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KUMMER COVERS AND BRAID MONODROMY

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Abstract In this work, we describe a method to construct the generic braid monodromy of the preimage of a curve by a Kummer cover. This method is interesting since it combines two techniques, namely, the construction of a highly non-generic braid monodromy and a systematic method to go from a non-generic to a generic braid monodromy. The latter process, called *generification*, is independent from Kummer covers, and it can be applied in more general circumstances since non-generic braid monodromies appear more naturally and are oftentimes much easier to compute. Explicit examples are computed using these techniques.

Keywords: Kummer covers; braid monodromy; plane curves

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Introduction

A Kummer cover is a map $\pi_n : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_n([x : y : z]) := [x^n : y^n : z^n]$. Kummer covers are a very useful tool in order to construct complicated algebraic curves starting from simple ones. Since Kummer covers are finite Galois covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\mathrm{Gal}(\pi_n) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, topological properties of the new curves can be obtained: Alexander polynomial, fundamental group, characteristic varieties, and so on (see [3, 4, 17, 41, 25, 19] for papers using these techniques).

On the other hand, the generic braid monodromy of a plane projective curve is a powerful invariant that provides a way to compute the fundamental group of its complement, and it was originally described as a formalization of the Zariski–van Kampen method [42, 26] (see [35, 23]; also [18] and references therein for a detailed exposition on the subject). However, generic braid monodromies are much more powerful invariants, since in fact they encode the topology of the embedding of the curve, as well as the isomorphism problem for surfaces whose branching locus over $\mathbb{P}^2$ is a given curve (see [13, 29, 37, 15, 14, 28, 9, 21, 1], to name only a few).

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In this work, we focus on a method to construct the generic braid monodromy of the preimage of a curve by a Kummer cover. This method is interesting in and of itself, since it combines two techniques, namely, the construction of a highly non-generic braid monodromy and a systematic method to go from a non-generic to a generic braid monodromy. The latter process, called *generification*, is independent from Kummer covers, and it can be applied in more general circumstances (see also [38, 6, 2]). The reason for this is that, oftentimes, non-generic braid monodromies are much easier to compute, since they involve either braids with fewer strands or simply fewer braids, or a combination of both.

This paper provides, in particular, a constructive proof of the following results (see § 1 for the appropriate definitions).

Theorem 1. *Let* $\mathcal{C} = \{f(x, y, z) = 0\}$ *be a plane curve such that* $P = [0 : 0 : 1] \notin \mathcal{C}$*, and assume that its Kummer cover* $\mathcal{C}_n = \{f(x^n, y^n, z^n) = 0\}$ *contains only* $\mathbb{A}_2$*-singularities. Let* ∇ *be the braid monodromy given by* $(\mathcal{C} \cup L_z, L_\infty, P)$ *for a generic* L_∞*, and denote by* σ_x *(respectively,* σ_y*) the braid associated with the meridian of* L_x *(respectively,* L_y*) by* ∇*. Then* $(\nabla, \sigma_x, \sigma_y)$ *determines the generic braid monodromy of* $\mathcal{C}_n$*.*

The fundamental group of the complement of a curve under a generic Kummer cover was computed by Uludağ [41]. The following result shows that in fact it is possible to recover its generic braid monodromy.

Theorem 2. *Let* $\mathcal{C} = \{f(x, y, z) = 0\}$ *be a plane curve which is transversal to the union of the coordinate axes. Then the generic braid monodromy given by* $(\mathcal{C}, L_\infty, P)$ *determines the generic braid monodromy of its Kummer cover* $\mathcal{C}_n$*.*

The core of the method proposed in this paper provides a constructive proof of the following result.

Theorem 3. *Let* $\mathcal{C} = \{f(x, y, z) = 0\}$ *be a plane curve such that* $P = [0 : 0 : 1] \notin \mathcal{C}$*, and let* ∇*,* σ_x*, and* σ_y *be as in Theorem 1. Then* $(\nabla, \sigma_x, \sigma_y)$ *determines the braid monodromy given by* $(\mathcal{C}_n, L_\infty, P)$*.*

As an application of this method, it is worth recalling that Libgober (see [34]) proposed a class of invariants for algebraic plane curves via representations of *generic* braid monodromies. In order to obtain such invariants, one needs to acquire both interesting braid representations and generic braid monodromies. In [34], the author uses the Burau representation to obtain basically the Alexander polynomial (up to a factor encoding only on the degree of the curve). Later, with the works of Bigelow, Lawrence, and Krammer [12, 27, 30], new (linear and faithful) representations of the braid group have been found; they can be used to obtain invariants using the above-mentioned method. This paper provides an effective tool to find a wide variety of generic braid monodromies of curves from simpler ones. The method presented here will provide a large range of highly non-trivial examples where these Libgober invariants can be successfully applied.

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We briefly recall that any direct computation of the braid monodromy of a curve (that is, using the Zariski–van Kampen method) requires one to calculate the discriminant Δ of a generic projection of the degree-d polynomial $f(x, y)$ defining the curve, with respect to a variable, say x. Once this is obtained, one needs a system of generators of the complement of Δ in $\mathbb{C}$. The braids are obtained as the 1-parametric collection of d distinct roots of $f(x(t), y)$ for $t \in [0, 1]$ and $x(t) \subset \mathbb{C} \setminus \Delta$ one of the aforementioned generators. This requires some kind of appropriate polynomial root approximation method ensuring the exactness of the result. This is in practice a very complicated issue, which can only be dealt with in particularly simple cases, such as curves whose equation can be given in $\mathbb{Z}[x, y, z]$ ([11, 13]), strongly real curves ([7, p. 17] and references therein), or line arrangements (see [8, 39, 20, 40, 21]). One of the applications of the method presented here is to avoid numerical issues of a direct computation of the braid monodromy.

The previous paragraph justifies a fairly detailed section devoted to describing a list of applications of the method. In 7.1, generic braid monodromies of smooth curves of any degree are computed providing a simpler approach to Moishezon's famous result [37]; the same applies to the Zariski sextics in 7.2. We also describe the generic braid monodromy of a sextic with six cusps not on a conic in 7.8; according to Degtyarev [22], this family is connected, and hence it completes the equisingular stratum of sextics with six cusps. As examples of highly non-real curves, we present the generic braid monodromies of a sextic with nine cusps (in 7.3) as well as the Hesse arrangement in 7.7 and generalized Ceva arrangements in 7.4.

The layout of the paper is as follows. After describing in § 1 the main objects to be used, such as generic and non-generic braid monodromies, Kummer covers and extended braid monodromies are described in § 2 as the main tool to recover a non-generic braid monodromy of a Kummer cover from a braid monodromy of the base. Useful *generification* techniques are described in § 4. Sections 3 and 5 are more technical. The first one describes useful connections between systems of generators for the braid group of k-strings, say on a base $\mathbb{C} \setminus \Delta$ ($k = \#\Delta$) and the braid group of kn-strings on a base $\mathbb{C} \setminus \pi^{-1}(\Delta)$, where π is the k-Kummer cover of $\mathbb{C}$, defined as $z \mapsto z^k$. The second one describes the properties and singularities of Kummer transforms. Theorems 1–3 are proved in § 6. Finally, a detailed account of numerous examples of the power of Kummer transforms is given in § 7, where generic braid monodromies of smooth curves, sextics with six cusps (on a conic and otherwise), sextics with nine cusps, Hesse, Ceva, and MacLane arrangements are provided.

1. Settings: braid monodromy

After the work of Zariski, braid monodromy was defined by Chisini [16], but it was necessary to wait for Moishezon [37] in order to get further applications of this invariant.

1.1. Construction

Let us fix a curve $\bar{\mathcal{C}} \subset \mathbb{P}^2$ of degree d, a point $P_y \in \mathbb{P}^2$, and a line $\bar{L}_\infty$ in such a way that $P_y \in \bar{L}_\infty$. We say that the curve is *horizontal* with respect to P_y if it does not contain

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any line through P_y; we assume $\bar{\mathcal{C}}$ to be horizontal. We consider a system of coordinates $[X:Y:Z]$ such that $P_y:=[0:1:0]$ and $\bar{L}_\infty:=\{Z=0\}$. We identify $\mathbb{C}^2\equiv\mathbb{P}^2\setminus\bar{L}_\infty$ with affine coordinates $(x,y)\equiv[x:y:1]$.

Let $F(x,y,z)=0$ be a reduced equation of $\bar{\mathcal{C}}$, $k:=\deg_y F$,

$$F(x,y,z)=\sum_{j=0}^{k}\bar{a}_{d-j}(x,z)y^j,\quad \bar{a}_{d-k}(x,z)\neq 0,\quad \bar{a}_j \text{ homogeneous of degree } j,$$

normalized such that the coefficient of the term of higher degree of $\bar{a}_{d-k}(x,z)$ in x is 1. The fact that $\bar{\mathcal{C}}$ is horizontal is equivalent to $\gcd(F,\bar{a}_{d-k})=1$.

The pencil of lines through P_y is identified with $\mathbb{P}^1\equiv\mathbb{C}\cup\{\infty\}$, where ∞ corresponds with $\bar{L}_\infty$. Following the previous notation, the lines in the pencil are denoted by $\bar{L}_t:=\{X-tZ=0\}$. Let us restrict our attention to the affine part. Let $\mathcal{C}:=\bar{\mathcal{C}}\cap\mathbb{C}^2$ and $L_t:=\bar{L}_t\cap\mathbb{C}^2$; the line L_t has equation $x=t$, while $\mathcal{C}$ has equation $f(x,y)=0$, where

$$f(x,y):=F(x,y,1)=\sum_{j=0}^{k}a_{d-j}(x)y^j,\quad a_j(x):=\bar{a}_j(x,1).$$

Let $\mathcal{B}:=\{t\in\mathbb{C}|\#(L_t\cap\mathcal{C})<k\}$; this is a finite set which consists of the roots of $a_{d-k}(x)$ (if any) and the values t such that $L_t\not\pitchfork\mathcal{C}$. The set $\mathcal{B}$ is the zero locus of the product of $a_{d-k}(x)$ and the discriminant of $f(x,y)$ with respect to y.

Let $\Sigma_k(\mathbb{C}):=\{A\subset\mathbb{C}|\#A=k\}$ be a configuration space of $\mathbb{C}$; for any $A:=\{x_1,\ldots,x_k\}\in\Sigma_k(\mathbb{C})$, the fundamental group $\pi_1(\Sigma_k(\mathbb{C});A)=:\mathbb{B}(x_1,\ldots,x_k)$ is isomorphic to the braid group $\mathbb{B}_k$ with the usual Artin presentation,

$$\mathbb{B}_k:=\left\langle \sigma_1,\ldots,\sigma_{k-1}\;\middle|\; \underset{1<i+1<j<k}{[\sigma_i,\sigma_j]=1},\quad \underset{1\leqslant i<k-1}{\sigma_i\cdot\sigma_{i+1}\cdot\sigma_i=\sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1}}\right\rangle. \tag{1.1}$$

For the next sections we need to describe a canonical identification between $\mathbb{B}_k$ and $\mathbb{B}(x_1,\ldots,x_k)$; the group $\pi_1(\Sigma_k(\mathbb{C});A)$ is identified with the homotopy classes of sets of arcs $\varphi_1,\ldots,\varphi_k:[0,1]\to\mathbb{C}$ starting and ending in A and such that $\#\{\varphi_1(t),\ldots,\varphi_k(t)\}=k$, $\forall t\in[0,1]$. Let us order the points of A, say $x_1,\ldots,x_k$, and consider a set I of simple segments A_i, $1\leqslant i<k$, such that $\partial A_i=\{x_i,x_{i+1}\}$, $A_i\cap A_{i+1}=\{x_{i+1}\}$ and the other intersections are empty; such a collection I will be called a *diagram system* for $(x_1,\ldots,x_k)$. Then we associate to σ_i the braid which is constant for $x_1,\ldots,x_{i-1},x_{i+2},\ldots,x_k$ and behaves in a small closed (topological) disk $N(A_i)$ such that A_i is a diameter as follows. The points x_i and x_{i+1} counterclockwise exchange along $\partial N(A_i)$. These generators may be understood as *half-twists* around A_i.

There is also a basis of the free group $\pi_1(\mathbb{C}\setminus A;x_0)$ if one chooses a simple edge A_0 from x_0 to x_1 intersecting $\bigcup_{i=1}^{k-1}A_i$ only at x_1. This basis $\mu_1,\ldots,\mu_k$ is obtained as follows. Take small disks Δ_i centered at x_i and assume that their intersection with $A_{i-1}\cup A_i$ are diameters with ends x_i^-,x_i^+. Then μ_i is defined as follows. Take a path α_i from x_0 to x_i^- running along $A_0\cup\cdots\cup A_{i-1}$ outside the interior of the disk Δ_j and going counterclockwise along $\partial\Delta_j$ from x_j^- to x_j^+, $1\leqslant j\leqslant i$. Let β_i be the closed path obtained by running counterclockwise along $\partial\Delta_j$ with base point x_i^-, and define $\mu_i:=\alpha_i\cdot\beta_i\cdot\alpha_i^{-1}$.

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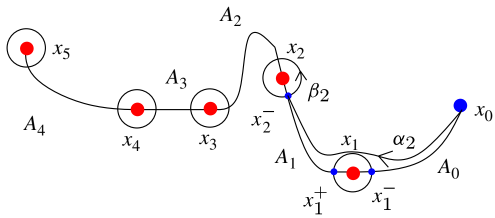

Figure 1. Diagram system, $k = 5$.

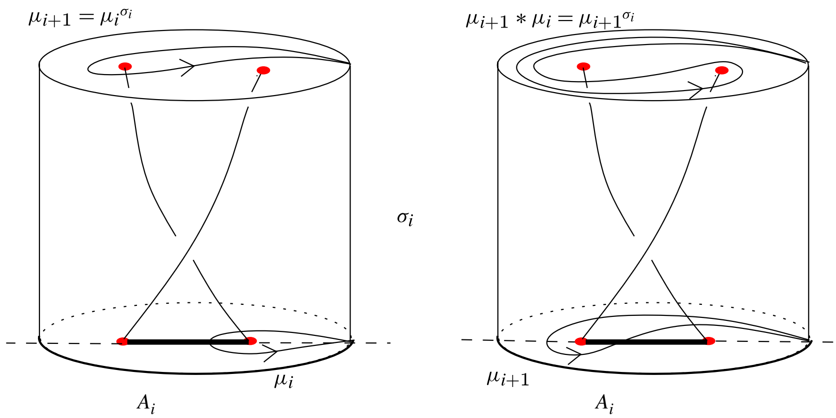

Figure 2. Geometric version of the action of $\mathbb{B}_k$ on $\mathbb{F}_k$.

The group $\mathbb{B}_k$ acts geometrically on the group $\mathbb{F}_k := \pi_1(\mathbb{C} \setminus A; x_0)$ as follows:

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } j = i, \\ \mu_i \cdot \mu_{i-1} \cdot \mu_i^{-1} & \text{if } j = i-1, \\ \mu_i & \text{otherwise.} \end{cases} \tag{1.2}$$

Notation 1.1. As usual, $a^b = b^{-1}ab$ is the conjugation of a by b. Also, for brevity, we will write $b * a = bab^{-1}$.

There are two important facts in these definitions: the element $\mu_\infty := (\mu_k \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of the point at infinity, and μ_∞ is a fixed point by the action of $\mathbb{B}_k$. We say that $(\mu_1, \ldots, \mu_k)$ is an *ordered geometric basis* of $\pi_1(\mathbb{C} \setminus A; x_0)$. As a general notation, if G is a group and $\mathbf{x} := (x_1, \ldots, x_k) \in G^k$, we define the *pseudo-Coxeter element* of $\mathbf{x}$ as $c_{\mathbf{x}} := x_k \cdot \ldots \cdot x_1$.

After this digression, note that f defines a map $\tilde{f} : \mathbb{C} \setminus \mathcal{B} \to \Sigma_k(\mathbb{C})$.

Definition 1.2. The *braid monodromy* of the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ is the morphism

$$\nabla : \pi_1(\mathbb{C} \setminus \mathcal{B}; t_0) \to \mathbb{B}_k, \quad t_0 \in \mathbb{C} \setminus \mathcal{B},$$

defined by $\tilde{f}$ on the fundamental group.

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Remark 1.3. Consider an ordered geometric basis $(\gamma_1, \ldots, \gamma_r)$ of $\pi_1(\mathbb{C} \setminus \mathcal{B}; t_0)$, and let c_∞ be its pseudo-Coxeter element. Note that ∇ is determined by $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r)) \in \mathbb{B}_k^r$ having as pseudo-Coxeter element $\nabla(c_\infty)$.

The braid monodromy measures the motions of the points of $\mathcal{C}$ along the affine lines L_t (identified with $\mathbb{C}$).

Given a triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ as above, the choice of an element of $\mathbb{B}_k^r$ determining the braid monodromy is not unique. It is not hard to check that these choices are given by the orbits of an action of $\mathbb{B}_k \times \mathbb{B}_r$ on $\mathbb{B}_k^r$ as follows. The action of $\mathbb{B}_k$ is given by simultaneous conjugation. The action of $\mathbb{B}_r$ is defined as follows. Let $h_1, \ldots, h_{r-1}$ be an Artin system of generators of $\mathbb{B}_r$. Then, if $(\tau_1, \ldots, \tau_r) \in \mathbb{B}_k^r$,

$$(\tau_1, \ldots, \tau_r)^{h_i} := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1} \cdot \tau_i \cdot \tau_{i+1}^{-1}, \tau_{i+2}, \ldots, \tau_r); \tag{1.3}$$

h_i is called a *Hurwitz move*. In particular, for a choice of $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$, two objects are unique and well defined: the conjugacy classes of the pseudo-Coxeter element and of the *monodromy group*, i.e., the group generated by $\nabla(\gamma_1), \ldots, \nabla(\gamma_r)$.

In light of the previous discussion, a braid monodromy ∇ of a triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ will sometimes be considered as a morphism (see Definition 1.2) or as a list of braids $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r))$, where $(\gamma_1, \ldots, \gamma_r)$ is an ordered geometric basis.

1.2. Applications

The first application of braid monodromy is the computation of the fundamental group; see [5].

Let ∇ be the braid monodromy of a triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$. Consider $\mathbb{C}^2 := \mathbb{P}^2 \setminus \bar{L}_\infty$, $\mathcal{C} := \bar{\mathcal{C}} \cap \mathbb{C}^2$, and L_t $(t \in \mathcal{B})$ the non-generic fibers of the pencil of lines through P_y as above. One has the following result.

Theorem 1.4. *The group $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C} \cup \bigcup_{t\in\mathcal{B}} L_t))$ has the following presentation:*

$$\left\langle \mu_1, \ldots, \mu_k, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \,\middle|\, \underset{1\leqslant i\leqslant r;\, 1\leqslant j\leqslant k}{\tilde{\gamma}_i^{-1} \cdot \mu_j \cdot \tilde{\gamma}_i = \mu_j^{\nabla(\gamma_i)}} \right\rangle.$$

A triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ (or simply a curve $\mathcal{C}$) is said to be *fully horizontal* if $\mathcal{C}$ has no vertical asymptotes, i.e., if $a_{d-k}(x) = 1$. In that case, for any $\mathcal{B}_0 \subset \mathcal{B}$ (which may be empty), corresponding to the generators $\{\gamma_{i_1}, \ldots, \gamma_{i_s}\}$, one has a new version of Zariski–van Kampen theorem, due to the fact that the above generators $\tilde{\gamma}_j$ are meridians of the corresponding lines.

Theorem 1.5. *The group $\pi_1(\mathbb{C}^2 \setminus (\mathcal{C} \cup \bigcup_{t\in\mathcal{B}_0} L_t))$ has the following presentation:*

$$\left\langle \mu_1, \ldots, \mu_k, \tilde{\gamma}_{i_1}, \ldots, \tilde{\gamma}_{i_s} \,\middle|\, \underset{1\leqslant j\leqslant s;\ 1\leqslant j\leqslant k}{\tilde{\gamma}_{i_j}^{-1} \cdot \mu_\ell \cdot \tilde{\gamma}_{i_j} = \mu_\ell^{\nabla(\gamma_{i_j})}}, \quad \underset{i\neq i_1,\ldots,i_s;\ 1\leqslant j<k}{\mu_j = \mu_j^{\nabla(\gamma_i)}} \right\rangle.$$

There is also a statement like Theorem 1.5 for curves with vertical asymptotes, but it is more technical; see [13] or [35].

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Let us assume now that $P_y \notin \bar{\mathcal{C}}$. In this case, the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ is automatically fully horizontal, and $d = k$. In this case, another version can be stated, which is closer to the original Zariski–van Kampen theorem (see [7, Corollary 1.11] for a proof).

Theorem 1.6. *The group $\pi_1(\mathbb{P}^2 \setminus \bar{\mathcal{C}})$ has the following presentation:*

$$\left\langle \mu_1, \ldots, \mu_d \,\middle|\, \underset{1 \leqslant i \leqslant r;\ 1 \leqslant j < d}{\mu_j = \mu_j^{\nabla(\gamma_i)}}, \quad \mu_d \cdot \ldots \cdot \mu_1 = 1 \right\rangle.$$

There is also a procedure to compute $\pi_1(\mathbb{P}^2 \setminus \bar{\mathcal{C}})$ when $P_y \in \bar{\mathcal{C}}$, but the general formula is not so closed since it depends heavily on the local singularity $(\bar{\mathcal{C}} \cup \bar{L}_\infty, P_y)$.

1.3. Generic braid monodromies

We finish this sequence of versions of the Zariski–van Kampen theorem with the generic cases. We will describe different notions of *genericity* of braid monodromies which will play a role in the forthcoming sections. By a *generic triple* we mean a triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ such that $P_y \notin \bar{\mathcal{C}}$, $\bar{L}_\infty \pitchfork \bar{\mathcal{C}}$ (i.e., they intersect at d distinct points), and, moreover, for each $t \in \mathcal{B}$ there is exactly one point $P_t \in \bar{L}_t \cap \bar{\mathcal{C}}$ where the intersection is not transversal and it satisfies

$$(\bar{\mathcal{C}} \cdot \bar{L}_t)_{P_t} = \begin{cases} 2 & \text{if } (\bar{\mathcal{C}}, P_t) \text{ is smooth}, \\ m_t & \text{if } (\bar{\mathcal{C}}, P_t) \text{ is singular}, \end{cases} \tag{1.4}$$

where m_t is the multiplicity of the germ $(\bar{\mathcal{C}}, P_t)$. In the case when P_t is singular, this condition means that $\bar{L}_t$ is not in the tangent cone of $(\bar{\mathcal{C}}, P_t)$ and, in the smooth case, it means that P_t is not an inflection point. In order to state the final form of the Zariski–van Kampen theorem, we need some notation. The above conditions imply that, for each γ_i, $1 \leqslant i \leqslant r$, we can express $\nabla(\gamma_i) = \eta_i \cdot \tau_i \cdot \eta_i^{-1}$, where $\eta_i, \tau_i \in \mathbb{B}_d$ and τ_i is a positive word in $\Sigma_i := \{\sigma_j\}_{j \in s_i}$, where $s_i \subset \{1, \ldots, d-1\}$ and $\# s_i = (\bar{\mathcal{C}} \cdot \bar{L}_t)_{P_t} - 1$.

For each i $(1 \leqslant i \leqslant r)$, the elements $\mu_j(i) := \mu_j^{(\eta_i^{-1})}$ represent another basis of $\mathbb{F}_d$. Let us denote by $\mu_j(i)^{\sigma_k(i)}$ the action (1.2) written in the basis $\mu_1(i), \ldots, \mu_d(i)$.

Theorem 1.7. *Let $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ be a generic triple. Then the group $\pi_1(\mathbb{P}^2 \setminus \bar{\mathcal{C}})$ admits the following presentation:*

$$\left\langle \mu_1, \ldots, \mu_d \,\middle|\, \underset{1 \leqslant i < r;\ j \in s_i}{\mu_j(i) = \mu_j(i)^{\tau_i(i)}}, \quad \mu_d \cdot \ldots \cdot \mu_1 = 1 \right\rangle.$$

The braid monodromy associated with a generic triple will be called a *generic braid monodromy*. In this case, the pseudo-Coxeter element of $(\nabla(\gamma_1), \ldots, \nabla(\gamma_r))$ is Δ_d^2, the positive generator of the central element of $\mathbb{B}_d$. This is why in the literature generic braid monodromies are also referred to as *factorizations* of Δ_d^2. The main point is that any generic braid monodromy is an invariant of $\bar{\mathcal{C}}$, that is, it is independent of the choice of P_y and $\bar{L}_\infty$ (of course, as long as $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ is a generic triple). Moreover, if two curves can be connected by a path of equisingular curves, then their generic braid monodromies coincide.

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A more general type of genericity occurs simply when $P_y \notin \bar{\mathcal{C}}$, $\bar{L}_\infty \pitchfork \bar{\mathcal{C}}$. We will refer to such a triple as *generic at infinity*. In this case, Theorem 1.7 is still true, but the sets s_i have to be replaced by a finite number of disjoint subsets $s_{i,1}, \ldots, s_{i,\kappa_i} \subset \{1, \ldots, d-1\}$.

An interesting case of genericity at infinity occurs when condition (1.4) holds, but each fiber L_t, $t \in \mathcal{B}$, is allowed to have more than one ramification point. We refer to this case as *local genericity*. From a locally generic braid monodromy, a generic braid monodromy might easily be obtained (see Proposition 4.3).

2. Pencils of lines and Kummer covers

Following notation from § 1, let us fix a curve $\bar{\mathcal{C}} \subset \mathbb{P}^2$ of degree k. We assume that $P_y := [0:1:0] \notin \bar{\mathcal{C}}$. Let $\bar{\mathcal{C}}_n := \pi_n^{-1}(\bar{\mathcal{C}})$ be its Kummer transform. Denote by $\bar{L}_t := \{X - tZ = 0\}$ the pencil of lines through P_y.

Let $\bar{L}_Y := \{Y = 0\} \subset \mathbb{P}^2$, and let L_Y be its affine part. We consider the set of lines through P_y which are not transversal to $\bar{\mathcal{C}} \cup \bar{L}_Y$ and in which we include the lines $\bar{L}_0$ and $\bar{L}_\infty$ if necessary (due to their special relation with π_n). Let

$$B := \{t \in \mathbb{P}^1 | \bar{L}_t \not\pitchfork (\bar{\mathcal{C}} \cup \bar{L}_Y)\}, \quad B^* := B \cap \mathbb{C}^*, \quad \tilde{B} := B^* \cup \{0\}, \quad B^\infty := B \cup \{\infty\}.$$

The braid monodromy of $\bar{\mathcal{C}} \cup \bar{L}_Y$ with respect to P_y and $\bar{L}_\infty$ is the morphism

$$\nabla : \pi_1(\mathbb{C} \setminus B; t_0) \to \mathbb{B}_{k+1}, \quad t_0 \in \mathbb{C} \setminus \tilde{B}.$$

Note that the image of ∇ is contained in $\mathbb{B}_{k,1}$, the subgroup of $\mathbb{B}_{k+1}$ given by all the braids whose associated permutation fixes a given point (see § 3 for a description of this group).

Definition 2.1. The map

$$\tilde{\nabla} : \pi_1(\mathbb{C} \setminus \tilde{B}; t_0) \to \mathbb{B}_{k,1} \tag{2.1}$$

is called the *extended braid monodromy* of the triple $(\bar{\mathcal{C}}, P_y, L_\infty)$ with respect to L_Y.

Note that if $0 \notin B$ then the image by $\tilde{\nabla}$ of a meridian around 0 is the trivial braid.

We define the analogous objects for $\bar{\mathcal{C}}_n$: $B_n, B_n^*, \tilde{B}_n, B_n^\infty, \nabla_n, \tilde{\nabla}_n$ for a choice of $t_{0,n}$, an nth root of t_0. The following lemmas are easy consequences of the properties of the Kummer cover.

Lemma 2.2. *Let $\phi_n : \mathbb{C} \to \mathbb{C}$ be the map defined by $\phi_n(t) := t^n$. Then, $B_n^* = \phi_n^{-1}(B^*)$ and $\tilde{B}_n = \phi_n^{-1}(\tilde{B})$.*

In order to avoid ambiguity, we will denote by $L_{0,n} = \{X = 0\}$ the preimage of $L_0 = \{X = 0\}$ by π_n. The same will occur with $L_{\infty,n}$ and $L_{Y,n}$. We will also denote by $L_{t,n} = \{X - tZ = 0\}$ the corresponding lines in the source of π_n.

Lemma 2.3. *The map π_n induces degree-n coverings $\varphi_n : L_{t,n} \mapsto L_{t^n}$. Moreover, the preimage of a line L_s by π_n is the disjoint union of $L_{t,n}$, $t^n = s$, each one inducing a covering φ_n as above.*

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As a consequence of these lemmas, the braid monodromy $\tilde{\nabla}_n$ can be thought of as follows. Consider the loops γ in $\mathbb{C} \setminus \tilde{B}$ which can be lifted as loops by ϕ_n, and replace the braid $\nabla(\gamma)$, where k points move in $\mathbb{C}^*$, by the braid obtained by the constant string 0 and the n-roots of these k points. This construction defines a map

$$\tilde{\rho}_{n,k} : \mathbb{B}_{k,1} \to \mathbb{B}_{nk,1},$$

which is described in § 3.

Let us summarize the results. If $r := \#B^*$, we denote by $\mathbb{F}_{r+1}$ the free group $\pi_1(\mathbb{C} \setminus \tilde{B}; t_0)$. Analogously, we denote by $\mathbb{F}_{nr+1}$ the free group $\pi_1(\mathbb{C} \setminus \tilde{B}_n; t_{0,n})$. Using covering theory, the map $(\phi_n)_*$ fits in a short exact sequence

$$0 \to \mathbb{F}_{nr+1} \to \mathbb{F}_{r+1} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

The following commutative diagram holds:

$$\begin{array}{ccc} \mathbb{F}_{r+1} & \xrightarrow{\tilde{\nabla}} & \mathbb{B}_{k,1} \\ \uparrow & & \downarrow \\ \mathbb{F}_{nr+1} & \xrightarrow{\tilde{\nabla}_n} & \mathbb{B}_{nk,1} \end{array} \qquad (2.2)$$

Using this diagram, one can recover ordered geometric bases and relate the central elements of the corresponding braid groups. The upcoming results will show this construction useful.

Summarizing, let $\tilde{\nabla} := (\tau_1, \ldots, \tau_{r+1}) \in \mathbb{B}_{k,1}^{r+1}$ be an extended braid monodromy for $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ with respect to L_Y, where τ_{r+1} is the braid corresponding to $\bar{L}_0$. Consider $\pi_n(X : Y : Z) = [X^n : Y^n : Z^n]$, the nth Kummer cover of $\mathbb{P}^2$, and denote by $\bar{\mathcal{C}}_n$ the preimage of $\bar{\mathcal{C}}$ by π_n. The braid monodromy $\tilde{\nabla}$ produces a list of braids $\tilde{\nabla}_n \in \mathbb{B}_{nk,1}^{nr+1}$, as described in diagram (2.2). One has the following proposition.

Proposition 2.4. *The element $\tilde{\nabla}_n$ described above is an extended braid monodromy for the triple $\pi_n^{-1}(\bar{\mathcal{C}}, P_y, \bar{L}_\infty) = (\bar{\mathcal{C}}_n, P_y, \bar{L}_{\infty,n})$ with respect to $\pi_n^{-1}(\bar{L}_Y) = \bar{L}_{Y,n}$.*

Proof. This is an immediate consequence of the previous construction and Lemmas 2.2 and 2.3. □

Remark 2.5. Note that the braid monodromy obtained from $\tilde{\nabla}_n$ via the forgetful map $\mathbb{B}_{nk,1} \to \mathbb{B}_{nk}$ might be a highly non-generic braid monodromy.

Lemma 2.6. *Let $(\gamma_1, \ldots, \gamma_{r+1})$ be an ordered geometric basis of $\mathbb{F}_{r+1}$ such that γ_{r+1} is a meridian of $0 \in \tilde{B}$. Then, the ordered list*

$$(v_0, \ldots, v_j, \ldots, v_{n-1}, \gamma_{r+1}^n),$$

where $v_j := (\gamma_1^{\gamma_{r+1}^j}, \ldots, \gamma_r^{\gamma_{r+1}^j})$ (see Notation 1.1), forms an ordered geometric basis of $\mathbb{F}_{nr+1}$.

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Lemma 2.7. *Let* $\Delta^2_{k+1} \in \mathbb{B}_{k,1}$ *be the positive generator of the center of* $\mathbb{B}_{k+1}$*. Then* $\Delta^{2n}_{k+1} = \Delta^2_{nk+1}$ *via the inclusion* $\mathbb{B}_{k,1} \hookrightarrow \mathbb{B}_{nk,1}$*.*

The purpose of the forthcoming sections will be to describe an effective way to obtain a generic braid monodromy for $\bar{\mathcal{C}}_n$ from the given computation of $\tilde{\nabla}_n$. This is an outline of the general strategy, which will be developed in what follows.

- If $\bar{L}_{Y,n} \not\subseteq \bar{\mathcal{C}}_n$, we compose $\tilde{\nabla}_n$ with the natural map $\mathbb{B}_{nk,1} \to \mathbb{B}_{nk}$ obtained by forgetting the constant string 0.
- If $0 \notin B_n$, then the map factorizes through $\mathbb{F}_{nr+1} \twoheadrightarrow \mathbb{F}_{nr}$, whose kernel is generated by the meridians around 0.
- If $\infty \in B_n$, we change the line at infinity; the image α of a meridian at infinity is obtained as follows. Let τ be the pseudo-Coxeter element of the braid monodromy; then $\alpha := \Delta^2 \tau^{-1}$, where Δ^2 is the positive generator of the center of the braid group.
- Move the projection point P_y slightly to obtain a generic projection.

3. The motion of *n*-roots

As was introduced in § 2, let $\mathbb{B}_{k,1}$ be the subgroup of $\mathbb{B}_{k+1}$ given by all the braids whose associated permutation fixes a given point in $\mathbb{C}$, say 0.

We identify $\mathbb{B}_{k+1}$ with the group of braids with ends in $k+1$ non-negative real points $x_1 > \cdots > x_k > x_{k+1} = 0$, and we consider the Artin generators $\sigma_1, \ldots, \sigma_k$ in $\mathbb{B}_{k+1}$ which are geometrically associated with the diagram system obtained from the paths joining these points in the real line; see § 1. Recall that any choice of a *diagram system* I of $k+1$ points in $\mathbb{C}$ induces an Artin system of generators (1.1) of $\mathbb{B}_{k+1}$. Note that the only half-twist σ_i that moves 0 is σ_k. The following result is well known.

Lemma 3.1. *The group* $\mathbb{B}_{k,1}$ *is generated by* $\sigma_1, \ldots, \sigma_{k-1}, \sigma_k^2$ *(as a subgroup of* $\mathbb{B}_{k+1}$*), and it is naturally isomorphic to the group of braids of* k *strands in* $\mathbb{C}^*$*.*

Let us consider the map $\rho_n : \mathbb{C}^* \to \mathbb{C}^*$, given by $\rho_n(z) := z^n$. Note that this map induces a morphism $\tilde{\rho}_n : \mathbb{B}_{k,1} \to \mathbb{B}_{nk,1}$ via the multivalued function ρ_n^{-1}. The goal of this section is to give explicit formulæ for this morphism. In order to do so, one needs to explicitly choose systems of generators for $\mathbb{B}_{k+1}$ and $\mathbb{B}_{nk+1}$.

Consider the diagram system I for $\mathbb{B}_{k,1}$ described above. The image of the half-twist σ_i, associated with A_i, $1 \leqslant i < k$, is a product of half-twists associated with the pairwise disjoint arcs $A_{i,1}, \ldots, A_{i,n}$ such that $\rho_n^{-1}(A_i) = A_{i,1} \cup \cdots \cup A_{i,n}$. The image of σ_k^2 is more complicated since $A_{k,1}, \ldots, A_{k,n}$ intersect at 0. In order to describe $\tilde{\rho}_n$, it is convenient to consider diagram systems on both sets of points in $\mathbb{C}$ in the source and in the target to fix a basis. However, note that the system of arcs $A_{i,j}$ obtained as the preimage of A_i does not produce a diagram system for $\mathbb{B}_{nk+1}$. In order to solve this situation, we will define diagram systems for $\mathbb{B}_{nk,1}$ from which the arcs $A_{i,j}$ can be easily described. For different purposes, different diagram systems might be more appropriate. Here we will focus on two particular diagram systems: the circular and the radial.

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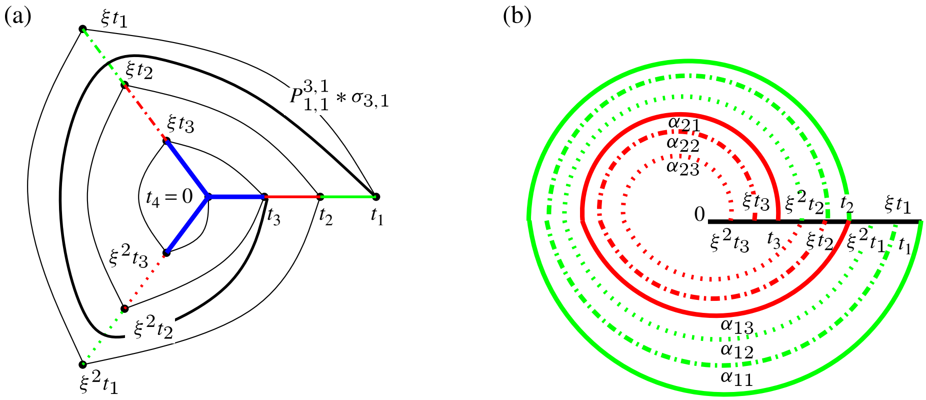

Figure 3. (a) Circular diagram system of $\mathbb{B}_{nk+1}$ for $k = n = 3$. (b) A *straightened* view of Figure 3.

3.1. Circular diagram systems

Let us denote $d := nk$. We identify $\mathbb{B}_{d+1}$ with $\mathbb{B}(\{\xi_n^j t_i | 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n\} \cup \{0\})$, where t_i is the non-negative nth root of $x_i \in \mathbb{R}_{\geqslant 0}$ and $\xi_n := \exp(\frac{2\pi\sqrt{-1}}{n})$. Consider the following arcs.

- $c_{i,j}$ is the counterclockwise arc joining $t_i \xi_n^{j-1}$ to $t_i \xi_n^j$ in the circle centered at 0, $1 \leqslant j < n$.
- $c_{i,n}$ is the segment joining $t_i \xi_n^{n-1}$ and t_{i+1}.

These arcs and segments are illustrated in Figure 3(a). The list of arcs $\{c_{i,j}\}_{i,j}$ with a left-lexicographic order produces a diagram system called the *circular diagram system*. The half-twist produced by an arc $c_{i,j}$ will be denoted by $\sigma_{i,j} \in \mathbb{B}_{nk,1}$. The half-twists associated with $A_{i,j}$ will be denoted by $\alpha_{i,j}$.

Figure 3(a) is obtained from Figure 3(a) after unwinding the circular diagram system into a straight line. This will help visualize the rewriting of the half-twists $\alpha_{i,j}$ in terms of the $\sigma_{i,j}$.

Consider $(i_1, j_1) < (i_2, j_2)$, where $<_\ell$ represents the left-lexicographic order (that is, either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$). We will define

$$\begin{aligned}
P^{i_2,j_2}_{i_1,j_1} &:= \sigma_{i_1,j_1} \cdot \ldots \cdot \sigma_{i_2,j_2} = \prod_{(i_1,j_1) \leqslant_\ell v \leqslant_\ell (i_2,j_2)} \sigma_v, \\
\bar{P}^{i_2,j_2}_{i_1,j_1} &:= \sigma^{-1}_{i_1,j_1} \cdot \ldots \cdot \sigma^{-1}_{i_2,j_2} = \prod_{(i_1,j_1) \leqslant_\ell v \leqslant_\ell (i_2,j_2)} \sigma_v^{-1}, \\
M^{i_2,j_2}_{i_1,j_1} &:= \sigma_{i_2,j_2} \cdot \ldots \cdot \sigma_{i_1,j_1} = \prod_{(i_2,j_2) \geqslant_\ell v \geqslant_\ell (i_1,j_1)} \sigma_v, \\
\bar{M}^{i_2,j_2}_{i_1,j_1} &:= \sigma^{-1}_{i_2,j_2} \cdot \ldots \cdot \sigma^{-1}_{i_1,j_1} = \prod_{(i_2,j_2) \geqslant_\ell v \geqslant_\ell (i_1,j_1)} \sigma_v^{-1},
\end{aligned} \tag{3.1}$$

where the order in the products is given by the lexicographic order described above.

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Remark 3.2. Note that $P_{i_1,j_1}^{i_2,j_2} = (\bar{M}_{i_1,j_1}^{i_2,j_2})^{-1}$, $(\bar{P}_{i_1,j_1}^{i_2,j_2})^{-1} = M_{i_1,j_1}^{i_2,j_2}$. Also, note that

$$P_{i_1,j_1}^{i_2,j_2} * \sigma_{i_2,j_2} = \bar{M}_{i_1,j_1}^{i_2,j_2} * \sigma_{i_1,j_1} \quad (\text{respectively, } \bar{P}_{i_1,j_1}^{i_2,j_2} * \sigma_{i_2,j_2}^{-1} = M_{i_1,j_1}^{i_2,j_2} * \sigma_{i_1,j_1}^{-1})$$

represents a half-twist (respectively, a negative half-twist) interchanging $t_{i_1}\xi_n^{j_1-1}$ and $t_{i_2}\xi_n^{j_2+1}$ along a spiral arc. The arc corresponding to $P_{1,1}^{3,1} * \sigma_{3,1} = \bar{M}_{1,1}^{3,1} * \sigma_{1,1}$ (or $\bar{P}_{1,1}^{3,1} * \sigma_{3,1}^{-1} = M_{1,1}^{3,1} * \sigma_{1,1}^{-1}$) is shown in Figure 3(a).

Lemma 3.3. *Under the above conditions, and for any* $(i_1, j_1) < (i, j) < (i_2, j_2)$*, one has*

(1) $P_{i_1,j_1}^{i_2,j_2} * \sigma_{i_2,j_2} = \left(\bar{M}_{(i,j)^+}^{i_2,j_2} \cdot P_{i_1,j_1}^{i,j}\right) * \sigma_{i,j}$*, where* $(i,j)^+$ *denotes the element following* (i,j) *in the left-lexicographic order.*

(2) $\bar{P}_{i_1,j_1}^{i_2,j_2} * \sigma_{i_2,j_2} = \left(M_{(i,j)^+}^{i_2,j_2} \cdot \bar{P}_{i_1,j_1}^{i,j}\right) * \sigma_{i,j}$.

(3) $\alpha_{i,j} = \left(P_{i+1,j}^{k,n}\bar{P}_{i,j}^{k,n}\right) * \sigma_{k,n} = \left(P_{i+1,j}^{k,n}\sigma_{k,n}^2 M_{(i_2,j_2)^+}^{k,n-1}\bar{P}_{i,j}^{i_2,j_2}\right) * \sigma_{i_2,j_2} = \left(P_{i+1,j}^{k,n-1}\sigma_{k,n}^2 M_{i,j}^{k,n-1}\right) * \sigma_{i,j}$.

Proof. Property (1) is immediate by induction, since

$$\begin{aligned} P_{i_1,j_1}^{i_2,j_2} * \sigma_{i_2,j_2} &= \left(P_{i_1,j_1}^{(i_2,j_2)-2} \cdot \sigma_{(i_2,j_2)^-}\right) * \sigma_{i_2,j_2} \\ &= \left(P_{i_1,j_1}^{(i_2,j_2)-2} \cdot \sigma_{i_2,j_2}^{-1}\right) * \sigma_{(i_2,j_2)^-} = \left(\sigma_{i_2,j_2}^{-1} \cdot P_{i_1,j_1}^{(i_2,j_2)^-}\right) * \sigma_{(i_2,j_2)^-}, \end{aligned}$$

where $(i_2, j_2)^-$ denotes the element immediately smaller than (i_2, j_2) and $(i_2, j_2) - 2$ denotes its second predecessor $((i_2, j_2)^-)^-$. Property (2) is analogous.

In order to show Property (3), note that $\alpha_{i,j}$ is a half-twist that exchanges $t_i\xi_n^{j-1}$ and $t_{i+1}\xi_n^{j-1}$. Therefore, according to Remark 3.2, one has

$$\alpha_{i,j} = \left(P_{i+1,j}^{k,n} * \sigma_{k,n}\right) \cdot \left(\bar{P}_{i,j}^{k,n} * \sigma_{k,n}\right) \cdot \left(P_{i+1,j}^{k,n} * \sigma_{k,n}^{-1}\right).$$

A similar induction argument shows that

$$\left(P_{i+1,j}^{k,n} * \sigma_{k,n}\right) \cdot \left(\bar{P}_{i,j}^{k,n} * \sigma_{k,n}\right) \cdot \left(P_{i+1,j}^{k,n} * \sigma_{k,n}^{-1}\right) = \left(P_{i+1,j}^{k,n}\bar{P}_{i,j}^{k,n}\right) * \sigma_{k,n}.$$

The other equalities are a consequence of Remark 3.2 and Properties (1) and (2). □

Proposition 3.4. *The map* $\tilde{\rho}_n : \mathbb{B}_{k,1} \to \mathbb{B}_{d,1}$ *is given by*

$$\begin{aligned} \sigma_i &\mapsto \prod_{j=1}^{n} \alpha_{i,j}, \qquad\qquad i \in 1, \ldots, k-1 \\ \sigma_k^2 &\mapsto \sigma_{k,n}^2\sigma_{k,n-1}\sigma_{k,n-2} \cdot \ldots \cdot \sigma_{k,1}. \end{aligned}$$

Proof. As mentioned at the beginning of this section, if $i \neq k$, the preimage of A_i by ρ_n is a disjoint union of segments $A_{i,1}, \ldots, A_{i,n}$. Therefore the image of the half-twist σ_i associated with A_i is the product of the n half-twists $\alpha_{i,1} \cdot \ldots \cdot \alpha_{i,n}$. Note that σ_k^2 corresponds to a full turn around 0, its preimage by ρ_n being a counterclockwise rotation of angle $\frac{2\pi}{n}$ of the points $\xi_n^j t_k$. This is nothing but $\sigma_{k,n}^2 \cdot \sigma_{k,n-1} \cdot \ldots \cdot \sigma_{k,1}$ (see Figure 4). □

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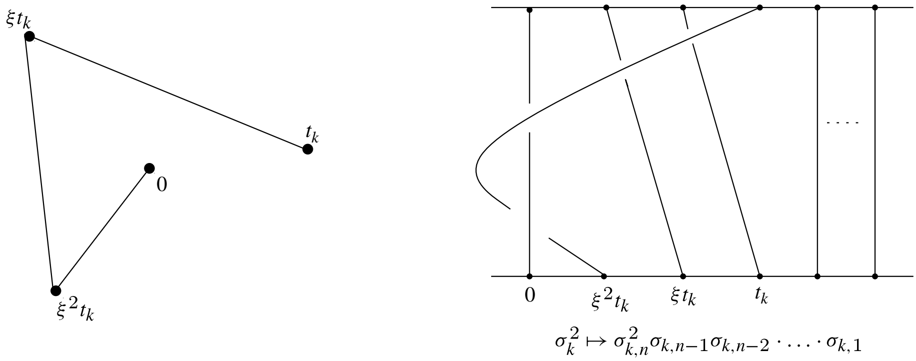

Figure 4. Image of σ_k^2, $n = 3$.

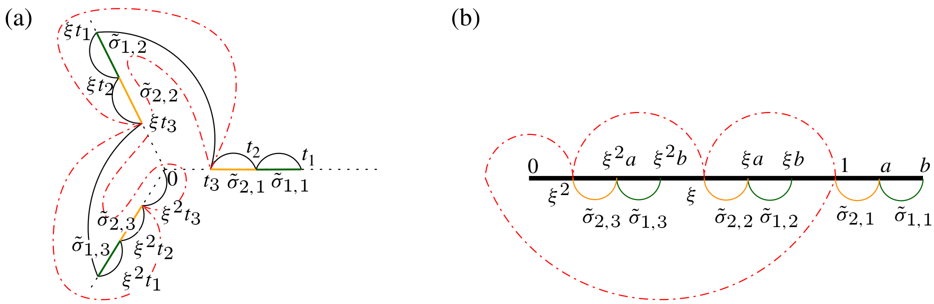

Figure 5. (a) Radial diagram system of $\mathbb{B}_{nk+1}$ for $k = n = 3$. (b) A *straightened* view of Figure 5(a).

Example 3.5. Let us consider the case $k = 1$ and the composition of $\tilde{\rho}_n$ with the natural projection of $\mathbb{B}_{n,1} \twoheadrightarrow \mathbb{B}_k$ given by forgetting the string 0. Then we have a map $\hat{\rho}_n : \mathbb{B}_{1,1} \to \mathbb{B}_n$ such that $\hat{\rho}_n(\sigma_1^2) = \sigma_{n-1}\sigma_{n-2} \cdot \ldots \cdot \sigma_1$.

Example 3.6. Next consider the case $k = 2$ and the map $\hat{\rho}_n : \mathbb{B}_{2,1} \to \mathbb{B}_{2n}$, as in Example 3.5. Then, according to Lemma 3.3(3),

$$\hat{\rho}_n(\sigma_1) = \prod_{j=1}^{n} \left(P_{2,j}^{2,n-1} M_{1,j}^{2,n-1} \right) * \sigma_{1,j} = \prod_{j=1}^{n} \left(\sigma_{2,j} \cdot \ldots \cdot \sigma_{2,n-1}^2 \cdot \ldots \cdot \sigma_{1,j-1} \right) * \sigma_{1,j}$$

and

$$\hat{\rho}_n(\sigma_2^2) = \sigma_{2,n-1} \cdot \ldots \cdot \sigma_{2,1}.$$

3.2. Radial diagram systems

For some applications, it is more useful to consider a different diagram system, which will be called *radial*; see Figure 5(a).

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We call the Artin generators for this system $\tilde{\sigma}_{i,j}$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$. According to this choice, these are the half-twists $\alpha_{i,j}$ associated with the segments $A_{i,j}$ for $1 \leqslant i \leqslant k-1$; that is,

$$\tilde{\sigma}_{i,j} = \alpha_{i,j}. \tag{3.2}$$

However, the element $\tilde{\sigma}_{k,j}$ corresponds to an arc joining $t_k \xi_n^j$ with $t_1 \xi_n^{j+1}$, as shown in Figure 5(a). Denote $\beta_j := \tilde{\sigma}_{k-1,j} \cdot \ldots \cdot \tilde{\sigma}_{1,j}$. Then we have

$$\sigma_{k,j} = \beta_{j+1} \tilde{\sigma}_{k,j} \beta_{j+1}^{-1} \quad (1 \leqslant j \leqslant n-1). \tag{3.3}$$

Moreover, $\sigma_{k,n} = \tilde{\sigma}_{k,n}$. The following result holds.

Proposition 3.7. *The map* $\tilde{\rho}_n : \mathbb{B}_{k,1} \to \mathbb{B}_{d,1}$ *is given by*

$$\begin{aligned} \sigma_i &\mapsto \prod_{j=1}^{n} \tilde{\sigma}_{i,j}, && 1 \leqslant i \leqslant k-1 \\ \sigma_k^2 &\mapsto \tilde{\sigma}_{k,n}^2 \prod_{(k-1,n) \geqslant_r v \geqslant_r (k,1)} \tilde{\sigma}_v \cdot \prod_{j=2}^{n} \beta_j^{-1}, \end{aligned}$$

where $\geqslant_r$ *represents the right-lexicographic order.*

Proof. The result follows from Proposition 3.4. The formula for the image of σ_i is a direct consequence of (3.2). The formula for the image of σ_k^2 can be obtained using (3.3) and the fact that β_j commutes with $\beta_{j'}$ and with $\tilde{\sigma}_{i,j'}$ for $j' \leqslant j$ and $(i, j') \neq (k, j-1)$ (see Figure 6). □

Example 3.8. Let us consider the case $n = 2$ and $\hat{\rho}_2 : \mathbb{B}_{k,1} \to \mathbb{B}_{2k}$ the composition of $\tilde{\rho}$ with the forgetful map as in Example 3.5. According to Proposition 3.7, one has

$$\begin{aligned} \sigma_i &\mapsto \tilde{\sigma}_{i,1} \tilde{\sigma}_{i,2}, && 1 \leqslant i \leqslant k-1 \\ \sigma_k^2 &\mapsto (\tilde{\sigma}_{k-1,2} \cdot \ldots \cdot \tilde{\sigma}_{1,2}) * \tilde{\sigma}_{k,1}. \end{aligned}$$

In fact, a more convenient description of these maps can be given for another choice of generators $\hat{\sigma}_i$ coming from the natural diagram system on the real line joining the preimages of x_k (which all lie on the real line, since $n = 2$).

$$\begin{array}{rccc} \rho_2' : & \mathbb{B}_{k,1} & \to & \mathbb{B}_{2k+1} \\ & \sigma_i & \mapsto & \hat{\sigma}_i \hat{\sigma}_{2k-i+1} \\ & \sigma_k^2 & \mapsto & \hat{\sigma}_{k+1} \hat{\sigma}_k \hat{\sigma}_{k+1} \end{array} \qquad \begin{array}{rccc} \hat{\rho}_2' : & \mathbb{B}_{k,1} & \to & \mathbb{B}_{2k} \\ & \sigma_i & \mapsto & \hat{\sigma}_i \hat{\sigma}_{2k-i} \\ & \sigma_k^2 & \mapsto & \hat{\sigma}_k. \end{array}$$

4. Deformations of braid monodromies

We are mostly interested in computing generic braid monodromies of curves (see § 1.3). However, it is usually easier (and oftentimes more natural) to compute non-generic braid monodromies. For instance, assume that $(\bar{\mathcal{C}}, P_y, L_\infty)$ is a generic triple, where

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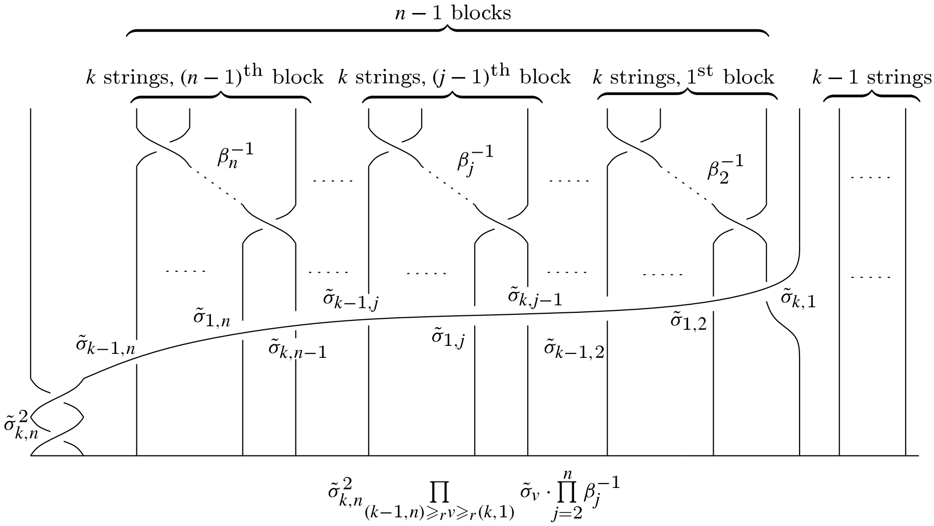

Figure 6. Image of σ_k^2 by Proposition 3.7.

$P_y=[0:1:0]$ and $L_\infty=\{Z=0\}$, and consider the Kummer cover $\pi_n:\mathbb{P}^2\to\mathbb{P}^2$ given by the equation $[X:Y:Z]\mapsto[X^n:Y^n:Z^n]$. Assume that the line $L_0:=\{X=0\}$ is tangent to $\bar{\mathcal{C}}$. Note that the preimage $L_{0,n}$ of L_0 (which is also a line by the equation $\{X=0\}$ on the source $\mathbb{P}^2$) will intersect the curve $\pi_n^{-1}(\bar{\mathcal{C}})$ at n singular points each with local equation $x^n-y^2=0$. The triple $(\bar{\mathcal{C}}_n,P_y,L_{\infty,n})$, where $\bar{\mathcal{C}}_n$ (respectively, $L_{\infty,n}$) is the preimage of $\bar{\mathcal{C}}$ (respectively, L_∞), is locally generic, but not generic.

4.1. From fully horizontal to generic at infinity

Let us consider the braid monodromy of a fully horizontal triple $(\bar{\mathcal{C}},P,\bar{L})$, where $P\notin\bar{\mathcal{C}}$ (e.g., the hypothesis in Theorem 1.6), where $\bar{L}$ and $\bar{\mathcal{C}}$ are not necessarily transversal.

Proposition 4.1. *Assume that $\bar{L}\not\pitchfork\bar{\mathcal{C}}$, let $(\tau_1,\dots,\tau_r)\in\mathbb{B}_d^r$ be a braid monodromy factorization for $(\bar{\mathcal{C}},P,\bar{L})$, and consider a line $\bar{L}'$ such that $P\in\bar{L}'$ and $\bar{L}'\pitchfork\bar{\mathcal{C}}$. Then $(\tau_1,\dots,\tau_r,\Delta_d^2(\tau_r\cdot\dots\cdot\tau_1)^{-1})$ is a braid monodromy factorization for $(\bar{\mathcal{C}},P,\bar{L}')$.*

Proof. Note that $\mathcal{B}'=\mathcal{B}\cup\{t_L\}$, where $\mathcal{B}$ (respectively, $\mathcal{B}'$) is the ramification set for $(\bar{\mathcal{C}},P,\bar{L})$ (respectively, for $(\bar{\mathcal{C}},P,\bar{L}')$). Since the line $\bar{L}'$ is transversal, the pseudo-Coxeter element of its braid monodromy factorization is Δ_d^2, and the result follows. □

This will be a common situation when studying Kummer covers for curves which are not transversal to the axes. In that case, we can be more explicit.

Let $T:=(\tau_1,\dots,\tau_r,\tau_{r+1})\in\mathbb{B}_{k,1}^{r+1}$ be an extended braid monodromy factorization for $(\bar{\mathcal{C}},P_y=[0:1:0],\bar{L}_\infty)$ with respect to $\bar{L}_Y$. Assume that $\bar{L}_0=\{X=0\}$ is not transversal to $\bar{\mathcal{C}}$, and that τ_{r+1} is the braid corresponding to $\bar{L}_0$. Let us denote by c_T its pseudo-Coxeter element.

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Consider $\pi_n(X:Y:Z)=[X^n:Y^n:Z^n]$ the nth Kummer cover of $\mathbb{P}^2$, and denote by $\bar{\mathcal{C}}_n$ the preimage of $\bar{\mathcal{C}}$ by π_n. The extended braid monodromy T produces an extended braid monodromy $T_n\in\mathbb{B}_{nk,1}^{nr+1}$ for $(\bar{\mathcal{C}}_n,\pi_n^{-1}P_y,\bar{L}_{\infty,n})$ (see Proposition 2.4).

Proposition 4.2. *Under the above conditions, a generic at infinity extended braid monodromy factorization is obtained by adding to* T_n *the braid* $\Delta_{k+1}^{2n}c_T^{-n}$*, using the inclusion* $\mathbb{B}_{k,1}\hookrightarrow\mathbb{B}_{nk,1}$ *(see diagram (2.2)).*

Proof. This is a consequence of Proposition 4.1 and Lemma 2.7. □

4.2. From locally generic to generic

As announced in § 1, some types of non-generic braid monodromies can be easily deformed into generic braid monodromies. This is the case with locally generic triples. Let $(\bar{\mathcal{C}},P_y,\bar{L}_\infty)$ be a locally generic triple (see § 1.3), and let $\mathcal{B}$ be the set of ramification values of the projection. As already mentioned in § 1.3, for each $t\in\mathcal{B}$, there exists a finite number of points $P_{t,1},\dots,P_{t,\kappa_t}\in L_t$ satisfying (1.4).

Proposition 4.3. *Under the above conditions, let* τ_t *be the braid associated with* L_t*,* $t\in\mathcal{B}$*. Then there is a factorization* $\tau_t=\tau_{t,\kappa_t}\cdot\ldots\cdot\tau_{t,1}$*, where* $\tau_{t,1},\dots,\tau_{t,\kappa}$ *are pairwise commuting braids and each one corresponds to one non-transversal point in* L_t*.*

Moreover, replacing τ *by* $(\tau_{t,1},\dots,\tau_{t,\kappa_t})$ *(the order does not matter) in the braid monodromy factorization of* $(\bar{\mathcal{C}},P_y,\bar{L}_\infty)$ *for each* $t\in\mathcal{B}$ *produces a generic braid monodromy factorization of* $\bar{\mathcal{C}}$*.*

Proof. Since $P_y\notin\bar{\mathcal{C}}$, the triple $(\bar{\mathcal{C}},P_y,\bar{L}_\infty)$ is fully horizontal; that is, there are no vertical asymptotes. Consider $P_{t,1},\dots,P_{t,\kappa_t}\in L_t$. Due to the conic structure of $(\mathbb{C}^2,\mathcal{C})_{P_{t,i}}$, one can find disjoint Milnor polydisks $\mathbb{B}_{P_{t,i}}:=\mathbb{D}_{t,x}\times\mathbb{D}_{t,y}$ around each $P_{t,i}$ such that $\mathcal{C}\cap\partial\mathbb{B}_{P_{t,i}}\subset\partial\mathbb{D}_{t,x}\times\mathbb{D}_{t,y}$ (we need condition (1.4) to ensure this). Let us denote by $\mu_{t,i}$ the local monodromy around each $P_{t,i}$ based at t_ε (close enough to t). Note that $\mu_{t,i}\in\mathbb{B}_m$, where $m=1$ if and only if $P_{t,i}$ is a transversal intersection of $\mathcal{C}$ and L_t. If $P_{t,i}$ is singular in $\mathcal{C}$, then the link produced by closing $\mu_{t,i}$ is the link of the singularity at $P_{t,i}$. If $P_{t,i}$ is a single tangency, then $\mu_{t,i}$ is a half-twist. Therefore, $\mu_{t,i}$ is trivial if and only if $P_{t,i}$ is a transversal intersection of $\mathcal{C}$ and L_t. We disregard these trivial braids, denote by κ_t the number of non-trivial braids, and denote them by $\mu_{t,1},\dots,\mu_{t,\kappa_t}$ after reordering. Define by β_t a path that joins t_ε and t, consider $\tilde{\beta}_t$ the open braid associated with β_t, and define $\tau_{t,i}:=\beta_t\cdot\mu_{t,i}\cdot\beta_t^{-1}$ (here, "$\cdot$" simply means juxtaposition). Since the strings inside each Milnor polydisk are different, the braids $\tau_{t,i}$ automatically commute.

For the *moreover* part, note that moving P_y generically on L_∞ causes just a small change of coordinates $X\mapsto X+\eta Y$, and hence each L_t splits into κ_t non-transversal vertical lines satisfying the genericity condition (1.4) at only one point. This shows the statement. □

4.3. From generic at infinity to locally generic

In the more general case when triples $(\bar{\mathcal{C}},P_y,\bar{L}_\infty)$ are just generic at infinity, the change of projection might lead to more complicated situations, but in general the result is of

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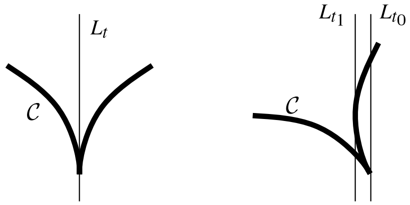

Figure 7. Changing the projection.

the same nature; that is, some braids produced by the monodromy around $\bar{L}_t$ in the non-generic braid monodromy need to be replaced by an appropriate factorization. To simplify notation, we say that a braid $\sigma = \nabla(\gamma)$ (image by the monodromy of a meridian γ in a certain ordered geometric basis) is *replaced by a factorization* $(\beta_1, \ldots, \beta_r)$ if there is a triple for which $\beta_i = \nabla(\gamma_i)$ and $\gamma = \gamma_r \cdots \gamma_1$, and hence $\sigma = \beta_r \cdots \beta_1$. We state some interesting particular cases.

Proposition 4.4. *Let us assume that L_t intersects $\bar{\mathcal{C}}$ transversally at $k-3$ points and that it is tangent to an ordinary cusp. The braid $\mu := (\sigma_2\sigma_1)^2$ can be seen as the image of the monodromy on a certain meridian around $t \in \mathbb{C}$, the projection of L_t.*

Moreover, μ can be replaced by the factorization $(\sigma_1^{\sigma_2}, \sigma_2^3)$ so that the monodromy becomes generic in a regular neighborhood of L_t.

Proof. The local equation of $\mathcal{C}$ at P is $y^3 - x^2 = 0$. Note that the local monodromy around $t = 0$ is given by the braid parameterized by $y^3 = e^{4\pi\sqrt{-1}\lambda}$, $\lambda \in [0, 1]$, that is, a rotation of angle $\frac{4}{3}\pi$ of the third roots of unity. This produces the braid $(\sigma_2\sigma_1)^2$. A small perturbation of the projection point along $\bar{L}_\infty$ produces a change of variable $f_\eta := y^3 - (x - 3\eta y)^2$. The discriminant of f_η with respect to y is $x^3(x - 4\eta^3)$, which implies that the non-transversal vertical line $\bar{L}_0$ splits into $\bar{L}_0$ and $\bar{L}_{t_1}$, where $t_1 = 4\eta^3$. It is straightforward to check that there is a cusp at $(0, 0)$ (whose tangent $x = 3\eta y$ is not vertical if $\eta \neq 0$), a simple point at $(0, 9\eta^2)$, a vertical tangency at $(t_1, 4\eta^2)$, and a simple point at (t_1, η^2) (see Figure 7).

Regardless of the value of η, there is an ordered geometric basis on a local disk $\mathbb{D}_{x,0}$ centered at $t = 0$ and a choice of generators σ_1 and σ_2 in $\mathbb{B}_3$ such that the local braid monodromy around $t = 0$ is given by σ_2^3. Therefore, since the product should be $(\sigma_2\sigma_1)^2$ and $\sigma_2^3 \cdot \sigma_1^{\sigma_2} = \sigma_2^2\sigma_1\sigma_2 = (\sigma_2\sigma_1)^2$, the local monodromy around $t = t_1$ should be given by $\sigma_1^{\sigma_2}$, which concludes the proof. □

Similar computations provide the following results for the simplest cases.

Proposition 4.5. *Let us assume that L_t intersects $\bar{\mathcal{C}}$ transversally at $k-3$ points and that it is tangent at one of the local branches of an ordinary double point. After conjugation, the monodromy around L_t produces the braid $\mu := \sigma_1\sigma_2\sigma_1$.*

Moreover, μ can be replaced by the factorization $(\sigma_2^{\sigma_1}, \sigma_1^2)$ so that the monodromy becomes generic in a regular neighborhood of L_t.

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Proposition 4.6. *Let us assume that L_t intersects $\bar{\mathcal{C}}$ transversally at $k-m$ points and that it is tangent to an inflection point of order m. After conjugation, the monodromy around L_t produces the braid $\mu := \sigma_{m-1} \dots \sigma_1$.*

Moreover, μ can be replaced by the factorization $(\sigma_1, \dots, \sigma_{m-1})$ so that the monodromy becomes generic in a regular neighborhood of L_t.

4.4. Line arrangements: from non-generic to generic

We are also interested in the case where $\bar{\mathcal{C}}$ is the union of a fully horizontal curve $\bar{\mathcal{C}}_0$ (such that $P_y \notin \bar{\mathcal{C}}$) and some lines $\bar{L}_1, \dots, \bar{L}_\ell$ passing through P_y and different from $\bar{L}_\infty$. In order to avoid technical problems, we restrict our attention to the case of line arrangements, which admits two approaches. One of them is to use wiring diagrams [8] (which is an invariant equivalent to braid monodromy). The other is more direct, but we need some definitions.

Definition 4.7. Let $n, k \in \mathbb{N}$, and let $\ell \in \{0, 1, \dots, k\}$. The *$\ell$-shift* of $\mathbb{B}_n$ into $\mathbb{B}_{n+k}$ is the inclusion $\rho_\ell : \mathbb{B}_n \to \mathbb{B}_{n+k}$ such that $\rho(\sigma_j) = \sigma_{j+\ell}$, $1 \leqslant j < n$.

Definition 4.8. The *partial Garside element* of the strings $i, \dots, j$, $i \leqslant j$, is the image $\Delta_{i,j}$ of the Garside element by the $(i-1)$-shift of $\mathbb{B}_{j-i+1}$ into $\mathbb{B}_n$.

Let $\bar{\mathcal{C}} = \bar{L}_1 \cup \dots \cup \bar{L}_n \cup \bar{L}_{n+1} \cup \dots \cup \bar{L}_{n+k}$ be a line arrangement such that P_y is a point of multiplicity k and $P_y \in \bar{L}_{n+1} \cap \dots \cap \bar{L}_{n+k}$. Let $\bar{L}_\infty$ be a generic line through P_y. We want to construct a generic braid monodromy for $\bar{\mathcal{C}}$ starting from the one of $(\bar{\mathcal{C}}_h, P_y, \bar{L}_\infty)$, where $\bar{\mathcal{C}}_h := \bar{L}_1 \cup \dots \cup \bar{L}_n$. Recall that this braid monodromy is obtained as a representation $\nabla : \pi_1(\mathbb{C} \setminus \mathcal{B}; t_0) \to \mathbb{B}_n$, where $\mathcal{B}$ is the set of x-coordinates of the multiple points of $\mathcal{C}_h$. Let $\mathcal{B}_0 := \mathcal{B} \cup \{t_1, \dots, t_k\}$, where $L_{n+i} = \{x = t_i\}$; since the braids associated with the meridians around the points in $\mathcal{B}_0 \setminus \mathcal{B}$ are trivial, the above mapping defines a representation $\nabla_0 : \pi_1(\mathbb{C} \setminus \mathcal{B}_0; t_0) \to \mathbb{B}_n$ (which will be referred to as the *augmented braid monodromy*). A choice of an ordered geometric basis $(\gamma_1, \dots, \gamma_r)$ allows us to represent this braid monodromy by $(\tau_1, \dots, \tau_r) \in \mathbb{B}_n^r$; let $1 \leqslant i_1 < \dots < i_k \leqslant r$ be the indices of the braids corresponding to the vertical lines $L_{n+1}, \dots, L_{n+k}$.

Proposition 4.9. *Under the above conditions, let us decompose τ_i as $\beta_i * \alpha_i$, where*

$$\alpha_i = \prod_{s=1}^{m(i)} \Delta^2_{a_s(i), a_{s+1}(i)-1}, \quad 1 = a_1(i) < \dots < a_{m(i)+1}(i) = n+1, \tag{4.1}$$

is a product of squares of partial Garside elements of $\mathbb{B}_n$. Then, a generic braid monodromy for $\bar{\mathcal{C}}$ is obtained by replacing the braids τ_i as follows.

(1) *If $i_j < i < i_{j+1}$ (by convention $i_0 = 0$, $i_{k+1} = r+1$), then replace τ_i by the sequence*

$$\left\{ \beta_i * \Delta^2_{a_s(i), a_{s+1}(i)-1} \right\}_{m(i) \geqslant s \geqslant 1},$$

and take out the trivial braids (for s such that $a_s(i) < a_{s+1}(i) - 1$).

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(2) *If* $i = i_j$*, then replace* τ_i *by*

$$\left\{ (\beta_i \cdot \sigma_{n+j-1}^{-1} \cdot \dots \cdot \sigma_{n+1}^{-1} \cdot \Delta_{a_{m(i)}(i), a_{m(i)+1}(i)} \cdot \dots \cdot \Delta_{a_{s+1}(i), a_{s+2}(i)}) * \Delta^2_{a_s(i), a_{s+1}(i)} \right\}_{m(i) \geqslant s \geqslant 1}.$$

(3) *Finally, add* $\Delta^2_{n+1,n+k}$.

Proof. For simplicity, we can assume that $r = k$, since we may add vertical lines to the arrangement, and once the generic braid monodromy is obtained we may forget the strings corresponding to the added lines; it is easily seen that this does not affect the final result.

The augmented braid monodromy ∇_0 described above may be computed as follows. Fix a generic line $\bar{L}_*$ through P_y close to $\bar{L}_\infty$, and fix a base point $P_* := (t_0, y_*) \in L_*$ close to P_y. We fix an ordered geometric basis $(\mu_1, \dots, \mu_n)$ of $\pi_1(L_* \setminus \mathcal{C}_h; P_*)$ such that μ_i is a meridian of L_i and its pseudo-Coxeter element is the boundary of a disk $\mathbb{D}_*$ in L_* surrounding $L_* \cap \mathcal{C}_h$ (which is also the negative boundary of a disk in $\bar{L}_*$ centered at P_y).

In the affine plane $\mathbb{C}^2 = \mathbb{P}^2 \setminus \bar{L}_\infty$, we consider the horizontal line H_* passing through P_*. We can choose P_* such that there is a disk $\mathbb{D}_x$ containing $\mathcal{B}_0 \times \{y_*\}$ in its interior. We fix an ordered geometric basis $(\tilde{\mu}_1, \dots, \tilde{\mu}_n, \tilde{\gamma}_1, \dots, \tilde{\gamma}_k)$ of $\pi_1(H_* \setminus \mathcal{C}; P_*)$ such that the following hold.

- The meridians $\tilde{\mu}_i$ and μ_i are equal in $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}; P_*)$ (and we identify them from now on). The product $\tilde{\mu}_n \cdot \dots \cdot \tilde{\mu}_1$ is the boundary of a disk $\tilde{\mathbb{D}}_*$ which is isotopic to $\mathbb{D}_*$ (in $\mathbb{C}^2 \setminus \mathcal{C}$) and disjoint to $\mathbb{D}_x$ (see Figure 8(a)).
- The meridians $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_k)$ are obtained as follows. Pick a base point in $\partial\mathbb{D}_x$; all the meridians have as common part a path joining P_* with this point avoiding (counterclockwise) the disk $\tilde{\mathbb{D}}_*$. These meridians project onto the meridians γ_i.

It is then clear that $(\tilde{\gamma}_k \cdot \dots \cdot \tilde{\gamma}_1) \cdot (\mu_d \cdot \dots \cdot \mu_1) = 1$ in $\pi_1(\mathbb{P}^2 \setminus \bar{\mathcal{C}}; P_*)$.

The next step is to deform the projection point in $\bar{L}_\infty$, which induces a family of coordinate changes $(x, y) \mapsto (x + s\varepsilon y, y)$ for a fixed $0 < |\varepsilon| \ll 1$ and $s \in (0, 1]$. For each new projection point, we obtain a braid monodromy $\nabla_s : \pi_1(\mathbb{C} \setminus \mathcal{B}_s; t_{0,s}) \to \mathbb{B}_{n+k}$, $s \in (0, 1]$. We can fix small pairwise disjoint disks $\mathbb{D}_1, \dots, \mathbb{D}_k$ (centered at $t_1, \dots, t_k$ resp.) such that $\mathcal{B}_s$ is contained in the interior of $\bigcup_{j=1}^k \mathbb{D}_j$. Note that $\#(\mathcal{B}_s \cap \mathbb{D}_j) = \#(\mathcal{C}_h \cap L_{n+j}) = m(j)$. In order to compute the braid monodromy ∇_s, we choose an ordered geometric basis $(\eta_{i,j})_{1 \leqslant i \leqslant m(j)}^{1 \leqslant j \leqslant k}$, such that $\prod_{i=1}^{m(j)} \eta_{i,j} = \gamma_j$. The generic base fiber is the new vertical line passing through P_*. In this vertical line we can construct disks isotopic to $\tilde{\mathbb{D}}_*$ and $\mathbb{D}_x$ containing respectively the intersections with $L_1 \cup \dots \cup L_n$ and $L_{n+1} \cup \dots \cup L_{n+k}$.

We start with the computation of $\nabla_1(\gamma_j)$. The decomposition $\tau_j = \beta_j * \alpha_j$ has a geometric meaning: β_j is the braid starting from the vertical line through P_* and ending in a vertical line close to L_{n+j}, while α_j is the braid obtained by turning around L_{n+j}, as (4.1) justifies. Let us produce a similar decomposition for $\nabla_1(\gamma_j) = \tilde{\beta}_j * \tilde{\alpha}_j$ in $\mathbb{B}_{n+k}$. The motion producing β_j induces now a braid $\tilde{\beta}_j$ as the product of two commuting braids: β_j and $\sigma_{n+j-1}^{-1} \cdot \dots \cdot \sigma_{n+1}^{-1}$; see Figure 8(b). The first one corresponds to the motion of the points in the disk $\tilde{\mathbb{D}}_*$, and the second one is obtained by considering the motion

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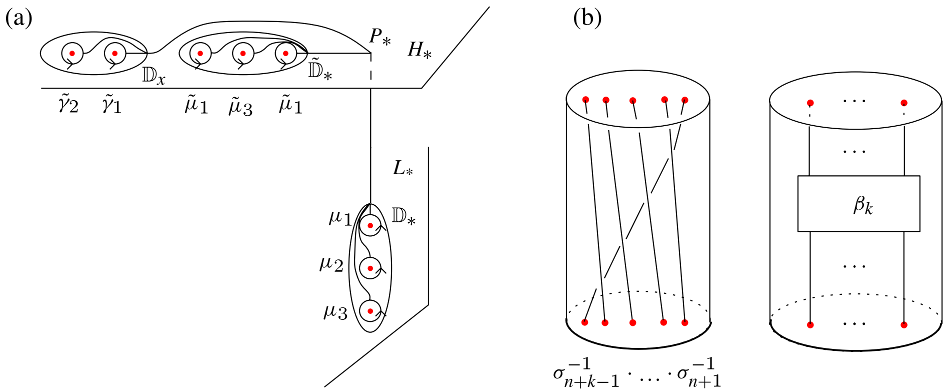

Figure 8. (a) $\pi_1(H_* \setminus \bar{\mathcal{C}}; P_*)$. (b) $\pi_1(L_* \setminus \bar{\mathcal{C}}; P_*)$.

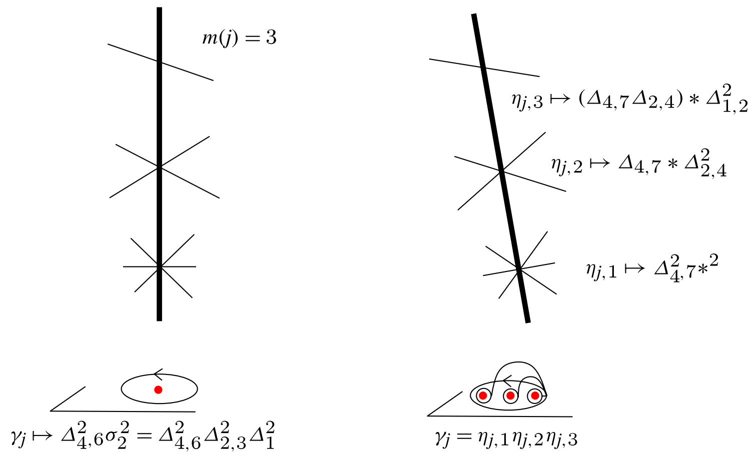

Figure 9. Braid $\tilde{\alpha}_j$.

of the ith point to the boundary (behind the other ones), and then how this point approaches $\tilde{\mathbb{D}}_*$.

The braid $\tilde{\alpha}_j$ is decomposed as a product of conjugate of partial Garside elements leading to the images of $\eta_{i,j}$ as in (1); see Figure 9.

For the case $k < r$, the decomposition in (1) obtained is the same one as that obtained by forgetting the strings corresponding to the verticals line not in $\mathcal{C}$.

The final step consists in the choice of a new line at infinity close to $\bar{L}_\infty$ (without changing the projection point); the last lines are no longer parallel, since the multiple point P_y becomes affine, and we must add the braid of (3). □

Example 4.10. Let us consider the line arrangement of Figure 10. In order to obtain the augmented braid monodromy ∇_0, it is necessary to add an extra vertical line L

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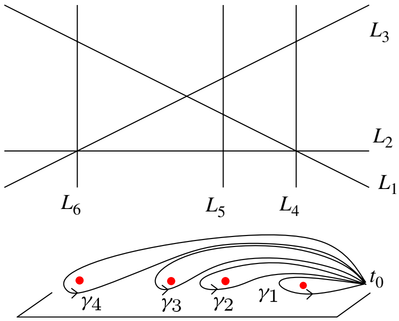

Figure 10. From non-generic to generic braid monodromy.

passing through the double point $L_1 \cap L_3$ of $\mathcal{C}$. It is easy to see that ∇_0 is given by $(\sigma_2^2, 1, \sigma_2 * \sigma_1^2, \sigma_1^2)$. The generic braid monodromy is given by

$$(\Delta_{2,4}^2, \Delta_{2,4} * \sigma_1^2, (\sigma_3^2)^{\sigma_4}, (\sigma_4^{-1}\sigma_3) * \sigma_2^2, (\sigma_4^{-1}\sigma_3\sigma_2) * \sigma_1^2, \sigma_2 * \sigma_1^2, (\sigma_3^2)^{\sigma_4\sigma_5},$$
$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3) * \Delta_{1,3}^2).$$

5. Transformations of curves by Kummer covers

Let $\bar{\mathcal{C}}$ be a (reduced) projective curve of degree k of equation $F_k(x, y, z) = 0$, and let $\bar{\mathcal{C}}_n$ be its transform by a Kummer cover π_n, $n > 1$. Note that $\bar{\mathcal{C}}_n$ is a projective curve of degree nk of equation $F_k(x^n, y^n, z^n) = 0$. Another obvious remark is that, if $\bar{\mathcal{C}}$ is reducible, then so is $\bar{\mathcal{C}}_n$. The converse is not true, as we will see in § 7.

We will briefly analyze the singularities of $\bar{\mathcal{C}}_n$ in terms of $\bar{\mathcal{C}}$. For convenience, we distinguish three types of point in $\mathbb{P}^2$.

Definition 5.1. Let $P \in \mathbb{P}^2$ such that $P := [x_0 : y_0 : z_0]$. We say that P is a point of *type* $(\mathbb{C}^*)^2$ (or simply of *type* 2) if $x_0 y_0 z_0 \neq 0$. If $x_0 = 0$ but $y_0 z_0 \neq 0$, the point is said to be of *type* $\mathbb{C}_x^*$ (types $\mathbb{C}_y^*$ and $\mathbb{C}_z^*$ are defined accordingly). Such points will also be referred to as *type*-1 points. The corresponding line (either $L_X := \{X = 0\}$, $L_Y := \{Y = 0\}$, or $L_Z := \{Z = 0\}$) that type-1 points lie on will be referred to as their *axis*. The remaining points $P_x := [1 : 0 : 0]$, $P_y := [0 : 1 : 0]$, and $P_z := [0 : 0 : 1]$ will be called *vertices* (or type-0 points), and their axes are the two lines (either L_X, L_Y, or L_Z) they lie on.

Remark 5.2. Note that a point of type ℓ, $\ell = 0, 1, 2$, in $\mathbb{P}^2$ has exactly n^ℓ preimages under π_n. It is also clear that the local type of $\mathcal{C}_n$ at any two points on the same fiber are analytically equivalent.

The following results are immediate, and we omit their proofs.

Lemma 5.3. *Let $P \in \mathbb{P}^2$ be a point of type ℓ, and let $Q \in \pi_n^{-1}(P)$. There exist local coordinates (u_0, v_0) and (u_1, v_1) centered at Q and P, respectively, such that the following hold.*

(1) *If $\ell = 2$, then $(u_1, v_1) = \pi_n(u_0, v_0) = (u_0, v_0)$.*

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(2) *If* $\ell = 1$, *then* $(u_1, v_1) = \pi_n(u_0, v_0) = (u_0^n, v_0)$, *where* $u_0 = 0$ *and* $u_1 = 0$ *are the local equations (at* Q *and* P*, respectively) of the axes containing the points.*

(3) *If* $\ell = 0$, *then* $(u_1, v_1) = \pi_n(u_0^n, v_0) = (u_0^n, v_0^n)$, *where* $u_i = 0$ *and* $v_i = 0$ *are the local equations of the axes containing* P, Q.

Proposition 5.4. *Let* $P \in \mathbb{P}^2$ *be a point of type* ℓ, *and let* $Q \in \pi_n^{-1}(P)$. *One has the following.*

(1) *If* $\ell = 2$, *then* $(\bar{\mathcal{C}}, P)$ *and* $(\bar{\mathcal{C}}_n, Q)$ *are analytically isomorphic.*

(2) *If* $\ell = 1$, *then* $(\bar{\mathcal{C}}_n, Q)$ *is a singular point of type* 1 *if and only if* $m > 1$, *where* $m := (\bar{\mathcal{C}} \cdot \bar{L})_P$ *and* $\bar{L}$ *is the axis of* P.

(3) *If* $\ell = 0$, *then* $(\bar{\mathcal{C}}_n, Q)$ *is a singular point.*

Example 5.5. In some cases, we can be more explicit about the singularity type of $(\bar{\mathcal{C}}_n, Q)$.

(1) If P is of type 1, $(\bar{\mathcal{C}}, P)$ is smooth, and $m := (\bar{\mathcal{C}} \cdot \bar{L})_P$, then $(\bar{\mathcal{C}}_n, Q)$ has the same topological type as $u_0^n - v_0^m = 0$. In particular, if $n = 2$, then $(\bar{\mathcal{C}}_n, Q)$ is of type $\mathbb{A}_{m-1}$.

(2) If P is of type 0, and $(\bar{\mathcal{C}}, P)$ is smooth and transverse to the axes, then $(\bar{\mathcal{C}}_n, Q)$ is an ordinary multiple point of multiplicity n.

In order to better describe singular points of type 0 and 1 of $\bar{\mathcal{C}}_n$, we will introduce some notation. Let $P \in \mathbb{P}^2$ be a point of type $\ell = 0, 1$, and let $Q \in \pi_n^{-1}(P)$ be a singular point of $\bar{\mathcal{C}}_n$. Denote by μ_P (respectively, μ_Q) the Milnor number of $\bar{\mathcal{C}}$ at P (respectively, $\bar{\mathcal{C}}_n$ at Q); also, denote by $\delta_1, \ldots, \delta_r$ the local branches of $\bar{\mathcal{C}}$ at P, and consider $\tilde{\delta}_i := \pi_n^{-1}(\delta_i)$. Define μ_{P,δ_i} (respectively, $\mu_{Q,\tilde{\delta}_i}$) as the Milnor number of the singularity (δ_i, P) (respectively, $(\tilde{\delta}_i, Q)$). Since $\ell = 0, 1$, P and Q belong to either exactly one or two axes. If P and Q belong to an axis $\bar{L}$, then $m_P^{\bar{L}} := (\bar{\mathcal{C}} \cdot \bar{L})_P$ and $m_{P,\delta_i}^{\bar{L}} := (\delta_i \cdot \bar{L})_P$ (there is analogous notation for Q and $\tilde{\delta}_i$). More specific details about singular points of type 0 and type 1 can be described as follows.

Proposition 5.6. *Under the above conditions and notation, one has the following.*

(1) *For* $\ell = 1$, P *belongs to a unique axis* $\bar{L}$, *and the following hold.*

  (a) $\mu_Q = n\mu_P + (m_P^{\bar{L}} - 1)(n-1).$,

  (b) $\mu_{Q,\tilde{\delta}_i} = n\mu_{P,\delta_i} + (m_{P,\delta_i}^{\bar{L}} - 1)(n-1)$.

  (c) *If* $i \neq j$ *then* $(\tilde{\delta}_i \cdot \tilde{\delta}_j)_Q = n(\delta_i \cdot \delta_j)_P$.

  (d) *If* $r_i := \gcd(n, m_{P,\delta_i}^{\bar{L}})$, *then* $\tilde{\delta}_i$ *has* r_i *irreducible components.*

(2) *For* $\ell = 0$, P *belongs to exactly two axes* $\bar{L}_1$ *and* $\bar{L}_2$, *and the following hold.*

  (a) $\mu_Q = n^2\mu_P + (n-1)(n(m_P^{\bar{L}_1} + m_P^{\bar{L}_2}) - 1)$.

  (b) $\mu_{Q,\tilde{\delta}_i} = n^2\mu_{P,\delta_i} + (n-1)(n(m_{P,\delta_i}^{\bar{L}_1} + m_{P,\delta_i}^{\bar{L}_2} - 1) - 1)$, $m_{Q,\tilde{\delta}_i}^{\bar{L}_j} = nm_{P,\delta_i}^{\bar{L}_j}$.

  (c) *If* $i \neq j$, *then* $(\tilde{\delta}_i \cdot \tilde{\delta}_j)_Q = n^2(\delta_i \cdot \delta_j)_P$.

  (d) *If* $r_i := \gcd(n, m_{P,\delta_i}^{\bar{L}_1}, m_{P,\delta_i}^{\bar{L}_2})$, *then* $\tilde{\delta}_i$ *has* nr_i *irreducible components, which are analytically isomorphic to each other for any fixed* i.

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Proof. For part (1), note that

$$\mu_Q = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(x^{n-1}f_x(x^n,y), f_y(x^n,y))} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(x^{n-1}, f_y(x,y))} + \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f_x(x^n,y), f_y(x^n,y))}$$
$$= (n-1)\dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(x, f_y(x,y))} + n\dim_{\mathbb{C}} \frac{\mathbb{C}\{x^n,y\}}{(f_x(x^n,y), f_y(x^n,y))} = (n-1)(m_P^{\bar{L}} - 1) + n\mu_P.$$

The same proof applies to $\mu_{Q,\tilde{\delta}_i}$. Also,

$$(\tilde{\delta}_i \cdot \tilde{\delta}_j)_Q = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f_i(x^n,y), f_j(x^n,y))} = n\dim_{\mathbb{C}} \frac{\mathbb{C}\{x^n,y\}}{(f_i(x^n,y), f_j(x^n,y))} = n(\delta_i \cdot \delta_j)_P,$$

where f_i (respectively, f_j) is a local equation for δ_i (respectively, δ_j).

Finally, we can describe the irreducible branch δ_i as a Puiseux factorization of type $f_i(x,y) = \prod_{j=1}^{\nu}(y - s(\xi_\nu^j x^{1/\nu}))$, where ν is a multiple of $m = m_{P,\delta_i}^{\bar{L}}$, the order of f_i in y. Note that $f_i(x^n,y) = \prod_{j=1}^{\nu}(y - s(\xi_\nu^j x^{n/\nu}))$. Assume that r is a common divisor, that is, $n = rn'$, $m = rm'$ (and hence $\nu = r\nu'$); then

$$\prod_{j=1}^{\nu'}(y - s(\xi_\nu^{rj} x^{rn'/\nu})) = \prod_{j=1}^{\nu'}(y - s(\xi_{\nu'}^j x^{n'/\nu'}))$$

is invariant under the Galois conjugation by ν'-roots of unity, and hence it is a convergent series in $\mathbb{C}\{x,y\}$, that is, a union of branches. Therefore the result follows.

The proof of part (2) is analogous. □

6. Proofs of main results

Proof of Theorem 1. By Proposition 5.4 (see also Example 5.5), the curve $\mathcal{C}$ can only have ordinary cusps as singularities. Moreover, it should intersect the axes transversally except if $n = 2$ (respectively, $n = 3$), where the axes can be tangent to ordinary flexes (respectively, ordinary tangents) eventually at several points. The monodromy given by $(\mathcal{C} \cup L_z, L_\infty, P)$ determines as shown in § 3 the braid monodromy given by $(\mathcal{C}_n, L_\infty, P)$, which might not be generic, but it is generic at infinity. Propositions 4.4 and 4.3 provide the result. □

Proof of Theorem 2. It is straightforward that the braid monodromy given by $(\mathcal{C}, L_\infty, P)$ determines the one given by $(\mathcal{C} \cup L_z, L_\infty, P)$ for generic choices of L_z, L_∞, and P. As above, the method described in § 3 provides a (non-generic) braid monodromy given by $(\mathcal{C}_n, L_\infty, P)$. The result follows from Proposition 4.3. □

Proof of Theorem 3. This is the content of § 3. □

7. Examples and applications

In this section, the results obtained in § § 2–3 will be applied to produce generic braid monodromies of curves. We will follow the conventions introduced in [5]. In the language of the previous section, diagram systems on $\mathbb{C}$ for the starting curve $\mathcal{C}$ will be chosen

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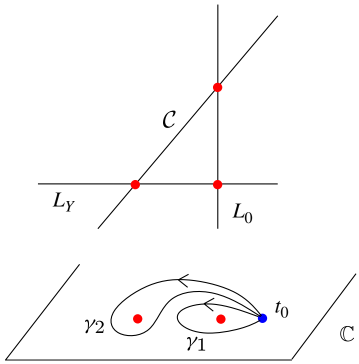

Figure 11. A generic line.

by joining the points with segments in a decreasing order according to the lexicographic order.

7.1. Smooth curves

We start with the braid monodromy of a smooth curve considered as a Fermat curve $\bar{\mathcal{C}}_n = \{x^n - y^n + z^n = 0\} \subset \mathbb{P}^2$, via the nth Kummer cover of the line $\bar{\mathcal{C}}$ of equation $x - y + z = 0$. Using the conventions of § 2, one can obtain an extended braid monodromy $\tilde{\nabla} : \pi_1(\mathbb{C} \setminus \tilde{B}; t_0) \to \mathbb{B}_{1,1}$, where γ_1, γ_2 is an ordered geometric basis of the free group $\pi_1(\mathbb{C} \setminus \tilde{B}; t_0)$ and $\tilde{\nabla}(\gamma_1) = 1$, $\tilde{\nabla}(\gamma_2) = \sigma_1^2$, as shown in Figure 11. Using Example 3.5 and the commutative diagram (2.2), one can obtain a braid monodromy $\nabla_n : \mathbb{F}_n \to \mathbb{B}_n$ for $\mathcal{C}_n$ as follows:

$$\nabla_n(\tilde{\gamma}_j) = \sigma_{n-1} \cdot \ldots \cdot \sigma_1, \tag{7.1}$$

where the basis $\tilde{\gamma}_j$, $j = 1, \ldots, n$, is obtained as in Lemma 2.6, by forgetting the last term, since $\bar{L}_0$ is transversal.

However, note that ∇_n is not a generic braid monodromy, since the projection from P_y contains n special fibers $\bar{L}_{\xi_n^i}$, $i = 0, \ldots, n-1$, with an order-n tangency at $[\xi_n^i : 0 : -\xi_n^i]$. Still, ∇_n is useful to compute the fundamental group of the complement of a smooth curve (using Theorem 1.6). This group is generated by $\mu_1, \ldots, \mu_d$ with a relation $\mu_d \cdot \ldots \cdot \mu_1$, and, since

$$\mu_i^{\sigma_{n-1} \cdot \ldots \cdot \sigma_1} = \mu_{i+1}, \quad 1 \leqslant i < d,$$

the group is cyclic of order d, as obtained by Zariski in [42].

By the previous discussion, in order to obtain a generic braid monodromy, it is enough to use Proposition 4.6 and obtain the following.

Proposition 7.1 ([37, Theorem 1, p. 120]). *Let $\bar{\mathcal{C}}$ be a smooth curve of degree n. Then $\bar{\mathcal{C}}$ induces a braid monodromy factorization of Δ_n^2 given by*

$$(\overbrace{\sigma_1, \ldots, \sigma_{n-1}}^{1^{st} package}, \ldots, \overbrace{\sigma_1, \ldots, \sigma_{n-1}}^{n^{th} package}).$$

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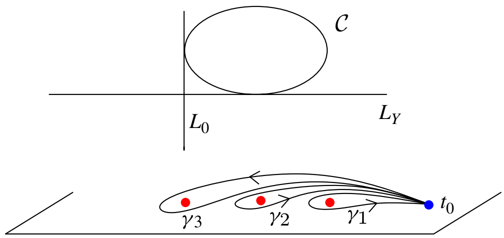

Figure 12. A conic tangent to two axes.

7.2. Zariski sextics

In [42], Zariski showed that the fundamental group of a sextic with six cusps on a conic is $\mathbb{Z}/2 * \mathbb{Z}/3$ and that the family of such sextics is irreducible, so they all have equivalent braid monodromies. We are going to compute one in two steps: first, a (fully horizontal) non-generic braid monodromy of such a sextic will be obtained from a simpler curve via a Kummer covering and second, a deformation will be performed to compute the desired generic braid monodromy.

Consider a conic as in Figure 12. It is tangent to two of the axes and transversal to the third one, e.g., the conic $\bar{\mathcal{C}} = \{x^2 + y^2 - 2xz - 2yz + z^2 = 0\}$. Using the conventions of § 2, the projection from P_y has three non-generic fibers at $\tilde{B} = \{0, 1, 2\}$. Consider $t_0 = 3$ and $\gamma_1, \gamma_2, \gamma_3$ meridians around 2, 1, and 0, respectively (see Figure 12), forming an ordered geometric basis of the free group $\pi_1(\mathbb{C} \setminus \tilde{B}; t_0)$. One can obtain an extended braid monodromy $\tilde{\nabla} : \pi_1(\mathbb{C} \setminus \tilde{B}; t_0) \to \mathbb{B}_{2,1}$ as follows:

$$\tilde{\nabla}(\gamma_1) = \sigma_1, \quad \tilde{\nabla}(\gamma_2) = \sigma_1 * \sigma_2^4, \quad \tilde{\nabla}(\gamma_3) = (\sigma_1\sigma_2^2) * \sigma_1.$$

After conjugating by the braid $(\sigma_1\sigma_2^2)$, the above representation becomes

$$\tilde{\nabla}(\gamma_1) = \sigma_1^{\sigma_2^2}, \quad \tilde{\nabla}(\gamma_2) = \sigma_2^4, \quad \tilde{\nabla}(\gamma_3) = \sigma_1.$$

Let us consider the third Kummer cover of $\bar{\mathcal{C}}$. The curve $\bar{\mathcal{C}}_3$ is a curve of degree 6. Since $\bar{\mathcal{C}}$ is smooth and intersects $\bar{L}_\infty$ transversally, the singular points are contained in $\bar{L}_0 \cup \bar{L}_Y$ (Proposition 5.4). Using Proposition 5.6, the curve $\bar{\mathcal{C}}_3$ possesses six ordinary cusps (which are in a conic, namely the union of the lines $\pi_3^{-1}(\bar{L}_0) = \bar{L}_0$ and $\pi_3^{-1}(\bar{L}_Y) = \bar{L}_Y$).

Using a radial system of generators as in § 3.2, the map $\hat{\rho}_3 : \mathbb{B}_{2,1} \to \mathbb{B}_6$ is given (see Proposition 3.7) by

$$\begin{aligned} \sigma_1 &\mapsto \sigma_1\sigma_3\sigma_5 \\ \sigma_2^2 &\mapsto \sigma_5\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_5^{-1} = (\sigma_5\sigma_4\sigma_2^{-1}) * (\sigma_4\sigma_3), \end{aligned}$$

where the generators σ_i on the right-hand side are defined as $\sigma_1 := \tilde{\sigma}_{1,1}, \sigma_2 := \tilde{\sigma}_{2,1}, \ldots, \sigma_5 := \tilde{\sigma}_{1,3}$ (with the right-lexicographic order) for simplicity. Let us conjugate

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the result by $\sigma_5\sigma_3$:

$$\begin{array}{rcl} \sigma_1 & \mapsto & \sigma_1\sigma_3\sigma_5 \\ \sigma_2^2 & \mapsto & \sigma_3^{-1}\sigma_4\sigma_3\sigma_2 = \sigma_4 * (\sigma_3\sigma_2). \end{array} \tag{7.2}$$

Proposition 7.2. *There is an ordered geometric basis, as in Lemma 2.6, such that* $\nabla_3 : \mathbb{F}_7 \to \mathbb{B}_6$ *is given as follows:*

$$\begin{array}{rclcll} \tilde{\gamma}_1 & = & \gamma_1 & \mapsto & (\sigma_1\sigma_3\sigma_5)^{\sigma_4*(\sigma_3\sigma_2)} = & (\sigma_1\sigma_3\sigma_5)^{\sigma_4\sigma_3\sigma_2} \\ \tilde{\gamma}_2 & = & \gamma_2 & \mapsto & \sigma_4 * (\sigma_3\sigma_2)^2 & \\ \tilde{\gamma}_3 & = & \gamma_1^{\gamma_3} & \mapsto & (\sigma_1\sigma_3\sigma_5)^{\sigma_4*(\sigma_3\sigma_2)\sigma_1\sigma_3\sigma_5} = & (\sigma_1\sigma_3\sigma_5)^{\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_5} \\ \tilde{\gamma}_4 & = & \gamma_2^{\gamma_3} & \mapsto & ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5} & \\ \tilde{\gamma}_5 & = & \gamma_1^{\gamma_3^2} & \mapsto & (\sigma_1\sigma_3\sigma_5)^{\sigma_4*(\sigma_3\sigma_2)\sigma_1^2\sigma_3^2\sigma_5^2} = & (\sigma_1\sigma_3\sigma_5)^{\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_3^2\sigma_5^2} \\ \tilde{\gamma}_6 & = & \gamma_2^{\gamma_3^2} & \mapsto & ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1^2\sigma_3^2\sigma_5^2} & \\ \tilde{\gamma}_7 & = & \gamma_3^3 & \mapsto & \sigma_1^3\sigma_3^3\sigma_5^3. & \end{array}$$

As mentioned above, this braid monodromy is not generic. However, the triple $(\bar{\mathcal{C}}_3, P_y, L_\infty)$ is fully horizontal and transversal at infinity. One can deform ∇_3 as described in § 4. First, a deformation is performed, as in § 4.3, to produce a locally generic triple. Then, a move as described in § 4.2 will produce a generic braid monodromy.

Also note that the non-generic fibers of the projection from P_y are

$$\tilde{B}_3 = \{t_1 = \sqrt[3]{2}, t_2 = 1, t_3 = \xi_3\sqrt[3]{2}, t_4 = \xi_3, t_5 = \xi_3^2\sqrt[3]{2}, t_6 = \xi_3^2, t_7 = 0\},$$

where $\tilde{\gamma}_i$ is a meridian around t_i, and the following hold.

(1) $\bar{L}_{t_i}$, $i = 1, 3, 5$, correspond to vertical lines containing three simple tangencies (hence locally generic fibers).

(2) $\bar{L}_{t_i}$, $i = 2, 4, 6$, correspond to vertical lines passing tangent to an ordinary cusp and transversal to three smooth points of $\bar{\mathcal{C}}_3$.

(3) $\bar{L}_0$ corresponds to a vertical line passing through three ordinary cusps (hence a locally generic fiber).

Step 1. This step is necessary at the fibers of type (2) above. According to Proposition 4.4, the braid obtained as the image of $\tilde{\gamma}_2$, that is, $\sigma_4 * (\sigma_3\sigma_2)^2 = ((\sigma_4 * \sigma_3) \cdot (\sigma_2))^2$ has to be replaced by

$$(\sigma_2^{\sigma_4*\sigma_3}, \sigma_4 * \sigma_3^3) = (\sigma_2^{\sigma_3\sigma_4^{-1}}, (\sigma_3^3)^{\sigma_4^{-1}}). \tag{7.3}$$

Analogously this can be done with $\nabla_3(\tilde{\gamma}_4)$ and $\nabla_3(\tilde{\gamma}_6)$.

Step 2. This is necessary at the locally generic fibers described above. Applying Proposition 4.3 to the fibers of types 1 and 3, one obtains that, for instance, $\nabla_3(\tilde{\gamma}_1) = (\sigma_1\sigma_3\sigma_5)^{\sigma_4\sigma_3\sigma_2}$ is replaced by

$$(\sigma_1^{\sigma_4\sigma_3\sigma_2}, \sigma_3^{\sigma_4\sigma_3\sigma_2}, \sigma_5^{\sigma_4\sigma_3\sigma_2}) = (\sigma_1^{\sigma_2}, \sigma_4, \sigma_5^{\sigma_4\sigma_3\sigma_2}). \tag{7.4}$$

Analogously this can be done with $\nabla_3(\tilde{\gamma}_3)$, $\nabla_3(\tilde{\gamma}_5)$, and $\nabla_3(\tilde{\gamma}_7)$.

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However, instead of working out each $\nabla_3(\tilde{\gamma}_i)$ separately, we can note that $\nabla_3(\tilde{\gamma}_{i+2j}) = \nabla_3(\tilde{\gamma}_i)^{\sigma_1^j\sigma_3^j\sigma_5^j}$ ($i=1,2$, $j=0,1,2$). Therefore one can also work out the first set of braids coming from $\nabla_3(\tilde{\gamma}_1)$ and $\nabla_3(\tilde{\gamma}_2)$ and conjugate to obtain the rest of the braids coming from $\nabla_3(\tilde{\gamma}_i)$ $i=3,\ldots,6$. Finally attach the braids $(\sigma_1^3,\sigma_3^3,\sigma_5^3)$ obtained from $\nabla_3(\tilde{\gamma}_7)$. From (7.3) and (7.4), one obtains

$$(\sigma_1^{\sigma_2},\sigma_4,\sigma_5^{\sigma_4\sigma_3\sigma_2},\sigma_2^{\sigma_3\sigma_4^{-1}},(\sigma_3^3)^{\sigma_4^{-1}}).$$

Using the sequence of Hurwitz moves $h_2^{-1}h_3^{-1}h_4^{-1}h_2h_3$ (see (1.3)) one obtains the first block:

$$(\sigma_1^{\sigma_2},\sigma_2^{\sigma_3},\sigma_3^3,\sigma_4^{\sigma_5^{-1}\sigma_3^{-1}},\sigma_4). \tag{7.5}$$

After conjugating (7.5) by $\sigma_1\sigma_3\sigma_5$ and applying h_2, one obtains the second block:

$$(\sigma_2^{\sigma_3},\sigma_3^3,\sigma_2^{\sigma_3^{-1}\sigma_1},\sigma_4,\sigma_4^{\sigma_3\sigma_5}). \tag{7.6}$$

The third block is obtained from (7.6) using conjugation by $\sigma_1\sigma_3\sigma_5$ and applying $h_1h_4^{-1}$:

$$(\sigma_3^3,\sigma_2^{\sigma_3^{-1}\sigma_1},\sigma_2^{\sigma_1^2},\sigma_4,\sigma_4^{\sigma_3\sigma_5}). \tag{7.7}$$

Proposition 7.3. *A generic braid monodromy for $\bar{\mathcal{C}}_3$ is given by*

$$(\sigma_1^{\sigma_2},\sigma_2^{\sigma_3},\sigma_3^3,\sigma_4^{\sigma_5^{-1}\sigma_3^{-1}},\sigma_4,\sigma_2^{\sigma_3},\sigma_3^3,\sigma_2^{\sigma_3^{-1}\sigma_1},\sigma_4,\sigma_4^{\sigma_3\sigma_5},\sigma_3^3,\sigma_2^{\sigma_3^{-1}\sigma_1},\sigma_1^3,\sigma_1^{\sigma_2},\\ \sigma_4,\sigma_4^{\sigma_3\sigma_5},\sigma_3^3,\sigma_5^3).$$

Remark 7.4. This result is also obtained in [37, Theorem 1(3), p. 160]. It is straightforward to compute the fundamental group of the complement of $\bar{\mathcal{C}}_3$ and to retrieve Zariski's computation. Moreover, this braid monodromy allows us to compute the homotopy type of $\mathbb{C}^2\setminus\mathcal{C}_3$ (for a generic choice of line at infinity) using the method in [33]. It is easy to see that

$$\mathbb{C}^2\setminus\mathcal{C}_3\simeq(\mathbb{S}^3\setminus\{\text{trefoil knot}\})\vee\bigvee_{i=1}^{13}\mathbb{S}^2.$$

7.3. Dual of a smooth cubic

The dual of a smooth cubic is a sextic with nine cusps. Kummer covers allow one to recover one of these curves easily.

Consider a conic $\bar{\mathcal{C}} := \{x^2+y^2+z^2-2(xy+xz+yz)=0\}$, as in Figure 13. Projecting from $P_y=[0:1:0]$ as usual, one obtains two non-generic fibers $\bar{L}_\infty$ and $\bar{L}_0$ with tangencies $P_3:=[1:1:0]$ and $P_1:=[0:1:1]$, respectively. Also note that $\mathcal{C}$ is tangent to $\bar{L}_Y$ at $P_2:=[1:0:1]$. Using the conventions of § 2, we obtain an extended braid monodromy $\tilde{\nabla}:\pi_1(\mathbb{C}\setminus\tilde{B};t_0)\to\mathbb{B}_{2,1}$, where $\tilde{B}=\{P_1,P_2,P_3\}$, γ_1,γ_2 are meridians around P_2 and P_1, respectively, forming an ordered basis of the free group $\pi_1(\mathbb{C}\setminus\tilde{B};t_0)$ and $\tilde{\nabla}(\gamma_1)=\sigma_2^4$, $\tilde{\nabla}(\gamma_2)=\sigma_2^2*\sigma_1$. For simplicity, we conjugate this monodromy by σ_2^2, and obtain

$$\gamma_1\mapsto\sigma_2^4,\quad \gamma_2\mapsto\sigma_1. \tag{7.8}$$

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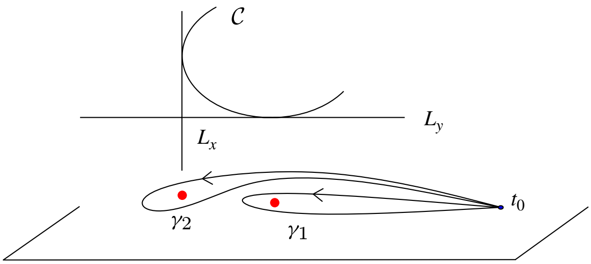

Figure 13. Tritangent conic.

The preimage of $\bar{\mathcal{C}}$ by the Kummer cover of order 3 is a sextic $\bar{\mathcal{C}}_3$ with nine cusps, as can be deduced from Example 5.5. Let us compute the braid monodromy $(\bar{\mathcal{C}}_3, P_y, \bar{L}_\infty)$. Following the ideas in § 7.2, one immediately obtains

$$\begin{array}{ccc} \gamma_1 & \mapsto & ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}} \\ \gamma_1^{\gamma_3} & \mapsto & ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5} \\ \gamma_1^{\gamma_3^2} & \mapsto & ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1^2\sigma_3^2\sigma_5^2} \\ \gamma_3^3 & \mapsto & \sigma_1^3\sigma_3^3\sigma_5^3. \end{array}$$

Since the line $\bar{L}_\infty$ is not generic, one can apply Proposition 4.2 to obtain a braid monodromy that is generic at infinity. The pseudo-Coxeter element for the monodromy (7.8) is $c := \sigma_1\sigma_2^4$, and hence $\Delta_3^6 c^{-3} = (\Delta_3^2 c^{-1})^3 = (\sigma_1^3)^{\sigma_2^2} \in \mathbb{B}_{2,1}$, whose image in $\mathbb{B}_6$ via $\hat{\rho}_3$ (see (7.2)) is $(\sigma_1^3\sigma_3^3\sigma_5^3)^{\sigma_4\sigma_3\sigma_2\sigma_4^{-1}}$. Therefore

$$(((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}}, ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5}, ((\sigma_3\sigma_2)^2)^{\sigma_4^{-1}\sigma_1^2\sigma_3^2\sigma_5^2}, \sigma_1^3\sigma_3^3\sigma_5^3, (\sigma_1^3\sigma_3^3\sigma_5^3)^{\sigma_4\sigma_3\sigma_2\sigma_4^{-1}})$$

is a braid monodromy factorization of Δ_6^2 generic at infinity for the Zariski sextic $\bar{\mathcal{C}}_3$. Finally, in order to obtain a generic braid monodromy, one needs to apply Propositions 4.2 and 4.3 to slightly turn the projection generic and perform Hurwitz moves to simplify the braids. One can check that the final generic braid monodromy of $\bar{\mathcal{C}}_3$ is

$$\sigma_2^3, \sigma_2^{\sigma_3\sigma_4^{-1}}, \sigma_2^{\sigma_3\sigma_4^{-1}\sigma_1\sigma_3\sigma_5}, \sigma_3^3, (\sigma_3^3)^{\sigma_4\sigma_5}, \sigma_5^3, \sigma_2^{\sigma_3^{-1}\sigma_4\sigma_1^2\sigma_5^{-1}}, (\sigma_5^3)^{\sigma_4}, \sigma_1^3, (\sigma_1^3)^{\sigma_2},$$
$$\sigma_4^3, (\sigma_5^3)^{\sigma_4\sigma_3\sigma_2}).$$

The fundamental group of this curve complement has been extensively studied by Zariski in [32]. He also used deformation arguments to recover the fundamental group studied in [42], see § 7.2, as well as to study the fundamental group of sextics with six cusps not on a conic.

Similar arguments can be used to describe the generic braid monodromy of any curve $\bar{\mathcal{C}}_n$, n odd, which is an irreducible curve of degree-$3n$ and degree-$9n$ singularities of type $\mathbb{A}_{n-1}$ (see [17]).

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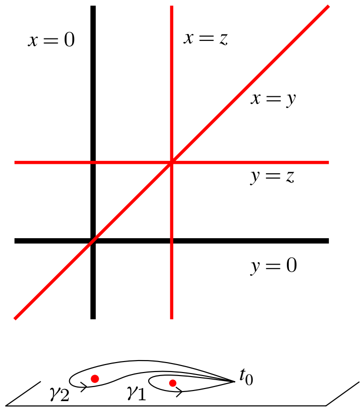

Figure 14. Ceva arrangement.

7.4. Ceva arrangement

Line arrangements provide a particularly interesting source of examples of arrangements of curves. Their braid monodromies, or their equivalent objects such as *braided wiring diagrams*, have been used and calculated in the literature (see [8, 39, 20, 40, 21]). Such braid monodromies are especially simple to obtain for real arrangements. In this section, we will use the classical Ceva arrangement (six lines joining four points in general positions) which is a real arrangement, to find the braid monodromy of the 9-Ceva arrangement

$$\mathcal{C} := \{(x^3-y^3)(y^3-z^3)(x^3-z^3)=0\},$$

since it can be obtained as a Kummer covering of three concurrent lines in a classical Ceva arrangement; see Figure 14.

The extended braid monodromy on $\mathbb{B}_{2,1}$ of Figure 14 is given by

$$\gamma_1 \mapsto \sigma_1^2, \quad \gamma_2 \mapsto \sigma_2^2.$$

Hence, after performing a Kummer cover of order 3, one obtains

$$\begin{array}{ccc} \gamma_1 & \mapsto & \sigma_1^2\sigma_3^2\sigma_5^2 \\ \gamma_1^{\gamma_2} & \mapsto & (\sigma_1^2\sigma_3^2\sigma_5^2)^{\sigma_4\sigma_3\sigma_2} \\ \gamma_1^{\gamma_2^2} & \mapsto & (\sigma_1^2\sigma_3^2\sigma_5^2)^{\sigma_4^2\sigma_3\sigma_2^2} \\ \gamma_2^3 & \mapsto & \sigma_4 * (\sigma_3\sigma_2)^3 \end{array}$$

(see (7.2)).

In order to make this monodromy generic at infinity, one applies Proposition 4.2, obtaining the new braid:

$$(\Delta_3^2 c^{-1})^3 = ((\sigma_3\sigma_2)^3)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5}.$$

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Using Proposition 4.9, one can obtain a generic braid monodromy as follows:

$$(\Delta^2_{5,7}, (\Delta^2_{3,5})^{\Delta^{-1}_{5,7}}, (\Delta^2_{1,3})^{\Delta^{-1}_{3,5}\Delta^{-1}_{5,7}}, (\Delta^2_{5,7})^{\sigma_4\sigma_3\sigma_2\sigma_7}, (\Delta^2_{3,5})^{\Delta^{-1}_{5,7}\sigma_4\sigma_3\sigma_2\sigma_7},$$
$$(\Delta^2_{1,3})^{\Delta^{-1}_{3,5}\Delta^{-1}_{5,7}\sigma_4\sigma_3\sigma_2\sigma_7}, (\Delta^2_{5,7})^{\sigma_4^2\sigma_3\sigma_2^2\sigma_7\sigma_8}, (\Delta^2_{3,5})^{\Delta^{-1}_{5,7}\sigma_4^2\sigma_3\sigma_2^2\sigma_7\sigma_8},$$
$$(\Delta^2_{1,3})^{\Delta^{-1}_{3,5}\Delta^{-1}_{5,7}\sigma_4^2\sigma_3\sigma_2^2\sigma_7\sigma_8}, \sigma_4 * (\sigma_3\sigma_2)^3, ((\sigma_3\sigma_2)^3)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5}).$$

Another direct application is the monodromy of the MacLane arrangement [36], which is obtained from $\bar{\mathcal{C}}_3$ by deleting one line. This results in deleting one of the strings, that is,

$$(\Delta^2_{5,7}, (\Delta^2_{3,5})^{\Delta^{-1}_{5,7}}, (\Delta^2_{1,3})^{\Delta^{-1}_{3,5}\Delta^{-1}_{5,7}}, (\Delta^2_{5,7})^{\sigma_4\sigma_3\sigma_2\sigma_7}, (\Delta^2_{3,5})^{\Delta^{-1}_{5,7}\sigma_4\sigma_3\sigma_2\sigma_7},$$
$$(\Delta^2_{1,3})^{\Delta^{-1}_{3,5}\Delta^{-1}_{5,7}\sigma_4\sigma_3\sigma_2\sigma_7}, (\sigma_5^2)^{\sigma_4^2\sigma_3\sigma_2^2}, (\sigma_3^2)^{\sigma_4^2\sigma_3\sigma_2^2}, (\sigma_1^2)^{\sigma_4^2\sigma_3\sigma_2^2},$$
$$\sigma_4 * (\sigma_3\sigma_2)^3, ((\sigma_3\sigma_2)^3)^{\sigma_4^{-1}\sigma_1\sigma_3\sigma_5}).$$

The computational difficulty of the braid monodromy of the 9-Ceva as well as the MacLane arrangements comes from the fact that they cannot be given by real equations; the MacLane arrangement is the smallest one with this property. A direct computation of the braid monodromy of the MacLane arrangement can be found in [20]. Using our construction, everything is reduced to computing the braid monodromy of a very simple real line arrangement.

Braid monodromy for generalized Ceva arrangements [10] ($n > 3$) are also obtained, but we omit their lengthy factorizations.

7.5. A useful nodal cubic

This section will serve as an important tool for the remaining examples. The main object will be the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$, where $\bar{\mathcal{C}}$ is a nodal cubic, $P_y \notin \bar{\mathcal{C}}$, $\bar{L}_\infty$ is tangent to $\bar{\mathcal{C}}$ at an inflection point, and the projection has only two non-generic fibers: another tangency at an inflection point and the line passing though P_y and the node. The purpose of this section is to calculate the extended non-generic braid monodromy of the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ with respect to the tangent line $\bar{L}_Y$ at the remaining inflection point. Let us consider the following cubic, $\bar{\mathcal{C}} = \{f(x, y, z) = 0\}$, $f(x, y, z) = (x + y + z)^3 - 27xyz$, whose real picture appears in Figure 15(a). The usual projection from P_y has four non-generic fibers $\bar{L}_t$ at $t = \infty, 1, 0$, where $\bar{L}_\infty$ and $\bar{L}_0$ are tangent lines at inflection points and $\bar{L}_1$ is the vertical line through the node (which is real, but whose branches are not). For this reason, the real picture is not enough to recover the braid monodromy of $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$. However, the extra information required can be obtained from drawing the real part of the missing complex conjugate branches (shown in Figure 15(b) as dotted lines).

In order to find these dotted lines, one can proceed as follows. Let $p(y) := y^3 - a_1 y + a_2 y^2 - a_3 \in \mathbb{R}[y]$, and assume that it has only one real root, say t_1. Let $t_2, t_3 \in \mathbb{C}$ be the remaining roots, $t_3 = \bar{t}_2$. Note that their common real part is $\frac{t_2+t_3}{2}$.

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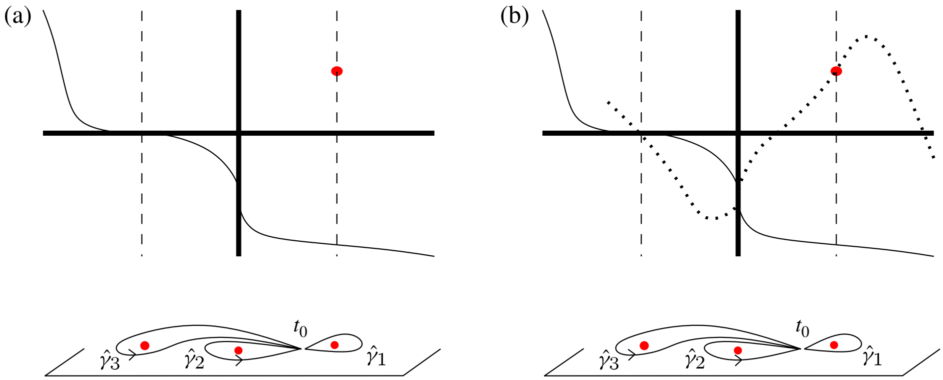

Figure 15. (a) The nodal cubic. (b) Nodal cubic with a *global* real picture.

Lemma 7.5. *Under the above conditions, the polynomial*

$$q(y) = y^3 - a_1 y^2 + \frac{a_1^2 + a_2}{4} y - \frac{a_1 a_2 - a_3}{8}$$

contains exactly one real root, which is the common real part $\frac{t_2+t_3}{2}$ *of the complex conjugate roots* t_2, t_3 *of* $p(y)$.

Proof. Consider $a_i = s_i(t_1, t_2, t_3)$ the symmetric polynomial of degree i; it is enough to show that $a_1 = s_1(\tilde t_1, \tilde t_2, \tilde t_3)$, $\frac{a_1^2+a_2}{4} = s_2(\tilde t_1, \tilde t_2, \tilde t_3)$, and $\frac{a_1 a_2 - a_3}{8} = s_3(\tilde t_1, \tilde t_2, \tilde t_3)$, where $\tilde t_i = \frac{t_j + t_k}{2}$, $\{i, j, k\} = \{1, 2, 3\}$, which is a simple exercise. □

Since the affine cubic $f(x_0, y, 1)$ satisfies the conditions of Lemma 7.5, for any fixed $\bar x \in \mathbb{R}$, the real parts of the complex roots of $f(x_0, y, 1)$ are given by the equation

$$\begin{aligned} 0 &= y^3 + 3(x_0+1)y^2 + 3\frac{4x_0^2 - x_0 + 4}{4} + (x_0+1)\frac{8x_0^2 - 65x_0 + 8}{8} \\ &= (y + x_0 + 1)^3 - \frac{27}{4} x_0 y - \frac{81}{8} x_0(x_0+1). \end{aligned}$$

This is enough to compute the required braid monodromy; see [6] for details.

In order to show the extended braid monodromy $\tilde\nabla$ of $\mathcal{C}$, note that L_Y is also tangent to $\mathcal{C}$ at the inflection point $(-1, 0)$. Hence $\tilde B = \{1, 0, -1\}$. Choose an ordered geometric basis as in Figure 15(b) and a system diagram on the fiber given by decreasing lexicographic order on $\mathbb{C}$. For instance, according to Figure 15(b), σ_1 is the half-twist exchanging the two complex conjugated roots of $f(t_0, y, 1)$, and σ_2 is the half-twist exchanging the complex conjugated root of $f(t_0, y, 1)$ of negative real part and 0. Then, $\tilde\nabla : \pi_1(\mathbb{C} \setminus \tilde B; t_0) \to \mathbb{B}_{3,1}$ is given by

$$\begin{aligned} \hat\gamma_1 &\mapsto \sigma_1^2 \\ \hat\gamma_2 &\mapsto (\sigma_2\sigma_3)^{\sigma_1^{-1}\sigma_2} \\ \hat\gamma_3 &\mapsto \sigma_2^6. \end{aligned}$$

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In order to meet the hypothesis of § 2, one needs the system diagram to be such that the last string is the one that remains constant. This means that one needs to conjugate $\tilde{\nabla}$ by σ_3^{-1}. Combining this with a conjugation by σ_2^{-1}, the Hurwitz move h_2^{-1} (changing $\hat{\gamma}_i$ to γ_i), and another conjugation by σ_1^{-1}, one obtains

$$\begin{array}{rcl} \gamma_1 & \mapsto & \sigma_2^2 \\ \gamma_2 & \mapsto & \sigma_2 * \sigma_3^6 \\ \gamma_3 & \mapsto & (\sigma_1\sigma_2)^{\sigma_3^2}. \end{array} \tag{7.9}$$

An easy computation gives that the product of Δ^2 and the inverse of the pseudo-Coxeter element is $(\sigma_1\sigma_2)^{\sigma_3^2\sigma_2}$.

7.6. The dual of the nodal quartic

The dual curve of a nodal quartic is a sextic curve with six cusps and four nodes. Moreover, this curve appears as a generic plane section of the discriminant of the polynomials of degree 4, and its fundamental group has been computed by Zariski [31]: it is the braid group on $\mathbb{S}^2$ with four strings.

This sextic can also be obtained from the cubic in § 7.5 using a Kummer cover π_2. Each inflection point in the axes produces two cusps, while the double point produces four nodes; see Example 5.5.

In order to give a braid monodromy for $(\mathcal{C}_2, P_y, \bar{L}_\infty)$ from (7.9), one needs to combine $\hat{\rho}'_2$ (see Example 3.8) and Lemma 2.6. We start with the first part of (2.2):

$$\begin{array}{ccccc} \mathbb{F}_5 & \hookrightarrow & \mathbb{F}_3 & \xrightarrow{\tilde{\nabla}} & \mathbb{B}_{3,1} \\ \tilde{\gamma}_1 & \mapsto & \gamma_1 & \mapsto & \sigma_2^2 \\ \tilde{\gamma}_2 & \mapsto & \gamma_2 & \mapsto & \sigma_2 * \sigma_3^6 \\ \tilde{\gamma}_3 & \mapsto & \gamma_1^{\gamma_3} & \mapsto & (\sigma_2^2)^{\sigma_3^{-2}\sigma_1\sigma_2\sigma_3^2} = (\sigma_2\sigma_3^2\sigma_1) * \sigma_2^2 \\ \tilde{\gamma}_4 & \mapsto & \gamma_2^{\gamma_3} & \mapsto & (\sigma_3^6)^{\sigma_2^{-1}\sigma_3^{-2}\sigma_1\sigma_2\sigma_3^2} = (\sigma_3^6)^{\sigma_2^{-1}\sigma_1} \\ \tilde{\gamma}_5 & \mapsto & \gamma_3^2 & \mapsto & \sigma_3^{-2} * (\sigma_1\sigma_2)^2. \end{array} \tag{7.10}$$

Using $\hat{\rho}'_2$, we obtain the braid monodromy of $(\bar{\mathcal{C}}_2, P_y, \bar{L}_\infty)$:

$$(\hat{\sigma}_2^2\hat{\sigma}_4^2, (\hat{\sigma}_2\hat{\sigma}_4) * \hat{\sigma}_3^3, (\hat{\sigma}_2\hat{\sigma}_4\hat{\sigma}_3\hat{\sigma}_5\hat{\sigma}_1) * (\hat{\sigma}_2^2\hat{\sigma}_4^2), (\hat{\sigma}_3^3)^{\hat{\sigma}_2^{-1}\hat{\sigma}_4^{-1}\hat{\sigma}_5\hat{\sigma}_1}, ((\hat{\sigma}_1\hat{\sigma}_2)^2(\hat{\sigma}_5\hat{\sigma}_4)^2)^{\hat{\sigma}_3}). \tag{7.11}$$

Using Proposition 4.2, the monodromy can be made generic at infinity adding the image by $\hat{\rho}'_2$ of $(\sigma_2^{-1}\sigma_3^2) * (\sigma_2\sigma_1)^3$, which is $((\hat{\sigma}_1\hat{\sigma}_2)^2(\hat{\sigma}_5\hat{\sigma}_4)^2)^{\hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_4}$.

In order to obtain a generic braid monodromy, we apply Propositions 4.3 and 4.4. Table 1 shows the decompositions.

We can compute the fundamental group of $\mathbb{C}^2 \setminus \mathcal{C}_2$. It is well known [31] that this group is obtained as a central extension of the braid group of $\mathbb{S}^2$ with four strings by $\mathbb{Z}$. More precisely, applying Theorem 1.7 (without the relation $\mu_6 \cdot \ldots \cdot \mu_1 = 1$), we obtain a

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$\hat\sigma_2^2\hat\sigma_4^2$	(σ_2^2,σ_4^2)
$(\hat\sigma_2\hat\sigma_4)*\hat\sigma_3^3$	$((\hat\sigma_2\hat\sigma_4)*\hat\sigma_3^3)$
$(\hat\sigma_2\hat\sigma_4\hat\sigma_3\hat\sigma_5\hat\sigma_1)*(\hat\sigma_2^2\hat\sigma_4^2)$	$((\hat\sigma_2\hat\sigma_4\hat\sigma_3\hat\sigma_1)*(\hat\sigma_2^2),(\hat\sigma_2\hat\sigma_4\hat\sigma_3\hat\sigma_5)*(\hat\sigma_4^2))$
$(\hat\sigma_3^3)^{\hat\sigma_2^{-1}\hat\sigma_4^{-1}\hat\sigma_5\hat\sigma_1}$	$((\hat\sigma_3^3)^{\hat\sigma_2^{-1}\hat\sigma_4^{-1}\hat\sigma_5\hat\sigma_1})$
$((\hat\sigma_1\hat\sigma_2)^2(\hat\sigma_5\hat\sigma_4)^2)^{\hat\sigma_3}$	$(\hat\sigma_2^{\hat\sigma_1\hat\sigma_3},\hat\sigma_1^3,\hat\sigma_4^{\hat\sigma_5\hat\sigma_3},\hat\sigma_5^3))$
$((\hat\sigma_1\hat\sigma_2)^2(\hat\sigma_5\hat\sigma_4)^2)^{\hat\sigma_3\hat\sigma_2\hat\sigma_4}$	$(\hat\sigma_2^{\hat\sigma_1\hat\sigma_3\hat\sigma_2\hat\sigma_4},(\hat\sigma_1^3)^{\hat\sigma_2},\hat\sigma_4^{\hat\sigma_5\hat\sigma_3\hat\sigma_2\hat\sigma_4},(\hat\sigma_5^3)^{\hat\sigma_4})$

TABLE 1. From generic at infinity to generic braid monodromy.

group with generators $\mu_1,\dots,\mu_6$ and relators

$$[\mu_4,\mu_5]=[\mu_2^2,\mu_5\mu_4^{-1}]=1,\quad \mu_2\mu_4\mu_2=\mu_4\mu_2\mu_4,\,\mu_2\mu_5\mu_2=\mu_5\mu_2\mu_5,$$
$$\mu_6=\mu_2^{\mu_5},\quad \mu_1=\mu_4*\mu_2,\,\mu_3=\mu_5^{\mu_2\mu_4},\quad \underbrace{1=1}_{7\text{ times}}.$$

This presentation is obtained using GAP4 [24].

From the original presentation, we obtain this one only using Tietze moves of type I and type II. Hence, by Libgober's result [33], one can verify that $\mathbb{C}^2\setminus\mathcal{C}_2$ has the homotopy type of $K\vee\bigvee_{j=1}^7\mathbb{S}^2$, where K is the 2-complex associated with the presentation

$$\langle x,y,z\,|\,[x,z]=[y^2,xz^{-1}]=1,\,xyx=yxy,\,yzy=zyz\rangle.$$

7.7. Hesse arrangement

The Hesse arrangement, that is, the arrangement of the 12 lines joining the inflection points of a smooth cubic, is a complex arrangement of lines with only double and quadruple singular points. It can be seen as the union of the four completely reducible fibers in a pencil of smooth cubics whose base points are the nine inflection points. Our purpose in this section is to obtain the generic braid monodromy of the Hesse arrangement. This problem was considered in [18] using a computer-based approach. For our approach, note that the Hesse arrangement can be obtained from the nodal cubic in § 7.5 using a Kummer cover π_3: it is the preimage of the cubic and the three ramification lines. It can be seen using Proposition 4.3 and Example 5.5, and also directly from the following equations:

$$(x^3+y^3+z^3)^3-27x^3y^3z^3=\prod_{\zeta_1^3=1}(x^3+y^3+z^3-3\zeta_1xyz).$$

Hence $xyz=0$, and the above factors give the four reducible members of the pencil of cubics generated by $x^3+y^3+z^3=0$ and $xyz=0$. The base points of this pencil are the nine common inflection points of the irreducible cubics of the pencil. The reducible fibers split as a product of lines:

$$x^3+y^3+z^3-3\zeta_1xyz=\prod_{\zeta_2^3=1}(x+\zeta_2y+\zeta_1\bar\zeta_2z).$$

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These are the lines joining the inflection points, i.e., we obtain the Hesse arrangement. We start with the braid monodromy with respect to $(\bar{\mathcal{C}}_3, P_y, \bar{L}_\infty)$: $(\sigma_2^2, \sigma_2 * \sigma_3^6, (\sigma_1\sigma_2)^{\sigma_3^2})$, and the braid at infinity is $(\sigma_1\sigma_2)^{\sigma_3^2\sigma_2}$. In order to simplify the braid monodromy of the Hesse arrangement, we conjugate by $\sigma_3^{-2}\sigma_2^{-1}$, and we obtain

$$((\sigma_2^2)^{\sigma_3^2}, \sigma_3^6, \sigma_2\sigma_1), \quad \infty \mapsto (\sigma_2\sigma_3^2\sigma_2^{-1}\sigma_3^{-2}) * (\sigma_1\sigma_2).$$

We proceed as in § 7.6, but we perform a Hurwitz move to the base of Lemma 2.6 in order to obtain simpler braids:

$$\begin{array}{ccccc}
\mathbb{F}_7 & \hookrightarrow & \mathbb{F}_3 & \overset{\tilde{\nabla}}{\to} & \mathbb{B}_{3,1} \\
\tilde{\gamma}_1 & \mapsto & \gamma_1 & \mapsto & (\sigma_2^2)^{\sigma_3^2} \\
\tilde{\gamma}_2 & \mapsto & \gamma_2 & \mapsto & \sigma_3^6 \\
\tilde{\gamma}_3 & \mapsto & \gamma_1^{\gamma_3} & \mapsto & (\sigma_2^2)^{\sigma_3^2\sigma_2\sigma_1} \\
\tilde{\gamma}_4 & \mapsto & \gamma_2^{\gamma_3} & \mapsto & (\sigma_3^6)^{\sigma_2\sigma_1} \\
\tilde{\gamma}_5 & \mapsto & \gamma_3^3 & \mapsto & (\sigma_2\sigma_1)^3 \\
\tilde{\gamma}_6 & \mapsto & \gamma_3 * \gamma_1 & \mapsto & (\sigma_2\sigma_1\sigma_3^{-2}) * \sigma_2^2 \\
\tilde{\gamma}_7 & \mapsto & \gamma_3 * \gamma_2 & \mapsto & \sigma_2 * \sigma_3^6.
\end{array} \tag{7.12}$$

Let us recall the map $\tilde{\rho}_3 : \mathbb{B}_{3,1} \to \mathbb{B}_{9,1}$ of Proposition 3.7:

$$\begin{array}{ccc}
\sigma_1 & \mapsto & \tilde{\sigma}_{1,1}\tilde{\sigma}_{1,2}\tilde{\sigma}_{1,3} \\
\sigma_2 & \mapsto & \tilde{\sigma}_{2,1}\tilde{\sigma}_{2,2}\tilde{\sigma}_{2,3} \\
\sigma_3^2 & \mapsto & \tilde{\sigma}_{3,3}^2\tilde{\sigma}_{2,3}\tilde{\sigma}_{1,3}\tilde{\sigma}_{3,2}\tilde{\sigma}_{2,2}\tilde{\sigma}_{1,2}\tilde{\sigma}_{3,1}\tilde{\sigma}_{1,2}^{-1}\tilde{\sigma}_{2,2}^{-1}\tilde{\sigma}_{1,3}^{-1}\tilde{\sigma}_{2,3}^{-1}.
\end{array} \tag{7.13}$$

For convenience, we rewrite this map in the usual generators:

$$\begin{array}{ccc}
\sigma_1 & \mapsto & \sigma_1\sigma_4\sigma_7 \\
\sigma_2 & \mapsto & \sigma_2\sigma_5\sigma_8 \\
\sigma_3^2 & \mapsto & \sigma_9^2\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_8^{-1} = (\sigma_9^2\sigma_8\sigma_7)^{\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6}.
\end{array} \tag{7.14}$$

It is worthwhile noting that $(\sigma_9^2\sigma_8\sigma_7)^3 = \Delta_{7,10}^2$. Let us denote $\tau := \sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6$ and $\Theta := (\sigma_9^2\sigma_8\sigma_7)^\tau$. The next step is to write down the braid monodromy of $(\bar{\mathcal{C}}_3, P_y, \bar{L}_\infty)$:

$$((\sigma_2^2\sigma_5^2\sigma_8^2)^\Theta, (\Delta_{7,10}^2)^\tau, (\sigma_2^2\sigma_5^2\sigma_8^2)^{\Theta\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, (\Delta_{7,10}^2)^{\tau\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, \Delta_{1,3}^2\Delta_{4,6}^2\Delta_{7,9}^2,$$
$$(\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7\Theta^{-1}) * (\sigma_2^2\sigma_5^2\sigma_8^2), (\sigma_2\sigma_5\sigma_8\tau^{-1}) * \Delta_{7,10}^2).$$

This monodromy becomes generic at infinity by adding $(\sigma_2\sigma_5\sigma_8\Theta\sigma_8^{-1}\sigma_5^{-1}\sigma_2^{-1}\Theta^{-1}) * (\Delta_{1,3}^2\Delta_{4,6}^2\Delta_{7,9}^2)$. Let us denote $\eta := \sigma_2\sigma_5\sigma_8\Theta\sigma_8^{-1}\sigma_5^{-1}\sigma_2^{-1}\Theta^{-1}$.

To compute a generic braid monodromy to this arrangement, one has to apply Proposition 4.9, where we have two vertical lines corresponding to the fifth and eighth

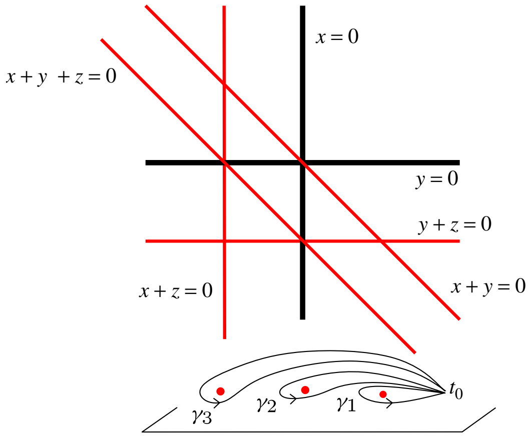

Figure 16. Arrangement $\bar{\mathcal{C}}$.

braids:

$$((\sigma_2^2)^{\Theta}, (\sigma_5^2)^{\Theta}, (\sigma_8^2)^{\Theta}, (\Delta_{7,10}^2)^{\tau}, (\sigma_2^2)^{\Theta\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, (\sigma_5^2)^{\Theta\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, (\sigma_8^2)^{\Theta\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, (\Delta_{7,10}^2)^{\tau\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7}, \sigma_{10}^2, \sigma_{10} * \Delta_{7,10}^2, (\sigma_{10}\Delta_{7,10}) * \Delta_{4,7}^2, (\sigma_{10}\Delta_{7,10}\Delta_{4,7}) * \Delta_{1,4}^2, (\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7\Theta^{-1}) * \sigma_2^2, (\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7\Theta^{-1}) * \sigma_5^2, (\sigma_2\sigma_1\sigma_5\sigma_4\sigma_8\sigma_7\Theta^{-1}) * \sigma_8^2, (\sigma_2\sigma_5\sigma_8\tau^{-1}) * \Delta_{7,10}^2, (\sigma_{11}^{-1}\eta) * \sigma_{10}^2, (\sigma_{11}^{-1}\eta\sigma_{10}) * \Delta_{7,10}^2, (\sigma_{11}^{-1}\eta\sigma_{10}\Delta_{7,10}) * \Delta_{4,7}^2, (\sigma_{11}^{-1}\eta\sigma_{10}\Delta_{7,10}\Delta_{4,7}) * \Delta_{1,4}^2, \sigma_{11}^2). \tag{7.15}$$

7.8. Sextics with six cusps outside a conic

The Kummer cover π_3 of the curve $\bar{\mathcal{C}}$ of equation $(x+y+z)(x+y)(y+z)(x+z) = 0$ produces a Fermat cubic (as the preimage of the line $x+y+z=0$), while the other three lines become the tangent lines to the nine inflection points. The non-generic extended braid monodromy associated with the triple $(\bar{\mathcal{C}}, P_y, \bar{L}_\infty)$ with respect to the line $\bar{L}_Y = \{y = 0\}$ is given by $(\sigma_2^2, \sigma_2 * (\sigma_1^2\sigma_3^2), (\sigma_2\sigma_3\sigma_2^{-1}) * \sigma_1^2)$. In order to apply the Kummer transformation to this monodromy, one needs to find an equivalent monodromy that meets the hypotheses of § 2. This can be achieved if the last generator in the ordered geometric basis corresponds to $x = 0$, and if the system diagram is such that the last string is the one that remains constant. One can meet the first requirement by using the Hurwitz move h_2^{-1}:

$$(\sigma_2^2, (\sigma_2\sigma_3^{-1}\sigma_1^{-1}) * \sigma_2^2, \sigma_2 * (\sigma_1^2\sigma_3^2)).$$

In order to switch the string $y = 0$ to the last position, one must perform the automorphism $\sigma_i \mapsto \sigma_{4-i}$, but these braids are invariant by the morphism. After these changes, the braid around infinity is $(\sigma_1^2\sigma_3^2)^{\sigma_2}$. The braid monodromy of the Kummer cover can be obtained using the appropriate transformation described in Proposition 3.7, or its version for $n = 3$ given in (7.13).

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In order to obtain a sextic with six cusps not on a conic, we will follow the method used in [3] by means of a two-fold Kummer cover where the three ramification lines are tangent lines to a smooth cubic at non-aligned inflection points. Hence we will restrict ourselves to the cubic previously constructed (the preimage of $x+y+z=0$), two preimages of $x+z=0$, one preimage of $y+z=0$, and none of $x+y=0$. Hence, we can forget the second string (recall the automorphism), and we obtain (the first braid disappears since it is trivial)

$$((\sigma_2^2)^{\sigma_1}, \sigma_1^2),$$

and hence the braid around infinity is σ_2^2. We have to forget some strings in the mapping $\hat{\rho}_3$. This causes the morphism to not be well defined in $\mathbb{B}_{2,1}$, but only in the pure braid group:

$$\begin{array}{ccc} \sigma_1^2 & \mapsto & \sigma_1^2 \\ \sigma_2^2 & \mapsto & 1 \\ (\sigma_2^2)^{\sigma_1} & \mapsto & (\sigma_3\sigma_2)^{\sigma_1}. \end{array}$$

Hence the braid monodromy of the cubic with the tangent lines is given by

$$((\sigma_3\sigma_2)^{\sigma_1}, (\sigma_3\sigma_2)^{\sigma_1^3}, (\sigma_3\sigma_2)^{\sigma_1^5}, \sigma_1^6),$$

where the first two braids correspond to the tangent lines in the curve, while the second string corresponds to the remaining tangent line. To simplify the braid monodromy, some operations will be performed, namely, conjugation by σ_1^{-3}, cyclic permutation, permutation of the order of the strings, and forgetting one braid (at infinity):

$$(\sigma_3^6, \sigma_3^2 * (\sigma_1\sigma_2), \sigma_1\sigma_2),$$

the braid at infinity being $(\sigma_1\sigma_2)^{\sigma_3^2}$. This result coincides with the one obtained using Carmona's program [13].

We first consider the braid monodromy factorization induced by this braid monodromy after a double cover of the base, that is, $\mathbb{F}_5 \hookrightarrow \mathbb{F}_3 \to \mathbb{B}_{3,1}$:

$$(\sigma_3^6, \sigma_3^2 * (\sigma_1\sigma_2), (\sigma_3^6)^{\sigma_2}, (\sigma_2^{-1}\sigma_3^2) * (\sigma_1\sigma_2), (\sigma_1\sigma_2)^2, ((\sigma_1\sigma_2)^2)^{\sigma_3^2}).$$

Then perform the morphism $\tilde{\rho}$ (see Example 3.8) and a generic deformation, resulting in

$$(\sigma_3^3, \sigma_1, \sigma_5, \sigma_3^{\sigma_2}, \sigma_3^{\sigma_4}, (\sigma_3^3)^{\sigma_2\sigma_4}, \sigma_1^{\sigma_2}, \sigma_5^{\sigma_4}, \sigma_3^{\sigma_2^2\sigma_4}, \sigma_3^{\sigma_2\sigma_4^2}, \sigma_2^{\sigma_1}, \sigma_1^3, \sigma_4^{\sigma_5}, \sigma_5^3, \sigma_2^{\sigma_1\sigma_3},$$
$$\sigma_1^3, \sigma_4^{\sigma_5\sigma_3}, \sigma_5^3).$$

A straightforward computation shows that $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_2) = \mathbb{Z}$. Using [33], one can check that

$$\mathbb{C}^2 \setminus \mathcal{C}_2 \simeq \mathbb{S}^1 \vee \bigvee_{i=1}^{13} \mathbb{S}^2.$$

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