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Construction of regular and singular Green's functions

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The Green function of singular limit-circle problems is constructed directly for the problem, not as a limit of sequences of regular Green's functions. This construction is used to obtain adjointness and self-adjointness conditions which are entirely analogous to the regular case. As an application, a new and explicit formula for the Green function of the classical Legendre problem is found.

1. Introduction

In this paper we construct the Green function of singular boundary-value problems (BVPs) consisting of general quasi-differential equations on an open, bounded or unbounded interval (a, b) of the real line and singular boundary conditions (BCs) at the end points. This applies for self-adjoint and non-self-adjoint problems. In contrast to the usual construction as, for example, in the well-known book by Coddington and Levinson [1], which involves a selection theorem to select a sequence of regular Green's functions on truncated intervals whose limit as the truncated end points approach the singular end points is the Green function of the singular problem, our construction is direct, elementary and explicit in terms of solutions. Our method is based on a simple transformation of the dependent variable which leaves the underlying interval unchanged and transforms the singular problem with limit-circle end points into a regular one. As an illustration we obtain a new and explicit formula for the Green function of the classical Legendre equation with arbitrary separated or coupled self-adjoint boundary conditions.

Necessary and sufficient conditions for two singular problems to be adjoint to each other and, in particular, self-adjoint then follow from the regular case.

Throughout this paper $n > 1$ is a positive integer, J denotes an open, bounded or unbounded interval of the real line

$$J = (a, b), \quad -\infty \leqslant a < b \leqslant \infty;$$

where $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ the complex numbers, $L(J) = L^1(J)$ is the set of complex-valued functions from J to $\mathbb{C}$ which are Lebesgue integrable, $L_{\text{loc}}(J) = L^1_{\text{loc}}(J)$ is the set of complex functions from J to $\mathbb{C}$ which are Lebesgue integrable on all compact subintervals of J and $M_n(S)$ denotes the $n \times n$ matrices with entries from an arbitrary set S. For $A \in M_n(\mathbb{C})$, A^* denotes the complex conjugate of A. For a vector or matrix function we write $F \in L(J)$ to mean that every component of F is in $L(J)$.

The paper is organized as follows: adjoint and self-adjoint Green's functions are discussed in § 2 for regular systems and in § 3 for regular scalar equations. The transformation from singular to regular problems is given in § 4 and is applied in § 5. The adjointness conditions are discussed in § 6 and illustrated for the second and fourth-order cases. In § 7 the Green function for the classical Legendre equation is constructed using the method of § 5. We believe this formula for the Legendre Green function is new.

2. Adjoint matrices and Green's functions for regular systems

In this section we define the concept of 'Lagrange adjoint' for first-order systems and establish the corresponding 'Lagrange Hermitian' properties of their Green matrices. Fundamental to this analysis is the 'adjointness lemma' (see below) and the above-mentioned construction of the Green matrices. The next section will contain applications of these results to scalar problems, and the following sections will discuss singular systems and singular scalar problems.

Let

$$E := ((-1)^i \delta_{i,n+1-j})_{1 \leqslant i,j \leqslant n}, \tag{2.1}$$

where δ is the Kronecker symbol, and note that

$$E^{-1} = E^* = (-1)^{n+1} E. \tag{2.2}$$

We first establish some general properties for singular systems before specializing to the regular case.

Definition 2.1. For $P \in L_{\text{loc}}(J)$ we define

$$P^+ := -E^{-1} P^* E. \tag{2.3}$$

We call P^+ the Lagrange adjoint matrix of P; note that $P^+ \in L_{\text{loc}}(J)$ and

$$\left.\begin{aligned} (P^+)^+ &= P, \\ (P + Q)^+ &= P^+ + Q^+, \\ (PQ)^+ &= -Q^+ P^+, \\ (cP)^+ &= \bar{c} P^+, \quad c \in \mathbb{C}. \end{aligned}\right\} \tag{2.4}$$

The terminology 'Lagrange adjoint' is introduced due to its relationship to boundary-value problems which are adjoint in the sense of Lagrange, as we will see below. An important special case arises when $P^+ = P$: this is the Lagrange Hermitian case which, as we will see below, generates symmetric differential operators.

REMARK 2.2. The relationship (2.3) can be described as follows: P^+ is obtained from P by performing the following three commutative operations.

(i) 'Flip' the components $p_{i,j}$ across the secondary diagonal:

$$p_{i,j} \to p_{n+1-j,n+1-i}.$$

(ii) Take conjugates:

$$p_{i,j} \to \bar{p}_{i,j}.$$

(iii) Change the sign for all the even positions:

$$p_{i,j} \to (-1)^{i+j+1} p_{i,j}.$$

DEFINITION 2.3 (primary fundamental matrix). Let $P \in M_n(L_{\mathrm{loc}}(J))$. For each $u \in J$, let $\Phi(t,u) = \Phi(t,u,P)$ denote the unique $n \times n$ matrix solution of the initial-value problem

$$Y' = PY, \qquad Y(u) = I, \tag{2.5}$$

where I is the identity. Then $\Phi(t,s)$ is defined for all $t, s \in J$ and is called the primary fundamental matrix of the system $Y' = PY$ [11], or just the primary fundamental matrix of P.

LEMMA 2.4. *Let $P \in M_n(L_{\mathrm{loc}}(J))$ and let $\Phi(t,s)$ be the primary fundamental matrix of P. Then for any $t, s, u \in J$ we have*

$$\Phi(t,u)\Phi(u,s) = \Phi(t,s). \tag{2.6}$$

Furthermore, if $P \in L(J)$, then (2.6) also holds when t, s, u are equal to a or b. Here $a = -\infty$ and $b = \infty$ are allowed.

Proof. This follows from the well-known representation

$$\Phi(t,s) = Y(t)Y^{-1}(s),$$

where Y is any fundamental matrix of $Y' = PY$. For the 'furthermore' statement see [11, theorem 1.5.2]. □

LEMMA 2.5. *Let $P \in M_n(L_{\mathrm{loc}}(J))$ and let $\Phi(t,s)$ be the primary fundamental matrix of P. Suppose $\operatorname{tr} P(t) = 0$ for $t \in J$. Then*

$$\det(\Phi(t,s)) = 1, \quad t, s \in J.$$

Proof. Fix $s \in J$. It is well known that $\det \Phi(t,s)' = \operatorname{tr}(P(t)) \det(\Phi(t,s))$ and the result follows. □

Next we establish a basic relationship between the primary fundamental matrices of P and P^+.

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Lemma 2.6 (adjointness lemma). *Suppose that $P \in M_n(L_{\mathrm{loc}}(J))$. Then we have $P^+ \in M_n(L_{\mathrm{loc}}(J))$. If $\Phi(t,s) = \Phi(t,s,P)$ and $\Psi(t,s) = \Phi(t,s,P^+)$ are the primary fundamental matrices of P and P^+, respectively, then*

$$\Psi(t,s) = E^{-1}\Phi^*(s,t)E, \quad \Phi(t,s) = E^{-1}\Psi^*(s,t)E, \quad t,s \in J. \tag{2.7}$$

Furthermore, if $P \in L(J)$, then $P^+ \in L(J)$ and (2.6), (2.7) hold for $a \leqslant s,t \leqslant b$, even when $a = -\infty$ or $b = +\infty$.

Proof. Fix $s \in J$ and let $X(t) = E^{-1*}\Phi^*(t,s)E^*\Psi(t,s)$ for $t \in J$. Then $X(s) = I$ and

$$\begin{aligned} X'(t) &= E(\Phi^*)'(t,s)E^*\Psi(t,s) + E\Phi^*(t,s)E^*\Psi'(t,s) \\ &= E(P(t)\Phi(t,s))^*E^*\Psi(t,s) + E\Phi^*(t,s)E^*P^+(t)\Psi(t,s) \\ &= E\Phi^*(t,s)P^*(t)E^*\Psi(t,s) + E\Phi^*(t,s)E^*(-E^{-1}P^*(t)E)\Psi(t,s) \\ &= E\Phi^*(t,s)P^*(t)E^*\Psi(t,s) - E\Phi^*(t,s)(-1)^{n+1}EE^{-1}P^*(t)(-1)^{n+1}E^*\Psi(t,s) \\ &= 0 \quad \text{for all } t \in J. \end{aligned}$$

Hence, for all $t \in J$, $X(t) = I$ and

$$\Psi(t,s) = E^{-1}(\Phi^*)^{-1}(t,s)E = E^{-1}(\Phi^{-1})^*(t,s)E = E^{-1}\Phi^*(s,t)E.$$

In the last step we used $\Phi^{-1}(t,s) = \Phi(s,t)$ which follows from (2.6). This completes the proof of the first part of the Lemma and the second part follows from the first and (2.2). The 'furthermore' statement follows by taking limits as t and s approach a or b. These limits exist and are finite by [11, theorem 1.5.2, p. 11]. □

For $\lambda \in \mathbb{C}$, consider the following vector matrix boundary-value problem:

$$Y' = (P - \lambda W)Y, \qquad AY(a) + BY(b) = 0, \quad A,B \in M_n(\mathbb{C}). \tag{2.8}$$

We now construct the Green matrix for regular systems (2.8).

Theorem 2.7. *Assume that $P, W \in M_n(L(J))$. Let $\lambda \in \mathbb{C}$ and let $\Phi(t,s,\lambda)$ be the primary fundamental matrix of $P - \lambda W$.*

(i) *The homogeneous boundary-value problem (2.8) has a non-trivial solution if and only if*

$$\det[A + B\Phi(b,a,\lambda)] = 0. \tag{2.9}$$

(ii) *If $\det[A + B\Phi(b,a,\lambda)] \neq 0$, then for every $F \in L(J)$ the inhomogeneous problem*

$$Y' = (P - \lambda W)Y + F, \qquad AY(a) + BY(b) = 0 \tag{2.10}$$

has a unique solution Y given by

$$Y(t) = \int_a^b K(t,s,\lambda)F(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b, \tag{2.11}$$

where

$$K(t,s,\lambda) = \begin{cases} \Phi(t,a,\lambda)U\Phi(a,s,\lambda), & a \leqslant t < s \leqslant b, \\ \Phi(t,a,\lambda)U\Phi(a,s,\lambda) + \Phi(t,s,\lambda), & a \leqslant s < t \leqslant b, \\ \Phi(t,a,\lambda)U\Phi(a,s,\lambda) + \frac{1}{2}\Phi(t,s,\lambda), & a \leqslant s = t \leqslant b, \end{cases} \tag{2.12}$$

with

$$U = -[A + B\Phi(b, a, \lambda)]^{-1} B\Phi(b, a, \lambda). \tag{2.13}$$

We call $K(t, s, \lambda) = K(t, s, \lambda, P, W, A, B)$ the Green matrix of the boundary-value problem (2.8).

Proof. See [11, ch. 3]. Although formula (2.12) is a slightly modified version of the corresponding formula in [11], the construction given there applies here. See also [2, 9, 10], where special cases of this construction are given. □

Consider the boundary-value problem

$$Z' = (P - \lambda W)^+ Z, \qquad CZ(a) + DZ(b) = 0, \quad C, D \in M_n(\mathbb{C}), \tag{2.14}$$

and its Green matrix $L(t, s, \bar{\lambda}) = L(t, s, \bar{\lambda}, P^+, W^+, C, D)$ given by

$$L(t, s, \bar{\lambda}) = \begin{cases} \Psi(t, a, \bar{\lambda}) V \Psi(a, s, \bar{\lambda}), & a \leqslant t < s \leqslant b, \\ \Psi(t, a, \bar{\lambda}) V \Psi(a, s, \bar{\lambda}) + \Psi(t, s, \bar{\lambda}), & a \leqslant s < t \leqslant b, \\ \Psi(t, a, \bar{\lambda}) V \Psi(a, s, \bar{\lambda}) + \frac{1}{2}\Psi(t, s, \bar{\lambda}), & a \leqslant s = t \leqslant b, \end{cases} \tag{2.15}$$

with

$$V = -[C + D\Psi(b, a, \bar{\lambda})]^{-1} D\Psi(b, a, \bar{\lambda}). \tag{2.16}$$

Lemma 2.8. *Let $\lambda \in \mathbb{C}$. Suppose that $P, W \in M_n(L(J))$. Let $\Phi(t, s, \lambda), \Psi(t, s, \bar{\lambda})$ be the primary fundamental matrices of $(P - \lambda W)$ and $(P - \lambda W)^+$, respectively. Let $K(t, s, \lambda) = K(t, s, \lambda, P, W, A, B)$ be the Green matrix of the boundary-value problem (2.8) and let $L(t, s, \bar{\lambda}) = L(t, s, \bar{\lambda}, P^+, W^+, C, D)$ be the Green matrix of the boundary-value problem (2.14). Assume that*

$$\det[A + B\Phi(b, a, \lambda)] \neq 0 \neq \det[C + D\Psi(b, a, \bar{\lambda})]. \tag{2.17}$$

Then the Green matrices $K(t, s, \lambda)$ of (2.8) and $L(t, s, \bar{\lambda})$ of (2.14) exist by theorem 2.7 and we have

$$K(t, s, \lambda) + E^{-1} L^*(s, t, \bar{\lambda}) E = \Phi(t, a, \lambda) \Gamma \Phi(a, s, \lambda), \quad a \leqslant s, t \leqslant b, \tag{2.18}$$

with

$$\Gamma = U + E^{-1} V^* E + I. \tag{2.19}$$

Proof. This follows from the construction (2.12), (2.13), (2.15), (2.16) using (2.6), (2.7) and (2.2) as follows.

$$\begin{aligned} K(t, s, \lambda) &+ E^{-1} L^*(s, t, \bar{\lambda}) E - \Phi(t, a, \lambda) U \Phi(a, s, \lambda) \\ &= E^{-1}[\Psi(s, a, \bar{\lambda}) V \Psi(a, t, \bar{\lambda})]^* E + E^{-1} \Psi^*(s, t, \bar{\lambda}) E \\ &= E^{-1}[\Psi^*(a, t, \bar{\lambda}) V^* \Psi^*(s, a, \bar{\lambda})] E + \Phi(t, s, \lambda) \\ &= \Phi(t, a, \lambda) E^{-1} V^* E \Phi(a, s, \lambda) + \Phi(t, a, \lambda) \Phi(a, s, \lambda). \end{aligned} \tag{2.20}$$

Similarly, (2.20) also holds for the cases $a \leqslant s < t \leqslant b$ and $a \leqslant t = s \leqslant b$. (The fraction $\frac{1}{2}$ in the constructions (2.12), (2.15) is used when $s = t$.) Clearly, (2.18), (2.19) follow from (2.20) and this completes the proof. □

Lemma 2.9. *Let the hypotheses and notation of lemma 2.8 hold. Then*

$$K(t,s,\lambda) = -E^{-1}L^*(s,t,\bar{\lambda})E \quad \text{for all } a \leqslant s,t \leqslant b \tag{2.21}$$

if and only if

$$U = -[E^{-1}V^*E + I]. \tag{2.22}$$

Proof. This follows from lemma 2.8 by noting that (2.21) holds if and only if $\Gamma = 0$ and $\Gamma = 0$ is equivalent to (2.22). □

Lemma 2.10. *Let the hypotheses and notation of lemma 2.8 hold. Then*

$$K(t,s,\lambda) = +E^{-1}L^*(s,t,\bar{\lambda})E \quad \text{for all } a \leqslant s,t \leqslant b, \tag{2.23}$$

if and only if

$$U = +[E^{-1}V^*E + I]. \tag{2.24}$$

Proof. The proofs of lemmas 2.8 and 2.9 can be easily adapted to prove this lemma. □

Theorem 2.11. *Let the hypotheses and notation of lemma 2.8 hold. Then the Green matrices $K(t,s,\lambda)$ of (2.8) and $L(t,s,\bar{\lambda})$ of (2.14) exist by theorem 2.7, and (2.21) holds if and only if*

$$AEC^* = BED^*. \tag{2.25}$$

Proof. By lemma 2.9 we only need to show that (2.22) holds. From (2.13) and (2.16) we have

$$\begin{aligned} -I &= U + E^{-1}V^*E \\ &= -[A + B\Phi(b,a,\lambda)]^{-1}B\Phi(b,a,\lambda) - E^{-1}\Psi^*(b,a,\bar{\lambda})D^*[C + D\Psi(b,a,\bar{\lambda})]^{-1*}E. \end{aligned} \tag{2.26}$$

Multiplying equation (2.26) on the left by $-[A + B\Phi(b,a,\lambda)]$ and on the right by $E^{-1}[C + D\Psi(b,a,\bar{\lambda})]^*$, we obtain the equivalent identity

$$\begin{aligned} &[A + B\Phi(b,a,\lambda)]E^{-1}[C + D\Psi(b,a,\bar{\lambda})]^* \\ &\quad = B\Phi(b,a,\lambda)E^{-1}[C + D\Psi(b,a,\bar{\lambda})]^* + [A + B\Phi(b,a,\lambda)]E^{-1}\Psi^*(b,a,\bar{\lambda})D^*. \end{aligned}$$

This simplifies to

$$\begin{aligned} AE^{-1}C^* &= B\Phi(b,a,\lambda)E^{-1}\Psi^*(b,a,\bar{\lambda})EE^{-1}D^* \\ &= B\Phi(b,a,\lambda)\Phi(a,b,\lambda)E^{-1}D^* \\ &= BE^{-1}D^*, \end{aligned}$$

which is equivalent to (2.25) using (2.2). In the previous two steps we used (2.6) and (2.7). This completes the proof. Special cases of this theorem were obtained in [2, 9, 10]. □

Theorem 2.12. *Let the hypotheses and notation of lemma 2.8 hold. If $U = V$, then, for each λ satisfying (2.17), there exist s, t, $a \leqslant s,t \leqslant b$, such that*

$$K(t,s,\lambda) \neq +E^{-1}L^*(s,t,\bar{\lambda})E. \tag{2.27}$$

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In particular, (2.27) holds if $P = P^+$, $W = W^+$, $A = C$, $B = D$, and $\lambda = \bar{\lambda}$ satisfies (2.17).

Proof. By lemma 2.10 we only need to show that (2.24) does not hold. In this case $U = V$. Assume (2.24) holds with $U = (u_{ij}) = V$. Then (see remark 2.2) we have $u_{11} = \bar{u}_{nn} + 1$ and $u_{nn} = \bar{u}_{11} + 1$. Hence, $1 = u_{11} - \bar{u}_{nn} = \bar{u}_{11} - u_{nn} = -1$. This contradiction completes the proof. □

3. Green's functions of regular scalar boundary-value problems

Although in this paper our primary focus is on singular problems, we now discuss regular problems and their Green functions. The traditional construction of the Green function $K(t, s, \lambda)$ of regular ordinary boundary-value problems involves a recipe which, among other things, prescribes a jump discontinuity of the derivative with respect to t for fixed s when $t = s$ (see, for example, [1]). Here we construct $K(t, s, \lambda)$ directly in such a way that this jump discontinuity does not have to be prescribed *a priori* but occurs naturally. This is accomplished by converting the scalar problem to a system, constructing a Green matrix for the system as in § 2, then extracting the scalar Green function from the Green matrix. The jump discontinuities along the diagonal $t = s$ are clearly apparent from this construction. This and a number of other features make this construction more direct and, we believe, more 'natural' than the traditional one found in textbooks. This construction is a modification of a construction used previously by Neuberger [8] for the second-order case and by Zettl [9,10] and Coddington and Zettl [2] for higher orders. It seems not to be widely known, and most current textbooks still use the 'recipe' construction mentioned above. Our construction of singular Green's functions in § 5 is based on our construction of regular Green's functions; thus, we give it here.

We now specialize to a subset of the matrices P and W which generate the scalar quasi-differential equations studied here. As in [6], let

$$Z_n(J) := \{P = (p_{rs}) \in M_n(L_{\mathrm{loc}}(J)),\quad p_{r,r+1} \neq 0 \text{ a.e. on } J \text{ for } 1 \leqslant r \leqslant n-1,\quad p_{rs} = 0 \text{ a.e. for } 2 \leqslant r+1 < s \leqslant n\}. \tag{3.1}$$

For $P \in Z_n(J)$ we define quasi-derivatives $y^{[r]}$ as follows:

$$V_0 := \{y\colon J \to \mathbb{C},\ y \text{ measurable}\} \quad \text{and } y^{[0]} := y \text{ for } y \in V_0.$$

Inductively, for $r = 1, \dots, n$, we define $V_r := \{y \in V_{r-1} : y^{[r-1]} \in AC_{\mathrm{loc}}(J)\}$ and

$$y^{[r]} := p_{r,r+1}^{-1}\biggl\{(y^{[r-1]})' - \sum_{s=1}^{r} p_{rs} y^{[s-1]}\biggr\} \quad \text{for } y \in V_r, \tag{3.2}$$

where $p_{n,n+1} := 1$, and $AC_{\mathrm{loc}}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of J. For V_n we also use the notation $V_n = D(P)$ to indicate its dependence on P. Finally, we set

$$My := M_P y := i^n y^{[n]} \quad \text{for } y \in V_n. \tag{3.3}$$

The expression $M = M_P$ is the quasi-differential expression generated by P, the function $y^{[r]} = y_P^{[r]}$ for $0 \leqslant r \leqslant n$ is called the rth quasi-derivative of y with respect to P and $D(P)$ is called the expression domain of M. To simplify the notation we omit the subscript P on $y^{[r]}$ and M when this is clear from the context. Clearly, these quasi-differential expressions include the classical expressions

$$Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y,$$

where $y^{(r)}$ denotes the classical derivative as a very special case. They also are much more general than the quasi-differential expressions discussed in [7]. For a more detailed discussion of the expressions $M_P y$ and their relationship to the classical expressions, the reader is referred to [2, 4, 6, 7].

Note that the operator M from $D(P)$ to $L_{\text{loc}}(J)$ is linear. Next we discuss the system formulation of the scalar equation $y_P^{[n]} = f$.

Lemma 3.1. *Let $P \in Z_n(J)$, $f \in L_{\text{loc}}(J)$ and set*

$$Y = \begin{bmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}. \tag{3.4}$$

Then the scalar equation

$$y_P^{[n]} = f \tag{3.5}$$

is equivalent to the first-order system

$$Y' = PY + F \tag{3.6}$$

in the sense that if y is a solution of (3.5) and Y, F are defined by (3.4), then (3.6) holds. Conversely, if Y is a solution vector of (3.6), then its first component y is a solution of (3.5).

Proof. This can be checked by a direct computation. For details, see [6, proposition 2.2]. □

Throughout the remainder of this section we assume that $P = (p_{ij}) \in Z_n(J)$ satisfies

$$p_{ij} \in L(J), \quad 1 \leqslant i \leqslant j, \quad j = 1, 2, \ldots, n; \qquad p_{j,j+1}^{-1} \in L(J), \quad j = 1, 2, \ldots, n-1. \tag{3.7}$$

Let $w \in L(J)$ and let W be the $n \times n$ matrix

$$W := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ w & 0 & \cdots & 0 \end{pmatrix}. \tag{3.8}$$

For any $\lambda \in \mathbb{C}$ and $f \in L(J)$ we now consider the equation

$$My = \lambda w y + f \tag{3.9}$$

and its equivalent systems formulation

$$Y' = (P - \lambda W)Y + F, \tag{3.10}$$

where Y and F are given by (3.4). For $A, B \in M_n(\mathbb{C})$, consider the two-point boundary-value problem consisting of (3.9) and the boundary conditions

$$AY(a) + BY(b) = 0. \tag{3.11}$$

Note that this is a well-defined problem since the quasi-derivatives $y^{[r]}$ exist as finite limits at both end points [11].

THEOREM 3.2. *Let $\lambda \in \mathbb{C}$, let W be given by (3.8) with $w \in L(J)$. Assume that $P = (p_{ij}) \in Z_n(J)$ satisfies (3.7). Let $\Phi(t, s, \lambda)$ be the primary fundamental matrix of $P - \lambda W$. Then we have the following.*

(i) *The homogeneous boundary-value problem*

$$Y' = (P - \lambda W)Y, \quad AY(a) + BY(b) = 0, \quad A, B \in M_n(\mathbb{C}), \tag{3.12}$$

has a non-trivial solution if and only if

$$\det[A + B\Phi(b, a, \lambda)] = 0. \tag{3.13}$$

(ii) *The homogeneous boundary-value problem*

$$My = \lambda wy, \qquad AY(a) + BY(b) = 0, \qquad M = M_P \tag{3.14}$$

has a non-trivial solution if and only if

$$\det[A + B\Phi(b, a, \lambda)] = 0. \tag{3.15}$$

(iii) *If $\det[A + B\Phi(b, a, \lambda)] \neq 0$, then for every $F \in L(J)$ the inhomogeneous problem*

$$Y' = (P - \lambda W)Y + F, \qquad AY(a) + BY(b) = 0, \tag{3.16}$$

has a unique solution Y given by

$$Y(t) = \int_a^b K(t, s, \lambda)F(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b, \tag{3.17}$$

where $K(t, s, \lambda)$ is given by (2.12), (2.13).

(iv) *Let $K(t, s, \lambda) = K_{ij}(t, s, \lambda)$, $1 \leqslant i, j \leqslant n$. If $\det[A + B\Phi(b, a, \lambda)] \neq 0$, then for every $f \in L(J)$ the inhomogeneous problem*

$$My = \lambda wy + f, \quad AY(a) + BY(b) = 0, \tag{3.18}$$

has a unique solution y given by

$$y(t) = \int_a^b K_{1n}(t, s, \lambda)f(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b. \tag{3.19}$$

Furthermore, K_{1n} is continuous on $[a, b] \times [a, b]$ and is unique.

Proof. All four parts follow from theorem 2.7. The continuity of K_{1n} is clear from (2.12) since $n > 1$ and the jump discontinuities of K occur only on the diagonal $s = t$. The proof of the uniqueness of K_{1n} is standard by using compact support function for f. □

Next we consider boundary-value problems for the system

$$Z' = (P - \lambda W)^{+}Z + F = (P^{+} - \bar{\lambda}W^{+})Z + F = (P^{+} - (-1)^{n}\overline{\lambda W})Z + F \quad (3.20)$$

and its equivalent scalar equation

$$M^{+}z = (-1)^{n}\bar{\lambda}\bar{w}z + f, \qquad M^{+} = M_{P^{+}}, \quad (3.21)$$

both with boundary condition

$$CZ(a) + DZ(b) = 0, \quad C, D \in M_n(\mathbb{C}). \quad (3.22)$$

THEOREM 3.3. *Let $\lambda \in \mathbb{C}$. Let $A, B, C, D \in M_n(\mathbb{C})$. Assume that $P = (p_{ij}) \in Z_n(J)$ satisfies (3.7) and W is given by (3.8) with $w \in L(J)$. Let E be given by (2.1). Then $P^{+} \in Z_n(J)$ and satisfies (3.7). Let $\Phi(t, s, \lambda)$ and $\Psi(t, s, \bar{\lambda})$ be the primary fundamental matrices of $P - \lambda W$ and $(P - \lambda W)^{+}$, respectively. If*

$$\det[A + B\Phi(b, a, \lambda)] \neq 0 \neq \det[C + D\Psi(b, a, \bar{\lambda})]. \quad (3.23)$$

Then the Green matrices $K(t, s, \lambda) = (K_{ij}(t, s, \lambda))$ of problem (3.12) and $L(t, s, \bar{\lambda}) = (L_{ij}(t, s, \bar{\lambda}))$ of problem (3.20) exist by theorem 2.7 and the following three statements are equivalent:

(i)
$$K(t, s, \lambda) = -E^{-1}L^{*}(s, t, \bar{\lambda})E, \quad a \leqslant s, t \leqslant b; \quad (3.24)$$

(ii)
$$K_{1,n}(t, s, \lambda) = (-1)^{n}\bar{L}_{1,n}(s, t, \bar{\lambda}), \quad a \leqslant s, t \leqslant b; \quad (3.25)$$

(iii)
$$AEC^{*} = BED^{*}. \quad (3.26)$$

Proof. From (2.18) and the non-singularity of the primary fundamental matrix Φ it follows that (3.24) holds if and only if $\Gamma = 0$. By theorem 2.11, (i) and (iii) are equivalent. Clearly, (i) implies (ii). To show that (ii) implies (i) we show that $\Gamma = 0$. This follows from (2.18) and the linear independence of $\phi_{1j}(t, a, \lambda)$, $j = 1, \dots, n$, as functions of t and the linear independence of $\phi_{jn}(a, s, \lambda)$, $j = 1, \dots, n$, as functions of s. Fix s and let $C(s) = \Gamma\Phi(a, s, \lambda)$. By (2.18) we have

$$\sum_{j=1}^{n} \phi_{1j}(t, a, \lambda)C_{jn}(s) = 0, \quad a \leqslant t \leqslant b.$$

Hence, $C_{jn}(s) = 0$, $j = 1, \dots, n$, by the linear independence of $\phi_{1j}(t, a, \lambda)$, $j = 1, \dots, n$. Thus, we have

$$C_{jn}(s) = \sum_{k=1}^{n} \Gamma_{jk}\Phi_{kn}(a, s, \lambda) = 0, \quad a \leqslant s \leqslant b,$$

and from the linear independence of $\Phi_{kn}(a,s,\bar{\lambda})$, $k=1,\dots,n$, as functions of s (see lemma 2.6) we conclude that $\Gamma_{jk}=0$, for $k=1,\dots,n$. Since this holds for each j, we may conclude that $\Gamma_{jk}=0$, for $j,k=1,\dots,n$, and this completes the proof of theorem 3.3. Special cases of this theorem were proved in [2, 9, 10]. □

THEOREM 3.4. *Let $A,B,C,D\in M_n(\mathbb{C})$. Assume that $P\in Z_n(J)$ satisfies (3.7) and W is given by (3.8) with $w\in L(J)$ and real valued. Suppose $P=P^+$, $W=W^+$, $A=C$, $B=D$ and $\lambda\in\mathbb{R}$. If $\det[A+B\Phi(b,a,\lambda)]\neq 0$, then there exist t and s, $a\leqslant t,s\leqslant b$, such that*

$$K_{1,n}(t,s,\lambda)\neq(-1)^{n+1}\bar{K}_{1,n}(s,t,\lambda). \tag{3.27}$$

Proof. Note that $(P-\lambda W)^+=P-\lambda W$ and thus (3.27) follows from theorem 2.12. □

REMARK 3.5. Together, theorems 3.3 and 3.4 say that for scalar problems when $(P-\lambda W)^+=P-\lambda W$, $\lambda\in\mathbb{R}$, the Green function cannot be symmetric when n is odd and it cannot be antisymmetric when n is even.

4. Regularization of singular problems

In this section we show that singular scalar equations with limit-circle end points can be 'regularized' in the sense that they can be transformed to regular problems. The end points may be finite or infinite and no oscillatory restrictions on the coefficients or solutions are assumed. Here the components of P and w are in $L_{\mathrm{loc}}(J)$ but not necessarily in $L(J)$. This transformation transforms the dependent variable and leaves the independent variable and the domain interval unchanged.

Let $P\in Z_n(J)$, let W be given by (3.8) with $w\in L_{\mathrm{loc}}(J)$ and let $\lambda\in\mathbb{C}$. Then $P^+\in Z_n(J)$. Let $M=M_P$, $M^+=M_{P+}$ and consider the scalar equations

$$My=\lambda wy \quad \text{on } J, \tag{4.1}$$

$$M^+z=(-1)^n\bar{\lambda}\bar{w}z \quad \text{on } J, \tag{4.2}$$

and their system formulations

$$Y'=(P-\lambda W)Y \qquad \text{on } J, \tag{4.3}$$

$$Z'=(P-\lambda W)^+Z \quad \text{on } J. \tag{4.4}$$

Let Φ, Ψ be the primary fundamental matrices of $(P-\lambda W)$ and $(P-\lambda W)^+$, respectively. Fix $c\in J$ and choose $r\in\mathbb{R}$ (this r can be chosen arbitrarily but, once chosen, it remains fixed). Then

$$\left.\begin{aligned} &U=\Phi(\cdot,c,r), && V=\Psi(\cdot,c,r)=(v_{ij}),\\ &U=(u_{ij})=(u_i^{[j-1]})^{\mathrm{T}}, && U^{-1}=(U_{ij}).\end{aligned}\right\} \tag{4.5}$$

THEOREM 4.1. *Assume that $\operatorname{tr}(P)=0$. Suppose that for $\lambda=r$ all solutions of (4.1) and (4.2) are in $L^2(J,|w|)$. For any $\lambda\in\mathbb{C}$ and any vector solution $Y=Y(\cdot,\lambda)$ of the system (4.3) let $X=X(\cdot,\lambda)$ be defined by*

$$X(t)=U^{-1}(t)Y(t), \quad t\in J. \tag{4.6}$$

Then

(i)

$$X' = (r - \lambda)wQX \quad \text{on } J, \tag{4.7}$$

where

$$Q = \begin{pmatrix} u_1U_{1n} & u_2U_{1n} & \cdots & u_nU_{1n} \\ u_1U_{2n} & u_2U_{2n} & \cdots & u_nU_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_1U_{nn} & u_2U_{nn} & \cdots & u_nU_{nn} \end{pmatrix}, \tag{4.8}$$

(ii) $wQ \in L^1(J)$,

(iii) *both limits*

$$X(a) = \lim_{t \to a^+} X(t), \qquad X(b) = \lim_{t \to b^-} X(t) \tag{4.9}$$

exist and are finite.

Proof.

$$\begin{aligned} X' &= -U^{-1}U'U^{-1}Y + U^{-1}Y' \\ &= U^{-1}[-(P - rW)UU^{-1}Y + (P - \lambda W)Y] \\ &= (r - \lambda)(U^{-1}WU)X \quad \text{on } J. \end{aligned}$$

Now observe that

$$U^{-1}WU = wQ \tag{4.10}$$

and the proof of (i) is complete. For the proof of (ii) the critical observation is to note that $\bar{U}_{jn}$ is a solution of the adjoint equation (4.2) for each $j = 1, 2, \dots, n$. Using the adjointness lemma, the fact that $\Phi(c, t) = \Phi^{-1}(t, c)$, which follows from the representation $\Phi(c, t) = Y(c)Y^{-1}(t)$ for any fundamental matrix Y of (4.3), and the hypothesis that $\operatorname{tr}(P) = 0$, which implies that $\det \Phi(c, t) = 1$ for all $t \in J$, we have

$$\Psi(t, c) = E^{-1}\Phi^*(c, t)E = E^{-1}\Phi^{-1*}(t, c)E = E^{-1}(U^{-1})^*E. \tag{4.11}$$

Therefore,

$$\Psi_{1,n+1-j}(t, c) = \pm\bar{U}_{j,n}, \quad j = 1, 2, \dots, n. \tag{4.12}$$

From the hypothesis that all solutions of (4.1) and (4.2) are in $L^2(J, |w|)$ and the Schwarz inequality we get that

$$\left(\int_J |wu_jU_{kn}|\right)^2 \leqslant \int_J |w||u_j|^2 \int_J |w||U_{kn}|^2 < \infty, \quad j, k = 1, 2, \dots, n. \tag{4.13}$$

This completes the proof of (ii). Part (iii) follows from (ii) (see [11]) and the proof of the theorem is complete. □

Corollary 4.2. *Let the hypotheses and notation of theorem 4.1 hold. Then all solutions of (4.1) and (4.2) are in $L^2(J, |w|)$ for every $\lambda \in \mathbb{C}$.*

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Proof. By (4.6), $Y(t,\lambda) = U(t)X(t,\lambda)$. By hypothesis, $u_{1,j} \in L^2(J,|w|)$ for $j = 1,\dots,n$. By theorem 4.1(iii), each component of $X(t,\lambda)$ has a finite limit at each end point and is therefore bounded in a neighbourhood of each end point and therefore the conclusion follows for equation (4.1). Since $(P^+)^+ = P$ this argument is symmetric with respect to P and P^+ and the conclusion follows also for equation (4.2). □

5. Green's functions of singular boundary-value problems

In this section we construct Green's functions for singular boundary-value problems on $J = (a,b)$ for the case when each end point is either a regular or singular limit-circle. The end points may be finite or infinite and no oscillatory restrictions on the coefficients or solutions are assumed. Here the components of P and w are in $L_{\text{loc}}(J)$ but not necessarily in $L(J)$.

THEOREM 5.1. *Let $w \in L_{\text{loc}}(J)$. Suppose $P \in Z_n(J)$, $\operatorname{tr}(P) = 0$. Let $M = M_P$ and $M^+ = M_{P^+}$ be the scalar nth-order differential expressions generated by P and P^+, respectively. Let U, V be determined by (4.5), X be determined by (4.6) and let Q be given by (4.8). Let $A,B,C,D \in M_n(\mathbb{C})$. We consider the following boundary-value problems:*

$$M_P y = \lambda w y \quad \text{on } J, \qquad \lambda \in \mathbb{C}, \quad AX(a) + BX(b) = 0, \quad \text{where } X = U^{-1}Y; \tag{5.1}$$

$$M_{P^+} z = (-1)^n \bar{\lambda} \bar{w} z \quad \text{on } J, \quad \bar{\lambda} \in \mathbb{C}, \quad C\Xi(a) + D\Xi(b) = 0, \quad \text{where } \Xi = V^{-1}Z, \tag{5.2}$$

and

$$Y' = (P - \lambda W)Y + F \qquad \text{on } J, \qquad AX(a) + BX(b) = 0, \tag{5.3}$$

$$Z' = (P^+ - (-1)^n \bar{\lambda}\bar{W})Z + G \quad \text{on } J, \qquad C\Xi(a) + D\Xi(b) = 0, \tag{5.4}$$

$$X' = (r - \lambda)wQX + U^{-1}F \qquad \text{on } J, \qquad AX(a) + BX(b) = 0, \tag{5.5}$$

$$\Xi' = (r - \bar{\lambda})\bar{w}Q^+\Xi + V^{-1}G \qquad \text{on } J, \qquad A\Xi(a) + B\Xi(b) = 0. \tag{5.6}$$

Note that if $Y' = (P - \lambda W)Y + F$ and $X = U^{-1}Y$, then from (5.3) we have

$$\begin{aligned} U'X + UX' &= (P - \lambda W)UX + F, \\ UX' &= [(P - \lambda W)U - U']X + F, \\ X' &= U^{-1}[(P - \lambda W)U - U']X + U^{-1}F = (r - \lambda)wQX + U^{-1}F. \end{aligned}$$

Similarly, (5.6) follows from (5.4) and the transformation $\Xi = V^{-1}Z$.

Assume that $r - \lambda$ is not an eigenvalue of the BVP (5.5) and $r - \bar{\lambda}$ is not an eigenvalue of the BVP (5.6). Then the Green matrices of the regular systems (5.5), (5.6)

$$K(t,s,r-\lambda,Q,W), \qquad K(t,s,r-\bar{\lambda},Q^+,\bar{W})$$

exist by theorem 2.7. Define

$$G(t,s,\lambda,P,W,A,B) = U(t)K(t,s,r-\lambda,Q,W,A,B)U^{-1}(s), \qquad s,t \in J,$$

$$G(t,s,\bar{\lambda},P^+,\bar{W},C,D) = V(t)K(t,s,r-\bar{\lambda},Q^+,\bar{W},C,D)V^{-1}(s), \quad s,t \in J,$$

where Q^+ is defined as Q but with P replaced by P^+. Then for each $U^{-1}F \in L(J)$ the regular boundary-value problem (5.5) has a unique solution X given by

$$X(t) = \int_a^b K(t, s, r - \lambda, Q, W)U^{-1}(s)F(s)\,\mathrm{d}s, \quad a < t < b,$$

and hence

$$U^{-1}(t)Y(t) = \int_a^b U^{-1}(t)G(t, s, \lambda, P, W, A, B)U(s)U^{-1}(s)F(s)\,\mathrm{d}s, \quad a < t < b.$$

Therefore,

$$Y(t) = \int_a^b G(t, s, \lambda, P, W, A, B)F(s)\,\mathrm{d}s, \quad a < t < b.$$

Similar solutions arise for (5.4) and (5.6). Then for each $V^{-1}G \in L(J)$ the regular boundary-value problem (5.4) has a unique solution Z given by

$$Z(t) = \int_a^b G(t, s, \bar{\lambda}, P^+, \bar{W}, C, D)G(s)\,\mathrm{d}s, \quad a < t < b.$$

Given the above, we prove that the following statements are equivalent:

(i) $AEC^* = BED^*$;

(ii) $K(t, s, Q, r - \lambda, A, B) = E^{-1}K^*(s, t, Q^+, r - \bar{\lambda}, C, D)E,\ s, t \in J$;

(iii) $G(t, s, \lambda, P, W, A, B) = -E^{-1}G^*(s, t, \bar{\lambda}, P^+, \bar{W}, C, D)E,\ s, t \in J$;

(iv) $K_{1n}(t, s, \lambda, P, W, A, B) = (-1)^n \bar{K}_{1n}(s, t, \bar{\lambda}, P^+, \bar{W}, C, D),\ s, t \in J$.

In particular, each of (i)–(iv) implies that

$$G_{1n}(t, s, \lambda, P, W, A, B) = (-1)^n \bar{G}_{1n}(s, t, \bar{\lambda}, P^+, \bar{W}, C, D), \quad s, t \in J.$$

Proof. Parts (i) and (ii) are equivalent by theorem 2.11. To show that (iii) is equivalent to (ii) we proceed as follows:

$$\begin{aligned}
-\,&E^{-1}G^*(s, t, \bar{\lambda}, P^+, \bar{W}, C, D)E \\
&= -E^{-1}[V(s)K(s, t, r - \bar{\lambda}, Q^+, \bar{W}, C, D)V^{-1}(t)]^*E \\
&= -E^{-1}V^{-1*}(t)EE^{-1}K^*(s, t, r - \bar{\lambda}, Q^+, \bar{W}, C, D)EE^{-1}V^*(s)E \\
&= U(t)K(t, s, r - \lambda, Q, W, A, B)U^{-1}(s) \\
&= G(t, s, \lambda, P, W, A, B).
\end{aligned}$$

In the last step we used the identities:

$$U(t) = E^{-1}V^{-1*}(t)E, \qquad V(s) = E^{-1}U^{-1*}(s)E.$$

These can be established by showing that both sides satisfy the same initial-value problem. The equivalence of (iii) and (iv) is established similarly to the corresponding result of theorem 3.3. □

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6. Construction of adjoint and self-adjoint boundary conditions

In this section we comment on the adjointness and self-adjointness conditions of theorem 3.3:

$$AEC^* = BED^*, \tag{6.1}$$

$$AEA^* = BEB^*. \tag{6.2}$$

and discuss a construction for these conditions for the cases $n = 2$ and $n = 4$. This construction is based on the method used in [1] but is more explicit because our Lagrange bracket is much simpler than the classical one used in [1]. In particular, it does not depend on the coefficients of the equation.

But first we review the Lagrange identity, which is fundamental to the study of boundary-value problems. Let $P \in Z_n(J)$, $M = M_P$, let $Q = P^+$, $M^+ = M_Q$ (this Q is not related to the Q used in § 4). For $y \in D(P)$, $z \in D(Q)$, define the Lagrange bracket $[\cdot,\cdot]$ by

$$[y,z] = (-1)^k \sum_{r=0}^{n-1} (-1)^{n+1-r} \bar{z}^{[n-r-1]} y^{[r]}. \tag{6.3}$$

Here we have omitted the subscript P on the quasi-derivatives of y and the subscript Q on the quasi-derivatives of z.

Lemma 6.1 (Lagrange identity). *For any $y \in D(P)$ and $z \in D(Q)$ we have*

$$\bar{z}My - y\overline{(M^+z)} = [y,z]'.$$

Proof. This is a special case of [6, lemma 3.3]. □

Assume that $P = (p_{ij}) \in Z_n(J)$ and p_{ij}, w satisfy

$$w, p_{ij} \in L(J), \quad 1 \leqslant i \leqslant j, \quad j = 1,\dots,n; \qquad p_{j,j+1}^{-1} \in L(J), \quad j = 1,\dots,n-1. \tag{6.4}$$

Then equations (4.1) and (4.2) are regular and therefore $y^{[r]}$ and $z^{[r]}$ are well defined at both end points a and b as finite limits [11].

Lemma 6.2. *Assume that (6.4) holds. For any $y \in D(P)$ and $z \in D(Q)$ we have*

$$\int_a^b \{\bar{z}My - y\overline{(M^+z)}\} = [y,z](b) - [y,z](a). \tag{6.5}$$

Proof. This follows from lemma 6.1 by integration. □

Remark 6.3. We comment on the difference between this Lagrange identity and the classical one as found, for example, in the well-known books by Coddington and Levinson [1] and Dunford and Schwartz [3]: the fundamental differences are that

(i) the matrix E is a simple constant matrix, whereas in the classical case it is a complicated non-constant function depending on the coefficients,

(ii) we assume only that the coefficients are locally Lebesgue integrable in contrast to [1, 3], where strong smoothness conditions are required.

The price we pay for this generalization and simplification is the use of 'messy' quasi-derivatives $y^{[r]}$, which depend on the coefficients, in place of the classical derivatives $y^{(r)}$.

Definition 6.4. Given matrices $A, B \in M_n(\mathbb{C})$ satisfying $\operatorname{rank}(A, B) = n$ and the boundary condition

$$AY(a) + BY(b) = 0 \tag{6.6}$$

and matrices $C, D \in M_n(\mathbb{C})$ satisfying $\operatorname{rank}(C, D) = n$ and boundary condition

$$CZ(a) + DZ(b) = 0, \tag{6.7}$$

we say that the boundary condition (6.7) is adjoint to (6.6) if $[y, z](b) - [y, z](a) = 0$ for all $y \in D(P)$ and $z \in D(Q)$. Note that (6.7) is adjoint to (6.6) if and only if (6.6) is adjoint to (6.7).

Example 6.5. Let $n = 2$. Consider the Sturm–Liouville equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J = (a, b), \quad -\infty \leqslant a < b \leqslant \infty,$$

and its adjoint equation

$$M^+z = -(\bar{p}z')' + \bar{q}z = \bar{\lambda}\bar{w}z \quad \text{on } J$$

with

$$\frac{1}{p}, q, w \in L^1(J, \mathbb{C}), \quad \lambda \in \mathbb{C}.$$

Here

$$P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix}, \qquad P^+ = \begin{bmatrix} 0 & 1/\bar{p} \\ \bar{q} & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and the Lagrange identity is

$$\bar{z}My - \overline{yM^+z} = [y, z]', \quad \text{where } [y, z] = y(p\bar{z}') - \bar{z}(py')$$

for all $y \in D(M)$, $z \in D(M^+)$.

The next lemma yields a construction for adjoint and, as we will see below, also self-adjoint boundary conditions.

Lemma 6.6. *Let $A, B \in M_2(\mathbb{C})$, the set of 2×2 matrices over the complex numbers, with*

$$\operatorname{rank}(A, B) = 2$$

and let

$$Y = \begin{bmatrix} y \\ py' \end{bmatrix}.$$

Choose any matrices α, β such that the block matrix

$$\begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix}$$

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is non-singular; then choose 2×2 *matrices* F, G, H, K *such that*

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix} \begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} -E & 0 \\ 0 & E \end{bmatrix}, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6.8}$$

Then the boundary conditions

$$G^*Z(a) + K^*Z(b) = 0, \quad \textit{with } Z = \begin{bmatrix} z \\ \bar{p}z' \end{bmatrix},$$

are adjoint to the conditions

$$AY(a) + BY(b) = 0.$$

Proof. Let $y \in D(M)$, $z \in D(M^+)$. Then

$$\bar{z}My - y\overline{M^+z} = \bar{z}[-(py')' + qy] - y\overline{[-(\bar{p}z')' + \bar{q}z]} = [yp\bar{z}' - \bar{z}py']'.$$

Note that

$$yp\bar{z}' - \bar{z}py' = [\bar{z}, p\bar{z}'] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ py' \end{bmatrix}.$$

Hence,

$$\begin{aligned} \int_a^b \{\bar{z}My - y\overline{M^+z}\} &= [yp\bar{z}' - \bar{z}py'](b) - [yp\bar{z}' - \bar{z}py'](a) \\ &= Z^*(b)EY(b) - Z^*(a)EY(a) \\ &= [Z^*(a), Z^*(b)] \begin{bmatrix} -E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix} \\ &= [Z^*(a), Z^*(b)] \begin{bmatrix} F & G \\ H & K \end{bmatrix} \begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix} \\ &= [Z^*(a)F + Z^*(b)H][AY(a) + BY(b)] \\ &\qquad + [Z^*(a)G + Z^*(b)K][\alpha Y(a) + \beta Y(b)], \end{aligned}$$

and $Z^*(a)G + Z^*(b)K = 0$ if and only if $G^*Z(a) + K^*Z(b) = 0$. This completes the proof. □

Illustration. Let $B = -I$, $\alpha = -I$ and $\beta = 0$. Then the following equations hold:

(i) $FA + G\alpha = -E$;

(ii) $FB + G\beta = 0$;

(iii) $HA + K\alpha = 0$;

(iv) $HB + K\beta = E$.

Hence,

$$C = G^* = E^* = -E, \qquad D = K^* = (-EA)^* = -A^*E^* = A^*E.$$

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Checking the 'Green function identities' condition we have

$$AEC^* = AE(-E)^* = AEE = -A,$$
$$BED^* = BE(A^*E)^* = BEE^*A = -IE(-E)A = -A.$$

Thus, we have constructed adjoint boundary conditions. Next we show that this construction produces all the self-adjoint conditions

$$\frac{1}{p}, q, w \in L^1(J, \mathbb{R}), \quad \lambda \in \mathbb{C}.$$

Case 1 (all real coupled self-adjoint BCs). Let $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$, $\det A = 1$; $B = -I$, $\alpha = 0$, $\beta = -E$. From

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix} \begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} -E & 0 \\ 0 & E \end{bmatrix},$$

we have

$$F = -EA^{-1}, \qquad G = -EA^{-1}E, \qquad H = 0, \qquad K = -I.$$

Hence,

$$AEC^* = AEG = AE(-EA^{-1}E) = E,$$
$$BED^* = BEK = -IEK = -E(-I) = E.$$

Note that

$$G = -EA^{-1}E = A^*, \qquad K = B^*,$$

so

$$AEA^* = BEB^*.$$

Case 2 (all complex coupled self-adjoint BCs). Set $A = \mathrm{e}^{\mathrm{i}\gamma}T$, where T satisfies $T = (t_{ij})$, $t_{ij} \in \mathbb{R}$, $\det T = 1$ and $-\pi < \gamma < 0$ or $0 < \gamma < \pi$; $B = -I$; $\alpha = 0$; $\beta = -E$. By

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix} \begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} -E & 0 \\ 0 & E \end{bmatrix},$$

we have $F = -EA^{-1}$, $G = -EA^{-1}E$, $H = 0$, $K = -I$. Hence,

$$AEC^* = AEG = AE(-EA^{-1}E) = E,$$
$$BED^* = BEK = -IE(-I) = E.$$

So $AEC^* = BED^*$. Note that

$$G = -EA^{-1}E = \mathrm{e}^{-\mathrm{i}\gamma}T^* = A^*, \qquad K = B^*,$$

Hence,

$$AEA^* = BEB^*.$$

Case 3 (separated self-adjoint BCs). Set

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad a_1, a_2 \in \mathbb{R}, \quad a_1 \neq 0;$$

$$B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad b_1, b_2 \in \mathbb{R}, \quad b_1 \neq 0;$$

$$\alpha = \begin{pmatrix} -a_1 & \dfrac{1}{a_1} - a_2 \\ 0 & 0 \end{pmatrix};$$

$$\beta = \begin{pmatrix} 0 & 0 \\ -b_1 & -\dfrac{1}{b_1} - b_2 \end{pmatrix}.$$

From (6.8), we obtain that

$$F = \begin{pmatrix} a_1 & 0 \\ a_2 - \dfrac{1}{a_1} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & b_1 \\ 0 & \dfrac{1}{b_1} + b_2 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}.$$

By a computation we then obtain

$$AEC^* = AEG = 0, \qquad BED^* = BEK = 0.$$

Note that $G = A^*$, $K = B^*$, so

$$AEA^* = BEB^*.$$

The other cases, $a_1 = 0$, $a_2 \neq 0$, $b_1 = 0$, $b_2 \neq 0$, are similar and hence omitted. The three cases combined show that the construction of this lemma generates all self-adjoint boundary conditions [11].

Example 6.7. Let $n = 4$. Consider the equation

$$My = [(p_2 y'')' + p_1 y']' + qy = \lambda w y \quad \text{on } J = (a, b), \quad -\infty \leqslant a < b \leqslant \infty,$$

and its adjoint equation

$$M^+ z = [(\bar{p}_2 z'')' + \bar{p}_1 z']' + \bar{q} z = \bar{\lambda} \bar{w} z \quad \text{on } J,$$

with

$$\frac{1}{p_2}, p_1, q, w \in L^1(J, \mathbb{C}), \quad \lambda \in \mathbb{C}.$$

Lemma 6.8. *Let $A, B \in M_4(\mathbb{C})$, the set of 4×4 matrices over the complex numbers, with*

$$\operatorname{rank}(A, B) = 4$$

and let

$$Y = \begin{bmatrix} y \\ y' \\ p_2 y'' \\ (p_2 y'')' + p_1 y' \end{bmatrix}, \qquad Z = \begin{bmatrix} z \\ z' \\ \bar{p}_2 z'' \\ (\bar{p}_2 z'')' + \bar{p}_1 z' \end{bmatrix}.$$

Choose any 4×4 matrices α, β such that the block matrix

$$\begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix}$$

is non-singular; then choose 4×4 matrices F, G, H, K such that

$$\begin{bmatrix} F & G \\ H & K \end{bmatrix}\begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} E_4 & 0 \\ 0 & -E_4 \end{bmatrix}, \qquad E_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6.9}$$

Then the boundary conditions

$$G^*Z(a) + K^*Z(b) = 0,$$

are adjoint to the conditions

$$AY(a) + BY(b) = 0.$$

Proof. Let $y \in D(M)$, $z \in D(M^+)$. Then

$$\begin{aligned} \bar{z}My - y\overline{M^+z} &= \bar{z}\{[(p_2y'')' + p_1y']' + qy\} - y\overline{\{[(\bar{p}_2z'')' + \bar{p}_1z']' + \bar{q}z\}} \\ &= \{\bar{z}[(p_2y'')' + p_1y'] - y[(p_2\bar{z}'')' + p_1\bar{z}'] - (p_2y'')\bar{z}' + p_2\bar{z}''y'\}'. \end{aligned}$$

However,

$$\begin{aligned} &\bar{z}[(p_2y'')' + p_1y'] - y[(p_2\bar{z}'')' + p_1\bar{z}'] - (p_2y'')\bar{z}' + p_2\bar{z}''y' \\ &\qquad = (\bar{z}\bar{z}'p_2\bar{z}''(p_2\bar{z}'')' + p_1\bar{z}')\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} y \\ y' \\ p_2y'' \\ (p_2y'')' + p_1y' \end{pmatrix} \\ &\qquad = -Z^*E_4Y. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_a^b \{\bar{z}My - y\overline{M^+z}\} \\ &\quad = \{\bar{z}[(p_2y'')' + p_1y'] - y[(p_2\bar{z}'')' + p_1\bar{z}'] - (p_2y'')\bar{z}' + p_2\bar{z}''y'\}(b) \\ &\qquad - \{\bar{z}[(p_2y'')' + p_1y'] - y[(p_2\bar{z}'')' + p_1\bar{z}'] - (p_2y'')\bar{z}' + p_2\bar{z}''y'\}(a) \\ &\quad = Z^*(a)E_4Y(a) - Z^*(b)E_4Y(b) \\ &\quad = [Z^*(a), Z^*(b)]\begin{bmatrix} E_4 & 0 \\ 0 & -E_4 \end{bmatrix}\begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix} \\ &\quad = [Z^*(a), Z^*(b)]\begin{bmatrix} F & G \\ H & K \end{bmatrix}\begin{bmatrix} A & B \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix} \\ &\quad = [Z^*(a)F + Z^*(b)H][AY(a) + BY(b)] \\ &\qquad + [Z^*(a)G + Z^*(b)K][\alpha Y(a) + \beta Y(b)], \end{aligned}$$

and $Z^*(a)G + Z^*(b)K = 0$ if and only if $G^*Z(a) + K^*Z(b) = 0$. □

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Illustration. By (6.9) we have

$$\left.\begin{aligned} FA + G\alpha &= E_4, & FB + G\beta &= 0,\\ HA + K\alpha &= 0, & HB + K\beta &= -E_4. \end{aligned}\right\} \tag{6.10}$$

CASE 1. Set $A = A$, $B = -I$, $\alpha = -E_4$, $\beta = 0$. Then, by (6.10), we have

$$G = -I, \qquad F = 0, \qquad K = -E_4AE_4, \qquad H = E_4.$$

Hence,

$$\begin{aligned} AE_4C^* &= AE_4G = AE_4(-I) = -AE_4,\\ BE_4D^* &= BE_4K = -IE_4(-E_4AE_4) = -AE_4. \end{aligned}$$

So

$$AE_4C^* = BE_4D^*.$$

In the following, we let

$$\frac{1}{p_2}, p_1, q, w \in L^1(J, \mathbb{R}).$$

CASE 2 (self-adjoint BCs). Set

$$A = \begin{pmatrix} I_2 & \gamma \\ 0 & I_2 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix},$$

where γ_{ij} satisfy $\gamma_{11} = -\bar{\gamma}_{22}$ and γ_{12}, γ_{21} are real numbers. Note that $\gamma = E_2\gamma^*E_2$, and set $B = I_4$, $\alpha = 0_4$, $\beta = -E_4$. By (6.10) we have

$$F = E_4A^{-1}, \qquad G = -E_4A^{-1}E_4, \qquad H = 0, \qquad K = I.$$

In terms of $\gamma = E_2\gamma^*E_2$, we can easily obtain that $AE_4A^* = E_4$.

Note that

$$G = -E_4A^{-1}E_4 = -E_4A^{-1}AE_4A^* = A^*, \qquad C = G^* = A, \qquad D = K^* = B.$$

So

$$\begin{aligned} AE_4C^* &= AE_4A^* = E_4,\\ BE_4D^* &= BE_4B^* = IE_4I = E_4. \end{aligned}$$

Therefore,

$$AE_4A^* = BE_4B^*.$$

CASE 3 (separated self-adjoint BCs). Set

$$A = \begin{pmatrix} I & A_1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ B_1 & I \end{pmatrix},$$

where I is the 2×2 unit matrix, 0 is the 2×2 zero matrix and

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

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satisfy

$$A_1 = E_2 A_1^* E_2, \qquad B_1 = E_2 B_1^* E_2,$$

i.e. a_2, a_3, b_2, b_3 are real numbers and $a_1 = -\bar{a}_4$, $b_1 = -\bar{b}_4$. In addition, set

$$\alpha = \begin{pmatrix} 0 & E_2 \\ E_2 & E_2 A_1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} E_2 B_1 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

Then, by (6.10), we have

$$C = G^* = A, \qquad D = K^* = B.$$

So

$$AE_4C^* = AE_4A^* = 0,$$
$$BE_4D^* = BE_4B^* = 0.$$

Hence,

$$AE_4A^* = BE_4B^*.$$

7. The Legendre Green function

As an illustration of some of the above results we construct the singular Legendre Green function in this section. This seems to be new even though the Legendre equation

$$-(py')' = \lambda y, \quad p(t) = 1 - t^2 \text{ on } J = (-1, 1), \tag{7.1}$$

is one of the simplest singular differential equations and there is a voluminous literature associated with it in pure and applied mathematics. Its potential function q is zero, its weight function w is the constant 1 and its leading coefficient p is a simple quadratic. It is singular at both end points -1 and $+1$. The singularities are due to the fact that $1/p$ is not Lebesgue integrable in left and right neighbourhoods of these points. In spite of its simple appearance, (7.1) and its associated self-adjoint operators exhibit a surprisingly wide variety of interesting phenomena.

The above construction of singular Green's functions is a five-step procedure.

1. Formulate the singular second-order scalar equation (7.1) as a first-order singular system.

2. 'Regularize' this singular system by constructing regular systems which are equivalent to it.

3. Construct the Green matrix for boundary-value problems of the regular system.

4. Construct the singular Green matrix for the equivalent singular system from the regular one.

5. Extract the upper right corner element from the singular Green matrix. This is the Green function for singular scalar boundary-value problems for equation (7.1).

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For $\lambda = 0$, two linearly independent solutions of (7.1) are given by

$$u(t) = 1, \qquad v(t) = -\tfrac{1}{2}\ln\left(\left|\frac{1-t}{t+1}\right|\right). \tag{7.2}$$

The standard system formulation of (7.1) has the form

$$Y' = (P - \lambda W)Y \quad \text{on } (-1, 1), \tag{7.3}$$

where

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1/p \\ 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.4}$$

Let u and v be given by (7.2) and let

$$U = \begin{pmatrix} u & v \\ pu' & pv' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}. \tag{7.5}$$

Note that $\det U(t) = 1$ for $t \in J = (-1, 1)$, and set

$$Z = U^{-1}Y. \tag{7.6}$$

Then

$$\begin{aligned} Z' &= (U^{-1})'Y + U^{-1}Y' \\ &= -U^{-1}U'U^{-1}Y + (U^{-1})(P - \lambda W)Y = -U^{-1}U'Z + (U^{-1})(P - \lambda W)UZ \\ &= -U^{-1}(PU)Z + U^{-1}(PU)Z - \lambda(U^{-1}WU)Z \\ &= -\lambda(U^{-1}WU)Z. \end{aligned}$$

Letting $G = (U^{-1}WU)$, we may conclude that

$$Z' = -\lambda GZ, \tag{7.7}$$

where

$$G = U^{-1}WU = \begin{pmatrix} -v & -v^2 \\ 1 & v \end{pmatrix}. \tag{7.8}$$

Definition 7.1. We call (7.7) a 'regularized' Legendre system.

The next theorem justifies this definition and gives the relationship between this 'regularized' system and equation (7.1).

Theorem 7.2. *Let $\lambda \in \mathbb{C}$ and let G be given by (7.8).*

(i) *Every component of G is in $L^1(-1, 1)$ and therefore (7.7) is a regular system.*

(ii) *For any $c_1, c_2 \in \mathbb{C}$ the initial-value problem*

$$Z' = -\lambda GZ, \qquad Z(-1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{7.9}$$

has a unique solution Z defined on the closed interval $[-1, 1]$.

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(iii) *If*

$$Y = \begin{pmatrix} y(t,\lambda) \\ (py')(t,\lambda) \end{pmatrix}$$

is a solution of (7.3) and

$$Z = U^{-1}Y = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix},$$

then Z is a solution of (7.7), and for all $t \in (-1,1)$ we have

$$y(t,\lambda) = uz_1(t,\lambda) + v(t)z_2(t,\lambda) = z_1(t,\lambda) + v(t)z_2(t,\lambda), \tag{7.10}$$

$$(py')(t,\lambda) = (pu')z_1(t,\lambda) + (pv')(t)z_2(t,\lambda) = z_2(t,\lambda). \tag{7.11}$$

(iv) *For every solution $y(t,\lambda)$ of the singular scalar Legendre equation (7.1) the quasi-derivative $(py')(t,\lambda)$ is continuous on the compact interval $[-1,1]$. More specifically, we have*

$$\lim_{t\to -1^+}(py')(t,\lambda) = z_2(-1,\lambda), \qquad \lim_{t\to 1^-}(py')(t,\lambda) = z_2(1,\lambda). \tag{7.12}$$

Thus, the quasi-derivative is a continuous function on the closed interval $[-1,1]$ for every $\lambda \in \mathbb{C}$.

(v) *Let $y(t,\lambda)$ be given by (7.10). If $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is unbounded at 1. If $z_2(-1,\lambda) \neq 0$, then $y(t,\lambda)$ is unbounded at -1.*

(vi) *Fix $t \in [-1,1]$ and let $c_1, c_2 \in \mathbb{C}$. If*

$$Z = \begin{pmatrix} z_1(t,\lambda) \\ z_2(t,\lambda) \end{pmatrix}$$

is the solution of (7.7) determined by the initial conditions $z_1(-1,\lambda) = c_1$, $z_2(-1,\lambda) = c_2$, then $z_i(t,\lambda)$ is an entire function of λ, $i = 1,2$. A similar solution can be obtained for the initial conditions $z_1(1,\lambda) = c_1$, $z_2(1,\lambda) = c_2$.

(vii) *For each $\lambda \in \mathbb{C}$ there is a non-trivial solution which is bounded in a (two-sided) neighbourhood of 1, and there is a (generally different) non-trivial solution which is bounded in a (two-sided) neighbourhood of -1.*

(viii) *A non-trivial solution $y(t,\lambda)$ of the singular scalar Legendre equation (7.1) is bounded at 1 if and only if $z_2(1,\lambda) = 0$. A non-trivial solution $y(t,\lambda)$ of the singular scalar Legendre equation (7.1) is bounded at -1 if and only if $z_2(-1,\lambda) = 0$.*

Proof. Part (i) follows from (7.8), part (ii) is a direct consequence of (i) and the theory of regular systems, $Y = UZ$ implies (iii) $\Longrightarrow$ (iv) and (v); part (vi) follows from (ii) and the basic theory of regular systems. For part (vii), determine solutions $y_1(t,\lambda)$, $y_{-1}(t,\lambda)$ by applying the Frobenius method to obtain power series solutions of (7.1) in the form [5, p. 5, with different notation]:

$$y_1(t,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(\lambda)(t-1)^n, \quad |t-1| < 2; \tag{7.13}$$

$$y_{-1}(t,\lambda) = 1 + \sum_{n=1}^{\infty} b_n(\lambda)(t+1)^n, \quad |t+1| < 2. \tag{7.14}$$

To prove (viii) it follows from (7.10) that if $z_2(1,\lambda) \neq 0$, then $y(t,\lambda)$ is not bounded at 1. Suppose $z_2(1,\lambda) = 0$. If the corresponding $y(t,\lambda)$ is not bounded at 1, then there are two linearly unbounded solutions at 1 and hence all non-trivial solutions are unbounded at 1. This contradiction establishes (viii) and completes the proof of the theorem. □

Remark 7.3. From theorem 7.2 we see that, for every $\lambda \in \mathbb{C}$, (7.1) has a solution y_1 which is bounded at 1 and has a solution y_{-1} which is bounded at -1.

It is well known that for $\lambda_n = n(n+1)\colon n \in \mathbb{N}_0 = \{0,1,2,\dots\}$ the Legendre polynomials P_n are solutions on $(-1,1)$ and hence are bounded at -1 and at $+1$.

For later reference we introduce the primary fundamental matrix of the system (7.7).

Definition 7.4. Fix $\lambda \in \mathbb{C}$. Let $\Phi(\cdot,\cdot,\lambda)$ be the primary fundamental matrix of (7.7); i.e. for each $s \in [-1,1]$, $\Phi(\cdot,s,\lambda)$ is the unique matrix solution of the initial-value problem:

$$\Phi(s,s,\lambda) = I, \tag{7.15}$$

where I is the 2×2 identity matrix. Since (7.7) is regular, $\Phi(t,s,\lambda)$ is defined for all $t,s \in [-1,1]$ and, for each fixed t,s, $\Phi(t,s,\lambda)$ is an entire function of λ.

We now consider two-point boundary conditions for (7.7); later we will relate these to singular boundary conditions for (7.1).

Let $A,B \in M_2(\mathbb{C})$, the set of 2×2 complex matrices, and consider the boundary-value problem

$$Z' = -\lambda G Z, \qquad AZ(-1) + BZ(1) = 0. \tag{7.16}$$

Lemma 7.5. *A complex number $-\lambda$ is an eigenvalue of (7.16) if and only if*

$$\Delta(\lambda) = \det[A + B\Phi(1,-1,-\lambda)] = 0. \tag{7.17}$$

Furthermore, a complex number $-\lambda$ is an eigenvalue of geometric multiplicity 2 *if and only if*

$$A + B\Phi(1,-1,-\lambda) = 0. \tag{7.18}$$

Proof. Note that a solution for the initial condition $Z(-1) = C$ is given by

$$Z(t) = \Phi(t,-1,-\lambda)C, \quad t \in [-1,1]. \tag{7.19}$$

The boundary-value problem (7.16) has a non-trivial solution for Z if and only if the algebraic system

$$[A + B\Phi(1,-1,-\lambda)]Z(-1) = 0 \tag{7.20}$$

has a non-trivial solution for $Z(-1)$.

To prove the 'furthermore' part, observe that two linearly independent solutions of the algebraic system (7.20) for $Z(-1)$ yield two linearly independent solutions $Z(t)$ of the differential system and vice versa. □

Given any $\lambda \in \mathbb{R}$ and any solutions y, z of (7.1), the Lagrange form $[y,z](t)$ is defined by

$$[y,z](t) = y(t)(p\bar{z}')(t) - \bar{z}(t)(py')(t).$$

So, in particular, we have

$$[u,v](t) = +1, \quad [v,u](t) = -1, \quad [y,u](t) = -(py')(t), \quad t \in \mathbb{R},$$
$$[y,v](t) = y(t) - v(t)(py')(t), \quad t \in \mathbb{R}, \quad t \neq \pm 1.$$

We will see below that, although v blows up at ± 1, the form $[y,v](t)$ is well defined at -1 and $+1$ since the limits

$$\lim_{t \to -1} [y,v](t), \qquad \lim_{t \to +1} [y,v](t)$$

exist and are finite from both sides. This holds for any solution y of (7.1) for any $\lambda \in \mathbb{R}$. Note that, since v blows up at 1, this means that y must blow up at 1 except, possibly, when $(py')(1) = 0$.

We are now ready to construct the Green function of the singular scalar Legendre problem consisting of the equation

$$My = -(py')' = \lambda y + h \quad \text{on } J = (-1,1), \quad p(t) = 1 - t^2, \ -1 < t < 1, \tag{7.21}$$

together with two-point boundary conditions

$$A \begin{bmatrix} (-py')(-1) \\ (ypv' - v(py'))(-1) \end{bmatrix} + B \begin{bmatrix} (-py')(1) \\ (ypv' - v(py'))(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7.22}$$

where u, v are given by (7.2) and A, B are 2×2 complex matrices. This construction is based on the system regularization discussed above and we will use the notation from above. Consider the regular non-homogeneous system

$$Z' = -\lambda GZ + F, \qquad AZ(-1) + BZ(1) = 0, \tag{7.23}$$

where

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad f_j \in L^1(J, \mathbb{C}), \ j = 1, 2. \tag{7.24}$$

Theorem 7.6. *Let $-\lambda \in \mathbb{C}$ and let $\Delta(-\lambda) = [A + B\Phi(1,-1,-\lambda)]$. Then the following statements are equivalent.*

(i) *For $F = 0$ on $J = (-1,1)$, the homogeneous problem (7.23) has only the trivial solution.*

(ii) *$\Delta(-\lambda)$ is non-singular.*

(iii) *For every $F \in L^1(-1,1)$ the non-homogeneous problem (7.23) has a unique solution Z and this solution is given by*

$$Z(t,-\lambda) = \int_{-1}^{1} K(t,s,-\lambda)F(s)\,\mathrm{d}s, \quad -1 \leqslant t \leqslant 1, \tag{7.25}$$

where

$$K(t,s,-\lambda) = \begin{cases} \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda), & -1 \leqslant t < s \leqslant 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \phi(t,s-\lambda), & -1 \leqslant s < t \leqslant 1, \\ \Phi(t,-1,-\lambda)\Delta^{-1}(-\lambda)(-B)\Phi(1,s,-\lambda) + \frac{1}{2}\phi(t,s-\lambda), & -1 \leqslant s = t \leqslant 1. \end{cases} \tag{7.26}$$

Proof. See theorem 2.7. □

Definition 7.7. Let

$$L(t,s,\lambda) = U(t)K(t,s,-\lambda)U^{-1}(s), \quad -1 \leqslant t,s \leqslant 1. \tag{7.27}$$

The next theorem shows that L_{12}, the upper right component of L, is the Green function of the singular scalar Legendre problem (7.21), (7.22).

Theorem 7.8. *Assume that $[A + B\Phi(1,-1,-\lambda)]$ is non-singular. Then, for every function h satisfying*

$$h, vh \in L^1(J,\mathbb{C}), \tag{7.28}$$

the singular scalar Legendre problem (7.21), (7.22) has a unique solution $y(\cdot,\lambda)$ given by

$$y(t,\lambda) = \int_{-1}^{1} L_{12}(t,s)h(s)\,\mathrm{d}s, \quad -1 < t < 1. \tag{7.29}$$

Proof. Let

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = U^{-1}H, \quad H = \begin{pmatrix} 0 \\ -h \end{pmatrix}. \tag{7.30}$$

Then $f_j \in L^1(J_2,\mathbb{C})$, $j = 1,2$. Since $Y(t,\lambda) = U(t)Z(t,-\lambda)$ we get from (7.25)

$$\begin{aligned} Y(t,\lambda) &= U(t)Z(t,-\lambda) \\ &= U(t)\int_{-1}^{1} K(t,s,-\lambda)F(s)\,\mathrm{d}s \\ &= \int_{-1}^{1} U(t)K(t,s,-\lambda)U^{-1}(s)H(s)\,\mathrm{d}s \\ &= \int_{-1}^{1} L(t,s,\lambda)H(s)\,\mathrm{d}s, \quad -1 < t < 1, \end{aligned} \tag{7.31}$$

and therefore

$$y(t,\lambda) = -\int_{-1}^{1} L_{12}(t,s,\lambda)h(s)\,\mathrm{d}s, \quad -1 < t < 1. \tag{7.32}$$

□

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