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DYNAMICAL SYSTEMS ON SOME ELLIPTIC MODULAR SURFACES VIA OPERATORS ON LINE ARRANGEMENTS

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Abstract. This paper further studies the matroid realization space of a specific deformation of the regular n-gon with its lines of symmetry. Recently, we obtained that these particular realization spaces are birational to the elliptic modular surfaces $\Xi_1(n)$ over the modular curve $X_1(n)$. Here, we focus on the peculiar cases when $n=7,8$ in more detail. We obtain concrete quartic surfaces in $\mathbb{P}^3$ equipped with a dominant rational self-map stemming from an operator on line arrangements, which yields K3 surfaces with a dynamical system that is semi-conjugated to the plane.

§1. Introduction

A *line arrangement* $\mathcal{C}=\ell_1+\cdots+\ell_k$ is a finite union of lines ℓ_j in the projective plane $\mathbb{P}^2$. Line arrangements are ubiquitous objects studied in various fields such as topology, algebra, algebraic geometry, see, for instance, [12], [16] for two surveys. In [9], the second author described a number of operators acting on line arrangements: if $\mathfrak{n},\mathfrak{m}$ are sets of integers at least 2, the operator $Ł_{\mathfrak{m},\mathfrak{n}}$ associates to a line arrangement $\mathcal{C}$ the line arrangement $\Lambda_{\mathfrak{m},\mathfrak{n}}(\mathcal{C})$ which is the union of the lines that contain $n\in\mathfrak{n}$ points among the m-points of $\mathcal{C}$, for $m\in\mathfrak{m}$ (recall that an m-point of $\mathcal{C}$ is a point where exactly m lines of $\mathcal{C}$ meet). For example $\Lambda_{\{2\},\{3\}}(\mathcal{C})$ is the union of the lines that contain exactly three double points of $\mathcal{C}$ (that line arrangement might be empty).

A labeled line arrangement $\mathcal{C}=(\ell_1,\dots,\ell_k)$ is a line arrangement for which one fixes the order of the lines. The configuration of a labeled line arrangement $\mathcal{C}$ is described by its associated *matroid* $M=M(\mathcal{C})$. Conversely, given a matroid M (a combinatorial object), one can look at line arrangements $\mathcal{C}$ for which $M(\mathcal{C})=M$. When such a $\mathcal{C}$ exists, one says that $\mathcal{C}$ is a realization of M. Let us denote by $\mathcal{R}=\mathcal{R}(M)$ the moduli space of realizations of M: a point of $\mathcal{R}$ is the orbit under the action of the projective general linear group PGL_3 of a realization of M. The space of all realizations of M is denoted by $\mathfrak{U}=\mathfrak{U}(M)$ and there is a natural quotient map $\mathfrak{U}\to\mathcal{R}$.

In [5], we constructed a realizable matroid M_n for any $n\geq 7$ that is based on the regular n-gon. Interestingly, there exists an operator Ł among the ones we described above (for example if $n=2k+1$ is odd, then $\Lambda=\Lambda_{\{2\},\{k\}}$) which acts non-trivially on $\mathfrak{U}(M_n)$: if $\mathcal{C}$ is a (generic) realization of M_n, then $\Lambda(\mathcal{C})$ is also a realization of M_n. We obtain in that way a dominant self-rational map l on the realization space $\mathcal{R}_n=\mathcal{R}(M_n)$.

The main result of [5] establishes that the realization space $\mathcal{R}_n$ is an open dense subscheme of the *elliptic modular surface* $\Xi_1(n)$, a well-studied surface, see, for example, Shioda's paper [11]. Recall that this surface $\Xi_1(n)$ parametrizes (up to isomorphisms) triples (E,t,p) of an elliptic curve and points t,p on E such that t has order n. The modular

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curve $X_1(n)$ parametrizes (up to isomorphisms) pairs (E,t), where E,t are as above. The map $(E,t,p) \to (E,t)$ defines an elliptic fibration on $\Xi_1(n)$, with fiber over the point (E,t) isomorphic to E. For any integer m, there is a natural multiplication by m rational map of the elliptic surface $\Xi_1(n)$. We obtain in [5] that, through the identification of $\mathcal{R}_n$ as an open subscheme of $\Xi_1(n)$, the rational self map λ induced by Λ is the multiplication by -2 map acting on $\Xi_1(n)$, in particular l has degree 4.

The aim of the present paper is to study the peculiar cases when $n = 7,8$ in more detail. In particular, we give another proof that the surface $\mathcal{R}_n$ is an open dense subscheme of $\Xi_1(n)$, and the degree of λ is 4 in these cases. From now on assume $n \in \{7,8\}$; in those cases, we obtain (singular) models of $\Xi_1(n)$ as quartic surfaces in $\mathbb{P}^3$. There is a natural section $\mathcal{R}_n \to \mathfrak{U}_n = \mathfrak{U}(M_n)$ of the quotient map $\mathfrak{U}_n \to \mathcal{R}_n$, so that one may consider $\mathcal{R}_n$ as contained in $\mathfrak{U}_n$, and therefore one may consider a class as a realization of M_n. Using that fact, we are able to give explicit polynomials for the action $\lambda = \lambda(n)$ of $\Lambda = \Lambda(n)$ on $\mathcal{R}_n \subset \mathbb{P}^3$.

Recall that a dynamical system is a pair (X,λ) of a variety X and a dominant rational map $\lambda : X \to X$. A dynamical system (X,λ) is called *semi-conjugated* to a dynamical system (Y,μ) if there exists a generically finite rational dominant map $\pi : X \to Y$ such that $\pi \circ \lambda = \mu \circ \pi$. A principal result of this article is the following.

THEOREM 1.1. *For $n \in \{7,8\}$, the dynamical system $(\mathcal{R}_n,\lambda)$ is semi-conjugated to $(\mathbb{P}^2,F)$ where $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is an explicitly described rational self map; the dominant rational map $\pi : \mathcal{R}_n \to \mathbb{P}^2$ such that $\pi \circ \lambda = F \circ \pi$ is a double cover of $\mathbb{P}^2$ branched along a sextic curve.*

The surfaces $\Xi_1(7), \Xi_1(8)$ are K3 surfaces; to our knowledge these are the first examples of a degree > 1 dynamical system on a K3 surfaces that is semi-conjugated to the plane.

Let us describe the structure of this paper and some further results. In §2, we start by describing the line operators Λ and general results on matroids. In §2.3, we study under which conditions a K3 surface which is the double cover of the $\mathbb{P}^2$ may be semi-conjugated to $\mathbb{P}^2$. Subsequently, we study the case $n = 7$ in §3: we start by recalling the definition of the matroid M_7 and then show that $\Lambda_{\{2\},\{3\}}$ induces a rational self map $\lambda_{\{2\},\{3\}}$ on the quartic surface $\mathcal{R}_7 \subset \mathbb{P}^3$. We then compute the degree of $\lambda_{\{2\},\{3\}}$ and prove that $\mathcal{R}_7$ is an open subset of the elliptic modular surface $\Xi_1(7)$. The automorphism group of the matroid M_7 is the order 42 Frobenius group. There is a natural action of that group on the surface $\mathcal{R}_7$. We show that this action is faithful. The quotient surface $\mathcal{R}_7/\operatorname{aut}(M_7)$ is the moduli space for unlabeled line arrangements coming from realizations of M_7: we obtain that this is a rational surface. In §3.6, we describe explicitly the semi-conjugacy of $\mathcal{R}_7$ (or equivalently $\Xi_1(7)$) with $\mathbb{P}^2$. The branch loci of the double cover $\Xi_1(7) \to \mathbb{P}^2$ is the union of a line and a singular quintic curve which we describe. §4 follows a similar pattern for the case $n = 8$. In that case, the branch loci of the double cover $\Xi_1(8) \to \mathbb{P}^2$ is union of a conic and a singular quartic curve. We moreover describe some 3-periodic line arrangements for Λ; their classes are fixed points for the action of λ on $\mathcal{R}_8$.

We remark that for $n = 9$, one may similarly obtain that $\mathcal{R}_9$ (contained as a sextic surface in $\mathbb{P}^3$) is birational to $\Xi_1(9)$. That elliptic surface is no longer a K3 surface and we could not find a semi-conjugacy with the plane.

Computations in this paper are based on Magma [1] and `OSCAR` [3]. The arXiv ancillary file of this paper contains some data related to these computations.

§2. Notations and definitions.

Throughout this article we assume to be working over the field $\mathbb{C}$.

2.1 Line arrangements and the operator $\Lambda_{\mathfrak{n},\mathfrak{m}}$.

A line arrangement $\mathcal{C} = \ell_1 + \cdots + \ell_n$ is a union of finitely many distinct lines in $\mathbb{P}^2$. A labeled line arrangement $\mathcal{C} = (\ell_1, \ldots, \ell_n)$ is a line arrangement with a numbering of the lines. We sometime put a superscript $^\ell$ (resp. ^{u}) when we want to emphasize that an arrangement or related objects has (resp. does not have) a labeling.

For an integer $k \geq 2$, a k-point of the line arrangement $\mathcal{C}$ is a point where exactly k lines of $\mathcal{C}$ meet. As in [9], for a subset $\mathfrak{n}$ of integers at least 2, let us denote by $\mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ the set of k-points of $\mathcal{C}$ for all $k \in \mathfrak{n}$. We denote by $t_k = t_k(\mathcal{C}) = |\mathcal{P}_{\{k\}}(\mathcal{C})|$ the number of k-points of $\mathcal{C}$. For a finite set of point $\mathcal{P}$ in $\mathbb{P}^2$ and $\mathfrak{n}$ as above, we denote by $\mathcal{L}_{\mathfrak{n}}(\mathcal{P})$ the set of lines which contain exactly n points in $\mathcal{P}$ for some $n \in \mathfrak{n}$.

For subsets $\mathfrak{n}, \mathfrak{m}$ of integers at least 2, let us denote by $\Lambda_{\mathfrak{n},\mathfrak{m}}(\mathcal{C}) = \mathcal{L}_{\mathfrak{m}} \circ \mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ the line arrangement that contains all lines of $\mathbb{P}^2$ containing exactly m points of $\mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ for $m \in \mathfrak{m}$. For example $\Lambda_{\{2\},\{3,4\}}(\mathcal{C})$ is the union of the lines that contain three or four double points of $\mathcal{C}$. The arrangement could be the empty arrangement if no such lines exists.

2.2 Matroids and the period map of the moduli of a matroid.

A matroid is a fundamental and actively studied object in combinatorics. Matroids generalize linear dependency in vector spaces as well as forests in graphs. See, for example, [8] for a comprehensive treatment of matroids. We just briefly mention a few concepts about matroids that are relevant for this article.

A *matroid* is a pair $M = (E, \mathcal{B})$, where E is a finite *ground set* of elements called atoms and $\mathcal{B}$ is a nonempty collection of subsets of E, called *bases*, satisfying an exchange property reminiscent from linear algebra.

The prime examples of matroids arise by choosing a finite set E of vectors in a vector space and declaring the maximal linearly independent subsets of E as bases. In our case we obtain matroids through line arrangements: If $\mathcal{C} = (\ell_1, \ldots, \ell_m)$ is a labeled line arrangement, the subsets $\{i, j, k\} \subseteq \{1, \ldots, m\}$ such that the lines ℓ_i, ℓ_j, ℓ_k meet in three distinct points are the bases of a matroid $M(\mathcal{C})$ over the set $\{1, \ldots, m\}$. We say that $M(\mathcal{C})$ is the matroid associated to $\mathcal{C}$.

We denote by $\mathrm{aut}(M)$ the *automorphism group* of the matroid M, that is, the set of isomorphisms from M to M.

A *realization* (over some field) of a matroid $M = (E, \mathcal{B})$ is a converse operation to the association $\mathcal{C} \to M(\mathcal{C})$: it is a $3 \times m$-matrix with non-zero columns $C_1, \ldots, C_m$, which are considered up to a multiplication by a scalar (thus as point in the projective plane) such that a subset $\{i_1, i_2, i_3\}$ of E of size 3 is a basis if and only if the 3×3 minor $|C_{i_1}, C_{i_2}, C_{i_3}|$ is nonzero. We denote by ℓ_i the line with normal vector the point $C_i \in \mathbb{P}^2$.

If $\mathcal{C} = (\ell_1, \ldots, \ell_m)$ is a realization of M and $\gamma \in PGL_3$, then $(\gamma\ell_1, \ldots, \gamma\ell_m)$ is another realization of M; we denote by $[\mathcal{C}]$ the orbit of $\mathcal{C}$ under that action of PGL_3. The *moduli space* $\mathcal{R}(M)$ of realizations of M parametrizes the orbits $[\mathcal{C}]$ of realizations. A more detailed introduction to these moduli spaces together with a description of a software package in `OSCAR` that can compute these spaces is given in [3].

In this article, we always assume that each subset of three elements of the first four atoms is a basis (otherwise, we replace M by a matroid isomorphic to it). Then in the moduli space $\mathcal{R}(M)$, one can always map the first four vectors of $\mathcal{C} \in [\mathcal{C}]$ to a fixed projective basis, so that each element $[\mathcal{C}]$ of $\mathcal{R}(M)$ has a canonical representative, which we will identify with $[\mathcal{C}]$.

A useful tool for the computations related to the moduli space $\mathcal{R} = \mathcal{R}(M)$ of realizations of a matroid M is what we call the period map: Let us denote by $\mathfrak{U} = \mathfrak{U}(M)$ the scheme of all realizations of M in $\mathbb{P}^2$. By analogy with similar objects, we call the quotient map

$$\mathfrak{q} : \mathfrak{U}(M) \to \mathcal{R}(M)$$

the period map; a point c of $\mathcal{R} = \mathcal{R}(M)$ is the class $c = [\mathcal{C}]$ of a realization $\mathcal{C}$. Once a basis is fixed, each class c has a unique representative $\mathcal{C}_0$ and we can (and we will) identify c with that representative.

It often occurs that $\mathcal{R}$ is embedded in a space $\mathbb{S} = \mathbb{S}(y_1,\dots,y_k)$ (affine or projective) of small dimension, like $\mathbb{P}^3$. The coordinates of the normal vectors $n^{(j)} = (n_1^{(j)} : n_2^{(j)} : n_3^{(j)})$ of $\mathcal{C}_0$ are then polynomials $n_1^{(j)} = P_1^{(j)}(y),\dots,n_3^{(j)} = P_3^{(j)}(y)$ in the coordinates $y_1,\dots,y_k$ of $\mathcal{R}$ in $\mathbb{S}$.

One often arrives at the natural question on computing the point $y = (y_1,\dots,y_k)$ in $\mathcal{R}$ from the knowledge of the normal vectors n. In other words, we need an explicit form of the period map $\mathfrak{q}$ as a map from $\mathfrak{U}$ to the scheme $\mathcal{R}$ embedded in the space $\mathbb{S}$. The answer to that problem are polynomials (or rational functions) $Q_1,\dots,Q_k$ in the coordinates of the normal vectors $n^{(1)},\dots,n^{(m)}$ etc.; here m is the number of lines in an arrangement.

2.3 Degree two K3 surfaces semi-conjugated to the plane.

Let $C_1 : Q_1 = 0$ be a sextic curve with at most ADE singularities, so that the desingularization X^s of the associated double cover

$$X = \{y^2 = Q_1(z_1,z_2,z_3)\} \hookrightarrow \mathbb{P}(3,1,1,1),$$

is a K3 surface. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be a rational self-map defined by coprime homogeneous polynomials (F_1,F_2,F_3) of degree m. Suppose that $F^*C_1 = C_1 + 2D$, for an effective divisor D; in algebraic terms, that means that we assume that

$$Q_1(F_1,F_2,F_3) = Q_1 \cdot R^2,$$

for some polynomial R. Then the following relation holds

$$(y\,R(z))^2 = Q_1(z)\,R(z)^2 = Q_1(F_1(z),F_2(z),F_3(z)),$$

where $z = (z_1 : z_2 : z_3) \in \mathbb{P}^2$. Hence, the rational map

$$\tilde{F} : (y;z) \dashrightarrow (y\,R(z); F_1(z) : F_2(z) : F_3(z)),$$

is a rational self-map acting on the K3 surface X^s. Let $\pi : X \to \mathbb{P}^2$ be the double cover map. The following diagram

$$\begin{array}{ccc} X & \overset{\tilde{F}}{\dashrightarrow} & X \\ \pi\downarrow & & \pi\downarrow \\ \mathbb{P}^2 & \overset{F}{\dashrightarrow} & \mathbb{P}^2, \end{array} \tag{2.1}$$

is commutative and, by analogy with other dynamical systems, we say that the dynamical system $(X,\tilde{F})$ is *semi-conjugated* to $(\mathbb{P}^2,F)$.

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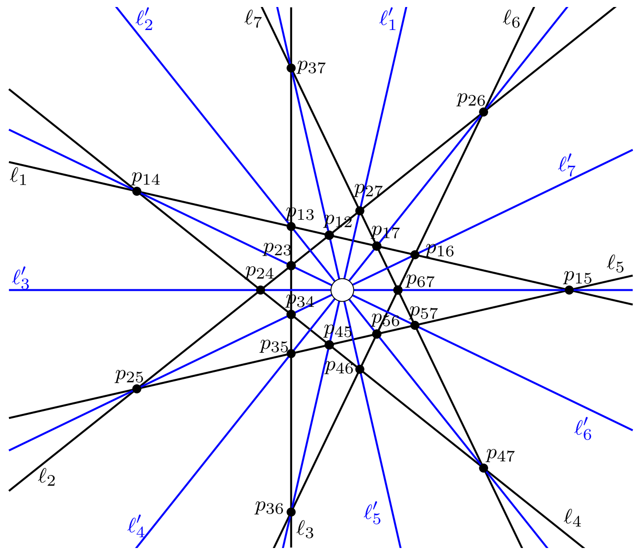

Figure 1.
The matroid M_7 whose construction is based on the regular heptagon.

EXAMPLE 2.1. Let C be an irreducible curve of degree 6 with 10 nodes. A Coble surface Y is the blow-up of $\mathbb{P}^2$ at the 10 nodal singularities of C. The group of birational transformations G preserving C is infinite, it is generated by Bertini involutions centered at the nodal points of C. When C is generic, the group G lifts to Y and the elements of G become automorphisms of Y. The automorphism group $G \subset \mathrm{aut}(Y)$ preserves the pull-back C' of C, thus taking the double cover of Y branched over C', one gets a smooth K3 surface X and the group G is in fact the automorphism group of X (see, e.g., [2]). The surface X is also the minimal desingularization of the double cover branched over C and the diagram (2.1) is commutative.

§3. The heptagon.

3.1 $\Lambda_{\{2\},\{3\}}$ is a rational self-map on $\mathcal{R}_7$ and $\mathfrak{U}_7$.

3.1.1. Definition of the matroid M_7.

The matroid M_7 has 14 atoms $1,\dots,7,1',\dots,7'$ and the bases are the triples $\{a,b,c\}$ with $\{a,b\} \subset \{1,\dots,7\}$ and $c \in \{1',\dots,7'\}$ such that $a+b \neq 2c \bmod 7$. A sketch of M_7 is described in Figure 1, where the atoms $i \in \{1,\dots,7\}$ and $j \in \{1',\dots,7'\}$ correspond to the lines ℓ_i and ℓ'_j, resp., and three lines form a basis if they do not meet in one point. Note that the central singularity of arrangement in Figure 1 is not part of the matroid and therefore removed.

Let $\mathcal{A}_1$ be a generic line arrangement realizing the matroid M_7. We write $\mathcal{A}_1 = \mathcal{C}_0 \cup \mathcal{C}_1$ where $\mathcal{C}_0$ are the first seven lines and $\mathcal{C}_1$ are the seven last ones. By the combinatorics of the matroid M_7 and the genericity assumption, the property $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ holds, and – that will be important for us – the image of $\mathcal{C}_0$ by the operator $\Lambda_{\{2\},\{3\}}$ has a natural labeling: for any $j \in \{1,\dots,7\}$, the six line arrangement

$$H_j = \sum_{k \in \{1,\dots,7\},\, k \neq j} \ell_k, \tag{3.1}$$

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is such that the line arrangement $\Lambda_{\{2\},\{3\}}(H_j)$ is a unique line ℓ'_j, moreover:

$$\mathcal{C}_1 = (\ell'_1, \dots, \ell'_7).$$

Since $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$, to shorten our notations, we will often speak of $\mathcal{C}_0$ as a realization of M_7 instead of $\mathcal{C}_0 \cup \mathcal{C}_1$.

The singularities of $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) are 21 double points. The 21 singularities on $\mathcal{C}_0$ become the triple points on $\mathcal{C}_0 \cup \mathcal{C}_1$, moreover $t_2(\mathcal{C}_0 \cup \mathcal{C}_1) = 28$.

3.1.2. Equation of the quartic surface Z_7 and realization space of M_7.

Consider Z_7, the quartic surface in $\mathbb{P}^3$ given by the equation

$$y_1^2y_2^2 + y_1^2y_2y_3 - y_1y_2^2y_3 - y_1y_2y_3^2 - y_1^2y_2y_4 - y_1y_2^2y_4 + y_1y_2y_3y_4 - y_2y_3^2y_4 + y_1y_2y_4^2 + y_3^2y_4^2 = 0. \tag{3.2}$$

The eight singularities of Z_7 are of type $4A_1 + A_2 + 3A_3$, at the points respectively

$$s_1 = (0:0:0:1), s_2 = (1:0:0:1), s_3 = (0:0:1:0), s_4 = (1:0:1:0),$$
$$s_5 = (0:1:0:0), s_6 = (0:1:0:1), s_7 = (1:-1:1:0), s_8 = (1:0:0:0).$$

The minimal desingularization of Z_7 is a K3 surface which we denote by Z_7^s. Let x_1, x_2, x_3 be the coordinates on the affine chart $y_4 \neq 0$. For a generic point $x = (x_1, x_2, x_3)$ on the surface Z_7 in the chart $y_4 \neq 0$, let us define the labeled arrangement of seven lines $\mathcal{C}_0 = \mathcal{C}_0(x)$ with normal vectors the points $p_1, \dots, p_7$ respectively defined by

$$\begin{gathered}(1:0:0), (0:1:0), (0:0:1), (-1:1:1)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 : x_1x_2 + x_1x_3 - x_1 : x_2 - 1)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 : x_1x_2 + x_1x_3 - x_1 + x_2^2\\ +x_2x_3 - 2x_2 - x_3 + 1 : x_2^2 + x_2x_3 - x_2 - x_3)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 + x_3^2 : x_1x_2 + x_1x_3 - x_1 - x_2x_3\\ -x_3^2 + x_3 : x_2^2 + x_2x_3 - x_2 - x_3).\end{gathered} \tag{3.3}$$

Let us also define the lines arrangement $\mathcal{C}_1 = \mathcal{C}_1(x)$ with normal vectors

$$\begin{gathered}(-x_1x_2^2 - x_1x_2x_3 + x_1x_2 + x_3^2 : x_1x_2^2 + 2x_1x_2x_3 - x_1x_2 + x_1x_3^2\\ -x_1x_3 - x_2^2x_3 - 2x_2x_3^2 + x_2x_3 - x_3^3 + x_3^2 : x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_1x_2 - x_1x_3 + x_1 : x_1x_2 + x_1x_3 - x_1 : x_2 - 1), (-x_2 : 1 : 0),\\ (-x_1x_2^3 - 2x_1x_2^2x_3 + x_1x_2^2 - x_1x_2x_3^2 + x_1x_2x_3 - x_2^2x_3 - x_2x_3^2 + x_2x_3\\ +x_3^2 : x_1x_2 + x_1x_3 - x_1 + x_2^2 + x_2x_3 - 2x_2 - x_3 + 1 : x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 : 0 : x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_2^2 - x_2x_3 + x_2 + x_3 : x_1x_2 + x_1x_3 - x_1 - x_2x_3 - x_3^2 + x_3 : x_2^2 + x_2x_3\\ -x_2 - x_3), (0:1:1).\end{gathered} \tag{3.4}$$

A computation in `OSCAR` yields the following concrete description of the moduli space $\mathcal{R}_7 = \mathcal{R}(M_7)$.

PROPOSITION 3.1. *The moduli space $\mathcal{R}_7$ is an open sub-scheme of Z_7: for $x \in \mathcal{R}_7$, the line arrangement $\mathcal{A} = \mathcal{C}_0(x) \cup \mathcal{C}_1(x)$ is a realization of M_7, and conversely any realization of M_7 is projectively equivalent to a unique such line arrangement.*

The complement of $\mathcal{R}_7$ in Z_7 is the union of 20 irreducible curves described in §3.2.

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From the definition of the matroid M_7, if $\mathcal{A}=\mathcal{C}_0\cup\mathcal{C}_1$ is a realization of M_7, one has $\Lambda_{\{2\},\{3\}}(\mathcal{C}_0)=\mathcal{C}_1$, but the following result on $\mathcal{C}_2=\Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ is unexpected

THEOREM 3.2. *Let $\mathcal{A}_0=\mathcal{C}_0\cup\mathcal{C}_1$ be a generic realization of M_7 and define $\mathcal{C}_2=\Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$. The labeled line arrangement $\mathcal{A}_1=\mathcal{C}_1\cup\mathcal{C}_2$ is again a realization of M_7. The operator $\Lambda_{\{2\},\{3\}}$ induces a rational self-map on the schemes $\mathfrak{U}_7$ of all realizations of M_7 and its moduli space $\mathcal{R}_7$.*

We denote by $\lambda_{\{2\},\{3\}}:Z_7\dashrightarrow Z_7$ the rational self-map on Z_7 induced by $\Lambda_{\{2\},\{3\}}$.

Proof. Up to projective automorphism, one can suppose that the line arrangement $\mathcal{A}_0$ is of the form $\mathcal{A}_0=\mathcal{C}_0(x)\cup\mathcal{C}_1(x)$ for x generic in Z_7: concretely, we use $x=(x_1,x_2,x_3)$, where $x_1,x_2,x_3\in\mathbb{C}(Z_7)$ are considered as rational functions. A direct computation (with Magma) then shows that $\mathcal{C}_2=\Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ is a line arrangement of seven lines. It has a canonical labeling as described in the previous Subsection and we then check that the matroid associated to $\mathcal{C}_1\cup\mathcal{C}_2$ is equal to M_7, so that $\mathcal{C}_1\cup\mathcal{C}_2$ is a realization of M_7. Using the period map, one computes $\lambda_{\{2\},\{3\}}$ and obtain that it is a dominant rational map. The reader can find the polynomials defining $\lambda_{\{2\},\{3\}}$ in an ancillary file of this paper on arXiv; it can be also retrieved from the polynomials given in §3.6. That describes action of $\Lambda_{\{2\},\{3\}}$ on the space of realization $\mathfrak{U}_7$ and on the moduli space $\mathcal{R}_7$. ∎

3.2 The open surface $\mathcal{R}_7$ inside Z_7.

The scheme $Z_7\setminus\mathcal{R}_7$ is the union of the following curves:

- The 12 lines

$$
\begin{array}{lll}
L_1: y_2=y_3=0, & L_2: y_1=y_3=0, & L_3: y_2=y_4=0,\\
L_4: y_1-y_3=y_4=0, & L_5: y_1=y_4=0, & L_6: y_2-y_4=y_3=0,\\
L_7: y_1-y_3-y_4=y_2+y_3=0, & L_8: y_1-y_3=y_2+y_3=0, & L_9: y_2+y_3=y_4=0,\\
L_{10}: y_1-y_4=y_3=0, & L_{11}: y_1-y_3=y_2-y_4=0, & L_{12}: y_1=y_2-y_4=0.
\end{array}
$$

  These lines are also the lines contained in the quartic surface Z_7 that contain at least two double points of Z_7.
- The conic C_o defined by $y_1y_3-y_3^2-y_1y_4=y_2+y_3-y_4=0$.
- Seven curves $E_1,\dots,E_7$ of geometric genus one. For example, one of these curves is given by

$$y_1^2-2y_1y_3+y_3^2-y_1y_4=y_2^2+y_2y_3+y_1y_4-y_3y_4-y_4^2=0.$$

The j-invariant of the normalizations of the curves E_i is equal to $-5^6/28$. The elliptic curve with this j-invariant is known as the modular curve $X_1(14)$ parameterizing pairs (E,t) where E is an elliptic curve and t is an order 14 torsion element of E. For a generic point p on the curves $E_1,\dots,E_7$, the line arrangement $\mathcal{C}_0(p)$ with normal vectors as in (3.3) is well-defined. The line arrangement $\mathcal{C}_1=\Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ has seven lines, but its singularities are $t_2=6,t_3=5$, and one has $\Lambda_{\{2\},\{3\}}(\mathcal{C}_1)=\emptyset$. Moreover, the singularities of $\mathcal{C}_0\cup\mathcal{C}_1$ are $t_2=13,t_3=26$.

The image of the curves $C_o,E_1,\dots,E_7$ under the map $\lambda_{\{2\},\{3\}}$ are lines L_k; when defined, the image of the lines L_k are lines $L_{k'}$ or points.

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3.3 The degree of $\lambda_{\{2\},\{3\}}$.

Recall that $\lambda_{\{2\},\{3\}}: Z \dashrightarrow Z$ denotes the action of the operator $\Lambda_{\{2\},\{3\}}$ on the K3 surface Z_7. One has:

THEOREM 3.3. *The operator $\lambda_{\{2\},\{3\}}$ acts on Z_7 as a degree 4 rational self-map.*

In order to prove Theorem 3.3, let us describe the period map: Let $\ell_1, ..., \ell_7$ be the lines of $\mathcal{C}_0$ with normal vectors as in Equation (3.3). Let us denote by $p_{i,j}$ the intersection point of the lines ℓ_i and ℓ_j. The point $p_{5,7}$ is $(1 : x_2 : x_3)$, so that one may recover x_2, x_3 from the knowledge of that point. Also the point $p_{1,7}$ is

$$(0 : -x_1x_2{}^2 - x_1x_2x_3 + x_1x_2 + x_2{}^2 - x_2 : x_2x_3 + x_3{}^2 - x_3), \tag{3.5}$$

this is linear in x_1, so that from the knowledge of $p_{5,7}$ and $p_{1,7}$, one may recover the point $(x_1, x_2, x_3) \in Z_7$.

Proof of Theorem 3.3. Let $A \in PGL_3(\mathbb{C})$ be the projective transformation that sends the first four lines of $\mathcal{C}_1$ to the four lines having the same normal vectors as the one of $\mathcal{C}_0$. Let $\mathcal{C}_1' = (\ell_1', \dots, \ell_7')$ be the image of $\mathcal{C}_1$ by A. Using the period map, one can determine the points $p_{5,7}'$ and $p_{1,7}'$ and we obtain a point $x' = (x_1', x_2', x_3')$ (in the function field of Z_7). The line arrangements $\mathcal{C}_0(x_1', x_2', x_3')$ and $\mathcal{C}_1'$ are equal, and the action of $\Lambda_{\{2\},\{3\}}$ on Z_7 is through the map

$$\lambda_{\{2\},\{3\}} : (x_1, x_2, x_3) \to (x_1', x_2', x_3').$$

The rational self-map $\lambda_{\{2\},\{3\}} : Z_7 \dashrightarrow Z_7$ is studied in §3.6.

Let us compute the degree of $\lambda_{\{2\},\{3\}}$; we apply the method from [15]. Let $f(x_1, x_2, x_3)$ be the equation of the quartic Z_7 in the chart $U_4 : y_4 \neq 0$. The space of global non-vanishing differential 2-forms is generated by a form ø, which one can choose so that on an open set of U_4 one has:

$$\omega = \frac{dx_2 \wedge dx_3}{\partial f / \partial x_1}.$$

The rational self-map $\lambda_{\{2\},\{3\}}$ preserves U_4, and by a direct computation one obtains that

$$\lambda_{\{2\},\{3\}}^* \omega = -2\omega.$$

The above expression shows that when applying $\lambda_{\{2\},\{3\}}$, the volume form $\omega\bar{\omega}$ is multiplied by 4, which gives the degree of $\lambda_{\{2\},\{3\}}$. ∎

3.4 Action of aut(M_7) on the K3 surface Z_7^s.

The automorphism group of M_7 is generated by the order 7 and 6 permutations

$$\sigma_1 = (1,7,4,3,6,5,2)(8,14,11,10,13,12,9) \text{ and } \sigma_2 = (1,3,5,6,7,2)(8,10,12,13,14,9).$$

with $1' = 8, \dots, 7' = 14$. This group is isomorphic to the the Frobenius group $F_7 = \mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/7\mathbb{Z}$. These automorphisms act on the K3 surface Z.

PROPOSITION 3.4. *The action of* aut(M_7) *on Z_7 is faithful.*

Proof. As in the proof of Theorem 3.3, let $\mathcal{C}_0 = \mathcal{C}_0(x_1, x_2, x_3)$ be the generic line arrangement in $\mathcal{R}_7$, where $x_1, x_2, x_3 \in \mathbb{C}(Z_7)$ are considered as rational functions.

For $\sigma \in$ aut(M_7), let $\mathcal{C}_0^\sigma$ be the image of $\mathcal{C}_0$ under the action of σ (that is just the permutation of the lines under σ). We apply the period map to the line arrangement $\mathcal{C}_0^\sigma$,

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where $\mathcal{C}_0 = \mathcal{C}_0(x)$. Using the period map, we obtain the point $\sigma(x) = (x'_1, x'_2, x'_3)$ which is a zero of the equation of Z_7 and such that $\mathcal{C}_0(\sigma(x))$ is projectively equivalent to $\mathcal{C}_0^\sigma$.

When $\sigma = \sigma_1$, the automorphism σ_1 acts on Z_7 through the map in $\mathbb{P}^3$ given by the ring homomorphism which to (y_1, y_2, y_3, y_4) associates

$$\begin{gathered}(y_1y_2^2y_3 + y_1y_2y_3^2 - y_2^2y_3^2 - y_2y_3^3 - y_1y_2y_3y_4 - y_2^2y_3y_4 + y_2y_3y_4^2 + y_3^2y_4^2,\\ y_1y_2^3 + y_1y_2^2y_3 + y_2^2y_3^2 + y_2y_3^3 - 2y_1y_2^2y_4 - y_1y_2y_3y_4 - 2y_2y_3^2y_4 - y_3^3y_4 + y_1y_2y_4^2 + y_3^2y_4^2,\\ y_1y_2^2y_3 + y_1y_2y_3^2 - y_2^2y_3^2 - y_2y_3^3 - y_1y_2y_3y_4 + y_2y_3^2y_4,\\ y_2^3y_3 + 2y_2^2y_3^2 + y_2y_3^3 - 2y_2^2y_3y_4 - 3y_2y_3^2y_4 - y_3^3y_4 + y_2y_3y_4^2 + y_3^2y_4^2).\end{gathered}$$

For σ_2, we obtain that it acts on the surface Z_7 through the map which to (y_1, y_2, y_3, y_4) associates

$$\begin{gathered}(-y_2^2y_3 - y_2y_3^2 + y_2y_3y_4, -y_1y_2y_3 + y_2y_3^2 + y_2y_3y_4 - y_3y_4^2,\\ y_1y_2^2 + y_1y_2y_3 - y_2^2y_3 - y_2y_3^2 - y_1y_2y_4 + y_2y_3y_4, y_2y_3y_4 - y_3y_4^2);\end{gathered}$$

this map is a birational transformation of $\mathbb{P}^3$. In order to check that the action of $\mathrm{aut}(M_7)$ is faithful on Z_7, it is then enough to check that the orbit of one point (for example the point $(-6 : -25/8 : 5 : 1)$ in Z_7) has 42 elements, which is a direct computation.

The fixed points under the order seven element σ_1 are the singularities s_5, s_7, s_8; (there is a unique conjugacy class of elements of order 7 in F_7).

The fixed points locus of σ_2 and the order 3 automorphism σ_2^2 acting on Z_7 are:

1. The A_3 singularity $(0 : 1 : 0 : 1)$,
2. The four points $p = (r^2 + 1 : r^2 - r + 2 : r : 1)$ where r is any complex root of $X^4 - X^3 + 3X^2 - X + 1$. These points are in $Z_7 \setminus \mathcal{R}_7$; they are periodic of period 2 for the rational self-map $\lambda_{\{2\},\{3\}}$, moreover the (unlabeled) line arrangements $\mathcal{C}_0(p), \mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0), \mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ have 7, 10, and 37 lines, respectively. It seems likely that the number of lines of the sequence $\mathcal{C}_{n+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_n)$ goes to infinity.
3. The points $(w+1 : -w : w : 1)$ where $w^2 + w + 1 = 0$, which are fixed by the rational self-map $\lambda_{\{2\},\{3\}}$; these two points are in $Z_7 \setminus \mathcal{R}_7$.

The fixed-point locus of the involution σ_2^3 acting on Z_7 is the union of the line L_7 and a curve E_j, which is in $Z_7 \setminus \mathcal{R}_7$ (see §3.2). There is a unique conjugacy class of involutions in F_7, so that similarly, any involution from $\mathrm{aut}(M_7)$ fixes a curve and a line. ∎

There is an open set in the quotient surface $Z_7/\mathrm{aut}(M_7)$ which parametrizes unlabeled line arrangements $\mathcal{C}_0^o$ associated to $\mathcal{C}_0$ in $\mathcal{R}_7$. One has:

Corollary 3.5. *The surface $Z_7/\mathrm{aut}(M_7)$ is rational.*

Proof. Since an involution of $\mathrm{aut}(M_7)$ fixes a one dimensional curve, it is non-symplectic (see [4]), thus $Z_7/\mathrm{aut}(M_7)$ is rational. ∎

For a labeled line arrangement $\mathcal{C}_0 = (\ell_1, \dots, \ell_k) \in \mathfrak{U}_7$ and $j \in \{1, \dots, 7\}$, let us denote by $H_j(\mathcal{C}_0)$ the line arrangement $H_j = \sum_{k \neq j} \ell_k$. The labeled line arrangement $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ is

$$\mathcal{C}_1 = (\Lambda_{\{2\},\{3\}}(H_1), \dots, \Lambda_{\{2\},\{3\}}(H_7)).$$

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An element $\sigma \in \mathrm{aut}(M_7)$ permutes the lines of $\mathcal{C}_0$: it can also be seen as a permutation of $\{1,\ldots,7\}$. The $\sigma(j)^{th}$ line of $\sigma.\mathcal{C}_1$ is $\Lambda_{\{2\},\{3\}}(H_{\sigma(j)}(\mathcal{C}_0))$. Since

$$H_{\sigma(j)}(\mathcal{C}_0) = \sum_{k\neq j} \ell_{\sigma(k)} = H_j(\sigma.\mathcal{C}_0),$$

the $\sigma(j)^{th}$ line of $\sigma.\mathcal{C}_1$ is $\Lambda_{\{2\},\{3\}}(H_j(\sigma.\mathcal{C}_0))$. Thus

$$\sigma.\Lambda_{\{2\},\{3\}}(\mathcal{C}_0) = (\Lambda_{\{2\},\{3\}}(H_{\sigma(1)}),\ldots,\Lambda_{\{2\},\{3\}}(H_{\sigma(7)})) = \Lambda_{\{2\},\{3\}}(\sigma.\mathcal{C}_0)$$

and we obtain that:

Proposition 3.6. *The action of* $\mathrm{aut}(M_7)$ *commutes with the action of* $\Lambda_{\{2\},\{3\}}$*, that is for all* $\sigma \in \mathrm{aut}(M_7)$ *it holds that*

$$\Lambda_{\{2\},\{3\}} \circ \sigma = \sigma \circ \Lambda_{\{2\},\{3\}}.$$

Remark 3.7. The group $\mathrm{aut}(M_7)$ acts faithfully on the surface $Z_7 \subset \mathbb{P}^3$, but does not extend canonically to a well-defined action on the ambient space $\mathbb{P}^3$. For example, the action of σ_2 we computed is the restriction of an order 6 birational map $\tilde{\sigma_2}$ of $\mathbb{P}^3$; in particular $\tilde{\sigma_2}^3$ is a birational involution of $\mathbb{P}^3$ defined by degree 5 coprime polynomials. If instead one starts with $\sigma_2^3 = (1,6)(2,5)(3,7)(8,13)(9,12)(10,14)$ and computes the action of $\widetilde{\sigma_2^3}$ on Z_7 as we did above for σ_1 and σ_2, one obtains that, surprisingly, the defining coprime polynomials of the rational map $\widetilde{\sigma_2^3} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ have degree 4, although the maps $\widetilde{\sigma_2^3}$ and $\tilde{\sigma_2}^3$ have the same effect on Z_7. Moreover although $(\widetilde{\sigma_2^3})^2$ is the identity on the surface Z_7, it is not the identity on $\mathbb{P}^3$ (it is defined by degree 6 coprime polynomials). Moreover, one can compute that the rational map $(\widetilde{\sigma_2^3})^4$ is defined by degree 21 coprime polynomials.

3.5 Fibration preserved by $\lambda_{\{2\},\{3\}}$ and the elliptic modular surface $\Xi_1(7)$.

The line $L_6 : y_2 - y_4 = y_3 = 0$ is contained in the surface Z_7. Let $\gamma : Z_7 \to \mathbb{P}^1$ be the elliptic fibration induced by the projection from that line. One obtains a smooth cubic affine model A in $\mathring{A}^2_{\mathbb{Q}(t)} = \mathring{A}^2_{\mathbb{Q}(t)}(x,y)$ of that elliptic fibration by substituting $(x, 1+ty, y, 1)$ in the equation of Z_7. A computation shows that $Z_7^s \to \mathbb{P}^1$ is (isomorphic to) the elliptic surface Y associated to the elliptic curve $E_{/\mathbb{Q}(t)}$ with Weierstrass model

$$E : y^2 = x^3 + \tfrac{(t^4-2t^3+3t^2+6t+1)}{(t+1)^2}x^2 + \tfrac{8t^3(t^2-t-1)}{(t+1)^3}x + 16\tfrac{t^6}{(t+1)^4}.$$

The map between A and Y sends $(0,0)$ to the zero section. The elliptic fibration $Y \to \mathbb{P}^1$ has singular fibers $3I_7 + 3I_1$ at the points

$$\infty, 0, -1, t^3 - 5t^2 - 8t - 1 = 0,$$

respectively.

We recall that the curve $X_1(7)$ parametrizes (up to isomorphisms) the pairs (E,p) where E is an elliptic curve and p is a torsion point of order 7 on E. A Weierstrass model E' of the elliptic modular surface $\Xi_1(7)$ over the curve $X_1(7) \simeq \mathbb{P}^1$ is computed in [13]. The j-invariant maps $j_E(t), j_{E'}(t) \in \mathbb{Q}$ of E and E' are related by the equality $j_{E'}(t) = j_E(-\frac{1}{t})$, which shows that E is isomorphic to E' and Z_7^s is isomorphic to the elliptic modular surface $X_1(7)$.

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The Mordell–Weil group of E is isomorphic to $\mathbb{Z}/7\mathbb{Z}$; it is generated by the point

$$p_t = (0 : 4t^3 : (t+1)^2) \in E.$$

We thus obtained the first part of the following theorem

THEOREM 3.8. 1. *The K3 surface Z_7^s is isomorphic to the modular elliptic surface $\Xi_1(7)$.*

2. *The rational self-map $\lambda_{\{2\},\{3\}}$ preserves the elliptic fibration $\gamma : Z_7 \to \mathbb{P}^1$ and acts on the base curve $\mathbb{P}^1$ through the order 3 map $t \to -1/(t+1)$. There exists an automorphism σ_0 coming from $\operatorname{aut}(M_7)$ such that $\sigma_0\lambda_{\{2\},\{3\}}$ preserves the fibration γ and acts on E as the multiplication by 2 map.*

The last property implies that the operator $\Lambda_{\{2\},\{3\}}$ preserves the moduli interpretation of $X_1(7)$.

Proof. Using the period map and the function field of A, one computes that the action of $\lambda_{\{2\},\{3\}}$ on the base $\mathbb{P}^1$ of the fibration $A \to \mathbb{P}^1$ is through the map $t \to -1/(t+1)$.

An automorphism $\sigma \in \operatorname{aut}(M_7)$ acts on the surface $Z_7 \cap \{y_4 \neq 0\}$ and on the affine model A. Using the period map and again the generic point of A, one computes the action of the rational self-maps $\sigma\lambda_{\{2\},\{3\}}$ ($\sigma \in \operatorname{aut}(M_7)$) on A. For 14 of these maps, the action on the base curve $\mathbb{P}^1$ is trivial. This is the case for example for

$$\sigma_0 = (1,2,4)(3,6,7)(8,9,11)(10,13,14).$$

The map $\mu = \sigma_0\lambda_{\{2\},\{3\}}$ also acts on E, one can thus compute its action on the generic point of E. Knowing that action, we are now able to compute the pull-back of a non-zero holomorphic one-form ω by μ, which is: $\mu^*\omega = 2\omega$. Using the seven torsion points, one computes that μ fixes the origin, thus $\mu = [2]$. ∎

Among the 12 lines contained in Z_7, in the complement of $\mathcal{R}_7$, eight are contained in the singular fibers of the fibration γ, and 4 are sections.

Using the pull-back to Z_7^s of the lines contained in Z_7 and the (-2)-curves of the desingularization, one may compute the Néron–Severi lattice of Z_7^s, and obtain that it has discriminant -7 and rank 20. The modular elliptic surface $\Xi_1(7)$ is well-known and studied; it is known as the unique K3 surface with Néron–Severi lattice of rank 20 and discriminant -7: we obtain that way another proof that Z^s is isomorphic to $\Xi_1(7)$. The inequivalent fibrations of $\Xi_1(7)$ have been classified (see [6]). Another remarkable fact is that $\Xi_1(7)$ is a ball-quotient surface: there exists a co-compact lattice Γ in the automorphism group of the unit ball $\mathbb{B}_2$ such that $\Xi_1(7) \simeq \mathbb{B}_2/\Gamma$ [7]. The automorphism group of $\Xi_1(7)$ is studied in [14].

3.6 The K3 surface Z_7^s is semi-conjugated to the plane.

The rational self-map $\lambda_{\{2\},\{3\}}$ acting on the quartic $Z_7 \hookrightarrow \mathbb{P}^3(y_1,\dots,y_4)$ is defined by

$$\lambda_{\{2\},\{3\}} = (P_1 : \cdots : P_4),$$

where $P_1,\dots,P_4$ are four homogeneous degree 11 polynomials computed via the period map. These polynomials are given in the ancillary file in the arXiv version of this paper; they also may be obtained from the polynomials Q_1,Q_2,Q_3 and R below. A remarkable fact about

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the polynomials $P_1,\dots,P_4$ is that

$$\deg_{y_1}(P_1)=1, \deg_{y_1}(P_2)=\deg_{y_1}(P_3)=\deg_{y_1}(P_4)=0, \tag{3.6}$$

where $\deg_{y_1}$ denote the degree relative to the variable y_1.

Let us define the polynomials $\tilde{P}_k = P_{k+1}(0,z_1,z_2,z_3)$ for $k\in\{1,2,3\}$ (where z_1,z_2,z_3 are the three coordinates on the plane $\mathbb{P}^2 : y_1=0$). The polynomials $\tilde{P}_k$, $k\in\{1,2,3\}$ define a rational self-map $F:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$; the base locus of the linear system generated by $\tilde{P}_1,\tilde{P}_2,\tilde{P}_3$ is the quintic curve B defined by

$$\begin{aligned} Q = {} & z_1^3z_2^2+2z_1^2z_2^3+z_1z_2^4+2z_1^3z_2z_3+4z_1^2z_2^2z_3+2z_1z_2^3z_3+z_1^3z_3^2 \\ & -4z_1^2z_2z_3^2-9z_1z_2^2z_3^2-4z_2^3z_3^2-2z_1^2z_3^3+2z_1z_2z_3^3+4z_2^2z_3^3+z_1z_3^4. \end{aligned}$$

That curve is irreducible, has geometric genus 1 and its normalization has j-invariant $-5^6/28$. By removing the base locus B, one obtains that the rational self-map F is defined by the following degree 6 polynomials

$$\begin{aligned} Q_1 = {} & z_1Q, \\ Q_2 = {} & -z_1^5z_2-3z_1^4z_2^2-3z_1^3z_2^3-z_1^2z_2^4+z_1^4z_2z_3+2z_1^3z_2^2z_3+z_1^2z_2^3z_3 \\ & +z_1^3z_2z_3^2+2z_1^2z_2^2z_3^2+z_1z_2^3z_3^2-z_1^2z_2z_3^3+z_2^3z_3^3-z_2^2z_3^4, \\ Q_3 = {} & 2z_1^4z_2z_3+4z_1^3z_2^2z_3+2z_1^2z_2^3z_3+z_1^4z_3^2-4z_1^3z_2z_3^2-8z_1^2z_2^2z_3^2-3z_1z_2^3z_3^2 \\ & -2z_1^3z_3^3+2z_1^2z_2z_3^3+4z_1z_2^2z_3^3+z_2^3z_3^3+z_1^2z_3^4, \end{aligned}$$

and the indeterminacy locus of $F=(Q_1:Q_2:Q_3)$ are the 8 points

$$\begin{aligned} & q_1=(0:0:1), q_2=(1:0:1), q_3=(0:1:0), q_4=(-1:1:0), \\ & q_5=(1:0:0), q_r=(-r^2+2r:r:1) \end{aligned}$$

where r is any root of X^3-4X^2+3X+1 (the field $\mathbb{Q}(r)$ is the degree 3 real subfield of $\mathbb{Q}(\zeta_7)$).

Let us define the projection map $\pi_1 : Z_7 \to \mathbb{P}^2(z_1,z_2,z_3)$ from the point $s_8 : y_2=y_3=y_4=0$ contained in surface Z_7. This point is an A_3 singularity on Z_7, in particular it has multiplicity 2, thus the map π_1 from the quartic to the plane has degree 2. One has:

LEMMA 3.9. *The branch loci of π_1 is the union of the quintic curve $B=\{Q=0\}$ and the line $L:z_1=0$.*

Proof. The ramification locus of π_1 is the discriminant of the equation of Z_7 (given in (3.2)) with respect to the variable y_1. The image of the ramification curve by π_1 is the curve $B+L$. ∎

The curve B has singularities of type A_4,A_4,A_2 at the points q_2,q_4,q_5, respectively. The union $L+B$ has singularities of type A_1,A_3,A_4,A_3,A_4,A_2 at the points $q_0=(0:1:1),q_1,q_2,q_3,q_4,q_5$, respectively.

A direct computation shows that

$$Q_1(Q_1,Q_2,Q_3)=Q_1R^2$$

for $R=\frac{1}{8}z_2^2(z_1-z_3)^2R_4R_7$, where

$$R_4 = z_1^4+2z_1^3z_2+z_1^2z_2^2-z_1^2z_3^2-z_1z_2z_3^2-z_2z_3^3,$$

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$$R_7 = z_1^6z_2 + 4z_1^5z_2^2 + 6z_1^4z_2^3 + 4z_1^3z_2^4 + z_1^2z_2^5 + z_1^6z_3 - 7z_1^4z_2^2z_3 - 11z_1^3z_2^3z_3 - 6z_1^2z_2^4z_3 - z_1z_2^5z_3 - z_1^5z_3^2 + 3z_1^3z_2^2z_3^2 + 2z_1^2z_2^3z_3^2 + 3z_1^2z_2^2z_3^3 + 5z_1z_2^3z_3^3 + 2z_2^4z_3^3 - 2z_1z_2^2z_3^4 - 2z_2^3z_3^4 - z_1z_2z_3^5.$$

The images of the curves $z_2 = 0$, $z_1 - z_3 = 0$ and $R_4 = 0$ by the rational self-map $F = (Q_1 : Q_2 : Q_3)$ of $\mathbb{P}^2$ are the indeterminacy points q_2, q_4, q_2, respectively. The image of the curve $R_7 = 0$ under the map F is the quintic curve B. The image of the quintic curve B under the map F is the line $L : z_1 = 0$. The rational map F preserves L and the action of F on L is through the map $(z_2 : z_3) \to (z_2 - z_3 : z_3)$.

From the above description and §2.3, the surface Z_7^s is the minimal desingularization of the double cover

$$X : \{y^2 = Q_1(z_1, z_2, z_3\} \hookrightarrow \mathbb{P}(3,1,1,1)$$

branched over $L + B$. The birational map between X and Z_7 is given by the equalities $y_{i+1} = z_i$ for $i \in \{1,2,3\}$ and

$$y_1 = \tfrac{1}{2}(y + z_2^2z_3 + z_2z_3^2 + z_2^2z_4 - z_2z_3z_4 - z_2z_4^2)/(z_2^2 + z_2z_3 - z_2z_4).$$

We continue to denote by $\lambda_{\{2\},\{3\}}$ the rational self-map

$$(y; z) \to (yR(z); F(z)).$$

Applying the results of §2.3, we obtain that:

THEOREM 3.10. *The dynamical system $(Z_7^s, \lambda_{\{2\},\{3\}})$ is semi-conjugated to $(\mathbb{P}^2, F)$.*

REMARK 3.11. a) The degrees of the coprime polynomials defining the rational maps F, F^2, F^3 are $6, 21, 82$, respectively.

b) It would be interesting to construct rational self-maps on some other degree two K3 surfaces.

§4. The Octagon and the operator $\Lambda_{\{2\},\{3,4\}}$.

4.1 The matroid M_8 constructed from the regular octagon.

Consider the 16 lines in Figure 2: the black lines $\ell_1, \dots, \ell_8$ are the 8 lines of the regular octagon $\mathcal{C}_1$ and the blue lines $\ell'_1, \dots, \ell'_8$ are the 8 lines symmetries of $\mathcal{C}_0$. The image of $\mathcal{C}_0 = \ell_1 + \cdots + \ell_8$ by the operator $\Lambda_{\{2\},\{3,4\}}$ is the line arrangement $\mathcal{C}_1 = \ell'_1 + \cdots + \ell'_8$.

The 8 lines ℓ_i, ℓ_j of $\mathcal{C}_0 = (\ell_1, \dots \ell_8)$ meet in 28 double points denoted by $p_{i,j}$ (some points are at infinity). The lines $\ell'_1, \dots, \ell'_8$ are the lines containing the points in sets $S_1, \dots, S_8$ which are respectively

$$\begin{gathered}\{p_{1,8}, p_{2,7}, p_{3,6}, p_{4,5}\}, \{p_{1,7}, p_{2,6}, p_{3,5}\}, \{p_{1,6}, p_{2,5}, p_{3,4}, p_{7,8}\}, \{p_{1,5}, p_{2,4}, p_{6,8}\}, \\ \{p_{1,4}, p_{2,3}, p_{5,8}, p_{6,7}\}, \{p_{1,3}, p_{4,8}, p_{5,7}\}, \{p_{1,2}, p_{3,8}, p_{4,7}, p_{5,6}\}, \{p_{2,8}, p_{3,7}, p_{4,6}\}.\end{gathered} \tag{4.1}$$

These sets S_k, $k = 1, \dots, 8$ form a partition of the 28 double points of $\mathcal{C}_0$; these 28 points are the triple points of $\mathcal{C}_0 \cup \mathcal{C}_1$. One has the relation $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0) = \mathcal{C}_1$ (as unlabeled line arrangements).

Let M_8 be the matroid associated to the incidences between the 16 labeled lines $\ell_1, \dots, \ell_8, \ell'_1, \dots, \ell'_8$ and the 28 triple points: it is obtained from the matroid associated to the labeled line arrangement $\mathcal{C}_0 \cup \mathcal{C}_1$, but we discard all non-bases coming from the central point, so that M_8 has 16 atoms and only 28 non-bases. We denote by $\mathcal{R}_8$ the moduli space of realizations of M_8 (over $\mathbb{C}$).

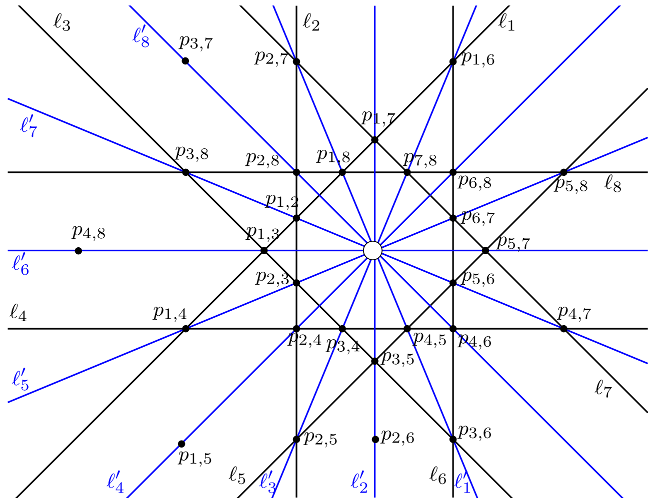

Figure 2.
Matroid M_8 (ℓ_i and ℓ_{i+4} meet at infinity at point $p_{i,i+4}$).

REMARK 4.1. A priori, there is no canonical choice for the labelings of the lines $\ell'_1, \dots, \ell'_8$ in the unlabeled line arrangement $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$. The choice we made in Equation (4.1) will be justified later, see Remark 4.6.

4.2 The moduli space $\mathcal{R}_8$ of M_8.

A direct computation in OSCAR shows that the moduli space $\mathcal{R}_8$ is two-dimensional, and an open sub-set of the quartic surface Z_8 in $\mathbb{P}^3$ with the equation

$$y_1y_2^2y_3 - y_1^2y_2y_4 + y_1y_2^2y_4 + y_1^2y_3y_4 - 2y_1y_2y_3y_4 - y_1y_3^2y_4 + y_1y_3y_4^2 - y_2y_3y_4^2 + y_3^2y_4^2 = 0.$$

The surface Z_8 has singularities $A_2, A_2, A_3, A_4, A_3, A_1$ at the respective points

$$(1:0:0:0):(0:1:0:0):(0:0:1:0):(0:0:0:1):(1:1:1:1):(1:0:1:0).$$

Its minimal desingularization Z_8^s is a K3 surface. The realization $\mathcal{A}(x)$ corresponding to a generic point $x = (x_1, x_2, x_3)$ of Z_8 in the affine chart $\mathbb{A}^3 = \{y_4 \neq 0\}$ is the union $\mathcal{A}(x) = \mathcal{C}_0(x) \cup \mathcal{C}_1(x)$, where $\mathcal{C}_0(x)$ is the line arrangement with eight lines with normal vectors the four vectors of the canonical basis and the following four vectors

$$\begin{gathered}(x_1 - x_2 : x_1^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1 - x_2x_3 - x_2 + x_3),\\ (x_1x_2 - x_1x_3 - x_2 + x_3 : x_1x_2^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1x_2x_3 - 2x_1x_3 + x_1 - x_2 + x_3)\\ (x_1 - 1 : x_1x_2 - x_2 : x_1 - x_2), (1 : x_1 : x_3).\end{gathered}$$

Moreover, $\mathcal{C}_1(x)$ is the line arrangement with normal vectors

$$\begin{gathered}(x_1x_2 - x_1x_3 - x_2 + x_3 : x_1x_2^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1x_2 - x_1x_3 - x_2^2 + x_2x_3),\\ (x_1^2x_2 - x_1^2x_3 - x_1x_2^2 + x_1x_2x_3 - x_1x_2 + x_1x_3 + x_2^2 - x_2x_3 : x_1^3x_2 - x_1^3x_3 - x_1^2x_2^2 + x_1^2x_3^2\\ +2x_1x_2x_3 - x_1x_2 - 2x_1x_3^2 + x_1x_3 + x_2^2 - 2x_2x_3 + x_3^2 : x_1^2x_2x_3 - x_1^2x_2 - x_1^2\\ x_3 + x_1^2 + x_1x_2^2 - 2x_1x_2 - x_1x_3^2 + 2x_1x_3 + x_2^2 - 2x_2x_3 + x_3^2),\\ (x_1 - x_2 : x_1 - x_2 : x_1 - x_2x_3 - x_2 + x_3), (x_3 : x_1x_2 : x_3), (0:1:1),\\ (x_1 - 1 : 0 : x_1 - x_3), (1 : x_1 : 0), (1 : x_2 : x_3).\end{gathered}$$

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From the definition of the matroid M_8, if $\mathcal{A}_0=\mathcal{C}_0\cup\mathcal{C}_1$ is a realization of M_8 and $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) denotes its first (resp. last) eight lines then, $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)=\mathcal{C}_1$ as unlabeled line arrangements. The following operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ gives a labeling to $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$:

DEFINITION 4.2. The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ associates to a labeled line arrangement L_8 of 8 lines $\ell_1,\dots,\ell_8$, the labeled line arrangement $\ell'_1,\dots,\ell'_8$ where ℓ'_j is the set of lines containing all the points in S_k defined in (4.1) (ℓ'_j is a line or the empty set).

For a generic arrangement L_8 of eight lines, one has $\Lambda^{\ell}_{\{2\},\{3,4\}}(L_8)=\emptyset$. The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ is constructed so that if $\mathcal{A}_0$ is any realization of M_8 and $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) denotes its first (resp. last) eight lines then $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_0)=\mathcal{C}_1$ as labeled line arrangements (and of course $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)=\mathcal{C}_1$ if one forgets the labels).

4.3 The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ acts as a rational self-map on $\mathcal{R}_8$.

A priori the line arrangement $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ could be empty, however:

THEOREM 4.3. *Suppose that $\mathcal{A}_0=\mathcal{C}_0\cup\mathcal{C}_1$ is generic among the realizations of M_8. Then the labeled line arrangement $\mathcal{C}_2=\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ has 8 lines and $\mathcal{A}_1=\mathcal{C}_1\cup\mathcal{C}_2$ is a realization of $\mathcal{R}_8$.*

Proof. Using the function field of $\mathcal{R}_8$, we realize the generic element of $\mathcal{R}_8$ using the formulas for $\mathcal{A}(x)$. Then we compute $\mathcal{C}_2=\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ and obtain eight lines. Finally we check that $\mathcal{C}_1\cup\mathcal{C}_2$ defines the same matroid as $\mathcal{A}_0$. ∎

The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ acts on realizations of M_8, sending $\mathcal{A}_0=\mathcal{C}_0\cup\mathcal{C}_1$ to $\mathcal{A}_1=\mathcal{C}_1\cup\mathcal{C}_2$, where $\mathcal{C}_2=\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$. It therefore acts on the moduli space Z_8: we denote by

$$\lambda_{\{2\},\{3,4\}}: Z_8 \dashrightarrow Z_8$$

that action. In order to obtain the explicit polynomials defining $\lambda_{\{2\},\{3,4\}}$, we remark that one may recover the coordinates x_1,x_2,x_3 of the line arrangement $\mathcal{A}_0(x)$ from the two last normal vectors $(1:x_1:0),(1:x_2:x_3)$ of $\mathcal{C}_1(x)$. Then one computes the unique line arrangement $\tilde{\mathcal{C}_1}\cup\tilde{\mathcal{C}_2}$ projectively equivalent to $\mathcal{A}_1(x)=\mathcal{C}_1(x)\cup\mathcal{C}_2(x)$ such that the first four normal vectors are the canonical basis. The image of x by $\lambda_{\{2\},\{3,4\}}$ is the point $x'=(x'_1,x'_2,x'_3)$ such that the two last normal vectors of $\tilde{\mathcal{C}_2}$ are $(1:x'_1:0),(1:x'_2:x'_3)$ (and $\tilde{\mathcal{C}_1}\cup\tilde{\mathcal{C}_2}=\mathcal{A}_0(x')$). Taking the homogenization to $\mathbb{P}^3$, one obtains that the map $\lambda_{\{2\},\{3,4\}}$ is defined by the four degree 10 coprime polynomials $P_1,\dots,P_4$ given in the ancillary file of the arXiv version of this paper. The base points of $\lambda_{\{2\},\{3,4\}}$ are

$$(-\sqrt{2}-1:\sqrt{2}+2:2\sqrt{2}+3:1),(\sqrt{2}-1:-\sqrt{2}+2:-2\sqrt{2}+3:1),$$
$$(i:0:1:1),(-i:0:1:1),(1:1:0:1),(0:1:1:0),(0:1:0:1).$$

The line arrangements $\mathcal{C}_0\cup\mathcal{C}_1$ associated to the first two points are the regular octagon and its lines of symmetries. The line arrangements $\mathcal{C}_0$ associated to the third and fourth points are such that $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$ is the Ceva line arrangement with 12 lines; it contains $\mathcal{C}_0$.

Using the explicit polynomials $P_1,\dots,P_4$, we obtain that:

PROPOSITION 4.4. *The degree of the rational self-map $\lambda_{\{2\},\{3,4\}}$ on Z_8^s is 4.*

Proof. We again apply the method from [15]. Let $f(x_1,x_2,x_3)$ be the equation of the quartic Z_8 in the chart $U_4: y_4\neq 0$. The space of global non-vanishing differential 2-forms

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is generated by a form ω, which one can choose so that on an open set of U_4 one has: $\omega = \frac{dx_2\wedge dx_3}{\partial f/\partial x_1}$. The rational self-map $\lambda_{\{2\},\{3,4\}}$ preserves U_4. A direct computation gives that $\lambda^*_{\{2\},\{3,4\}}\omega = -2\omega$. The pull-back by $\lambda_{\{2\},\{3,4\}}$ of the volume form $\omega\bar{\omega}$ is therefore $4\omega\bar{\omega}$, thus the degree of $\lambda_{\{2\},\{3,4\}}$ is 4. ∎

4.4 The dynamical system $(Z_8,\lambda_{\{2\},\{3,4\}})$ is semi-conjugated to the plane.

The four polynomials $P_1,\dots,P_4$ such that $\lambda_{\{2\},\{3,4\}} = (P_1 : \dots : P_4)$ verify $\deg_{y_1}(P_1) = 1$ and $\deg_{y_k}(P_1) = 0$ for $k \geq 2$. Let $\pi : Z_8 \dashrightarrow \mathbb{P}^2$ be the double cover obtained by projecting from the double point $(1:0:0:0)$ of Z_8.

LEMMA 4.5. *The branch curve B of π is the union of the conic $C = \{z_1^2 - z_2z_3 = 0\}$ and the quartic curve*

$$Q = \{z_1^2z_2^2 + 2z_1^2z_2z_3 - 4z_1z_2^2z_3 - z_2^3z_3 + z_1^2z_3^2 - 4z_1z_2z_3^2 + 6z_2^2z_3^2 - z_2z_3^3\}.$$

Proof. The ramification locus of π_1 is the discriminant of the equation of Z_8 with respect to the variable y_1. The image of the ramification curve by π_1 is the curve B. ∎

The quartic Q has geometric genus 0 and is singular at the points $(1:0:0),(1:1:1)$ with singularities A_3 and A_1. The curve $B = C + Q$ is singular at the points

$$(1:0:0),(0:1:0),(0:0:1),(1:1:1)$$

with singularities A_3, A_5, A_5, D_4, respectively.

Let us define the polynomials $Q_k = P_{k+1}(0,z_1,z_2,z_3)$ $(k = 1,2,3)$ and the rational self-map $\mu : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $\mu = (Q_1 : Q_2 : Q_3)$. One has $\mu^*(B) = B + 2D$ for a degree 27 curve D. Using §2.3, the double cover of $\mathbb{P}^2$ branched over $B = C + Q$ is birational to the surface Z_8 and $(Z_8,\lambda_{\{2\},\{3,4\}})$ is semi-conjugated to $(\mathbb{P}^2,\mu)$.

The indetermination points of μ are the 9 points

$$\begin{gathered}(1:0:0),(0:1:0),(0:0:1),(1:1:1),(1:0:1),(1:1:0),\\ (0:1:1),(-\sqrt{2}+2:-2\sqrt{2}+3:1),(\sqrt{2}+2:2\sqrt{2}+3:1).\end{gathered}$$

The image by μ of Q is the conic C; the rational map μ restricts to the identity on C.

REMARK 4.6. The choice for the labelings of the lines in the unlabeled line arrangement $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$ was made so that the defining polynomials of the rational self-map $\lambda_{\{2\},\{3,4\}}$ are of low degree. Moreover, for the other choices we tried, the degrees of the polynomials defining the analog of $\lambda_{\{2\},\{3,4\}}$ with respect to any variables y_i were never $1,0,0,0$, so that it was not possible to understand that rational self-map $\lambda_{\{2\},\{3,4\}}$ as a semi-conjugacy with the plane.

4.5 The K3 surface Z_8 and the modular surface $\Xi_1(8)$.

One has:

PROPOSITION 4.7. *The K3 surface Z_8^s is the unique K3 surface with discriminant -8 and Picard number 20.*

Proof. The eight lines with equations

$$\begin{gathered}(y_1 = y_3 = 0),(y_1 = y_4 = 0),(y_2 = y_3 = 0),(y_2 = y_4 = 0),(y_3 = y_4 = 0),\\ (y_1 - y_4 = y_2 - y_4 = 0),(y_1 - y_3 = y_2 - y_4 = 0),(y_2 - y_4 = y_3 - y_4 = 0)\end{gathered}$$

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are contained in the surface Z_8. Using Magma, one can compute that their strict transforms on Z_8^s together with the $15(-2)$-curves coming from the resolutions of the singularities of Z_8, generate a rank 20 lattice with discriminant -8. There is no K3 surface with Picard number 20 and discriminant -2 and there is a unique K3 surface with Picard number 20 and discriminant -8 (see, e.g., [10]) which yields the conclusion. ∎

PROPOSITION 4.8. *The surface Z_8 is (isomorphic to) the elliptic modular surface $\Xi_1(8)$ above the modular curve $X_1(8)$.*

Proof. The projection map from the line $y_2 - y_4 = y_3 - y_4 = 0$ induces a fibration $Z_8 \to \mathbb{P}^1$. By evaluating the Equation of Z_8 at $(X, 1+t(Y-1), Y, 1)$, one gets the cubic affine model

$$(t-1)X^2 - t^2XY^2 + XY + (t-1)^2X + (t-1)Y = 0,$$

of the generic fiber, where t is the parameter of $\mathbb{P}^1$. One computes that the Weierstrass model of it is the elliptic curve

$$E : y^2 = x^3 + (4t^4 - 8t^3 + 4t^2 + 1)/t^4x^2 + 8(t-1)^2/t^6x + 16(t-1)^4/t^8.$$

The associated elliptic surface is a smooth model of the K3 surface Z_8: it is isomorphic to Z_8^s. One computes that the singular fibers of the fibration are $2I_8 + I_4 + I_2 + 2I_1$, at the points $1, 0, \infty, 1/2, t^2 - t - 1/4 = 0$, respectively.

By [13, §2.3.3], the equation of a Weierstrass model of the elliptic surface $\Xi_1(8)$ above the modular curve $X_1(8)$ is

$$E' : \eta^2 = \xi^3 + (2 - s^2)\xi^2 + \xi,$$

where $s = 2t^2/(t^2-1)$. To check that $\Xi_1(8)$ is isomorphic to Z_8^s, one just has to compare the two j-invariants $j(E)(t) \in \mathbb{Q}(t)$ and $j(E')(t) \in \mathbb{Q}(t)$. We compute that $j(E)(\frac{1}{2}(1-\frac{1}{t})) = j(E')(t)$, therefore E is isomorphic to E', and $\Xi_1(8) \simeq Z_8^s$. ∎

4.6 Action of aut(M_8).

The automorphism group of M_8 is generated by the involutions

$$s_1 = (2,4)(3,7)(6,8)(9,11)(10,14)(13,15), s_2 = (2,6)(4,8)(9,13)(11,15),$$
$$s_3 = (1,2)(3,8)(4,7)(5,6)(9,13)(10,12)(14,16).$$

The group aut(M_8) is the semi-direct product $\mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. One computes that it acts faithfully on the K3 surface Z_8. The map s_2 (acting on Z_8) is given in the ancillary file of the arXiv version of this paper. It is a birational involution of $\mathbb{P}^3$.

The group of elements σ commuting with the action of $\lambda_{\{2\},\{3,4\}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The involution $s = (1,5)(2,6)(3,7)(4,8)$ is the unique automorphism of aut(M_8) such that $\lambda_{\{2\},\{3,4\}} \circ s = \lambda_{\{2\},\{3,4\}}$.

4.7 Periodic line arrangements.

Let us prove:

PROPOSITION 4.9. *The surface Z_8 contains a curve C_3 of geometric genus 5 such that each point of C_3 is fixed by $\lambda_{\{2\},\{3,4\}}$ and for a generic point x of C_3, the associated line arrangement $\mathcal{C}_0(x)$ in $\mathbb{P}^2$ is periodic of period 3 for the action of $\Lambda^{\ell}_{\{2\},\{3,4\}}$.*

REMARK 4.10. We recall that $\Lambda_{\{2\},\{3,4\}}$ is an operator acting on line arrangements, whereas $\lambda_{\{2\},\{3,4\}}$ is the rational self-map induced by $\Lambda_{\{2\},\{3,4\}}$: it acts on line arrangements

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modulo projective transformations. In particular, Proposition 4.9 implies that for a line arrangement $\mathcal{C}$ corresponding to a point on the curve C_3, one has $(\Lambda_{\{2\},\{3,4\}})^{\circ 3}(\mathcal{C}) = \mathcal{C}$ with $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}) \neq \mathcal{C}$, but $\Lambda_{\{2\},\{3,4\}}(\mathcal{C})$ is projectively equivalent to $\mathcal{C}$. The union of the line arrangements $(\Lambda_{\{2\},\{3,4\}})^{\circ k}(\mathcal{C})$, $k = 0,1,2$ has 24 lines with 84 triple points, 24 double points, and no other singularities.

Proof. We searched by random an example of a λ-fixed point x over a finite field and we found the point $x = (794 : 582 : 116 : 1) \in \mathbb{P}^3(\mathbb{F}_{1013})$ in the surface $(\mathcal{R}_8)_{/\mathbb{F}_{1013}}$. The corresponding line arrangement $\mathcal{C}_0 \cup \mathcal{C}_1$ is 3-periodic for the operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$: the line arrangements $\mathcal{C}_0 \cup \mathcal{C}_1$, $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_2 \cup \mathcal{C}_0$ are realizations of M_8 (over $\mathbb{F}_{1013}$), and $\mathcal{C}_{k+1\, mod\, 3} = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_k)$. One computes that the matroid N_{24} associated to $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ has an irreducible one dimensional moduli space $\mathcal{R}(N_{24})$ over $\mathbb{C}$ and that the geometric genus of the compactification of $\mathcal{R}(N_{24})$ is 5. Let $\mathcal{C}'_0 \cup \mathcal{C}'_1 \cup \mathcal{C}'_2$ be a realization (over $\mathbb{C}$) of N_{24}. From the combinatorics of M_8 and N_{24}, the line arrangements $\mathcal{C}'_0 \cup \mathcal{C}'_1$, $\mathcal{C}'_1 \cup \mathcal{C}'_2$ and $\mathcal{C}'_2 \cup \mathcal{C}'_0$ are realizations of M_8, and $\mathcal{C}'_{k+1 \text{ mod } 3} = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}'_k)$ if the realization is generic. The natural map $\mathcal{R}(N_{24}) \to \mathcal{R}_8$, which to a realization $\mathcal{C}'_0 \cup \mathcal{C}'_1 \cup \mathcal{C}'_2$ of N_{24} associates $\mathcal{C}'_0 \cup \mathcal{C}'_1$ is one-to-one onto its image (a curve denoted C_3) in $\mathcal{R}_8$, since one may recover $\mathcal{C}'_2$ (and therefore $\mathcal{C}'_0 \cup \mathcal{C}'_1 \cup \mathcal{C}'_2$) as $\mathcal{C}'_2 = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}'_1)$. A computer computation gives that C_3 has genus 5 and $\Lambda_{\{2\},\{3,4\}}(\eta)$ is projectively equivalent to η, where η is the generic point of C_3, thus any specialization η' is such that $\lambda_{\{2\},\{3,4\}}(\eta') = \eta'$, and the curve C_3 is point-wise fixed by $\lambda_{\{2\},\{3,4\}}$. ∎

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