t = c\}$ in $\mathbb{U}_p$ with $c \in \mathbb{T}_p$ is a *plaque*. The property of the above coordinate changes ensures that the plaques in different flow boxes are compatible in the intersection of the boxes. Two plaques are *adjacent* if they have non-empty intersection.

A *leaf* L is a minimal connected subset of M such that if L intersects a plaque, it contains that plaque. So a leaf L is a Riemann surface immersed in M which is a union of plaques. A leaf through a point x of this foliation is often denoted by L_x. A *transversal* is a Riemann surface immersed in X which is transverse to the leaves of $\mathscr{F}$.

A *holomorphic foliation with singularities* is the data $(M, \mathscr{F}, E)$, where M is a complex surface, E a closed subset of M and $(M \setminus E, \mathscr{F})$ is a holomorphic foliation. Each point in E is said to be a *singular point,* and E is said to be *the set of singularities* of the foliation. We always assume that $\overline{M \setminus E} = M$; see, for example, [**2**, **5**, **6**] for more details. A leaf L of the foliation is said to be *hyperbolic* if it is a hyperbolic Riemann surface, that is, it is uniformized by the unit disc $\mathbb{D}$. The foliation is said to be *hyperbolic* if its leaves are all hyperbolic.

Consider a holomorphic foliation $(M, \mathscr{F}, E)$ with a discrete set of singularities E on a complex surface M. We say that a singular point $x \in E$ is *linearizable* if there is a (local) holomorphic coordinate system of M on an open neighborhood $\mathbb{U}_x$ of x on which $(\mathbb{U}_x, x)$ is identified with $(\mathbb{D}^2, 0)$ and the leaves of $(M, \mathscr{F}, E)$ are, under this identification, integral curves of a linear vector field $\Phi(z, w) = \mu z\partial/(\partial z) + \lambda w\partial/(\partial w)$ with non-zero complex numbers λ, μ. Such a neighborhood $\mathbb{U}_x$ is called a *singular flow box* of x. Moreover, we say that a linearizable singular point $x \in E$ is *hyperbolic* if $\lambda/\mu \notin \mathbb{R}$.

Let $\mathscr{C}_{\mathscr{F}}$ (respectively, $\mathscr{C}^{1,1}_{\mathscr{F}}$) denote the space of functions (respectively, forms of bidegree (1, 1)) defined on leaves of the foliations and compactly supported on $M \setminus E$

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which are leafwise smooth and transversally continuous. A form $\alpha \in \mathscr{C}^{1,1}_{\mathscr{F}}$ is said to be *positive* if its restriction to every plaque is a positive (1, 1)-form in the usual sense.

Definition 2.1. A *harmonic current T directed by the foliation $\mathscr{F}$* (or equivalently, a *directed harmonic current T on $\mathscr{F}$*) is a linear continuous form on $\mathscr{C}^{1,1}_{\mathscr{F}}$ which satisfies the following two conditions:

(i) $i\partial\overline{\partial}T = 0$ in the weak sense, that is, $T(i\partial\overline{\partial}f) = 0$ for all $f \in \mathscr{C}_{\mathscr{F}}$, where in the expression $i\partial\overline{\partial}f$, we only consider $\partial\overline{\partial}$ along the leaves;

(ii) T is positive, that is, $T(\alpha) \geq 0$ for all positive forms $\alpha \in \mathscr{C}^{1,1}_{\mathscr{F}}$.

Let $\mathbb{U} \simeq \mathbb{B} \times \mathbb{T}$ be a flow box. By identifying $\mathbb{T}$ with a fiber of the natural projection of $\mathbb{U}$ onto $\mathbb{B}$, we may regard $\mathbb{T}$ as a transversal. Then a harmonic current T in $\mathbb{U}$ can be decomposed as

$$T = \int_{\alpha\in\mathbb{T}} h_\alpha [P_\alpha]\, d\nu(\alpha), \tag{1}$$

where ν is a positive measure on $\mathbb{T}$, and, for ν-almost every $\alpha \in \mathbb{T}$, P_α is the plaque in $\mathbb{U}$ passing through α and h_α denotes the harmonic function associated to the current T on P_α.

Recall from Skoda [**12**] that the Lelong number of T at a point $x \in M$ is

$$\mathcal{L}(T, x) := \lim_{r\to 0+} \frac{1}{\pi r^2} \int_{B(x,r)} T \wedge i\partial\overline{\partial}\|y\|^2, \tag{2}$$

where we identify, through a biholomorphic change of coordinates, a neighborhood of x in M with an open neighborhood of 0 in $\mathbb{C}^2$, and $B(x, r)$ is thus identified with the Euclidean ball with center 0 and radius r. In fact, the Lelong number $\mathcal{L}(T, x)$ is independent of the choice of local coordinates. The reader can find a more general notion (Dinh–Sibony cohomology classes) recently introduced and studied in [**4**].

In this work the letters c, c', c'', c_0, c_1, c_2, etc. denote positive constants, not necessarily the same at each occurrence. The symbols $\gtrsim$ and $\lesssim$ denote inequalities up to a multiplicative constant, whereas we write $\approx$ when both inequalities are satisfied.

3. *Main estimates*

We retain the hypotheses of Theorem 1.2. Suppose without loss of generality that the foliation $\mathscr{F}$ is defined on the bidisc of radius 2, that is, $(2\mathbb{D})^2$ in place of $\mathbb{D}^2$, and that the constant μ is equal to 1. Let $\mathscr{L}$ be the foliation in $\mathbb{C}^2$ associated to the vector field $\Phi(z, w) = z\partial/(\partial z) + \lambda w\partial/(\partial w)$ with some complex number $\lambda = a + ib$, $b \neq 0$. So $\mathscr{L} = \mathscr{F}$ on $(2\mathbb{D})^2$. Note that if we flip z and w, we replace λ by $1/\lambda = \bar{\lambda}/|\lambda|^2 = a/(a^2 + b^2) - ib/(a^2 + b^2)$. Therefore, we may assume without loss of generality that $b > 0$. We now describe a general leaf of $\mathscr{L}$. There are two separatrices, $(w = 0)$, $(z = 0)$. Other than that, a leaf L of $\mathscr{L}$ is equal to $L_{(1,\alpha)} =: L_\alpha$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Following [**7**, §2], L_α can be parametrized by

$$(z, w) = \psi_\alpha(\zeta), \quad z = e^{i(\zeta+(\log|\alpha|)/b)}, \quad \zeta = u + iv, \quad w = \alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)}, \tag{3}$$

because $\psi_\alpha(-\log|\alpha|/b) = (1, \alpha)$. Setting $t := bu + av$, we have that

$$|z| = e^{-v}, \quad |w| = e^{-bu-av} = e^{-t}. \tag{4}$$

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Observe that as we follow z once counterclockwise around the origin, u increases by 2π, so the absolute value of $|w|$ decreases by the multiplicative factor of $e^{-2\pi b}$. Hence, we cover all leaves of $\mathscr{F}|_{\mathbb{D}^2}$ by restricting the values of α so that $e^{-2\pi b} = |\alpha| < 1$. We notice that with the above parametrization, the intersection with the unit bidisc $\mathbb{D}^2$ of the leaf is given by the domain $\{(u, v) \in \mathbb{R}^2 : \ v > 0, u > -av/b\}$. The main point of this special parametrization is that the above domain is independent of α. In the (u, v)-plane this domain corresponds to a sector S_λ with corner at 0 and given by $0 < \theta < \arctan(-b/a)$ where the arctan is chosen to have values in $(0, \pi)$. Let $\gamma := \pi/\arctan(-b/a)$. It is important to note that $\gamma > 1$. Then the map

$$\phi : \tau = u + iv \mapsto \tau^\gamma = (u + iv)^\gamma =: U + iV \tag{5}$$

maps this sector to the upper half plane with coordinates (U, V).

The local leaf clusters on both separatrices. To investigate the clustering on the z-axis, we use a transversal $\mathbb{T}_{z_0} := \{(z_0, w) : |w| < 1\}$ for some $|z_0| = 1$. We can normalize so that $h_\alpha(z_0, w) = 1$ where (z_0, w) is the point on the local leaf with $e^{-2\pi b} \le |w| < 1$. So $(z_0, w) = \psi_\alpha(\zeta_0) = \psi_\alpha(u_0 + iv_0)$ with $v_0 = 0$ and $0 < u_0 \le 2\pi$ determined by the equations $|z_0| = e^{-v_0} = 1$ and $e^{-2\pi b} \le |w| = e^{-bu_0 - av_0} < 1$. Let T be a harmonic current of mass 1 directed by $\mathscr{F}$. Let $\mathbb{U}$ be a flow box which admits $\mathbb{T}_{z_0}$ as a transversal. Then by (1) we can write in $\mathbb{U}$,

$$T = \int h_\alpha [P_\alpha] \, d\nu(\alpha), \tag{6}$$

where, for ν-almost every α satisfying $e^{-2\pi b} \le |\alpha| \le 1$, h_α denotes the harmonic function associated to the current T on the plaque P_α which is contained in the leaf L_α. We still denote by h_α its harmonic continuation along L_α. Define $\tilde{h}_\alpha(\zeta) := h_\alpha(e^{i(\zeta + (\log|\alpha|)/b)}, \alpha e^{i\lambda(\zeta + (\log|\alpha|)/b)})$ on S_λ. Consider the harmonic function $\tilde{H}_\alpha := \tilde{h}_\alpha \circ \phi^{-1}$ defined in the upper half plane $\{U + iV : \ V > 0\}$, where ϕ is given in (5). Recall the following result from [**7**].

LEMMA 3.1. *The harmonic function $\tilde{H}_\alpha$ is the Poisson integral of its boundary values. So in the upper half plane $\{U + iV : \ V > 0\}$,*

$$\tilde{H}_\alpha(U + iV) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{H}_\alpha(y) \frac{V}{V^2 + (y - U)^2} \, dy$$

for ν-almost every α. Moreover,

$$\int_{e^{-2\pi b} \le |\alpha| \le 1} \int_{-\infty}^{\infty} \tilde{H}_\alpha(y)(1 + |y|)^{1/\gamma - 1} dy \, d\nu(\alpha) < \infty.$$

Proof. The lemma follows from [**7**, Proposition 1 and Remark 1]. The finiteness of the integral follows from the finiteness of the total mass of the harmonic currents on the disjoint flow boxes crossed when we follow a path around the two separatrices, but away from the singularity 0. □

For $0 < r < 1$, let

$$F(r) := \int_{B_r} T \wedge i\partial\overline{\partial}\|x\|^2, \tag{7}$$

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where B_r denotes the ball centered at 0 with radius r in $\mathbb{C}^2$. Consider also the function

$$G(r) := \frac{1}{r^2} F(r). \tag{8}$$

By Skoda [**12**], $G(r)$ decreases as $r \searrow 0$, and $\lim_{r\to 0} G(r)$ is the Lelong number $\mathcal{L}(T, 0)$ of T at 0. On the other hand, for each $s > 0$, consider two domains

$$D_s^* := \{(u, v) \in S_\lambda : \min\{v, bu + av\} \geq s\} \quad \text{and} \quad D_s := \{(t, v) \in \mathbb{R}^2 \ : \ \min\{t, v\} \geq s\},$$

and the function $K_s : \mathbb{R} \to \mathbb{R}^+$ given by

$$K_s(y) := \int_{D_s^*} \frac{e^{2s-2\min\{v,bu+av\}} V}{V^2 + (y - U)^2}\, du\, dv = \frac{1}{b} \int_{D_s} \frac{e^{2s-2\min\{v,t\}} V}{V^2 + (y - U)^2}\, dt\, dv, \qquad y \in \mathbb{R}. \tag{9}$$

Here the last equality holds since $t = bu + av$ by (4).

In what follows the letters c, c', c_1, c_2, etc. denote positive constants, not necessarily the same at each occurrence. For two positive-valued functions A and B, we write $A \approx B$ if there is a constant c such that $c^{-1}A \leq B \leq cA$.

LEMMA 3.2. *There is a constant $c > 0$ such that, for every $0 < r < 1$,*

$$G(r) \leq c \int_{e^{-2\pi b} \leq |\alpha| \leq 1} \bigg(\int_{-\infty}^{\infty} K_{-\log r}(y) \tilde{H}_\alpha(y)\, dy \bigg)\, d\nu(\alpha).$$

Proof. Using (6), (7) and the parametrization (3), and the assumption that T does not give mass to any of the two separatrices $(z = 0)$ and $(w = 0)$, we have, for $0 < r < 1$, that

$$F(r) = \int_{e^{-2\pi b} \leq |\alpha| \leq 1} \int_{\zeta \in S_\lambda :\ \|\psi_\alpha(\zeta)\| \leq r} h_\alpha(\psi_\alpha(\zeta)) \|\psi'_\alpha(\zeta)\|^2 i\, d\zeta \wedge d\bar{\zeta} d\nu(\alpha).$$

On the other hand, we infer from (4) that $\|(z, w)\| = \|\psi_\alpha(\zeta)\| \leq r$ implies $\min\{v, bu + av\} \geq -\log r$. Moreover, using (3) and (4) again, we get that

$$\|\psi'_\alpha(\zeta)\| = \sqrt{|z|^2 + |\lambda w|^2} \leq (1 + |\lambda|) e^{-\min\{v,bu+av\}}.$$

Consequently,

$$F(r) \leq c \int_{e^{-2\pi b} \leq |\alpha| \leq 1} \int_{(u,v) \in D^*_{-\log r}} h_\alpha(\psi_\alpha(u + iv)) e^{-2\min\{v,bu+av\}}\, du\, dv\, d\nu(\alpha).$$

Writing $U + iV = (u + iv)^\gamma$ as in (5), an application of Lemma 3.1 yields that

$$h_\alpha(\psi_\alpha(u + iv)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{H}_\alpha(y) \frac{V}{V^2 + (y - U)^2}\, dy$$

for ν-almost every α. Substituting this into the last estimate for $F(r)$, taking (8) into account and writing $r^{-2} = e^{-2\log r}$, the lemma follows. □

The next lemma studies the behavior of the Poisson kernel $V/(V^2 + (y - U)^2)$ in terms of u and v.

LEMMA 3.3. *There are constants $c_1, c_2, c_3 > 1$ large enough, with $c_3 > c_2$, such that the following properties hold for all $(u, v) \in \mathbb{R}^2$ with $\min\{v, bu + av\} \geq 1$.*

(1)
$$\frac{1}{c_1} \leq \frac{(\max\{v, bu + av\})^\gamma}{\sqrt{V^2 + U^2}} \leq c_1 \textit{ and}$$
$$\frac{1}{c_1} \leq \frac{(\max\{v, bu + av\})^{\gamma-1} \min\{v, bu + av\}}{V} \leq c_1.$$

(2) *If* $\max\{v, bu + av\} \geq c_2(1 + |y|)^{1/\gamma}$, *then*
$$\frac{1}{c_1} \frac{\min\{v, bu + av\}}{(\max\{v, bu + av\})^{\gamma+1}} \leq \frac{V}{V^2 + (y - U)^2} \leq c_1 \frac{\min\{v, bu + av\}}{(\max\{v, bu + av\})^{\gamma+1}}.$$

(3) *If* $\max\{v, bu + av\} \leq c_2^{-1}(1 + |y|)^{1/\gamma}$, *then*
$$\frac{1}{c_1} \frac{V}{(1 + |y|)^2} \leq \frac{V}{V^2 + (y - U)^2} \leq c_1 \frac{V}{(1 + |y|)^2}.$$

(4) *If* $c_2^{-1}(1 + |y|)^{1/\gamma} \leq v, bu + av \leq c_2(1 + |y|)^{1/\gamma}$, *then*
$$\frac{1}{c_1} \frac{1}{(1 + |y|)} \leq \frac{V}{V^2 + (y - U)^2} \leq c_1 \frac{1}{(1 + |y|)}.$$

(5) *If* $\min\{v, bu + av\} \leq c_3^{-1}(1 + |y|)^{1/\gamma}$ *and* $c_2^{-1}(1 + |y|)^{1/\gamma} \leq \max\{v, bu + av\} \leq c_2(1 + |y|)^{1/\gamma}$, *then*
$$\frac{1}{c_1} \leq \frac{V}{V^2 + (y - U)^2} : \frac{(1 + |y|)^{1/\gamma-1} \min\{v, bu + av\}}{(\min\{v, bu + av\})^2 + (\max\{v, bu + av\} - \rho)^2} \leq c_1,$$
where ρ is a real number which depends only on y and $\min\{v, bu + av\}$, *and which satisfies* $c_2^{-1}(1 + |y|)^{1/\gamma} \leq \rho \leq c_2(1 + |y|)^{1/\gamma}$.

Proof. (1) The first inequality of part (1) follows from the equality $|U + iV| = |u + iv|^\gamma$. To prove the second inequality of part (1) we use some elementary trigonometric arguments. Let O denote the origin in the (u, v)-plane and let M denote the point $u + iv$. Recall that the sector S_λ is delimited by two rays emanating from O which correspond to two lines $v = 0$ and $bu + av = 0$. Let A (respectively, B) be the unique point lying on the ray corresponding to $v = 0$ (respectively, $bu + av = 0$) such that $OA = 1$ (respectively, $OB = 1$). Let $\theta := \measuredangle\widehat{AOM}$ and $\vartheta := \measuredangle\widehat{MOB}$. Then $\theta, \vartheta \geq 0$ and $\theta + \vartheta = \arctan(-b/a) \in (0, \pi)$. A geometric argument gives that
$$\sin\theta = v/OM \quad \text{and} \quad \sin\vartheta = (bu + av)/OM.$$

Moreover,
$$\max\{v, bu + av\} \leq OM \leq |u| + |v| \leq (1 + (1 + |a|)b^{-1}) \max\{v, bu + av\}.$$

Consequently,
$$\sin\theta \approx \frac{v}{\max\{v, bu + av\}} \quad \text{and} \quad \sin\vartheta \approx \frac{bu + av}{\max\{v, bu + av\}}. \tag{10}$$

Let N be the point $U + iV$ in the (U, V)-plane. Let C (respectively, D) be the image of A (respectively, B) by the map $\phi\colon\ \tau \mapsto \tau^\gamma$ given in (5). Clearly, $\measuredangle\widehat{CON} = \gamma\theta$

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and $\measuredangle\widehat{NOD} = \gamma\vartheta$ and $\measuredangle\widehat{CON} + \measuredangle\widehat{NOD} = \gamma\theta + \gamma\vartheta = \pi$. Suppose without loss of generality that $\theta \le \vartheta$, or equivalently $v \le bu + av$. Then $0 \le \theta \le \pi/2$ and $\measuredangle\widehat{CON} = \gamma\theta \le \pi/2$. Combining this with the well-known estimate $2/\pi \le (\sin t)/t \le 1$ for $0 < t \le \pi/2$, we get that

$$\sin(\gamma\theta) \approx \gamma\theta \approx \gamma \sin\theta \approx \sin\theta \approx \frac{v}{\max\{v, bu + av\}},$$

where the last estimate holds by (10). On the other hand, a geometric argument shows that

$$V = \sqrt{U^2 + V^2} \cdot \sin \measuredangle\widehat{CON} = \sqrt{U^2 + V^2} \cdot \sin(\gamma\theta).$$

This, coupled with the last estimate for $\sin(\gamma\theta)$ and the first estimate (for $\sqrt{U^2 + V^2}$) in part (1), implies the second estimate of this part.

(2) We will show that there is a constant $c > 1$ such that

$$c^{-1}(U^2 + V^2) \le V^2 + (y - U)^2 \le c(U^2 + V^2). \tag{11}$$

Taking (11) for granted, part (2) follows from combining (11) with part (1).

Now we turn to the proof of (11). Using the first estimate of part (1) and the assumption of part (2), we have that

$$\sqrt{U^2 + V^2} \ge c_1^{-1}(\max\{v, bu + av\})^\gamma \ge c_1^{-1} c_2^\gamma (1 + |y|). \tag{12}$$

Therefore,

$$V^2 + (y - U)^2 \le V^2 + 2U^2 + 2y^2 \le (2 + 2c_1^2 c_2^{-2\gamma})(U^2 + V^2),$$

which proves the right-hand-side estimate of (11) for $c := 2 + 2c_1^2 c_2^{-2\gamma}$.

To prove the left-hand-side estimate of (11), consider two cases. If $V \ge |U|$ then $V^2 + (y - U)^2 \ge V^2 \ge \frac{1}{2}(U^2 + V^2)$. If $V < |U|$ then for $c_2 > 1$ large enough, (12) yields that $|U| \ge 2|y|$, which in turn implies that $V^2 + (y - U)^2 \ge V^2 + \frac{1}{4}U^2 \ge \frac{1}{4}(U^2 + V^2)$. This completes the proof of (11).

(3) Using the first estimate of part (1) and the assumption of part (3), we have that

$$\sqrt{U^2 + V^2} \le c_1(\max\{v, bu + av\})^\gamma \le c_1 c_2^{-\gamma}(1 + |y|).$$

We fix $c_2 > 1$ large enough so that the last line gives $|y| \ge 2c_1 \cdot \max\{|U|, V\}$. This gives, using the first estimate of part (1),

$$|y| \ge c_1\sqrt{U^2 + V^2} \ge (\max\{v, bu + av\})^\gamma \ge 1.$$

Consequently, we get, using $|y| > 2|U|$, that

$$\tfrac{1}{12}(1 + |y|)^2 \le \tfrac{1}{4}y^2 \le V^2 + (y - U)^2 \le V^2 + 2U^2 + 2y^2 \le 4(1 + |y|)^2,$$

which completes part (3).

(4) By the assumption of part (4), $v \approx bu + av$. Consequently, we deduce from (10) that $\theta, \vartheta \approx 1$, which in turn implies that $V, U \approx \sqrt{U^2 + V^2}$. This, combined with the assumption of part (4) and the first estimate of part (1), yields that

$$V, U, \sqrt{U^2 + V^2} \approx 1 + |y|.$$

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Using this and the inequalities

$$(1+|y|)^2 \approx V^2 \le V^2 + (y-U)^2 \le V^2 + 2U^2 + 2y^2 \approx (1+|y|)^2,$$

part (4) follows.

(5) Let (u, v) as in the assumption of part (5) and let $c_1, c_2 > 1$ be constants given by part (1). Suppose without loss of generality that $v \le t$, where $t := bu + av$ as usual. Fix $c_3 \ge c_2$ large enough so that for every $1 \le v \le c_3^{-1}(1+|y|)^{1/\gamma}$, there exists a solution $u := u(y, v)$ of the equation

$$U = y \quad \text{where } U + iV' = (u+iv)^\gamma,$$

which satisfies

$$c_2^{-1}(1+|y|)^{1/\gamma} \le u(y, v),\ \rho(y, v) \le c_2(1+|y|)^{1/\gamma},$$

where $\rho(y, v) := bu(y, v) + av$. Let $\rho = \rho(y, v)$. Note that

$$\min\{v, t\} \approx \min\{v, \rho\} \quad \text{and} \quad \max\{v, t\} \approx \max\{v, \rho\} \approx (1+|y|)^{1/\gamma}.$$

Therefore, we deduce from the second inequality of part (1) that

$$V \approx V'. \tag{13}$$

Note also that for $c_3 \ge c_2$ large enough, $u = b^{-1}(\rho - av) \approx (1+|y|)^{1/\gamma}$ and $v \ll (1+|y|)^{1/\gamma}$. In particular, we get that

$$0 < \arg(u+iv),\ \arg(u(y, v)+iv) < \frac{\pi}{2\gamma}, \tag{14}$$

where arg denotes the argument of a non-zero complex number. We need the following elementary fact.

LEMMA 3.4. *Let $c', \gamma > 1$ be two constants. Then there is a constant c'' such that, for all $w, w' \in \mathbb{C} \setminus \{0\}$ satisfying*

$$c'^{-1} \le |w'|/|w| \le c' \quad \textit{and} \quad 0 \le \arg w,\ \arg w' < \frac{\pi}{2\gamma},$$

we have that

$$c''^{-1}|w-w'|(|w|+|w'|)^{\gamma-1} \le |w^\gamma - w'^\gamma| \le c''|w-w'|(|w|+|w'|)^{\gamma-1}.$$

Proof. Suppose without loss of generality that $0 < \arg w \le \arg w' < \pi/2\gamma$. We consider two cases.

Case 1. $|w-w'| \ge \frac{1}{2}\min\{|w|, |w'|\}$. In this case it is easy to show that $|w^\gamma - w'^\gamma| \approx (|w|+|w'|)^{\gamma-1}$.

Case 2. $|w-w'| \le \frac{1}{2}\min\{|w|, |w'|\}$. Let w'' be the complex number such that $|w''| = |w|$ and $\arg w'' = \arg w'$, that is, $w'' := |w|e^{i \arg w'}$. It is not difficult to show in this case that $|w^\gamma - w'^\gamma| \approx |w''^\gamma - w'^\gamma|$. So it remains to estimate $|w''^\gamma - w'^\gamma|$. Since $\arg w' = \arg w''$, we can reduce the estimate to the case where $w', w'' > 0$ by multiplying both w' and w'' by $e^{-i \arg w'}$. The lemma is then an immediate consequence of the elementary inequality

$$\gamma|w'-w''|(\min\{w', w''\})^{\gamma-1} \le |w'^\gamma - w''^\gamma| \le \gamma|w'-w''|(\max\{w', w''\})^{\gamma-1}, \quad w', w'' > 0. \qquad \square$$

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We now return the proof of part (5). Recall estimate (14) and the equations

$$U + iV = (u + iv)^\gamma \quad \text{and} \quad y + iV' = (u(y, v) + iv)^\gamma.$$

Next, we deduce from (13) that $V \gtrsim |V - V'|$. Consequently, applying Lemma 3.4 to $w := u + iv$ and $w' := u(y, v) + iv$ yields that

$$\begin{aligned}
|V| + |U - y| &\approx V + |(U + iV) - (y + iV')| \\
&= V + |(u + iv)^\gamma - (u(y, v) + iv)^\gamma| \\
&\approx V + |u - u(y, v)|(u + u(y, v))^{\gamma - 1} \\
&\approx V + |u - u(y, v)|(1 + |y|)^{1 - 1/\gamma} \\
&\approx V + |(bu + av) - (bu(y, v) + av)|(1 + |y|)^{1 - 1/\gamma} \\
&= V + |t - \rho(y, v)|(1 + |y|)^{1 - 1/\gamma}.
\end{aligned}$$

This, combined with the second inequality of part (1), implies part (5). □

The following elementary estimate is needed.

LEMMA 3.5. *For every* $s_0 \geq 1$, $\int_{s_0}^{\infty} s e^{2s_0 - 2s} \, ds = s_0/2 + 1/4$.

Proof. Integration by parts gives

$$1/2 = \int_{s_0}^{\infty} e^{2s_0 - 2s} \, ds = [s e^{2s_0 - 2s}]_{s = s_0}^{s = \infty} + 2 \int_{s_0}^{\infty} s e^{2s_0 - 2s} \, ds = 2 \int_{s_0}^{\infty} s e^{2s_0 - 2s} \, ds - s_0.$$

□

We now come to the main estimate of this section, concerning the precise behavior of $K_s(y)$ when the leaves get close to the separatrices.

PROPOSITION 3.6. *There is a constant* $c > 0$ *such that, for all* $s > 0$ *and* $y \in \mathbb{R}$,

$$\frac{K_s(y)}{(1 + |y|)^{1/\gamma - 1}} \leq c \min\left\{1, \left[\frac{(1 + |y|)^{1/\gamma}}{s}\right]^{\gamma - 1}\right\}.$$

Proof. Let c_2, c_3 be the constants with $c_3 > c_2 > 1$ given by Lemma 3.3. We consider three cases.

Case 1. $s \geq c_2(1 + |y|)^{1/\gamma}$. By part (2) of Lemma 3.3 and by formula (9), we have that

$$K_s(y) \leq c \int_{D_s} e^{2s - 2\min\{v, t\}} \frac{\min\{v, t\}}{(\max\{v, t\})^{\gamma + 1}} \, dt \, dv \leq c' \bigg(\int_{t = s}^{\infty} \frac{dt}{t^{\gamma + 1}} \bigg) \bigg(\int_{v = s}^{\infty} v e^{2s - 2v} dv \bigg).$$

The first integral on the right-hand side is equal to $\gamma^{-1} s^{-\gamma}$, while the second one is, by Lemma 3.5, equal to $s/2 + 1/4$. Hence, $K_s(y) \leq c s^{1 - \gamma}$. This completes case 1.

Case 2. $c_2^{-1} \leq s/(1 + |y|)^{1/\gamma} \leq c_2$. Write $D_s = D_s' \cup D_s''$, where

$$\begin{aligned}
D_s' &:= \{(t, v) \in D_s : \ \max\{t, v\} \leq c_2(1 + |y|)^{1/\gamma}\}, \\
D_s'' &:= \{(t, v) : \ s \leq \min\{t, v\} \text{ and } c_2(1 + |y|)^{1/\gamma} \leq \max\{t, v\}\}.
\end{aligned}$$

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Consequently, formula (9) gives that

$$K_s(y)=\frac{1}{b}\bigg(\int_{D'_s}+\int_{D''_s}\bigg)\frac{e^{2s-2\min\{v,t\}}V}{V^2+(y-U)^2}\,dt\,dv=:I+II. \tag{15}$$

To estimate I, we apply part (4) of Lemma 3.3 and obtain

$$I\le c\int_{D'_s}e^{2s-2\min\{v,t\}}\frac{dt\,dv}{(1+|y|)}.$$

The integral is bounded by a constant times

$$\bigg(\int_{c_2^{-1}(1+|y|)^{1/\gamma}}^{c_2(1+|y|)^{1/\gamma}}\frac{dt}{(1+|y|)}\bigg)\bigg(\int_{v=c_2^{-1}(1+|y|)^{1/\gamma}}^{\infty}e^{2s-2v}\,dv\bigg).$$

The left integral is equal to $(c_2+c_2^{-1})(1+|y|)^{1/\gamma-1}$, while the right integral is bounded by $1/2$. Hence, $I\le c(1+|y|)^{1/\gamma-1}$.

To estimate II, we apply part (2) of Lemma 3.3 and obtain

$$II\le c\int_{D''_s}e^{2s-2\min\{v,t\}}\frac{\min\{t,v\}\,dt\,dv}{(\max\{t,v\})^{\gamma+1}}.$$

The integral in the last line is smaller than a constant times

$$\bigg(\int_{t=c_2^{-1}(1+|y|)^{1/\gamma}}^{\infty}\frac{dt}{t^{\gamma+1}}\bigg)\bigg(\int_{v=c_2^{-1}(1+|y|)^{1/\gamma}}^{\infty}ve^{2s-2v}dv\bigg).$$

The left integral is equal to $\gamma^{-1}c_2^{-\gamma}(1+|y|)^{-1}$, while the right integral is, by Lemma 3.5, equal to $(c_2^{-1}/2)(1+|y|)^{1/\gamma}+1/4$. Hence, $II\le c(1+|y|)^{1/\gamma-1}$.

Substituting the above estimates for I and II into (15), we obtain the desired estimate for $K_s(y)$ in the second case.

Case 3. $s\le c_2^{-1}(1+|y|)^{1/\gamma}$. Write $D_s=D_s^1\cup D_s^2\cup D_s^3$, where

$$\begin{aligned}
D_s^1 &:= \{(t,v):\ s\le t,\ v\le c_2^{-1}(1+|y|)^{1/\gamma}\},\\
D_s^2 &:= \{(t,v):\ s\le\min\{t,v\}\text{ and }c_2(1+|y|)^{1/\gamma}\le\max\{t,v\}\},\\
D_s^3 &:= \{(t,v):\ \max\{s,c_3^{-1}(1+|y|)^{1/\gamma}\}\le\min\{t,v\}\\
&\qquad\text{and }c_2^{-1}(1+|y|)^{1/\gamma}\le\max\{t,v\}\le c_2(1+|y|)^{1/\gamma}\},\\
D_s^4 &:= \{(t,v):\ s\le\min\{t,v\}\le c_3^{-1}(1+|y|)^{1/\gamma}\\
&\qquad\text{and }c_2^{-1}(1+|y|)^{1/\gamma}\le\max\{t,v\}\le c_2(1+|y|)^{1/\gamma}\}.
\end{aligned}$$

Consequently, we get, similarly to (15), that

$$K_s(y)=\frac{1}{b}\bigg(\int_{D_s^1}+\int_{D_s^2}+\int_{D_s^3}+\int_{D_s^4}\bigg)\frac{e^{2s-2\min\{v,t\}}V}{V^2+(y-U)^2}\,dt\,dv=:I+II+III+IV.$$

To estimate I, we apply parts (1) and (3) of Lemma 3.3. Consequently, we obtain that

$$I\le c\int_{D_s^1}(\max\{v,t\})^{\gamma-1}\min\{v,t\}e^{2s-2\min\{v,t\}}\frac{dt\,dv}{(1+|y|)^2}.$$

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The integral is bounded by a constant times

$$\bigg(\int_{t=s}^{c_2^{-1}(1+|y|)^{1/\gamma}} \frac{t^{\gamma-1}dt}{(1+|y|)^2}\bigg)\bigg(\int_{v=s}^{c_2^{-1}(1+|y|)^{1/\gamma}} ve^{2s-2v}dv\bigg).$$

The left integral is bounded by $\gamma^{-1}c_2^{-\gamma}(1+|y|)^{-1}$, while the right integral is, by Lemma 3.5, bounded by $s/2+1/4$. Hence, $I \leq cs(1+|y|)^{-1}$.

To estimate II, we apply part (2) of Lemma 3.3 and obtain

$$II \leq c\int_{D_s^2} e^{2s-2\min\{v,t\}} \frac{\min\{t,\, v\}\, dt\, dv}{(\max\{t,\, v\})^{\gamma+1}}.$$

The integral in the last line is smaller than a constant times

$$\bigg(\int_{t=c_2(1+|y|)^{1/\gamma}}^{\infty} \frac{dt}{t^{\gamma+1}}\bigg)\bigg(\int_{v=s}^{\infty} ve^{2s-2v}dv\bigg).$$

The left integral is equal to $\gamma^{-1}c_2^{-\gamma}(1+|y|)^{-1}$, while the right integral is, by Lemma 3.5, equal to $s/2+1/4$. Hence, $II \leq cs(1+|y|)^{-1}$.

To estimate III, we apply part (4) of Lemma 3.3 and argue as in case 2. Consequently, we can show that $III \leq c(1+|y|)^{1/\gamma-1}$.

To estimate IV, we apply part (5) of Lemma 3.3 and obtain

$$IV \leq c\int_{D_s^4} e^{2s-2\min\{v,t\}} \frac{(1+|y|)^{1/\gamma-1}\min\{v,\, bu+av\}\, dt\, dv}{(\min\{v,\, bu+av\})^2 + (\max\{v,\, bu+av\}-\rho)^2}.$$

We infer from this estimate that

$$IV \leq c\int_{v=s}^{\infty}\bigg(\int_{t=c_2^{-1}(1+|y|)^{1/\gamma}}^{c_2(1+|y|)^{1/\gamma}} \frac{(1+|y|)^{1/\gamma-1}v\, dt}{v^2+(t-\rho(y,\, v))^2}\bigg)e^{2s-2v}\, dv,$$

where $\rho(y,\, v)$ satisfies $c_2^{-1}(1+|y|)^{1/\gamma} \leq \rho(y,\, v) \leq c_2(1+|y|)^{1/\gamma}$. The inner integral is bounded by $IV_1 + IV_2$, where

$$IV_1 = \int_{|t-\rho(y,v)|\leq v} \frac{(1+|y|)^{1/\gamma-1}v\, dt}{v^2+(t-\rho(y,\, v))^2} \leq \int_{|t-\rho(y,v)|\leq v} \frac{(1+|y|)^{1/\gamma-1}\, dt}{v}$$
$$\leq c(1+|y|)^{1/\gamma-1},$$

and

$$IV_2 \leq \int \frac{(1+|y|)^{1/\gamma-1}v\, dt}{v^2+(t-\rho(y,\, v))^2} \leq \int \frac{(1+|y|)^{1/\gamma-1}v\, dt}{(t-\rho(y,\, v))^2} \leq c(1+|y|)^{1/\gamma-1}.$$

Here the integrals in the last line are taken over the region

$$\{t\in\mathbb{R}:\ c_2^{-1}(1+|y|)^{1/\gamma} \leq t \leq c_2(1+|y|)^{1/\gamma} \text{ and } |t-\rho(y,\, v)| \geq v\}.$$

So the inner integral is less than or equal to $c(1+|y|)^{1/\gamma-1}$. Hence, $IV \leq c(1+|y|)^{1/\gamma-1}$.

Combining the estimates for I, II, III and IV, and using the assumption $s \leq c_2^{-1}(1+|y|)^{1/\gamma}$, we infer that

$$K_s(y) = I+II+III+IV \leq c's(1+|y|)^{-1} + c'(1+|y|)^{1/\gamma-1} \leq c(1+|y|)^{1/\gamma-1}.$$

The proof of case 3, and hence the proposition, is thus complete. □

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4. *Proofs of the main results*

End of the proof of Theorem 1.2. By Proposition 3.6 the family of functions $(g_s)_{s>0}: \mathbb{R} \to \mathbb{R}^+$, where g_s is given by

$$g_s(y) := \frac{K_s(y)}{(1+|y|)^{1/\gamma-1}}, \quad y \in \mathbb{R},$$

is uniformly bounded. Moreover, $\lim_{s\to\infty} g_s(y) = 0$ for $y \in \mathbb{R}$.

On the other hand, consider the measure χ on $\mathbb{R}$, given by

$$\int_{\mathbb{R}} \varphi d\chi = \int_{e^{-2\pi b}\leq|\alpha|\leq 1} \bigg(\int_{-\infty}^{\infty} \varphi(y)\tilde{H}_\alpha(y)(1+|y|)^{1/\gamma-1} dy \bigg) d\nu(\alpha)$$

for every continuous bounded test function φ on $\mathbb{R}$. By Lemma 3.1, χ is a finite positive measure. Consequently, we get, by dominated convergence, that $\lim_{s\to\infty} \int_{\mathbb{R}} g_s d\chi = 0$. This, combined with Lemma 3.2, implies that

$$0 \leq \lim_{r\to 0+} G(r) \leq c \cdot \lim_{s\to\infty} \int_{\mathbb{R}} g_s d\chi = 0,$$

which, coupled with (7)–(8), gives that $\mathcal{L}(T, 0) = 0$, as desired. □

End of the proof of Theorem 1.3. Let $x \in M$ be a point. Consider two cases.

Case 1. $x \notin E$. Let $\mathbb{U}$ be a regular flow box with transversal $\mathbb{T}$ which contains x. By (1) we can write in $\mathbb{U}$,

$$T = \int h_\alpha [V_\alpha]\, d\nu(\alpha),$$

where, for ν-almost every $\alpha \in \mathbb{T}$, h_α denotes the positive harmonic function associated to the current T on the plaque V_α. By Harnack's inequality, there is a constant $c > 0$ independent of α such that

$$c^{-1} h_\alpha(z) \leq h_\alpha(w) \leq c h_\alpha(z), \quad z, w \in V_\alpha.$$

Using this and the above local description of T on $\mathbb{U}$ and formula (2), we infer easily that $\mathcal{L}(T, x) \leq c\nu(\{x\})$. On the other hand, by Theorem 1.1, $\nu(\{x\}) = 0$. Consequently, $\mathcal{L}(T, x) = 0$.

Case 2. $x \in E$. Fix a (local) holomorphic coordinate system of M on a singular flow box $\mathbb{U}_x$ of x such that $(\mathbb{U}_x, x)$ is identified with $(\mathbb{D}^2, 0)$ and the leaves of $(M, \mathscr{F}, E)$ are integral curves of the linear vector field $\Phi(z, w) = \mu z \partial/(\partial z) + \lambda w \partial/(\partial w)$ with non-zero complex numbers λ, μ such that $\lambda/\mu \notin \mathbb{R}$. On the other hand, it follows from Theorem 1.1 that T gives no mass to each single leaf. In particular, T does not give mass to any of the two separatrices $(z = 0)$ and $(w = 0)$. Consequently, we are able to apply Theorem 1.2. Hence, $\mathcal{L}(T, x) = 0$. □

5. *Application: recurrence of generic leaves*

Let $(M, \mathscr{F}, E)$ be a hyperbolic foliation with the set of singularities E in a Hermitian compact complex surface (M, ω). Let dist be the distance on M induced by the Hermitian metric. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Let T be a non-zero directed harmonic current on $(X, \mathscr{L}, E)$. The existence of such a current has been established by Berndtsson and Sibony in [**1**, Theorem 1.4], and Fornæss and Sibony in [**6**, Corollary 3]. Assume, in addition, that T is extremal (in the convex set of all directed harmonic currents). Let ω_P be the Poincaré metric on $\mathbb{D}$, given by

$$\omega_P(\zeta) := \frac{2}{(1-|\zeta|^2)^2} i\, d\zeta \wedge d\overline{\zeta}, \quad \zeta \in \mathbb{D}.$$

For any point $a \in M \setminus E$ consider a universal covering map $\phi_a : \mathbb{D} \to L_a$ such that $\phi_a(0) = a$. This map is uniquely defined by a up to a rotation on $\mathbb{D}$. Then, by pushing forward the Poincaré metric ω_P on $\mathbb{D}$ via ϕ_a, we obtain the so-called *Poincaré metric* on L_a which depends only on the leaf. The latter metric is given by a positive $(1, 1)$-form on L_a that we also denote by ω_P for the sake of simplicity. Since the measure $m_P := T \wedge \omega_P$ is, by [**2**], of finite mass, we may assume without loss of generality that m_P is a probability measure. So, m_P is a harmonic measure on X with respect to ω_P.

In this section we study the following problem. Given a point $x \in M$ and an m_P-generic point $a \in M \setminus E$, how often does the leaf L_a visit the ball $B(x, r)$ as $r \searrow 0$? Here $B(x, r)$ (respectively, $\overline{B}(x, r)$) denotes the open (respectively, closed) ball with center x and radius r with respect to the metric dist. The purpose of this section is to apply Theorem 1.3 in order to obtain a partial answer to this question.

Let us introduce some more notation and terminology. Denote by $r\mathbb{D}$ the disc of center 0 and of radius r with $0 < r < 1$. In the Poincaré disc $(\mathbb{D}, \omega_P)$, $r\mathbb{D}$ is also the disc of center 0 and of radius

$$R := \log \frac{1+r}{1-r}\cdot$$

So, we will also denote this disc by $\mathbb{D}_R$ and its boundary by $\partial\mathbb{D}_R$.

Together with Dinh and Sibony, we introduce the following indicator.

Definition 5.1. For each $r > 0$, the *visibility of a point $a \in M \setminus E$ within distance r from a point $x \in M$* is the number

$$N(a, x, r) = \limsup_{R\to\infty} \frac{1}{R} \int_0^R \left(\int_{\theta=0}^1 \mathbf{1}_{B(x,r)}(\phi_a(s_t e^{2\pi i\theta}))\, d\theta \right) dt \in [0, 1],$$

where $\mathbf{1}_{B(x,r)}$ is the characteristic function associated to the set $B(x, r)$, and s_t is defined by the relation $t = \log\,(1 + s_t)/(1 - s_t)$, that is, $s_t\mathbb{D} = \mathbb{D}_t$.

Geometrically, $N(a, x, r)$ is the average, as $R \to \infty$, over the hyperbolic time $t \in [0, R]$ of the Lebesgue measure of the set $\{\theta \in [0, 1] : \phi_a(s_t e^{2\pi i\theta}) \in B(x, r)\}$. The last quantity may be interpreted as the portion which hits $B(x, r)$ of the Poincaré circle of radius t with center a spanned on the leaf L_a.

We will see in Lemma 5.4 that the lim sup in Definition 5.1 above can be replaced by a true limit for m_P-almost every $a \in M \setminus E$. Moreover, Definition 5.1 can also be applied to singular holomorphic foliations (by hyperbolic Riemann surfaces) in arbitrary dimensions.

We are now in a position to state the main result of this section.

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THEOREM 5.2. *We retain the above hypothesis and notation. Then for m_P-almost every point $a \in M \setminus E$ and for every point $x \in M$,*

$$N(a, x, r) = \begin{cases} o(r^2), & x \in M \setminus E, \\ o(|\log r|^{-1}), & x \in E. \end{cases}$$

For the proof of this theorem we need some more preparatory results. For all $0 < R < \infty$, consider the following measure on M:

$$m_{a,R} := \frac{1}{M_R} (\phi_a)_* \bigg(\bigg(\log \frac{1}{|\zeta|} \omega_P \bigg) \bigg|_{\mathbb{D}_R} \bigg),$$

where ω_P denotes also the Poincaré metric on $\mathbb{D}$ and

$$M_R := \int_{\mathbb{D}_R} \log \frac{1}{|\zeta|} \omega_P = \int_{\zeta \in \mathbb{D}_R} \log \frac{1}{|\zeta|} \frac{2}{(1 - |\zeta|^2)^2} i \, d\zeta \wedge d\bar{\zeta}.$$

So, $m_{a,R}$ is a probability measure which depends on a and R but does not depend on the choice of ϕ_a. Recall the following geometric Birkhoff ergodic theorem.

THEOREM 5.3. (Dinh, Nguyen and Sibony [**2**]) *Under the above hypothesis and notation, for almost every point $a \in X$ with respect to the measure m_P, the measure $m_{a,R}$ defined above converges to m_P when $R \to \infty$.*

The above theorem gives the following connection between $N(a, x, r)$ and m_P.

LEMMA 5.4. *For m_P-almost every $a \in M \setminus E$ and for all $x \in M$, the* lim sup *in Definition 5.1 is in fact a true limit. Moreover, if $m_P(\partial B(x, r)) = 0$, then*

$$N(a, x, r) = \lim_{R \to \infty} m_{a,R}(B(x, r)) = m_P(B(x, r)).$$

Notice that there is a number $r_0 > 0$ small enough such that for every $x \in M$ and every $0 < r < r_0$, we have that $m_P(\partial B(x, r)) = 0$.

Proof. Let l_R be the length in the Poincaré metric of the circle $\partial\mathbb{D}_R$. For a continuous test function φ on M,

$$\frac{1}{R} \int_0^R \bigg(\int_{\theta=0}^1 \varphi(\phi_a(s_t e^{2\pi i\theta})) \, d\theta \bigg) dt = \frac{1}{R} \int_0^R \bigg(\int_{(\phi_a)_*[\partial\mathbb{D}_t]} \frac{\varphi \cdot d\sqrt{\omega_P}}{l_t} \bigg) dt,$$

where $d\sqrt{\omega_P}$ is the length element associated to the metric ω_P. Moreover, using polar coordinates, we get that

$$\bigg| \frac{1}{R} \int_0^R \bigg(\int_{(\phi_a)_*[\partial\mathbb{D}_t]} \frac{\varphi \cdot d\sqrt{\omega_P}}{l_t} \bigg) dt - \int \varphi \cdot \frac{1}{2\pi R} (\phi_a)_* \bigg(\bigg(\log \frac{1}{|\zeta|} \omega_P \bigg) \bigg|_{\mathbb{D}_R} \bigg) \bigg|$$
$$\leq \frac{\|\varphi\|_\infty}{R} \int_{t=0}^R |l_t (2\pi)^{-1} \log(1/s_t) - 1| \, dt.$$

Since $|l_t (2\pi)^{-1} \log(1/s_t) - 1| \approx e^{-t}$, the right-hand side tends to 0 as $R \to \infty$.

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Next, a direct computation shows that $|M_R - 2\pi R|$ is bounded by a constant. Consequently,

$$\int \varphi \cdot \frac{1}{2\pi R}(\phi_a)_*\bigg(\bigg(\log\frac{1}{|\zeta|}\omega_P\bigg)\bigg|_{\mathbb{D}_R}\bigg) - \int_M \varphi m_{a,R}$$

tends to 0 as $R\to\infty$. On the other hand, we infer from Theorem 5.3 that $\lim_{R\to\infty}\int\varphi m_{a,R} = \int\varphi m_P$ for m_P-almost every $a\in M\setminus E$. Putting these estimates altogether, we obtain that

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R\bigg(\int_{\theta=0}^1 \varphi(\phi_a(s_t e^{2\pi i\theta}))\,d\theta\bigg)\,dt = \int_M \varphi m_P.$$

Writing $\mathbf{1}_{B(x,r)}$ (respectively, $\mathbf{1}_{\overline{B}(x,r)}$) as the limit of an increasing (respectively, decreasing) sequence of continuous test functions φ and using that $m_P(\partial B(x,r)) = 0$, the lemma follows from the last equality. □

For simplicity we still denote by ω the Hermitian metric on leaves of the foliation $(M\setminus E, \mathscr{F})$ induced by the ambient Hermitian metric ω. Consider the function $\eta: M\setminus E\to [0,\infty]$ defined by

$$\eta(x) := \sup\{\|D\phi(0)\| : \phi : \mathbb{D}\to L_x \text{ holomorphic such that } \phi(0) = x\}.$$

Here, for the norm of the differential $D\phi$ we use the Poincaré metric on $\mathbb{D}$ and the Hermitian metric ω on L_x. We obtain the following relation between ω and the Poincaré metric ω_P on leaves:

$$\omega = \eta^2\omega_P. \tag{16}$$

We record here the following precise estimate on the function η.

LEMMA 5.5. *We keep the above hypotheses and notation. Then there exists a constant $c > 1$ such that $\eta\le c$ on M, $\eta\ge c^{-1}$ outside the singular flow boxes $\bigcup_{x\in E}\frac{1}{4}\mathbb{U}_x$ and*

$$c^{-1}s\log^\star s \le \eta(x) \le cs\log^\star s$$

for $x\in M\setminus E$ and $s := \mathrm{dist}(x, E)$. Here $\log^\star(\cdot) := 1 + |\log(\cdot)|$ is a log-type function, and $\frac{1}{4}\mathbb{U}_x := (\frac{1}{4}\mathbb{D})^2$ for $\mathbb{U}_x = \mathbb{D}^2$.

Proof. Since there exists no holomorphic non-constant map $\mathbb{C}\to M$ such that out of E the image of $\mathbb{C}$ is locally contained in leaves, it follows from [**6**, Theorem 15] that there is a constant $c > 0$ such that $\eta(x)\le c$ for all $x\in M\setminus E$. In other words, the foliation is *Brody hyperbolic* following the terminology of our joint work with Dinh and Sibony [**3**]. Therefore, the lemma is a direct consequence of [**3**, Proposition 3.3]. □

End of the proof of Theorem 5.2. Let $x\in M$ be a point. Consider two cases.

Case 1. $x\notin E$. Let $\mathbb{U}$ be a regular flow box with transversal $\mathbb{T}$ which contains x. By (1) we can write in $\mathbb{U}$,

$$T = \int h_\alpha[P_\alpha]\,d\nu(\alpha),$$

where, for ν-almost every $\alpha\in\mathbb{T}$, h_α denotes the positive harmonic function associated to the current T on the plaque P_α. On the other hand, since $\mathbb{U}$ is away from the set of

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singularities E, we deduce from Lemma 5.5 that $c^{-1} \le \eta(y) \le c$ for $y \in \mathbb{U}$. Using this and (16) and the above expression for T, we infer easily that

$$m_P(y) = (T \wedge \omega_P)(y) = \eta(y)(T \wedge \omega)(y) \approx (T \wedge \omega)(y) \approx T \wedge i\partial\overline{\partial}\|y\|^2, \quad y \in \mathbb{U}.$$

This, combined with formula (2), implies that

$$\lim_{r\to 0} r^{-2} m_P(B(x,r)) \le \lim_{r\to 0} cr^{-2} \int_{B(x,r)} T \wedge i\partial\overline{\partial}\|y\|^2 = c\mathcal{L}(T,x).$$

By Theorem 1.1, $\mathcal{L}(T,x) = 0$. On the other hand, we know from Lemma 5.4 that

$$\lim_{r\to 0} r^{-2} N(a,x,r) = \lim_{r\to 0} r^{-2} m_P(B(x,r))$$

for m_P-almost every $a \in M \setminus E$. Putting the last three estimates together, we obtain that $N(a,x,r) = o(r^2)$.

Case 2. $x \in E$. Fix a (local) holomorphic coordinate system of M on a singular flow box $\mathbb{U}_x$ of x such that $(\mathbb{U}_x, x)$ is identified with $(\mathbb{D}^2, 0)$ and the leaves of $(M, \mathscr{F}, E)$ are integral curves of the linear vector field $\Phi(z,w) = \mu z\partial/(\partial z) + \lambda w\partial/(\partial w)$ with non-zero complex numbers λ, μ such that $\lambda/\mu \notin \mathbb{R}$.

Suppose without loss of generality that the metric ω coincides with the standard metric $i\partial\overline{\partial}\|y\|^2$ on $\mathbb{D}^2$. Next, recall from (16) that

$$i\partial\overline{\partial}\|y\|^2 = \eta^2(y) g_P(y) \approx \|y\|^2 (\log\|y\|)^2 g_P(y) \quad \text{for } 0 < \|y\| < 1/2.$$

where the estimate $\approx$ holds by Lemma 5.5. Therefore, we infer that

$$m_P(y) := (T \wedge \omega_P)(y) \approx \frac{T \wedge i\partial\overline{\partial}\|y\|^2}{\|y\|^2(\log\|y\|)^2} \quad \text{on } B_{1/2}.$$

Consequently, for $0 < r < 1/2$,

$$\int_{B_r} m_P(y) \approx \int_{B_r} \frac{T \wedge i\partial\overline{\partial}\|y\|^2}{\|y\|^2(\log\|y\|)^2} = \int_0^r \frac{F'(s)\,ds}{s^2(\log s)^2},$$

where the last equality follows from (7). So case 2 will follow if we can show that $\int_0^r (F'(s)\,ds/s^2(\log s)^2) = o(|\log r|^{-1})$ as $r \to 0$. Since we know from (7) to (8) that $(s^2 G(s))' = F'(s)$, integration by parts on the last expression yields that

$$\begin{aligned}\int_0^r \frac{F'(s)\,ds}{s^2(\log s)^2} &= \int_0^r \frac{d(s^2G(s))}{s^2(\log s)^2}\\ &= \left[\frac{G(s)}{(\log s)^2}\right]_0^r + 2\int_0^r \frac{G(s)\,ds}{s(\log s)^2} + 2\int_0^r \frac{G(s)\,ds}{s(\log s)^3}.\end{aligned}$$

On the other hand, we know from (7)–(8) that $G(r)$ decreases, as $r \searrow 0$, to $\mathcal{L}(T,x)$, which is equal to 0 by Theorem 1.3. Therefore, a straightforward computation shows that all three terms on the right-hand side of the last line areof order $o(|\log r|^{-1})$ as $r \to 0$, as desired. This completes the proof of case 2, and hence of the theorem. □

Remark 5.6. We conclude this paper with some remarks and open questions.

(1) It seems to be of interest to investigate the main theorem in the case where the singularity 0 is only linearizable (see [**2**]).

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(2) A natural question arises whether the main results of this paper can be generalized to higher dimensions. We postpone this issue to forthcoming work.

(3) Now let $(M, \mathscr{F}, E)$ be a hyperbolic foliation with the set of singularities E in a Hermitian compact complex manifold (M, ω) of arbitrary dimension. Assume that all the singularities are linearizable. Using the finiteness of the Lelong number of a positive harmonic current [**12**], and applying [**2**] and arguing as at the end of the proof of Theorem 5.2, we can show the following weak form of this theorem (but in higher dimension). For m_P-almost every point $a \in M \setminus E$ and for every point $x \in M$, we have that

$$N(a, x, r) = \begin{cases} O(r^2), & x \in M \setminus E, \\ O(|\log r|^{-1}), & x \in E. \end{cases}$$

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