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Continuous-time quantum walks on the threshold network model

YUSUKE IDE[†] and NORIO KONNO[‡§]

[†]*Department of Information Systems Creation, Faculty of Engineering,*
Kanagawa University,
Kanagawa, Yokohama 221-8686, Japan.
Email: ide@kanagawa-u.ac.jp
[‡]*Department of Applied Mathematics, Faculty of Engineering,*
Yokohama National University,
Hodogaya, Yokohama 240-8501, Japan.
Email: konno@ynu.ac.jp

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It is well known that many real world networks have a power-law degree distribution (the scale-free property). However, there are no rigorous results for continuous-time quantum walks on such realistic graphs. In this paper, we analyse the space–time behaviour of continuous-time quantum walks and random walks on the threshold network model, which is a reasonable candidate model having the scale-free property. We show that the quantum walker exhibits localisation at the starting point, although the random walker tends to spread uniformly.

1. Introduction

Continuous-time quantum walks, which are the quantum counterparts of classical random walks, have been widely studied on various deterministic graphs, such as the line (Konno 2005), star graph (Salimi 2009; Xu 2009), cycle graph (Ahmadi *et al.* 2003; Inui *et al.* 2005; Mülken and Blumen 2006), dendrimers (Mülken *et al.* 2006), spidernet graphs (Salimi 2010), the Dual Sierpinski Gasket (Agliari *et al.* 2008), the direct product of Cayley graphs (Salimi and Jafarizadeh 2009), quotient graphs (Salimi 2008a), odd graphs (Salimi 2008b), trees (Konno 2006a; Jafarizadeh and Salimi 2007) and ultrametric spaces (Konno 2006b). For further information, see reviews such as Konno (2008) and Venegas-Andraca (2008). There are also simulation-based studies of continuous-time quantum walks on probabilistic graphs, such as small-world networks (Mülken *et al.* 2007) and the Erdős–Rényi random graph (Xu and Liu 2008). However, there are no rigorous results for continuous-time quantum walks on such probabilistic graphs. In this paper, we focus on the continuous-time quantum walk on a random graph called the threshold network model.

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Many real-world networks (graphs) are characterised by small diameters, high clustering and power-law (scale-free) degree distributions (Albert and Barabási 2002; Newman 2003; Boccaletti *et al.* 2006). The threshold network model belongs to the so-called hidden variable models (Caldarelli *et al.* 2002; Söderberg 2002) and is known for being capable of generating scale-free networks. Their mean behaviour (Caldarelli *et al.* 2002; Söderberg 2002; Boguñá and Pastor-Satorras 2003; Masuda *et al.* 2004; Servedio *et al.* 2004; Hagberg *et al.* 2006; Fujihara *et al.* 2010) and limit theorems (Konno *et al.* 2005; Ide *et al.* 2007; Fujihara *et al.* 2009; Ide *et al.* 2010a) for the degree, the clustering coefficients, the number of subgraphs and the average distance have been analysed. The strong law of large numbers and central limit theorem for the rank of the adjacency matrix of the model with self-loops are given by Bose and Sen (2007). Eigenvalues and eigenvectors of the adjacency matrix (Ide *et al.* 2010b) and the Laplacian matrix (Merris 1994; Merris 1998) of the model have been studied. See also Mahadev and Peled (1995), Konno *et al.* (2005), Masuda *et al.* (2005), Masuda and Konno (2006), Ide *et al.* (2007), Diaconis *et al.* (2009) and Ide *et al.* (2010a) for related work.

This paper is organised as follows. We define the threshold network model and give a brief review of the hierarchical structure of the graph in Section 2. In Section 3, we define the continuous-time quantum walk on the threshold network model and a special setting called the binary threshold model. The main results are presented in this section and the proofs are given in Section 4, and results for continuous-time random walks on the models are given in Section 5. Finally, we summarise our results in Section 6.

2. The threshold network model

The *threshold network model* $\mathcal{G}_n(X,\theta)$ is a random graph on the vertex set

$$V = \{1, 2, \ldots, n\}.$$

Let $\{X_1, X_2, \ldots, X_n\}$ be independent copies of a random variable X with distribution $\mathbb{P}$. We draw an edge between two distinct vertices $i, j \in V$ if $X_i + X_j > \theta$ where $\theta \in \mathbb{R}$ is a constant called a threshold. From now on we will use $\mathbb{P}^\infty$ as the distribution of $\{X_i\}_{i=1}^\infty$.

Each sample graph $G \in \mathcal{G}_n(X,\theta)$ has a hierarchical structure described by the so-called creation sequence (Diaconis *et al.* 2009; Hagberg *et al.* 2006), which is defined as follows. Let $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$ be a rearranged sequence of random variables $X_1, X_2, \ldots, X_n$ in increasing order. If $X_{(1)} + X_{(n)} > \theta$, we have

$$\theta < X_{(1)} + X_{(n)} \leqslant X_{(2)} + X_{(n)} \leqslant \cdots \leqslant X_{(n-1)} + X_{(n)},$$

which means that the vertex corresponding to $X_{(n)}$ is connected to the $n-1$ other vertices. Otherwise, we have

$$\theta \geqslant X_{(1)} + X_{(n)} \geqslant \cdots \geqslant X_{(1)} + X_{(3)} \geqslant X_{(1)} + X_{(2)},$$

which means that the vertex corresponding to $X_{(1)}$ is isolated. We set $s_n = 1$ or $s_n = 0$ according to whether the former or latter case occurs. Then, depending on the case, we

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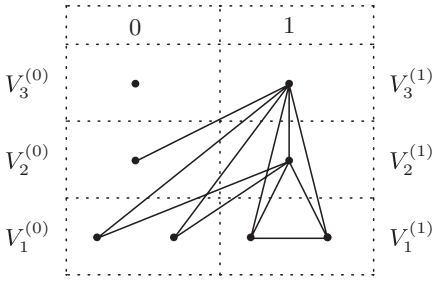

Fig. 1. A threshold graph G corresponding to $S_G = \{1,1,0,0,1,0,1,0\}$. $V_1^{(1)}$ is a set of two vertices located at the bottom right-hand corner. Similarly, $V_1^{(0)}$ is a set of two vertices located at the bottom left-hand corner, and so on.

remove the random variable $X_{(n)}$ or $X_{(1)}$, and then continue, using a similar procedure, to define $s_{n-1},\ldots,s_2$. Finally, we set $s_1 = s_2$ and obtain a $\{0,1\}$-sequence $\{s_1,s_2,\ldots,s_n\}$, which is called the *creation sequence* of G and is denoted by S_G.

Given a creation sequence S_G, let k_i and l_i denote the number of consecutive bits of 1's and 0's, respectively, as follows:

$$S_G = \{\overbrace{1,\ldots,1}^{k_1},\overbrace{0,\ldots,0}^{l_1},\overbrace{1,\ldots,1}^{k_2},\overbrace{0,\ldots,0}^{l_2},\ldots,\overbrace{1,\ldots,1}^{k_m},\overbrace{0,\ldots,0}^{l_m}\}. \tag{2.1}$$

It may happen that $k_1 = 0$ or $l_m = 0$, but we have $k_2,\ldots,k_m,l_1,\ldots,l_{m-1} \geqslant 1$ and $m \geqslant 1$. Moreover, by definition, we have two cases:

(a) $k_1 = 0$ (equivalently, $s_1 = 0$) and $l_1 \geqslant 2$;

(b) $k_1 \geqslant 2$ (equivalently, $s_1 = 1$).

For example, if $S_G = \{1,1,0,0,1,0,1,0\}$, then $k_1 = 2$, $l_1 = 2$, $k_2 = 1$, $l_2 = 1$, $k_3 = 1$, $l_3 = 1$, and Figure 1 shows the shape of G.

The creation sequence S_G gives rise to a partition of the vertex set:

$$V = \bigcup_{i=1}^{m} V_i^{(1)} \cup \bigcup_{i=1}^{m} V_i^{(0)} \qquad \sharp V_i^{(1)} = k_i\,, \quad \sharp V_i^{(0)} = l_i\,,$$

where $\sharp A$ is the number of elements in a set A. The subgraph induced by $V_i^{(1)}$ is the complete graph (that is, every pair of vertices is connected) of k_i vertices, and that induced by $V_i^{(0)}$ is the null graph (that is, the graph with no edges) of l_i vertices. Moreover, every vertex in $V_i^{(1)}$ is connected to, and every vertex in $V_i^{(0)}$ is disconnected from, all vertices in

$$V_1^{(1)} \cup \cdots \cup V_i^{(1)} \cup V_1^{(0)} \cup \cdots \cup V_{i-1}^{(0)}\,.$$

In general, a graph possessing the above hierarchical structure is called a *threshold graph* (Mahadev and Peled 1995).

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3. Our model and results

Let A_G be the adjacency matrix and D_G be the diagonal matrix of degrees (the sum of the rows of A_G) of $G \in \mathcal{G}_n(X, \theta)$. The Laplacian matrix L_G of G is then given by $L_G = D_G - A_G$. The time evolution operator $U^G_{n,t}$ of a continuous-time quantum walk on G is defined by

$$U^G_{n,t} = e^{itL_G} \equiv \sum_{k=0}^{\infty} \frac{(it)^k}{k!} L^k_G. \tag{3.2}$$

Let $\{\Psi^G_{n,t}\}_{t \geqslant 0}$ be the probability amplitude of the quantum walk, that is,

$$\Psi^G_{n,t} = U^G_{n,t} \Psi^G_{n,0},$$

and let $Y_{n,t}$ denote the position of the quantum walker at time t. The probability that the quantum walker on G is in position $y \in V$ at time t with initial condition $\Psi^G_{n,0}$ is then defined by

$$P^G_{n,t}(Y_{n,t} = y) \equiv |\Psi^G_{n,t}(y)|^2,$$

where $\Psi^G_{n,t} = {}^T\left[\Psi^G_{n,t}(1) \ \cdots \ \Psi^G_{n,t}(n)\right]$. In this paper, we use ${}^T A$ as the transpose of a matrix A to distinguish between the transpose and the power of a matrix. From now on, we will write $P^G_{n,t}(y)$ instead of $P^G_{n,t}(Y_{n,t} = y)$ for simplicity.

The time evolution operator is obtained as follows.

Theorem 3.1. Suppose $G \in \mathcal{G}_n(X, \theta)$ is connected. The time evolution operator $U^G_{n,t}$ of the continuous-time quantum walk on G is given by

$$\left(U^G_{n,t}\right)_{v,w} = \begin{cases} \left(U^G_{n,t}\right)^{(1,i)}_{v,w} & \text{if } v \in V^{(1)}_i, \ w \in \left\{\bigcup_{j=1}^{i-1} \left(V^{(1)}_j \cup V^{(0)}_j\right)\right\} \cup V^{(1)}_i \\ \left(U^G_{n,t}\right)^{(0,i)}_{v,w} & \text{if } v \in V^{(0)}_i, w \in \bigcup_{j=1}^{i} \left(V^{(1)}_j \cup V^{(0)}_j\right). \end{cases}$$

Here $A_{v,w}$ denotes the (v, w) element of a matrix A and

$$\begin{aligned} \left(U^G_{n,t}\right)^{(1,i)}_{v,w} = & \left(I_{\{v\}}(w) - \frac{1}{D_{k_i} - D_{l_i} + 1}\right) e^{it(D_{k_i}+1)} \\ & + \sum_{j=i+1}^{m} \frac{(D_{l_{j-1}} - D_{l_j}) e^{it(D_{k_j}+1)}}{(D_{k_j} - D_{l_{j-1}} + 1)(D_{k_j} - D_{l_j} + 1)} \\ & + \sum_{j=i}^{m-1} \frac{(D_{k_{j+1}} - D_{k_j}) e^{itD_{l_j}}}{(D_{k_j} - D_{l_j} + 1)(D_{k_{j+1}} - D_{l_j} + 1)} + \frac{1}{n} \end{aligned}$$

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$$\left(U_{n,t}^{G}\right)_{v,w}^{(0,i)} = \left(I_{\{v\}}(w) - \frac{1}{D_{k_{i+1}} - D_{l_i} + 1}\right) e^{itD_{l_i}} + \sum_{j=i}^{m} \frac{(D_{l_{j-1}} - D_{l_j}) e^{it(D_{k_j}+1)}}{(D_{k_j} - D_{l_{j-1}} + 1)(D_{k_j} - D_{l_j} + 1)} + \sum_{j=i+1}^{m-1} \frac{(D_{k_{j+1}} - D_{k_j}) e^{itD_{l_j}}}{(D_{k_j} - D_{l_j} + 1)(D_{k_{j+1}} - D_{l_j} + 1)} + \frac{1}{n}.$$

Where D_{k_i} and D_{l_i} denote the degrees of the vertices in $V_i^{(1)}$ and $V_i^{(0)}$, respectively, and $I_A(x)$ is the indicator function of a set A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ otherwise.

Because $U_{n,t}^{G}$ is a symmetric matrix, Theorem 3.1 covers all elements of the matrix.

Theorem 3.1 shows that we can, at least in principle, obtain the probability of the quantum walker in position $y \in V$ at time t for any initial conditions. However, in general it is hard to obtain the probability. In this paper, we analyse the behaviour of quantum walks starting from a vertex v, that is, for

$$\Psi_{n,0}(s) = \begin{cases} 1 & \text{if } s = v \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2. Suppose G is connected. The limit of the probability of the quantum walker starting from a vertex $v \in V_m^{(1)}$ is given by

$$\lim_{n\to\infty} P_{n,t}^{G}(y) = \begin{cases} 1 & \text{if } y = v \\ 0 & \text{otherwise.} \end{cases}$$

Because $P_{n,t}^{G}$ does not converge for $t \to \infty$, we study the time-averaged probability $\bar{P}_n^{G}(y)$ defined by

$$\bar{P}_n^{G}(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T P_{n,t}^{G}(X_{n,t} = y) dt.$$

Theorem 3.3. Suppose $G \in \mathcal{G}_n(X, \theta)$ is connected. The time-averaged probability $\bar{P}_n^{G}(y)$ of the quantum walker on G starting from a vertex $v \in V_m^{(1)}$ is

$$\bar{P}_n^{G}(y) = \begin{cases} \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} & \text{if } y = v \\ \frac{2}{n^2} & \text{otherwise.} \end{cases}$$

If we consider graphs containing vertices with degree $n-1$, then, as long as we start from the vertex, we obtain the same results as Theorems 3.2 and 3.3. These results state that a strong localisation occurs at the starting point. An interesting problem for future work is to find intuitive insights explaining this localisation; we are currently studying an extension to the general starting point. But if we consider a special setting of the model discussed in the next paragraph, we can obtain the transition probabilities for any starting points.

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In order to study more detailed properties of the quantum walk on the model, we focus on the threshold network model $\mathcal{G}_n(X,\theta)$ defined by Bernoulli trials with success probability $p \in (0,1)$, that is, $\mathbb{P}(X=1) = 1-\mathbb{P}(X=0) = p$, and a threshold $\theta \in [0,1)$. We call this model the *binary threshold model* $\mathcal{G}_n(p)$. For each $G \in \mathcal{G}_n(p)$, that is, realisation G of $\mathcal{G}_n(p)$, we consider the following partition of the vertex set V:

$$V = V_G^{(1)} \cup V_G^{(0)}, \quad V_G^{(1)} = \{i : X_i = 1\}, \quad V_G^{(0)} = \{i : X_i = 0\}.$$

It is easy to see that the subgraph induced by $V_G^{(1)}$ is the complete graph on $k_G \equiv \sharp V_G^{(1)}$ vertices, and the subgraph induced by $V_G^{(0)}$ is the null graph on $l_G \equiv \sharp V_G^{(0)}$ vertices. Moreover, every vertex in $V_G^{(1)}$ is connected to all vertices in $V_G^{(0)}$. Note that this is the case of $m=2$, $k_1 = l_2 = 0$, $k_2 = k_G$ and $l_1 = l_G$ in Equation (2.1).

The time evolution operator of the continuous-time quantum walk on the binary threshold model is obtained as follows.

Theorem 3.4. The time evolution operator $U_{n,t}^G$ of the continuous-time quantum walk on $G \in \mathcal{G}_n(p)$ is given by

$$U_{n,t}^G = \begin{bmatrix} U_{k_G,k_G} & U_{k_G,l_G} \\ U_{l_G,k_G} & U_{l_G,l_G} \end{bmatrix}.$$

The elements of $U_{n,t}^G$ are

$$\begin{aligned} U_{k_G,k_G} &= e^{int} I_{k_G} + \frac{1-e^{int}}{n} \mathbf{1}_{k_G,k_G} \\ U_{k_G,l_G} &= \frac{1-e^{int}}{n} \mathbf{1}_{k_G,l_G} \\ U_{l_G,k_G} &= \frac{1-e^{int}}{n} \mathbf{1}_{l_G,k_G} \\ U_{l_G,l_G} &= e^{ik_G t} I_{l_G} + \left(\frac{1}{n} + \frac{k_G e^{int}}{n l_G} - \frac{e^{ik_G t}}{l_G} \right) \mathbf{1}_{l_G,l_G}, \end{aligned}$$

where $\mathbf{1}_{i,j}$ is the $i \times j$ matrix consisting of only 1, and I_i is the $i \times i$ identity matrix.

On the binary threshold model, a strong localisation is observed at any starting point as follows.

Proposition 3.5. The limit of the probability of the quantum walker starting from a vertex $v \in V$ is given by

$$\lim_{n\to\infty} P_{n,t}^G(y) = \begin{cases} 1 & \text{if } y = v \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \mathbb{P}^\infty\text{-almost every } G.$$

Note that if the quantum walker starts from a vertex $v \in V_G^{(1)}$, the statement of Proposition 3.5 is the same as Theorem 3.2.

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The time-averaged probability of the quantum walker on the binary threshold model is obtained as follows.

Proposition 3.6. The time-averaged probability $\bar{P}_n^G(y)$ of the quantum walker on $G \in \mathcal{G}_n(p)$ starting from a vertex $v \in V_G^{(0)}$ is given by

$$\bar{P}_n^G(y) = \begin{cases} \left(1 - \frac{1}{l_G}\right)^2 + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2} & \text{if } y = v \\ \frac{1}{l_G^2} + \left(\frac{k_G}{nl_G}\right)^2 + \frac{1}{n^2} & \text{if } y \in V_G^{(0)} \setminus \{v\} \\ \frac{2}{n^2} & \text{otherwise.} \end{cases}$$

Note that when the quantum walker starts from a vertex $v \in V_G^{(1)}$, we can use Theorem 3.3.

4. Proofs

The Laplacian matrix L_G of $G \in \mathcal{G}_n(p)$ is given by

$$L_G = \begin{bmatrix} nI_{k_G} - \mathbf{1}_{k_G,k_G} & -\mathbf{1}_{k_G,l_G} \\ -\mathbf{1}_{l_G,k_G} & kI_{l_G} \end{bmatrix}.$$

The eigenvalues and eigenvectors of L_G are known (Hagberg *et al.* 2006; Merris 1994; Merris 1998), and are as follows:

eigenvalue	eigenvectors
n	$\mathbf{v}_j \equiv \frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{n-j-1,1} \end{bmatrix}$ $(1 \leqslant j \leqslant k_G - 1)$,
	$\mathbf{v}_{k_G} \equiv \frac{1}{\sqrt{nk_Gl_G}} \begin{bmatrix} l_G\mathbf{1}_{k_G,1} \\ -k_G\mathbf{1}_{l_G,1} \end{bmatrix}$
k_G	$\mathbf{w}_j \equiv \frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{k_G,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{l_G-j-1,1} \end{bmatrix}$ $(1 \leqslant j \leqslant l_G - 1)$
0	$\mathbf{w}_{l_G} \equiv \frac{1}{\sqrt{n}} \begin{bmatrix} \mathbf{1}_{n,1} \end{bmatrix}$,

where $\mathbf{0}_{i,j}$ is the $i \times j$ zero matrix. Note that the set of these eigenvectors forms an orthonormal basis of $\mathbb{R}^n$. Thus we can define an orthogonal matrix B_G corresponding to the eigenvectors as follows:

$$B_G = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_{k_G} & \mathbf{w}_1 & \dots & \mathbf{w}_{l_G} \end{bmatrix}. \tag{4.3}$$

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Because $U^G_{n,t}$ is an $n \times n$ (finite) matrix, using Equations (3.2) and (4.3), it can be represented by

$$U^G_{n,t} = B_G \begin{bmatrix} e^{int} I_{k_G} & \mathbf{0}_{k_G,l_G} & \mathbf{0}_{k_G,1} \\ \mathbf{0}_{l_G-1,k_G} & e^{ik_G t} I_{l_G-1} & \mathbf{0}_{l_G-1,1} \\ \mathbf{0}_{1,k_G} & \mathbf{0}_{1,l_G-1} & 1 \end{bmatrix} {}^T B_G.$$

It is easy to see that

$$U^G_{n,t} = \begin{bmatrix} e^{int} I_{k_G} + \left[\left(-\frac{1}{k_G} + \frac{l_G}{nk_G} \right) e^{int} + \frac{1}{n} \right] \mathbf{1}_{k_G,k_G} & \frac{1}{n} \left(1 - e^{int} \right) \mathbf{1}_{k_G,l_G} \\ \frac{1}{n} \left(1 - e^{int} \right) \mathbf{1}_{l_G,k_G} & e^{ik_G t} I_{k_G} + \left[-\frac{1}{l_G} e^{ik_G t} + \frac{k_G}{nl_G} e^{int} + \frac{1}{n} \right] \mathbf{1}_{l_G,l_G} \end{bmatrix}.$$

Using the relation $-1/k_G + l_G/nk_G = -1/n$, we can then obtain Theorem 3.4.

By simple calculations, we then have

$$\left| \left(1 - \frac{1}{n} \right) e^{int} + \frac{1}{n} \right|^2 = 1 - \frac{2}{n} \left(1 - \frac{1}{n} \right) (1 - \cos nt)$$

$$\left| \frac{1}{n} \left(1 - e^{int} \right) \right|^2 = \frac{2}{n^2} (1 - \cos nt)$$

$$\begin{aligned} f \left| \left(1 - \frac{1}{l_G} \right) e^{ik_G t} + \frac{k_G e^{int}}{nl_G} + \frac{1}{n} \right|^2 &= \left(1 - \frac{1}{l_G} \right)^2 + \left(\frac{k_G}{nl_G} \right)^2 + \frac{1}{n^2} \\ &\quad + \frac{2k_G}{nl_G} \left(1 - \frac{1}{l_G} \right) \left(\cos l_G t + \frac{l_G}{k_G} \cos k_G t \right) \\ &\quad + \frac{2k_G}{n^2 l_G} \cos nt \end{aligned}$$

$$\begin{aligned} \left| -\frac{e^{ik_G t}}{l_G} + \frac{k_G e^{int}}{nl_G} + \frac{1}{n} \right|^2 &= \frac{1}{l_G^2} + \left(\frac{k_G}{nl_G} \right)^2 + \frac{1}{n^2} \\ &\quad - \frac{2k_G}{nl_G^2} \left(\cos l_G t + \frac{l_G}{k_G} \cos k_G t - \frac{l_G}{n} \cos nt \right). \end{aligned}$$

On the other hand, $\lim_{n\to\infty} k_G/n = p$ and $\lim_{n\to\infty} l_G/n = 1 - p$, $\mathbb{P}^\infty$-almost surely by the strong law of large numbers for the i.i.d. Bernoulli sequence. Combining these facts and Theorem 3.4, we have Proposition 3.5. Also, we can obtain Proposition 3.6 immediately from

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \cos nt \; dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos k_G t \; dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos l_G t \; dt = 0.$$

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In the case of a connected $G \in \mathscr{G}_n(X, \theta)$, eigenvalues and eigenvectors of L_G are also known (Hagberg *et al.* 2006; Merris 1994; Merris 1998), and are as follows:

eigenvalue	eigenvectors
$D_{k_i} + 1$ $(2 \leqslant i \leqslant m)$	$\frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_i+k_i-j-1,1} \end{bmatrix}$ $(1 \leqslant j \leqslant k_i - 1)$ $\frac{1}{\sqrt{(k_i+d_i)k_i d_i}} \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ d_i \mathbf{1}_{k_i,1} \\ -k_i \mathbf{1}_{d_i,1} \end{bmatrix}$
D_{l_i} $(2 \leqslant i \leqslant m-1)$	$\frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{u_i,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_{i+1}-j-1,1} \end{bmatrix}$ $(1 \leqslant j \leqslant l_i - 1)$ $\frac{1}{\sqrt{(k_i+d_i)l_i d_{i+1}}} \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ (k_i + d_i)\mathbf{1}_{l_i,1} \\ -l_i \mathbf{1}_{k_i+d_i,1} \end{bmatrix}$
$D_{k_1} + 1$	$\frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{u_i+l_i} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{k_1-j-1,1} \end{bmatrix}$ $(j = 1, \ldots, k_1 - 1)$ if $k_1 \neq 0$
D_{l_1}	$\frac{1}{\sqrt{j(j+1)}} \begin{bmatrix} \mathbf{0}_{u_1,1} \\ \mathbf{1}_{j,1} \\ -j \\ \mathbf{0}_{d_2-j-1,1} \end{bmatrix}$ $(j = 1, \ldots, l_1 - 1)$
D_{l_1}	$\frac{1}{\sqrt{(k_1+d_1)l_1 d_2}} \begin{bmatrix} \mathbf{0}_{u_1+l_1} \\ (k_1 + d_1)\mathbf{1}_{l_1,1} \\ -l_1 \mathbf{1}_{k_1+d_1,1} \end{bmatrix}$ if $k_1 \neq 0$
0	$\frac{1}{\sqrt{n}} \left[\mathbf{1}_{n,1}\right]$,

where $u_i = \sum_{j>i}(k_j + l_j)$ and $d_i = \sum_{j<i}(k_j + l_j)$. We then have Theorem 3.1 using the same argument as in the proof of Theorem 3.4 and the following relations:

$$k_1 = D_{k_1} - D_{l_1} + 1 \text{ (if } k_1 \neq 0)$$
$$k_i = D_{l_{i-1}} - D_{l_i} \ (2 \leqslant i \leqslant m)$$
$$l_i = D_{k_{i+1}} - D_{k_i} \ (1 \leqslant i \leqslant m-1).$$

Note that $D_{k_m} = n - 1$ and $D_{l_m} = 0$ by assumption. Comparing Theorem 3.1 with Theorem 3.4, the transition probability of the walker on $G \in \mathscr{G}_n(p)$ starting from a vertex $v \in V_m^{(1)}$ is the same as that on $G \in \mathscr{G}_n(p)$ starting from a vertex $v \in V_G^{(1)}$. Thus we have Theorems 3.2 and 3.3.

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5. Classical case

The time evolution operator $\mathscr{U}_{n,t}^G$ of a continuous-time random walk on $G \in \mathscr{G}_n(X,\theta)$ is defined by

$$\mathscr{U}_{n,t}^G = e^{-tL_G} \equiv \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} L_G^k.$$

Let $\{\mathscr{P}_{n,t}^G\}_{t\geqslant 0}$ be the probability distribution of the random walk, that is, $\mathscr{P}_{n,t}^G = \mathscr{U}_{n,t}^G \mathscr{P}_{n,0}^G$. The yth element $P_{n,t}^G(y)$ of $P_{n,t}^G$ denotes the probability that the random walker is in position $y \in V$ at time t. By the same observation as in the previous section, we have the same results for $\mathscr{U}_{n,t}^G$ as Theorems 3.1 and 3.4 by exchanging it of $U_{n,t}^G$ for t. Using these results, we have the following.

Proposition 5.1. The limit of the probability that the random walker starting from a vertex $v \in V_m^{(1)}$ is given by

$$\lim_{n\to\infty} n\mathscr{P}_{n,t}^G(y) = 1 \quad \text{for all } y \in V \text{ and for } \mathbb{P}^\infty\text{-almost every } G.$$

Proposition 5.2. The long-time limit of the probability of the random walk on $G \in \mathscr{G}_n(p)$ starting from a vertex $v \in V$ is given by

$$\lim_{t\to\infty} \mathscr{P}_{n,t}^G(y) = \frac{1}{n} \quad \text{for all } y \in V.$$

We can also estimate the time-averaged probability $\bar{\mathscr{P}}_n^G(y)$. By simple calculation, we have $\bar{\mathscr{P}}_n^G(y) = 1/n$ for a random walk on $G \in \mathscr{G}_n(p)$ starting from a vertex $v \in V$.

6. Summary

In this paper, we have studied the continuous-time quantum and random walks on the threshold network model. By comparing Theorem 3.2 with Proposition 5.1, we have quite different limit behaviours as $n \to \infty$ for the two types of walks starting from a vertex with degree $n-1$. Although quantum walkers exhibit strong localisation at the starting point, random walkers tend to spread uniformly.

Theorem 3.3 and Proposition 3.6 show that the time-averaged probabilities of quantum walkers are not the uniform distribution (unlike the case for random walks). Furthermore, the time-averaged probability shows localisation at this starting point as $n \to \infty$. In the case of the binary threshold model, the rates of convergence are slightly different for the two starting points. Indeed, we have

$$\lim_{n\to\infty} n(1 - \bar{P}_n^G(v_1)) = 2 < \lim_{n\to\infty} n(1 - \bar{P}_n^G(v_0)) = 2/(1-p),$$

$\mathbb{P}^\infty$-almost surely for $v_0 \in V_G^{(0)}$ and $v_1 \in V_G^{(1)}$. A study covering the more general setting is now in progress. These results might be useful for data searching on some complex data structures or networks.

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