THE REVIEW OF SYMBOLIC LOGIC
Volume 13, Number 4, December 2020

NON–WELL-FOUNDED DERIVATIONS IN THE GÖDEL-LÖB PROVABILITY LOGIC

DANIYAR SHAMKANOV

Steklov Mathematical Institute of Russian Academy of Sciences and National Research University Higher School of Economics

Abstract. We consider Hilbert-style non–well-founded derivations in the Gödel-Löb provability logic GL and establish that GL with the obtained derivability relation is globally complete for algebraic and neighbourhood semantics.

§1. Introduction. The Gödel-Löb provability logic GL is a modal logic describing all universally valid principles of the formal provability in Peano arithmetic [20]. A proof-theoretic presentation of this logic in a form of a sequent calculus allowing non–well-founded proofs was given recently in [7, 15]. In this article, we consider Hilbert-style non–well-founded derivations in GL and study algebraic and neighbourhood semantics of GL with the obtained derivability relation. This article is an extension of a conference article [16].

The Gödel-Löb provability logic GL can be additionally defined as the logic of the class of Magari algebras [10, 19]. In this article, we introduce a notion of $\Box$-founded Magari algebra and show that GL enriched with non–well-founded derivations is strongly sound and complete for its algebraic interpretation over this class of algebras.

Neighbourhood semantics is a generalization of Kripke semantics independently developed by D. Scott and R. Montague in [14] and [12]. A neighbourhood frame can be defined as a pair $(X, \Box)$, where X is a set and $\Box$ is an unary operation in $\mathcal{P}(X)$. The Gödel-Löb provability logic GL is compact for its neighbourhood interpretation, which immediately implies that GL is strongly neighbourhood complete (see [2, 17]). However, this completeness result holds for the case of the so-called local semantic consequence relation. Recall that, over neighbourhood GL-models, a formula A is a local semantic consequent of Γ if for any neighbourhood GL-model $\mathcal{M}$ and any world x of $\mathcal{M}$

$$(\forall B \in \Gamma \quad \mathcal{M}, x \vDash B) \Rightarrow \mathcal{M}, x \vDash A.$$

A formula A is a global semantic consequent of Γ if for any neighbourhood GL-model $\mathcal{M}$

$$(\forall B \in \Gamma \quad \mathcal{M} \vDash B) \Rightarrow \mathcal{M} \vDash A.$$

Notice that this global semantic consequence relation coincides with the following one: A is a consequent of Γ if for any neighbourhood GL-model $\mathcal{M}$, any world x of $\mathcal{M}$ and any

Received: August 24, 2018.
2010 *Mathematics Subject Classification*: 03B65.
Key words and phrases: provability logic, algebraic semantics, neighbourhood semantics, scattered topological spaces, global consequence relations, non–well-founded derivations.

doi:10.1017/S1755020319000613

open neighbourhood U of x, that is $x \in U \subset \Box U$,

$$(\forall B \in \Gamma\ \forall y \in U\ \ \mathcal{M}, y \vDash B) \Rightarrow \mathcal{M}, x \vDash A.$$

We show that GL enriched with non–well-founded derivations is strongly complete with respect to the global semantic consequence relation over neighbourhood GL-frames. We also show that this system is complete for its neighbourhood interpretation over the single neighbourhood GL-frame defined by the first uncountable ordinal equipped with the interval topology. This latter observation resembles the result of M. Abashidze [1] and A. Blass [4] that GL is locally complete for its neighbourhood interpretation over the ordinal ω^ω (see also [2]).

Now it remains to stress that the ordinary global syntactic consequence relation in GL, which is a derivability relation standardly defined without non–well-founded derivations, is not neighbourhood complete (see Corollary 7.6 in [9]).

The plan of the article is as follows. In §2, we recall the Gödel-Löb provability logic GL and define non–well-founded derivations in it. In §3, we define $\Box$-founded Magari algebras and establish algebraic completeness of GL with non–well-founded derivations. In §4, we recall neighbourhood semantics of GL and consider a connection between scattered topological spaces and corresponding neighbourhood frames. In the next section, we show that the global consequence relation over neighbourhood GL-frames is determined by the set of countable ordinals equipped with the interval topology. In §6, we obtain a form of neighbourhood compactness result using the ultrabouquet construction from [17]. In §7, we present a sequent calculus for GL allowing non–well-founded proof trees. In the final section, we establish neighbourhood completeness for GL with non–well-founded derivations.

§2. Non–well-founded derivations in GL. In this section we recall the Gödel-Löb provability logic GL and define local and global derivability relations for the given system.

Formulas of GL (also called *modal formulas*) are built from the countable set of variables $PV = \{p, q, \dots\}$ and the constant $\bot$ using propositional connectives $\to$ and $\Box$. We treat other Boolean connectives and the modal connective $\Diamond$ as abbreviations:

$$\neg A := A \to \bot, \qquad \top := \neg\bot, \qquad A \wedge B := \neg(A \to \neg B),$$
$$A \vee B := \neg A \to B, \qquad A \leftrightarrow B := (A \to B) \wedge (B \to A), \qquad \Diamond A := \neg\Box\neg A.$$

In the sequel, the set of modal formulas is denoted by Fm.

The Gödel-Löb provability logic GL is defined via its Hilbert-style axiomatization.

Axiom schemes:

(i) the tautologies of classical propositional logic;

(ii) $\Box(A \to B) \to (\Box A \to \Box B)$;

(iii) $\Box A \to \Box\Box A$;

(iv) $\Box(\Box A \to A) \to \Box A$.

Inference rules:

$$\mathsf{mp}\frac{A \quad A \to B}{B}, \qquad \mathsf{nec}\frac{A}{\Box A}.$$

A relation of derivability from assumptions in GL is inductively defined in the following way. A formula A *is derivable from the set of assumptions* Γ (cf. [6]), if A is in Γ, or A is one of the axioms of GL, or follows from derivable formulas through applications of the

https://doi.org/10.1017/S1755020319000613 Published online by Cambridge University Press

inference rules (mp) and (nec) so that the rule (nec) can be applied only to derivations without assumptions. We denote this derivability relation by $\vdash_l$, where l stands for 'local'.

Global derivability relations are defined via non–well-founded derivations in GL. An ∞-*derivation* in GL is a (possibly infinite) tree whose nodes are marked by modal formulas and that is constructed according to the rules (mp) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec). An *assumption leaf* of an ∞-derivation is a leaf that is not marked by an axiom of GL. An assumption leaf is *boxed* if there is an application of the rule (nec) on the path from this leaf to the root of the tree.

The *main fragment* of an ∞-derivation is a finite tree obtained from the ∞-derivation by cutting every infinite branch at the nearest to the root application of the rule (nec). The *local height* $|\pi|$ *of an* ∞*-derivation* π is the length of the longest branch in its main fragment. An ∞-derivation only consisting of a single formula has height 0.

For example, consider the following ∞-derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Box p_3}\quad \Box p_3 \to p_2}{p_2}\,\textsf{mp}}{\Box p_2}\,\textsf{nec}\quad \Box p_2 \to p_1}{\dfrac{p_1}{\Box p_1}\,\textsf{nec}}\,\textsf{mp}\quad \Box p_1 \to p_0}{p_0}\,\textsf{mp}\ ,
$$

where assumption leafs are marked by formulas of the form $\Box p_{n+1} \to p_n$. The local height of this ∞-derivation equals to 1 and its main fragment has the form

$$
\textsf{mp}\ \dfrac{\Box p_1 \qquad \Box p_1 \to p_0}{p_0}\ .
$$

We set $\Gamma \vdash_g A$ if there is an ∞-derivation with the root marked by A in which all assumption leafs are marked by some elements of Γ. We also set $\Sigma; \Gamma \vdash A$ if there is an ∞-derivation with the root marked by A in which all boxed assumption leafs are marked by some elements of Σ and all nonboxed assumption leafs are marked by some elements of Γ.

Note that the relation $\vdash$ is a generalization of $\vdash_l$ and $\vdash_g$ since $\emptyset; \Gamma \vdash A \Leftrightarrow \Gamma \vdash_l A$ and $\Gamma; \Gamma \vdash A \Leftrightarrow \Gamma \vdash_g A$. The only nontrivial implication (whether $\emptyset; \Gamma \vdash A$ implies $\Gamma \vdash_l A$) will be checked in the next sections.

§3. Algebraic semantics. In this section we consider local and global semantic consequence relations over $\Box$-founded Magari algebras and obtain completeness results connecting semantic and previously introduced syntactic consequence relations.

A *Magari algebra* $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box)$ is a Boolean algebra $(X, \wedge, \vee, \to, 0, 1)$ together with a unary map $\Box\colon X \to X$ satisfying the identities:

$$
\Box 1 = 1, \quad \Box(x \wedge y) = \Box x \wedge \Box y, \quad \Box(\Box x \to x) = \Box x.
$$

A *valuation in* $\mathcal{A}$ is a function $\theta\colon Fm \to X$ such that $\theta(\bot) = 0$, $\theta(A \to B) = \theta(A) \to \theta(B)$, and $\theta(\Box A) = \Box\theta(A)$.

Note that, for any Magari algebra $\mathcal{A}$, the mapping $\Box$ is monotone with respect to the order (of the Boolean part of) $\mathcal{A}$. Indeed, if $a \leqslant b$, then $a \wedge b = a$, $\Box a \wedge \Box b = \Box(a \wedge b) = \Box a$, and $\Box a \leqslant \Box b$. In addition, we recall that, in any Magari algebra, an inequality $\Box x \leqslant \Box\Box x$ holds.

A filter F of (the Boolean part of) a Magari algebra $\mathcal{A}$ is called *open* if $\Box a \in F$, whenever $a \in F$. Let us remember that any open filter F of $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box)$ defines a congruence $\sim_F = \{(a, b) \in X \times X \mid (a \leftrightarrow b) \in F\}$ on $\mathcal{A}$ (see Lemma 3.1.5. from [8]). We will write $\mathcal{A}/F$ instead of $\mathcal{A}/\sim_F$.

We call a Magari algebra $\Box$-*founded (or Pakhomov-Walsh-founded)*[1] if, for every sequence of its elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$, we have $a_0 = 1$. This notion can be defined in terms of the binary relation $\prec$ on a Magari algebra $\mathcal{A}$:

$$a \prec b \Longleftrightarrow \Box a \leqslant b.$$

PROPOSITION 3.1. *For any Magari algebra $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box)$, the relation $\prec$ is a strict partial order on $X \backslash \{1\}$.*

Proof. We prove $a \nprec a$ for $a \neq 1$ by *reductio ad absurdum*. If $a \neq 1$ and $a \prec a$, then $(\Box a \to a) = 1$ and $\Box a = \Box(\Box a \to a) = \Box 1 = 1$. Consequently, $a = 1$. We obtain a contradiction with the assumption $a \neq 1$.

Now we check the transitivity condition. Suppose $b \prec c$ and $c \prec d$. We have $\Box b \leqslant c$ and $\Box c \leqslant d$. Hence, $\Box b \leqslant \Box\Box b \leqslant \Box c \leqslant d$. □

PROPOSITION 3.2. *For any Magari algebra $\mathcal{A} = (X, \wedge, \vee, \to, 0, 1, \Box)$, the algebra $\mathcal{A}$ is $\Box$-founded if and only if the partial order $\prec$ on $X \backslash \{1\}$ is well-founded.*

Proof. Suppose the algebra $\mathcal{A}$ is $\Box$-founded. We prove that the partial order $\prec$ on $X \backslash \{1\}$ is well-founded by *reductio ad absurdum*. If there is a descending sequence $a_0 \succ a_1 \succ \dots$ of elements of $X \backslash \{1\}$, then we obtain a sequence of elements such that $\Box a_{i+1} \leqslant a_i$. Since the algebra $\mathcal{A}$ is $\Box$-founded, we have $a_0 = 1$. We obtain a contradiction with the assumption that $a_i \neq 1$ for any $i \in \mathbb{N}$.

Now suppose that the partial order $\prec$ on $X \backslash \{1\}$ is well-founded. We prove that the algebra $\mathcal{A}$ is $\Box$-founded by *reductio ad absurdum*.

Consider a sequence of elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$ and $a_0 \neq 1$. We claim that $a_i \neq 1$ for any $i \in \mathbb{N}$ and prove it by induction on i. We have $a_0 \neq 1$. If $i = j + 1$, then $a_j \neq 1$ by induction hypothesis and $\Box a_i \leqslant a_j$. Thus, $\Box a_i \neq 1$ and $a_i \neq 1$. The claim is checked.

We have a descending sequence $a_0 \succ a_1 \succ \dots$ of elements of $X \backslash \{1\}$. We obtain a contradiction with the assumption that $\prec$ is well-founded on $X \backslash \{1\}$. □

Below a series of examples of $\Box$-founded Magari algebras is given. We call a Magari algebra σ-*complete* if its underlying Boolean algebra is σ-complete that is every its countable subset S has the least upper bound $\bigvee S$. An equivalent condition is that every countable subset S has the greatest lower bound $\bigwedge S$.

PROPOSITION 3.3. [2] *Any σ-complete Magari algebra is $\Box$-founded.*

Proof. Assume we have a σ-complete Magari algebra $\mathcal{A}$ and a sequence of its elements $(a_i)_{i\in\mathbb{N}}$ such that $\Box a_{i+1} \leqslant a_i$. We shall prove that $a_0 = 1$.

Put $b = \bigwedge\limits_{i\in\mathbb{N}} a_i$. For any $n \in \mathbb{N}$, we have $b \leqslant a_{n+1}$ and $\Box b \leqslant \Box a_{n+1} \leqslant a_n$. Hence,

$$\Box b \leqslant b, \qquad \Box b \to b = 1, \qquad \Box b = \Box(\Box b \to b) = \Box 1 = 1, \qquad b = 1.$$

We obtain that $a_0 = 1$. □

[1] This notion has been inspired by an article of F. Pakhomov and J. Walsh [13].

[2] This statement has been inspired by a correspondence with T. Litak (see also the proof of Theorem 2.15 from [9]).

REMARK. *Let us mention an example of $\Box$-founded Magari algebra of a different kind without going into details. If we consider the second-order arithmetical theory Σ^1_1- AC_0 extended with all true Σ^1_1-sentences, then its provability algebra forms a $\Box$-founded Magari algebra. This obseravtion can be obtained following the lines of Theorem 3.2 from [13].*

Now we define semantic consequence relations over $\Box$-founded Magari algebras corresponding to derivability relations $\vdash_l$, $\vdash_g$ and $\vdash$. For a subset S of a Magari algebra $\mathcal{A}$, the filter of (the Boolean part of) $\mathcal{A}$ generated by S is denoted by $\langle S \rangle$.

Given a set of modal formulas Γ and a formula A, we set $\Gamma \Vdash_l A$ if for any $\Box$-founded Magari algebra $\mathcal{A}$ and any valuation θ in $\mathcal{A}$

$$\theta(A) \in \langle \{\theta(B) \mid B \in \Gamma\} \rangle.$$

We also set $\Gamma \Vdash_g A$ if for any $\Box$-founded Magari algebra $\mathcal{A}$ and any valuation θ in $\mathcal{A}$

$$(\forall B \in \Gamma \;\; \theta(B) = 1) \Rightarrow \theta(A) = 1.$$

In addition, we set $\Sigma; \Gamma \Vdash A$ if for any $\Box$-founded Magari algebra $\mathcal{A}$ and any valuation θ in $\mathcal{A}$

$$(\forall C \in \Sigma \;\; \Box\theta(C) = 1) \Rightarrow \theta(A) \in \langle \{\theta(B) \mid B \in \Gamma\} \rangle.$$

The relation $\Vdash$ is a generalization of $\Vdash_l$ and $\Vdash_g$ since $\varnothing; \Gamma \Vdash A \Leftrightarrow \Gamma \Vdash_l A$ and $\Gamma; \Gamma \Vdash A \Leftrightarrow \Gamma \Vdash_g A$. The only nontrivial implication is the following.

PROPOSITION 3.4. *For any set of modal formulas Γ, and for any modal formula A, if $\Gamma \Vdash_g A$, then $\Gamma; \Gamma \Vdash A$.*

Proof. Assume $\Gamma \Vdash_g A$ and, in addition, we have a $\Box$-founded Magari algebra $\mathcal{A} = (X, \wedge, \vee, \rightarrow, 0, 1, \Box)$ together with a valuation θ in $\mathcal{A}$ such that $\Box\theta(B) = 1$ for any $B \in \Gamma$. We shall prove that $\theta(A) \in \langle \{\theta(B) \mid B \in \Gamma\} \rangle$.

We denote the filter $\langle \{\theta(B) \mid B \in \Gamma\} \rangle$ of $\mathcal{A}$ by F. Let us check that F is an open filter that is $\Box a \in F$, whenever $a \in F$. If $a \in F$, then $\theta(B_1) \wedge \cdots \wedge \theta(B_k) \leqslant a$ for a finite set of formulas $\{B_1, \ldots, B_k\} \subset \Gamma$. Consequently, $\Box\theta(B_1) \wedge \cdots \wedge \Box\theta(B_k) \leqslant \Box a$. Since $\Box\theta(B) = 1$ for any $B \in \Gamma$, we obtain $\Box a = 1$ and $\Box a \in F$.

Thus, we obtain the Magari algebra $\mathcal{A}/F$ and the canonical epimorphism $\nu \colon \mathcal{A} \to \mathcal{A}/F$. We claim that the algebra $\mathcal{A}/F$ is $\Box$-founded. Assume we have a sequence of elements $(a_i)_{i \in \mathbb{N}}$ of $\mathcal{A}$ such that $\Box\nu(a_{i+1}) \leqslant \nu(a_i)$. We have $\nu(\Box a_{i+1} \rightarrow a_i) = 1$ and $(\Box a_{i+1} \rightarrow a_i) \in F$. There is a sequence $(S_i)_{i \in \mathbb{N}}$ of finite subsets of Γ such that $\bigwedge\{\theta(B) \mid B \in S_i\} \leqslant (\Box a_{i+1} \rightarrow a_i)$ in $\mathcal{A}$. Consequently, $\bigwedge\{\theta(B) \mid B \in S_i\} \wedge \Box a_{i+1} \leqslant a_i$ and $\bigwedge\{\Box\theta(B) \mid B \in S_i\} \wedge \Box\Box a_{i+1} \leqslant \Box a_i$. Since $\Box\theta(B) = 1$ for any $B \in \Gamma$, we have $\Box\Box a_{i+1} \leqslant \Box a_i$ in $\mathcal{A}$. From $\Box$-foundedness of $\mathcal{A}$, we obtain $\Box a_i = 1$ for any $i \in \mathbb{N}$. Since $\bigwedge\{\theta(B) \mid B \in S_i\} \wedge \Box a_{i+1} \leqslant a_i$, we obtain $\bigwedge\{\theta(B) \mid B \in S_i\} \leqslant a_i$. Thus, $a_i \in F$ for any $i \in \mathbb{N}$ and $\nu(a_0) = 1$. The algebra $\mathcal{A}/F$ is $\Box$-founded.

Now consider the valuation $\nu \circ \theta$ in $\mathcal{A}/F$. We see that $(\nu \circ \theta)(B) = 1$ in $\mathcal{A}/F$ for any $B \in \Gamma$. From the assumption $\Gamma \Vdash_g A$, we conclude that $(\nu \circ \theta)(A) = 1$ and $\theta(A) \in F = \langle \{\theta(B) \mid B \in \Gamma\} \rangle$. $\Box$

LEMMA 3.5. *For any set of modal formulas Γ, and for any modal formula A, if $\Gamma \vdash_g A$, then $\Gamma \Vdash_g A$.*

Proof. Assume π is an ∞-derivation with the root marked by A in which all assumption leafs are marked by some elements of Γ. In addition, assume we have a $\Box$-founded Magari

https://doi.org/10.1017/S1755020319000613 Published online by Cambridge University Press

algebra $\mathcal{A} = (X, \wedge, \vee, \rightarrow, 0, 1, \Box)$ together with a valuation θ in $\mathcal{A}$ such that $\theta(B) = 1$ for any $B \in \Gamma$. We shall prove that $\theta(A) = 1$.

By N_π, we denote the set of nodes of the ∞-derivation π. For any of its nodes w, let π_w be the subtree of π with the root w. Also, put $r(w) = |\pi_w|$. In addition, let B_w be the formula of the node w. A node w belongs to the $(n+1)$-th slice of π if there are n applications of the rule (nec) on the path from this node to the root of π. By c_n, we denote the element $\bigwedge\{\theta(B_w) \mid w \text{ belongs to the } (n+1)\text{-th slice of } \pi\}$.

We claim that $\Box c_{n+1} \leqslant c_n$ for any $n \in \mathbb{N}$. It is sufficient to prove that $\Box c_{n+1} \leqslant \theta(B_w)$ whenever w belongs to the $(n+1)$-th slice of π. The proof is by induction on $r(w)$.

If B_w is an axiom of GL or an element of Γ, then we immediately obtain the required statement. Otherwise, B_w is obtained by an application of an inference rule in π.

If B_w is obtained by the rule (nec), then this formula has the form $\Box B_v$, where v is the premise of w. We see that v belongs to the $(n+2)$-th slice of π. Consequently, $c_{n+1} \leqslant \theta(B_v)$ and $\Box c_{n+1} \leqslant \theta(B_w)$.

Suppose B_w is obtained by the rule (mp). Consider the premises v_1 and v_2 of w. We have $r(v_1) < r(w)$ and $r(v_2) < r(w)$. By induction hypotheses, we obtain $\Box c_{n+1} \leqslant \theta(B_{v_1}) \wedge \theta(B_{v_2}) \leqslant \theta(B_w)$.

Now we see that $\Box c_{n+1} \leqslant c_n$ for any $n \in \mathbb{N}$. Applying $\Box$-foundedness of $\mathcal{A}$, we note that $c_0 = 1$. Since the root of the ∞-derivation π belongs to the first slice of π, we conclude that $c_0 \leqslant \theta(A)$ and $\theta(A) = 1$. □

PROPOSITION 3.6 (Algebraic soundness). *For any sets of modal formulas Σ and Γ, and for any modal formula A, if $\Sigma; \Gamma \vdash A$, then $\Sigma; \Gamma \Vdash A$.*

Proof. Assume π is an ∞-derivation with the root marked by A in which all boxed assumption leafs are marked by some elements of Σ and all nonboxed assumption leafs are marked by some elements of Γ. In addition, assume we have a $\Box$-founded Magari algebra $\mathcal{A}$ together with a valuation θ such that $\Box\theta(C) = 1$ for any $C \in \Sigma$. By induction on $|\pi|$ we prove that $\theta(A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$.

If A is an axiom of GL or an element of Γ, then we obtain the required statement immediately. Otherwise, consider the lowermost application of an inference rule in π.

Case 1. Suppose that π has the form

$$\mathsf{mp}\,\dfrac{\begin{matrix}\pi' \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}\pi'' \\ \vdots \\ D \rightarrow A\end{matrix}}{A}\,.$$

By the induction hypotheses for π' and π'', we have $\theta(D) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$ and $\theta(D \rightarrow A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$. We have $\theta(D) \wedge \theta(D \rightarrow A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$ and $\theta(D) \wedge \theta(D \rightarrow A) \leqslant \theta(A)$. Consequently, $\theta(A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$.

Case 2. Suppose that π has the form

$$\mathsf{nec}\,\dfrac{\begin{matrix}\pi' \\ \vdots \\ D\end{matrix}}{\Box D}\,,$$

where $\Box D = A$. We see that $\Sigma \vdash_g D$. By the previous lemma, we have $\Sigma \Vdash_g D$. From Proposition 3.4, we obtain $\Sigma, \Sigma \Vdash D$. It follows that $\theta(D) \in \langle\{\theta(C) \mid C \in \Sigma\}\rangle$ and there

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is a finite subset S of Σ such that $\bigwedge\{\theta(C) \mid C \in S\} \leqslant \theta(D)$. Therefore $\bigwedge\{\Box\theta(C) \mid C \in S\} \leqslant \Box\theta(D)$. We obtain $\theta(A) = \Box\theta(D) = 1$ from the assumption $\Box\theta(C) = 1$ for any $C \in \Sigma$. We conclude that $\theta(A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$. $\Box$

THEOREM 3.7. *For any sets of modal formulas Σ and Γ, and for any modal formula A, we have*

$$\Sigma; \Gamma \vdash A \iff \Sigma; \Gamma \Vdash A.$$

Proof. The left-to-right implication follows from Proposition 3.6. We prove the converse. Assume $\Sigma; \Gamma \Vdash A$. Consider the set $T = \{G \in Fm \mid \Sigma; \varnothing \vdash G\}$. We see that T contains axioms of GL and is closed under the rules (mp) and (nec). We define an equivalence relation $\sim_T$ on the set of modal formulas Fm by putting $D \sim_T E$ if and only of $(D \leftrightarrow E) \in T$. Let us denote the equivalence class of D by $[D]_T$. Applying the Lindenbaum-Tarski construction, we obtain a Magari algebra $\mathcal{L}_T$ on the set of equivalence classes of formulas, where $[D]_T \wedge [E]_T = [D \wedge E]_T$, $[D]_T \vee [E]_T = [D \vee E]_T$, $[D]_T \to [E]_T = [D \to E]_T$, $0 = [\bot]_T$, $1 = [\top]_T$ and $\Box[D] = [\Box D]$.

Let us check that the algebra $\mathcal{L}_T$ is $\Box$-founded. Assume we have a sequence of formulas $(D_i)_{i\in\mathbb{N}}$ such that $\Box[D_{i+1}]_T \leqslant [D_i]_T$. We have $[\Box D_{i+1} \to D_i]_T = 1$ and $(\Box D_{i+1} \to D_i) \in T$. For every $i \in \mathbb{N}$, there exists an ∞-derivation π_i for the formula $\Box D_{i+1} \to D_i$ such that all assumption leafs of π_i are boxed and marked by some elements of Σ. We obtain the following ∞-derivation for the formula D_0:

$$\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\begin{matrix}\vdots\\ \Box D_3\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdots\\ \Box D_3 \to D_2\end{matrix}}{D_2}}{\Box D_2}\qquad \begin{matrix}\pi_1\\ \vdots\\ \Box D_2 \to D_1\end{matrix}}{D_1}}{\Box D_1}\qquad \begin{matrix}\pi_0\\ \vdots\\ \Box D_1 \to D_0\end{matrix}}{D_0},$$

where all assumption leafs are boxed and marked by some elements of Σ. Hence, $D_0 \in T$ and $[D_0]_T = [\top]_T = 1$. We see that the Magari algebra $\mathcal{L}_T$ is $\Box$-founded.

Consider the valuation $\theta \colon E \mapsto [E]_T$ in the Magari algebra $\mathcal{L}_T$. Since $\{\Box C \mid C \in \Sigma\} \subset T$, we have $\Box\theta(C) = 1$ for any $C \in \Sigma$. From the assumption $\Sigma; \Gamma \Vdash A$, we obtain that $\theta(A) \in \langle\{\theta(B) \mid B \in \Gamma\}\rangle$. Consequently, there is a finite subset Γ_0 of Γ such that $\bigwedge\{\theta(B) \mid B \in \Gamma_0\} \leqslant \theta(A)$ in $\mathcal{L}_T$. We have $(\bigwedge\{[B]_T \mid B \in \Gamma_0\} \to [A]_T) = 1$ and $(\bigwedge\Gamma_0 \to A) \in T$. In other words, $\Sigma; \varnothing \vdash \bigwedge\Gamma_0 \to A$. Notice that $\varnothing; \Gamma \vdash \bigwedge\Gamma_0$. Thus, $\Sigma; \Gamma \vdash \bigwedge\Gamma_0$ and $\Sigma; \Gamma \vdash \bigwedge\Gamma_0 \to A$. Applying an inference rule (mp), we obtain $\Sigma; \Gamma \vdash A$. $\Box$

§4. Neighbourhood semantics. In this section we consider neighbourhood semantics of the Gödel-Löb provability logic and recall a connection between scattered topological spaces and corresponding neighbourhood frames.

An *Esakia frame*, or *neighbourhood* GL-*frame*, $\mathcal{X} = (X, \Box)$ is a set X together with a mapping $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ such that the Boolean algebra of subsets of X enriched with the mapping $\Box$ forms a Magari algebra. Elements of X are called *worlds* of the frame $\mathcal{X}$. An *Esakia model*, or *neighbourhood* GL-*model*, is a pair $\mathcal{M} = (\mathcal{X}, \theta)$, where $\mathcal{X}$ is an

Esakia frame and θ is a valuation in the powerset Magari algebra of $\mathcal{X}$. A formula A is *true at a world x of a model $\mathcal{M}$*, written as $\mathcal{M}, x \models A$, if $x \in \theta(A)$. In addition, a formula A is called *true in $\mathcal{M}$*, written as $\mathcal{M} \models A$, if A is true at all worlds of $\mathcal{M}$.

We briefly recall a connection between scattered topological spaces and Esakia frames (cf. [3]). Note that we allow Esakia frames and topological spaces to be empty.

In a topological space, an open set U containing a point x is called a *neighbourhood* of x. A set U is a *punctured neighbourhood* of x if $x \notin U$ and $U \cup \{x\}$ is open. For a topological space (X, τ) and its subset V, the *derivative set $d_\tau(V)$ of V* is the set of limit points of V:

$$x \in d_\tau(V) \iff \forall U \in \tau \ (x \in U \Rightarrow \exists y \neq x \ (y \in U \cap V)) .$$

The *co-derivative set $cd_\tau(V)$ of V* is defined as $X \backslash d_\tau(X \backslash V)$. By definition, $x \in cd_\tau(V)$ if and only if there is a punctured neighbourhood of x entirely contained in V. Notice that $cd_\tau(V)$ is open if V is open. In addition, V is open if and only if $V \subset cd_\tau(V)$.

In a topological space, a point having an empty punctured neighbourhood is called *isolated*. A topological space (X, τ) is called *scattered* if each nonempty subset of X (as a topological space with the inherited topology) has an isolated point. Notice that any ordinal with the standard (interval) topology gives us a natural example of scattered space.

It turns out that scattered topological spaces and Esakia frames are essentially the same notions.

PROPOSITION 4.1 (L. Esakia [5]). *If $(X, \Box)$ is an Esakia frame, then X bears a unique topology τ for which $\Box = cd_\tau$. Moreover, the space (X, τ) is scattered.*

PROPOSITION 4.2 (H. Simmons [18], L. Esakia [5]). *If (X, τ) is a scattered topological space, then (X, cd_τ) is an Esakia frame.*

In the sequel, we don't distinguish Esakia frames and corresponding topological spaces so that we use the topological terminology referring to (X, τ) for the frame (X, cd_τ). For example, we say that a subset U is *open* in $(X, \Box)$ if it is open in the corresponding topological space (which is equivalent to $U \subset \Box U$).

Notice that an open set in a scattered topological space is scattered (as a topological space with the inherited topology). Hence an open set in an Esakia frame $\mathcal{X}$ defines an Esakia frame, which is called an *open subframe of $\mathcal{X}$*.

LEMMA 4.3. *If $(X_0, \Box_0)$ is an open subframe of an Esakia frame $(X_1, \Box_1)$, then $\Box_0 V = X_0 \cap \Box_1 V$ for any $V \subset X_0$.*

LEMMA 4.4 (see Lemma 6 from [17]). *For Esakia models $(\mathcal{X}_0, \theta_0)$ and $(\mathcal{X}_1, \theta_1)$, where $\mathcal{X}_0$ is an open subframe of $\mathcal{X}_1$ and $\theta_0(p) = X_0 \cap \theta_1(p)$ for $p \in PV$, we have that $\theta_0(A) = X_0 \cap \theta_1(A)$ for any formula A.*

For a topological space (X, τ), we define transfinite iterations of the co-derivative-set operator by

- $cd^0_\tau(V) = V$, $cd^{\alpha+1}_\tau(V) = cd_\tau(cd^\alpha_\tau(V))$,
- $cd^\alpha_\tau(V) = \bigcup\limits_{\beta<\alpha} (cd^\beta_\tau(V))$ if α is a limit ordinal.

PROPOSITION 4.5 (Cantor). *A topological space (X, τ) is scattered if and only if $cd^\alpha_\tau(\varnothing) = X$ for some α.*

For a scattered topological space (X, τ) and a point $x \in X$, the *rank $\rho_\tau(x)$ of x* is the least ordinal α such that $x \in cd^{\alpha+1}_\tau(\varnothing)$.

Let us consider an example of an ordinal α with the left topology τ, where open sets have the form $\{\gamma \mid \gamma < \beta\}$ for $\beta \leqslant \alpha$. The pair (α, τ) is a scattered topological space. We have $cd_\tau^\beta(\varnothing) = \beta \cap \alpha$ and $\rho_\tau(\gamma) = \gamma$ for $\gamma \in \alpha$.

LEMMA 4.6 (cf. Lemma 3.11 from [2]). *In a scattered topological space (X, τ), we have*

- $cd_\tau^\alpha(\varnothing) \subset cd_\tau^\beta(\varnothing)$ *if* $\alpha \leqslant \beta$,
- $cd_\tau^\alpha(\varnothing)$ *is an open set for any* α.

LEMMA 4.7. *For any scattered topological space (X, τ) and any $x \in X$, the set $\{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$ is a punctured neighbourhood of x.*

Proof. Let us check that $\{x\} \cup \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$ is an open set. We have

$$\{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\} = cd_\tau^{\rho_\tau(x)}(\varnothing), \quad \{x\} \subset cd_\tau^{\rho_\tau(x)+1}(\varnothing),$$
$$cd_\tau^{\rho_\tau(x)}(\varnothing) \subset cd_\tau^{\rho_\tau(x)+1}(\varnothing).$$

Hence

$$\{x\} \cup \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\} \subset cd_\tau(\{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}) \subset$$
$$\subset cd_\tau(\{x\} \cup \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}).$$

Notice that, in any topological space, a set U is open if and only if $U \subset cd_\tau(U)$. Thus $\{x\} \cup \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$ is an open set. □

A topological space is T_d ($T_{\frac{1}{2}}$, or *local* T_1) if any point of the space is closed in some of its neighbourhoods.

LEMMA 4.8. *Any scattered topological space (X, τ) is T_d.*

Proof. Consider any point $x \in X$. By Lemma 4.7, the open set $cd_\tau^{\rho_\tau(x)}(\varnothing) = \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$ is a punctured neighbourhood of x. Hence the point x is closed in its neighbourhood $\{x\} \cup \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$. □

Let us remark that, for any Esakia frame $\mathcal{X}$, the powerset Magari algebra of $\mathcal{X}$ is σ-complete. Consequently this algebra is $\Box$-founded by Proposition 3.3. In a similar way to the definitions of $\Vdash_l$, $\Vdash_g$ and $\Vdash$, we define consequence relations $\Vdash_l^*$, $\Vdash_g^*$ and $\Vdash^*$ by considering only powerset Magari algebras of Esakia frames instead of arbitrary $\Box$-founded algebras.

The global consequence relation $\Vdash_g^*$ can be reformulated in terms related to Esakia frames as follows: $\Gamma \Vdash_g^* A$ if and only if for any Esakia model $\mathcal{M}$

$$(\forall B \in \Gamma \;\; \mathcal{M} \vDash B) \Rightarrow \mathcal{M} \vDash A.$$

We also denote this relation by $\vDash_g$.

In addition, we give another definition for the local consequence relation $\Vdash_l^*$. We will check that these two definitions are equivalent later in this section. Given a set of modal formulas Γ and a formula A, we set $\Gamma \vDash_l A$ if for any Esakia model $\mathcal{M}$ and any of its worlds x

$$(\forall B \in \Gamma \; \mathcal{M}, x \vDash B) \Rightarrow \mathcal{M}, x \vDash A.$$

REMARK. *Since any Esakia frame can be considered as a topoloigcal space, it is more natural to call the relation $\vDash_l$ pointwise rather than local. However we follow modal*

https://doi.org/10.1017/S1755020319000613 Published online by Cambridge University Press

logical tradition and call relations $\vDash_l$ and $\vDash_g$ local and global respectively (see [21] or p. 103 from [8]).

The following strong completeness result was obtained by V. Shehtman using the so-called ultrabouquet construction (see [17]).

THEOREM 4.9 (Local neighbourhood completeness). *For any set of modal formulas Γ, and for any modal formula A, we have*

$$\Gamma \vdash_l A \Longleftrightarrow \Gamma \vDash_l A.$$

We set $\Sigma; \Gamma \vDash A$ if for any Esakia model $\mathcal{M}$ and any of its worlds x

$$((\forall B \in \Gamma \ \ \mathcal{M}, x \vDash B) \wedge (\forall y \neq x \ \ \forall C \in \Sigma \ \ \mathcal{M}, y \vDash C)) \Rightarrow \mathcal{M}, x \vDash A.$$

We see that $\varnothing; \Gamma \vDash A \Leftrightarrow \Gamma \vDash_l A$ and $\Gamma; \Gamma \vDash A \Leftrightarrow \Gamma \vDash_g A$.

We will show that relations $\Vdash^*$ and $\vDash$ coincide by proving $\Sigma; \Gamma \Vdash^* A \Leftrightarrow \Sigma; \Gamma \vDash A$. The left-to-right implication is obtained below. The converse will be shown in the final section.

PROPOSITION 4.10. *For any sets of modal formulas Σ and Γ, and for any modal formula A, if $\Sigma; \Gamma \Vdash^* A$, then $\Sigma; \Gamma \vDash A$.*

Proof. Assume $\Sigma; \Gamma \Vdash^* A$. In addition, assume we have an Esakia model $\mathcal{M} = ((X_1, \Box_1), \theta_1)$ and its world x such that

$$\forall B \in \Gamma \ \ \mathcal{M}, x \vDash B \text{ and } \forall y \neq x \ \forall C \in \Sigma \ \ \mathcal{M}, y \vDash C.$$

We shall prove that $\mathcal{M}, x \vDash A$.

Put $\mathcal{X}_1 = (X_1, \Box_1)$. We have

$$x \in \bigcap \{\theta_1(B) \mid B \in \Gamma\}.$$

Also, $X_1 \backslash \{x\}$ is a punctured neighbourhood of x in $\mathcal{X}_1$ and

$$X_1 \backslash \{x\} \subset \bigcap \{\theta_1(C) \mid C \in \Sigma\}.$$

From the definition of co-derivative set, we see

$$x \in \Box_1 \bigcap \{\theta_1(C) \mid C \in \Sigma\}.$$

We denote the set $\Box_1 \bigcap \{\theta_1(C) \mid C \in \Sigma\}$ by X_0. We obtain

$$x \in X_0 \cap \bigcap \{\theta_1(B) \mid B \in \Gamma\} \subset X_0 \cap \bigcap \{X_0 \cap \theta_1(B) \mid B \in \Gamma\}. \tag{1}$$

Since an inequality $\Box a \leqslant \Box\Box a$ holds in any Magari algebra, we have $X_0 \subset \Box_1 X_0$. Consequently, the set X_0 is open in $\mathcal{X}_1$. Therefore the set X_0 defines an open subframe $\mathcal{X}_0 = (X_0, \Box_0)$ of $\mathcal{X}_1$. We define a valuation θ_0 over $\mathcal{X}_0$ obtained by restricting θ_1 to X_0. By Lemma 4.4, we have $\Box_0 \theta_0(E) = \theta_0(\Box E) = X_0 \cap \theta_1(\Box E) = X_0 \cap \Box_1 \theta_1(E)$ for any formula E. Notice that $X_0 \subset \Box_1 \theta_1(C)$ for any $C \in \Sigma$. Hence $\Box_0 \theta_0(C) = X_0 \cap \Box_1 \theta_1(C) = X_0$ for any $C \in \Sigma$. Applying the assumption $\Sigma; \Gamma \Vdash^* A$ to the Magari algebra of subsets of $\mathcal{X}_0$ and the valuation θ_0, we obtain

$$\theta_0(A) \in \langle \{\theta_0(B) \mid B \in \Gamma\} \rangle.$$

Therefore there is a finite subset S of Γ such that

$$X_0 \cap \bigcap \{\theta_0(B) \mid B \in S\} \subset \theta_0(A).$$

From 1 and Lemma 4.4, we have

$$x \in X_0 \cap \bigcap\{X_0 \cap \theta_1(B) \mid B \in \Gamma\} = X_0 \cap \bigcap\{\theta_0(B) \mid B \in \Gamma\} \subset$$
$$\subset X_0 \cap \bigcap\{\theta_0(B) \mid B \in S\} \subset \theta_0(A) = X_0 \cap \theta_1(A) \subset \theta_1(A).$$

We obtain $\mathcal{M}, x \vDash A$. □

The following corollary shows all local consequence relations considered so far coincide.

COROLLARY 4.11. *For any set of modal formulas* Γ, *and for any modal formula A, we have*

$$\Gamma \Vdash_l A \Longleftrightarrow \Gamma \Vdash_l^* A \Longleftrightarrow \Gamma \vDash_l A \Longleftrightarrow \Gamma \vdash_l A \Longleftrightarrow \varnothing; \Gamma \vdash A.$$

Proof. Assume $\Gamma \Vdash_l A$. We immediately obtain $\Gamma \Vdash_l^* A$. Hence $\varnothing; \Gamma \Vdash^* A$. By Proposition 4.10, we have $\varnothing; \Gamma \vDash A$. It follows that $\Gamma \vDash_l A$. Consequently, $\Gamma \vdash_l A$ by Theorem 4.9. We obtain $\varnothing; \Gamma \vdash A$. All the left-to-right implications are proved.

It is sufficient to prove that an assertion $\varnothing; \Gamma \vdash A$ implies $\Gamma \Vdash_l A$. Assume $\varnothing; \Gamma \vdash A$. By Proposition 3.6, we have $\varnothing; \Gamma \Vdash A$. Hence $\Gamma \Vdash_l A$. □

In the next section, we will show that it is sufficient to consider only countable Esakia frames in order to determine the relation $\vDash$. In fact, countable ordinals with the interval topology are enough. However, we can not restrict ourselves to Esakia frames where ranks of all worlds are bounded by a fixed countable ordinal.

Let us consider the following example. For a countable ordinal α, put $\Gamma_\alpha = \{p_\beta \to \Diamond p_\gamma \mid \gamma < \beta \leqslant \alpha\}$. Let $\mathcal{M} = ((\alpha + 1, \tau), \theta)$ be the Esakia model, where τ is the left topology on $\alpha + 1$ and $\mathcal{M}, x \vDash p_\beta \Leftrightarrow x = \beta$. We see $\mathcal{M} \vDash p_\beta \to \Diamond p_\gamma$ for any $\gamma < \beta \leqslant \alpha$ and $\mathcal{M}, \alpha \vDash p_\alpha$. Hence, $\Gamma_\alpha \nvDash_g \neg p_\alpha$.

PROPOSITION 4.12. *For any Esakia model* $\mathcal{M}$ *and any of its worlds x, if* $\mathcal{M}, x \vDash p_\alpha$ *and* $\forall B \in \Gamma_\alpha$ $\mathcal{M} \vDash B$, *then the rank of x is greater or equal than* α.

Proof. Let $\mathcal{M} = (\mathcal{X}, \theta)$ be an Esakia model, where $\forall B \in \Gamma_\alpha$ $\mathcal{M} \vDash B$, and τ be the scattered topology on the frame $\mathcal{X}$. Assume $\mathcal{M}, x \vDash p_\alpha$. We prove $\alpha \leqslant \rho_\tau(x)$ by transfinite induction on α.

Since $\mathcal{M}, x \vDash p_\alpha$, we have $\mathcal{M}, x \vDash \Diamond p_\beta$ for every $\beta < \alpha$. By Lemma 4.7, an open set $\{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\} = cd_\tau^{\rho_\tau(x)}(\varnothing)$ is a punctured neighbourhood of x. Thus, for every $\beta < \alpha$, there is a world $x_\beta \in \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$ such that $\mathcal{M}, x_\beta \vDash p_\beta$. By induction hypotheses for β, $\beta \leqslant \rho_\tau(x_\beta)$ for every $\beta < \alpha$. We have $\beta \leqslant \rho_\tau(x_\beta) < \rho_\tau(x)$ for all $\beta < \alpha$. Hence, $\alpha = \sup\{\beta + 1 \mid \beta < \alpha\} \leqslant \rho_\tau(x)$. □

§5. Tree-like frames. Recall that any ordinal equipped with the interval topology is a scattered topological space. This means that any ordinal yields an Esakia frame. In this section, we show that our semantic consequence relation $\vdash$ over Esakia frames is determined by Esakia frames obtained from countable ordinals.

Let $\mathbb{N}^*$ denote the set of finite sequences of natural numbers including the empty sequence λ. For $v \in \mathbb{N}^*$, its prefix of length (at most) k is denoted by $v[k]$. Note that $v[0] = \lambda$.

A *tree* is a subset of $\mathbb{N}^*$ that contains all prefixes of all its elements. For a tree T and a sequence $v \in T$ of length k, put $T_v = \{w \in T \mid w[k] = v\}$. A sequence $v \in T$ is a *leaf of T* if $T_v = \{v\}$.

A tree T is *ω-branching* if it satisfies the additional condition: if $n_1, \dots, n_k, n_{k+1} \in T$, then $n_1, \dots, n_k, m \in T$ for any $m \in \mathbb{N}$. For an ω-branching tree T, any of its elements $v = n_1, \dots, n_k$ and a natural number m, put

$$T_m(v) = \begin{cases} \{v\} & \text{if } v \text{ is a leaf of } T, \\ \{v\} \cup \bigcup\limits_{m \leqslant a} T_{n_1,\dots,n_k,a} & \text{otherwise.} \end{cases}$$

Any ω-branching tree T bears the topology σ_T defined from the basis

$$\{T_m(v) \mid v \in T,\ m \in \mathbb{N}\}.$$

A tree T is *well-founded* if there is no infinite branch in T, i.e., there is no infinite sequence of natural numbers with all its finite prefixes being elements of T. A nonempty ω-branching well-founded tree T together with the topology σ_T is called *ω-bouquet*[3].

LEMMA 5.1. *Any ω-bouquet is a countable compact Hausdorff space.*

Proof. Assume we have an ω-bouquet (T, σ_T). Obviously, it is a countable Hausdorff topological space.

We claim that T_v is compact for any $v \in T$. Let $v = n_1, \dots, n_k \in T$. Since T is a well-founded tree, we can proceed by transfinite induction on the ordinal rank of v. If $T_v = \{v\}$, then T_v is compact. Now consider the case when v is not a leaf of T. By induction hypotheses, $T_{n_1,\dots,n_k,n}$ is compact for every $n \in \mathbb{N}$. Assume $\mathcal{U}$ is an open cover of $T_{n_1,\dots,n_k}$. There is an open set $U \in \mathcal{U}$ such that $v \in U$. By definition of the topology on T, we have $v \in T_m(v) \subset U$ for some $m \in \mathbb{N}$. In addition, we have a finite cover $\mathcal{U}_n \subset \mathcal{U}$ of $T_{n_1,\dots,n_k,n}$ for every $n \in \mathbb{N}$. We see that $\{U\} \cup \bigcup\limits_{n=0}^{m-1} \mathcal{U}_n$ is the required finite subcover of $T_{n_1,\dots,n_k}$. Now we can conclude that T_v is compact for any $v \in T$. Hence, $T = T_\lambda$ is compact. □

Now we recall the following classical result.

THEOREM 5.2 (Mazurkiewicz-Sierpiński [11]). *Every countable compact Hausdorff space is homeomorphic to an ordinal with the interval topology.*

COROLLARY 5.3. *Any ω-bouquet is a scattered topological space. Moreover, this space is homeomorphic to a countable ordinal with the interval topology.*

LEMMA 5.4. *For any Esakia model $\mathcal{M}$, any of its worlds x and any neighbourhood U of x, there exists an Esakia model $\mathcal{K}$ over an ω-bouquet (T, σ_T) such that for any formula A*

$$\forall y \in U \backslash \{x\}\ \ \mathcal{M}, y \vDash A \Longrightarrow \forall v \in T \backslash \{\lambda\}\ \ \mathcal{K}, v \vDash A,$$
$$\mathcal{M}, x \vDash A \Longleftrightarrow \mathcal{K}, \lambda \vDash A.$$

Proof. Assume we have an Esakia model $\mathcal{M} = ((X, \Box), \theta)$. Let τ be the topology on X corresponding to the Esakia frame $(X, \Box)$. Assume also that x is a world of X and U is a neighbourhood of x. We construct the required ω-bouquet (T, σ_T) and the valuation ψ over it by transfinite induction on $\rho_\tau(x)$.

Suppose $\mathcal{M}, x \vDash \Box\bot$. Set $T = \{\lambda\}$. Define a valuation ψ on the set of propositional variables by letting $\lambda \in \psi(p) \Leftrightarrow x \in \theta(p)$. Put $\mathcal{K} = ((T, \sigma_T), \psi)$. We see that $T \backslash \{\lambda\} = \varnothing$

[3] We adopt the term ω-bouquet from the article [2]. However, we have slightly changed the corresponding notion.

and

$$\mathcal{M}, x \vDash A \Longleftrightarrow \mathcal{K}, \lambda \vDash A$$

for any formula A. Thus, we obtain the required ω-bouquet (T, σ_T) and the valuation ψ over it.

Suppose now that $\mathcal{M}, x \nvDash \Box\bot$. Let $\{\Box C_n \mid n \in \mathbb{N}\}$ be the set of all formulas of the form $\Box B$ that are true at x. Let $\{\Box E_n \mid n \in \mathbb{N}\}$ be the (nonempty) set of all formulas of the form $\Box B$ that are false at x. In addition, let every element $\Box E_n$ of $\{\Box E_n \mid n \in \mathbb{N}\}$ has infinitely many indices $n \in \mathbb{N}$. Put $D_n = \bigwedge_{i=0}^{n} C_i$. Notice that $\mathsf{GL} \vdash \bigwedge_{i=0}^{n} \Box C_i \leftrightarrow \Box D_n$.

For any $n \in \mathbb{N}$, we have $\mathcal{M}, x \vDash \Box D_n$ and $\mathcal{M}, x \nvDash \Box E_n$. Thus there is a neighbourhood V_n of x such that $\forall y \in V_n \backslash \{x\}\ \ \mathcal{M}, y \vDash D_n$. Now define the open set $U_n = U \cap V_n \cap \{y \in X \mid \rho_\tau(y) < \rho_\tau(x)\}$. The set U_n is, in addition, a punctured neighbourhood of x by Lemma 4.7. From the condition $\mathcal{M}, x \nvDash \Box E_n$, there is a world $x_n \in U_n$ such that $\mathcal{M}, x_n \nvDash E_n$. We have

$$\rho_\tau(x_n) < \rho_\tau(x), \quad \forall y \in U_n\ \ \mathcal{M}, y \vDash D_n, \quad \text{and} \quad \mathcal{M}, x_n \nvDash E_n.$$

By the induction hypothesis for x_n and U_n, there is an Esakia model $\mathcal{K}_n = ((L_n, \sigma_{L_n}), \varphi_n)$, where (L_n, σ_{L_n}) is an ω-bouquet, such that for any formula A

$$\forall y \in U_n \backslash \{x_n\}\ \ \mathcal{M}, y \vDash A \Longrightarrow \forall v \in L_n \backslash \{\lambda\}\ \ \mathcal{K}_n, v \vDash A, \tag{2}$$
$$\mathcal{M}, x_n \vDash A \Longleftrightarrow \mathcal{K}_n, \lambda \vDash A. \tag{3}$$

For $n \in \mathbb{N}$ and $v \in \mathbb{N}^*$, let $n \cdot v$ be the finite sequence of natural numbers with the first member n and the remainder v. Consider the ω-bouquet $T = \{\lambda\} \cup \bigcup_{n \in \mathbb{N}} \{n \cdot v \mid v \in L_n\}$ and the valuation ψ over it defined on the set of propositional variables as follows:

- $n \cdot v \in \psi(p)$ if and only if $v \in \varphi_n(p)$;
- $\lambda \in \psi(p)$ if and only if $x \in \theta(p)$.

Notice that $T_n = \{w \in T \mid w[1] = n\} = \{n \cdot v \mid v \in L_n\}$ is an open subframe of T. Let us denote the restriction of the valuation ψ on T_n by ψ_n. We see that every ω-bouquet L_n is homeomorphic to the open subframe T_n. Furthermore, by Lemma 4.4, every valuation φ_n can be identified with respect to this homeomorphism with the valuation ψ_n.

Put $\mathcal{K} = ((T, \sigma_T), \psi)$. Now we shall check that for any formula A

$$\forall y \in U \backslash \{x\}\ \ \mathcal{M}, y \vDash A \Longrightarrow \forall v \in T \backslash \{\lambda\}\ \ \mathcal{K}, v \vDash A, \tag{4}$$
$$\mathcal{M}, x \vDash A \Longleftrightarrow \mathcal{K}, \lambda \vDash A. \tag{5}$$

First, we see

$$\begin{aligned}
\forall y \in U \backslash \{x\}\ \ \mathcal{M}, y \vDash A &\Longrightarrow \forall n \in \mathbb{N}\ \ \forall y \in U_n\ \ \mathcal{M}, y \vDash A \\
&\Longrightarrow \forall n \in \mathbb{N}\ \ \forall v \in L_n\ \ \mathcal{K}_n, v \vDash A \text{ (from 2 and 3)} \\
&\Longrightarrow \forall n \in \mathbb{N}\ \ \forall v \in T_n\ \ ((T_n, \sigma_T \upharpoonright T_n), \psi_n), v \vDash A \\
&\Longrightarrow \forall v \in T \backslash \{\lambda\}\ \ \mathcal{K}, v \vDash A \text{ (by Lemma 4.4).}
\end{aligned}$$

We prove 5 by induction on the structure of A. Let us consider only the main case when A has the form $\Box B$. We have

$$\begin{aligned}\mathcal{M}, x \vDash \Box B &\Longrightarrow \Box B \in \{\Box C_n \mid n \in \mathbb{N}\}\\ &\Longrightarrow \exists m \in \mathbb{N} \;\; B = C_m\\ &\Longrightarrow \exists m \in \mathbb{N} \;\; \forall n \in \mathbb{N} \;\; (m \leqslant n \Rightarrow \forall y \in U_n \;\; \mathcal{M}, y \vDash B)\\ &\Longrightarrow \exists m \in \mathbb{N} \;\; \forall n \in \mathbb{N} \;\; (m \leqslant n \Rightarrow \forall v \in L_n \;\; \mathcal{K}_n, v \vDash B) \text{ (from 2 and 3)}\\ &\Longrightarrow \exists m \in \mathbb{N} \;\; \forall n \in \mathbb{N} \;\; (m \leqslant n \Rightarrow \forall v \in T_n \;\; ((T_n, \sigma_T \upharpoonright T_n), \psi_n), v \vDash B)\\ &\Longrightarrow \exists m \in \mathbb{N} \;\; \forall v \in T_m(\lambda)\backslash\{\lambda\} \;\; \mathcal{K}, v \vDash B \text{ (by Lemma 4.4)}\\ &\Longrightarrow \lambda \in cd_{\sigma_T}(\psi(B))\\ &\Longrightarrow \mathcal{K}, \lambda \vDash \Box B.\end{aligned}$$

We prove the converse

$$\begin{aligned}\mathcal{M}, x \nvDash \Box B &\Longrightarrow \Box B \in \{\Box E_n \mid n \in \mathbb{N}\}\\ &\Longrightarrow \exists^{\infty} m \in \mathbb{N} \;\; B = E_m\\ &\Longrightarrow \exists^{\infty} m \in \mathbb{N} \;\; \mathcal{M}, x_m \nvDash B\\ &\Longrightarrow \exists^{\infty} m \in \mathbb{N} \;\; \mathcal{K}_m, \lambda \nvDash B \text{ (from 3)}\\ &\Longrightarrow \exists^{\infty} m \in \mathbb{N} \;\; ((T_m, \sigma_T \upharpoonright T_m), \psi_m), m \nvDash B\\ &\Longrightarrow \exists^{\infty} m \in \mathbb{N} \;\; \mathcal{K}, m \nvDash B \text{ (by Lemma 4.4)}\\ &\Longrightarrow \lambda \notin cd_{\sigma_T}(\psi(B))\\ &\Longrightarrow \mathcal{K}, \lambda \nvDash \Box B,\end{aligned}$$

where $\exists^{\infty}$ means "there exist infinitely many".

Both Conditions 4 and 5 are satisfied. Hence, we obtain the required ω-bouquet (T, σ_T) and the valuation ψ over it. □

In a similar way to the definition of $\vDash$, we obtain a consequence relation $\vDash^*$ by considering only Esakia models over ω-bouquets.

THEOREM 5.5. *For any sets of modal formulas* Σ *and* Γ*, and for any modal formula* A*, we have*

$$\Sigma; \Gamma \vDash A \Longleftrightarrow \Sigma; \Gamma \vDash^* A.$$

Proof. The left-to-right implication is obvious. We prove the converse by *reductio ad absurdum*. Assume $\Sigma; \Gamma \nvDash A$ and $\Sigma; \Gamma \vDash^* A$. Then there exist an Esakia model $\mathcal{M}$ and a world x such that

$$\forall y \neq x \; \forall C \in \Sigma \;\; \mathcal{M}, y \vDash C, \quad \forall B \in \Gamma \;\; \mathcal{M}, x \vDash B \quad \text{and} \quad \mathcal{M}, x \nvDash A.$$

By Lemma 5.4, there is an Esakia model $\mathcal{K}$ over an ω-bouquet (T, σ_T) such that

$$\forall v \in T\backslash\{\lambda\} \; \forall C \in \Sigma \;\; \mathcal{K}, v \vDash C, \quad \forall B \in \Gamma \;\; \mathcal{K}, \lambda \vDash B \quad \text{and} \quad \mathcal{K}, \lambda \nvDash A.$$

We obtain a contradiction with the assumption $\Sigma; \Gamma \vDash^* A$. □

§6. Neighbourhood compactness. In this section we prove that if $\Sigma; \Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Sigma; \Gamma_0 \vDash A$. This neighbourhood compactness result is obtained by applying the ultrabouquet construction from [17].

For any $n \in \mathbb{N}$, let $\mathcal{X}_n = (X_n, \tau_n)$ be a topological space and x_n be a closed point in it. Let $\mathcal{U}$ be a nonprincipal ultrafilter in $\mathbb{N}$. The *ultrabouquet* $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ is a topological space

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obtained as a set from the disjoint union $\bigsqcup_{n\in\mathbb{N}} X_n$ by identifying all points x_n. A set U is *open in* $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ if and only if

- for any $n \in \mathbb{N}$ the set $U \cap (X_n \backslash \{x_n\})$ is open in $\mathcal{X}_n$,
- $\{n \in \mathbb{N} \mid U \cap X_n \text{ is open in } \mathcal{X}_n\} \in \mathcal{U}$ whenever $x_* \in U$,

where x_* is the point of $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ obtained by identifying points x_n.

It can be checked that an ultrabouquet of scattered topological spaces is a scattered topological space. Hence we can construct an Esakia frame as an ultrabouquet of a countable family of Esakia frames.

For $n \in \mathbb{N}$, let θ_n be a valuation over an Esakia frame $\mathcal{X}_n = (X_n, \Box_n)$. Let θ be a valuation over $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ defined on the set of propositional variables as follows:

- the restriction of $\theta(p)$ to $X_n \backslash \{x_n\}$ is equal to $\theta_n(p)$;
- $x_* \in \theta(p)$ if and only if $\{n \in \mathbb{N} \mid x_n \in \theta_n(p)\} \in \mathcal{U}$.

We denote this valuation θ by $\bigvee_{\mathcal{U}}(\theta_n, x_n)$.

LEMMA 6.1 (see Lemmas 22 and 27 from [17]). *For any $n \in \mathbb{N}$, let $(\mathcal{X}_n, \theta_n)$ be an Esakia model and x_n be a closet point in it. Let $\mathcal{U}$ be a nonprincipal ultrafilter in $\mathbb{N}$ and $\theta = \bigvee_{\mathcal{U}}(\theta_n, x_n)$. Then for any formula A we have*

- *$\theta(A) \cap (X_n \backslash \{x_n\}) = \theta_n(A)$ for any $n \in \mathbb{N}$;*
- *$x_* \in \theta(A)$ if and only if $\{n \in \mathbb{N} \mid x_n \in \theta_n(A)\} \in \mathcal{U}$.*

THEOREM 6.2. *For any sets of modal formulas Σ and Γ, and for any modal formula A, if $\Sigma; \Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Sigma; \Gamma_0 \vDash A$.*

Proof. Assume $\Sigma; \Gamma \vDash A$. We prove that there exists the required finite subset Γ_0 of Γ by *reductio ad absurdum*.

Suppose that for any finite subset Γ_0 of Γ we have $\Sigma; \Gamma_0 \nvDash A$. Let $\Gamma = \{B_n \mid n \in \mathbb{N}\}$ and $C_n = \bigwedge_{i=0}^{n} B_i$. Then, for any $n \in \mathbb{N}$, there exists an Esakia frame $\mathcal{X}_n = (X_n, \Box_n)$, a valuation θ_n over it and a world x_n such that

$$\forall y \neq x_n \ \forall D \in \Sigma \ \ (\mathcal{X}_n, \theta_n), y \vDash D, \quad (\mathcal{X}_n, \theta_n), x_n \vDash C_n \quad \text{and} \quad (\mathcal{X}_n, \theta_n), x_n \nvDash A.$$

By Lemma 4.8, any point of an Esakia frame is closed in some of its neighbourhoods. Let $\mathcal{Y}_n$ be an open subframe of $\mathcal{X}_n$, in which x_n is closed. We define a valuation ψ_n over $\mathcal{Y}_n$ obtained by restricting θ_n to $\mathcal{Y}_n$. By Lemma 4.4, we have

$$\forall y \neq x_n \ \forall D \in \Sigma \ \ (\mathcal{Y}_n, \psi_n), y \vDash D, \quad (\mathcal{Y}_n, \psi_n), x_n \vDash C_n \quad \text{and} \quad (\mathcal{Y}_n, \psi_n), x_n \nvDash A.$$

We take an nonprincipal ultrafilter $\mathcal{U}$ in $\mathbb{N}$ and consider the ultrabouquet $\mathcal{Y} = \bigvee_{\mathcal{U}}(\mathcal{Y}_n, x_n)$ together with the valuation $\psi = \bigvee_{\mathcal{U}}(\psi_n, x_n)$ over $\mathcal{Y}$. From Lemma 6.1, we have

$$\forall y \neq x_n \ \forall D \in \Sigma \ \ (\mathcal{Y}, \psi), y \vDash D, \quad \forall B \in \Gamma \ \ (\mathcal{Y}, \psi), x_n \vDash B \quad \text{and} \quad (\mathcal{Y}, \psi), x_n \nvDash A.$$

We obtain a contradiction with the assumption $\Sigma; \Gamma \vDash A$. Therefore there exists a finite subset Γ_0 of Γ such that $\Sigma; \Gamma_0 \vDash A$. □

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§7. Sequent calculus. In this section we define a calculus corresponding to the global consequence relation $\models$.

A *sequent* is an expression of the form $\Sigma; \Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas, and Σ is an arbitrary set of formulas. For a multiset of formulas $\Gamma = A_1, \dots, A_n$, we set $\Box\Gamma := \Box A_1, \dots, \Box A_n$.

Initial sequents and inference rules of the sequent calculus S have the following form:

$$\Sigma; \Gamma, p \Rightarrow p, \Delta, \qquad \Sigma; \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Sigma; \Gamma, B \Rightarrow \Delta \qquad \Sigma; \Gamma \Rightarrow A, \Delta}{\Sigma; \Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Sigma; \Gamma, A \Rightarrow B, \Delta}{\Sigma; \Gamma \Rightarrow A \to B, \Delta},$$

$$\Box \frac{\Sigma; \Sigma_0, \Gamma, \Box\Gamma \Rightarrow A}{\Sigma; \Pi, \Box\Gamma \Rightarrow \Box A, \Delta} \ (\Sigma_0 \text{ is a finite subset of } \Sigma).$$

An ∞*-proof in* S is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus. Notice that any infinite branch in an ∞-proof of S has infinitely many applications of the rule ($\Box$). A sequent $\Sigma; \Gamma \Rightarrow \Delta$ is *provable in* S if there is an ∞-proof π with the root marked by $\Sigma; \Gamma \Rightarrow \Delta$. In this case π is called an ∞*-proof of* $\Sigma; \Gamma \Rightarrow \Delta$.

A sequent $\Sigma; \Gamma \Rightarrow \Delta$ is called *valid* if $\Sigma; \{\bigwedge \Gamma\} \models \bigvee \Delta$.

LEMMA 7.1. *If a sequent* $\Sigma; \Gamma, A \to B \Rightarrow \Delta$ *is valid, then sequents* $\Sigma; \Gamma, B \Rightarrow \Delta$ *and* $\Sigma; \Gamma \Rightarrow A, \Delta$ *are valid. If* $\Sigma; \Gamma \Rightarrow A \to B, \Delta$ *is valid, then* $\Sigma; \Gamma, A \Rightarrow B, \Delta$ *is also valid.*

Proof. Let us check the first statement of the lemma. Assume a sequent $\Sigma; \Gamma, A \to B \Rightarrow \Delta$ is valid. In other words, we have $\Sigma; \{\bigwedge \Gamma \wedge (A \to B)\} \models \bigvee \Delta$. We shall prove that sequents $\Sigma; \Gamma, B \Rightarrow \Delta$ and $\Sigma; \Gamma \Rightarrow A, \Delta$ are valid. Consider any Esakia model $\mathcal{M}$ and any of its worlds x such that

$$\mathcal{M}, x \models \bigwedge \Gamma \qquad \forall y \neq x \ \ \forall C \in \Sigma \ \ \mathcal{M}, y \models C.$$

We claim $\mathcal{M}, x \models A \vee \bigvee \Delta$. If $\mathcal{M}, x \models A$, then we immediately obtain $\mathcal{M}, x \models A \vee \bigvee \Delta$. Otherwise, suppose $\mathcal{M}, x \not\models A$. It follows that $\mathcal{M}, x \models A \to B$. We obtain $\mathcal{M}, x \models \bigvee \Delta$ from the assumption $\Sigma; \{\bigwedge \Gamma \wedge (A \to B)\} \models \bigvee \Delta$. Thus, $\mathcal{M}, x \models A \vee \bigvee \Delta$. We see that $\Sigma; \{\bigwedge \Gamma\} \models A \vee \bigvee \Delta$. The sequent $\Sigma; \Gamma \Rightarrow A, \Delta$ is valid.

Now suppose that $\mathcal{M}, x \models B$. We obtain $\mathcal{M}, x \models A \to B$. Consequently, $\mathcal{M}, x \models \bigvee \Delta$ from the assumption $\Sigma; \{\bigwedge \Gamma \wedge (A \to B)\} \models \bigvee \Delta$. We see that $\Sigma; \{\bigwedge \Gamma \wedge B\} \models \bigvee \Delta$. The sequent $\Sigma; \Gamma, B \Rightarrow \Delta$ is valid.

The second statement of the lemma is obtained analogously. So we omit further details. $\Box$

A sequent $\Sigma; \Gamma \Rightarrow \Delta$ is called *saturated* if Γ and Δ do not contain formulas of the form $A \to B$.

LEMMA 7.2. *If* $\Sigma; \Pi, \Box\Gamma \Rightarrow \Delta$ *is a valid noninitial saturated sequent, where* Π *consists only of propositional variables, then there are a finite subset* Σ_0 *of* Σ *and a formula* $\Box A$ *from* Δ *such that* $\Sigma; \Sigma_0, \Gamma, \Box\Gamma \Rightarrow A$ *is a valid sequent.*

Proof. Assume $\Sigma; \Pi, \Box\Gamma \Rightarrow \Delta$ is a valid noninitial saturated sequent, where Π consists only of propositional variables. We claim that there is a formula $\Box A$ from Δ such that $\Sigma; \Sigma \cup \{\bigwedge \Gamma \wedge \bigwedge \Box\Gamma\} \vDash A$. We prove this claim by *reductio ad absurdum*.

Let $\Box A_1, \ldots, \Box A_n$ be all elements from Δ of the form $\Box A$. Suppose that for any $i \in \{1, \ldots, n\}$ there exist an Esakia frame $\mathcal{X}_i = (X_i, \Box_i)$, a valuation θ_i over it and a world x_i such that

$$\forall y \in X_i\ \forall B \in \Sigma\ (\mathcal{X}_i, \theta_i), y \vDash B, \quad (\mathcal{X}_i, \theta_i), x_i \vDash \bigwedge \Gamma \wedge \bigwedge \Box\Gamma \quad \text{and} \quad (\mathcal{X}_i, \theta_i), x_i \nvDash A_i.$$

We consider a topological space $\mathcal{X}$ obtained from the disjoint union of $\mathcal{X}_i$ by adding a new world x. A subset U of $\mathcal{X}$ is open if and only if the following conditions hold:

- the restriction of U to $\mathcal{X}_i$ is open for any $i \in \{1, \ldots, n\}$;
- if $x \in U$, then $x_i \in U$ for any $i \in \{1, \ldots, n\}$.

Clearly, the topological space $\mathcal{X}$ is scattered. Hence we can consider it as an Esakia frame. Let θ be a valuation over $\mathcal{X}$ defined on the set of propositional variables as follows:

- the restriction of $\theta(p)$ to $\mathcal{X}_i$ is equal to $\theta_i(p)$ for any $i \in \{1, \ldots, n\}$;
- $x \in \theta(p)$ if and only if $p \in \Pi$.

We shall show that

$$\forall y \neq x\ \forall B \in \Sigma\ (\mathcal{X}, \theta), y \vDash B, \quad (\mathcal{X}, \theta), x \vDash \bigwedge \Pi \wedge \bigwedge \Box\Gamma \quad \text{and} \quad (\mathcal{X}, \theta), x \nvDash \bigvee \Delta.$$

Every Esakia frame $\mathcal{X}_i$ is an open subframe of $\mathcal{X}$. Thus, by Lemma 4.4, the condition $\forall y \neq x\ \forall B \in \Sigma\ (\mathcal{X}, \theta), y \vDash B$ follows from $\forall i \in \{1, \ldots, n\}\ \forall y \in X_i\ \forall B \in \Sigma\ (\mathcal{X}_i, \theta_i), y \vDash B$. Further, we have $(\mathcal{X}, \theta), x \vDash p$ for $p \in \Pi$ and $(\mathcal{X}, \theta), x \nvDash p$ for $p \in \Delta$ by definition of θ.

Let us check that $(\mathcal{X}, \theta), x \vDash \bigwedge \Box\Gamma$. For any formula C from Γ and any $i \in \{1, \ldots, n\}$ we have $(\mathcal{X}, \theta), x_i \vDash C \wedge \Box C$ by Lemma 4.4. This yields that for any $i \in \{1, \ldots, n\}$ there is a neighbourhood U_i of x_i such that $U_i \subset \theta(C)$. We have that $\bigcup\limits_{1 \leqslant i \leqslant n} U_i \subset \theta(C)$, where $\bigcup\limits_{1 \leqslant i \leqslant n} U_i$ is a punctured neighbourhood of x. Hence $(\mathcal{X}, \theta), x \vDash \Box C$ for any $C \in \Gamma$.

It remains to check that $(\mathcal{X}, \theta), x \nvDash \Box A_i$ for $i \in \{1, \ldots, n\}$. For any punctured neighbourhood U of x, there is a world $x_i \in U$ such that $(\mathcal{X}, \theta), x_i \nvDash A_i$. Hence $(\mathcal{X}, \theta), x \nvDash \Box A_i$.

We obtain that the sequent $\Sigma; \Pi, \Box\Gamma \Rightarrow \Delta$ is not valid, which is a contradiction. Therefore there is a formula $\Box A$ from Δ such that $\Sigma; \Sigma \cup \{\bigwedge \Gamma \wedge \bigwedge \Box\Gamma\} \vDash A$. In addition, by Theorem 6.2, there is a finite subset Σ_0 of Σ such that $\Sigma; \Sigma_0 \cup \{\bigwedge \Gamma \wedge \bigwedge \Box\Gamma\} \vDash A$. Hence we find the required valid sequent $\Sigma; \Sigma_0, \Gamma, \Box\Gamma \Rightarrow A$. □

THEOREM 7.3. *Any valid sequent is provable in* S.

Proof. Let us consider a valid sequent $\Sigma; \Gamma \Rightarrow \Delta$. If this sequent is not saturated, then it can be obtained by an application of the rule $(\rightarrow_{\mathsf{R}})$ or $(\rightarrow_{\mathsf{L}})$ from other valid sequents using Lemma 7.1. If this sequent is saturated, then it is initial or can be obtained by an application of the rule $(\Box)$ from another valid sequent using Lemma 7.2. Therefore any valid sequent is an initial sequent of the sequent calculus S or can be obtained by an application of an inference rule from other valid sequents. Thus, for any valid sequent, its ∞-proof in S is immediately defined travelling upwards from conclusions to premises by co-recursion. □

COROLLARY 7.4. *For any set of modal formulas Σ, any finite set of modal formulas Γ and any modal formula A, if $\Sigma; \Gamma \models A$, then the sequent $\Sigma; \Gamma \Rightarrow A$ is provable in* S.

Proof. If $\Sigma; \Gamma \models A$, then the sequent $\Sigma; \Gamma \Rightarrow A$ is valid by definition. Hence this sequent is provable in S by Theorem 7.3. □

§8. Neighbourhood completeness.

THEOREM 8.1. *For any set of modal formulas Σ, and for finite multisets of modal formulas Γ and Δ, if a sequent $\Sigma; \Gamma \Rightarrow \Delta$ is provable in* S, *then $\Sigma; \varnothing \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.*

Proof. Assume π is an ∞-proof of $\Sigma; \Gamma \Rightarrow \Delta$ in S. We define the required ∞-derivation $f(\pi)$ in GL travelling upwards from conclusions to premises by co-recursion.

If $\Sigma; \Gamma \Rightarrow \Delta$ is an initial sequent of the sequent calculus S, then the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is provable in GL by a finite proof. Let $f(\pi)$ be such a proof.

Otherwise, consider the final application of an inference rule in π.

Case 1. If π has the form

$$\rightarrow_{\mathsf{L}} \frac{\begin{matrix}\pi' \\ \vdots \\ \Sigma; \Gamma, B \Rightarrow \Delta\end{matrix} \qquad \begin{matrix}\pi'' \\ \vdots \\ \Sigma; \Gamma, \Rightarrow A, \Delta\end{matrix}}{\Sigma; \Gamma, A \rightarrow B \Rightarrow \Delta},$$

then we define $f(\pi)$ as

$$\mathsf{mp}\ \frac{\begin{matrix}f(\pi') \\ \vdots \\ G\end{matrix} \qquad \mathsf{mp}\ \dfrac{\begin{matrix}f(\pi'') \\ \vdots \\ F\end{matrix} \qquad \begin{matrix}\xi \\ \vdots \\ F \rightarrow (G \rightarrow H)\end{matrix}}{G \rightarrow H}}{H},$$

where $F = \bigwedge \Gamma \rightarrow \bigvee(\{A\} \cup \Delta)$, $G = \bigwedge(\Gamma \cup \{B\}) \rightarrow \bigvee \Delta$, $H = \bigwedge(\Gamma \cup \{A \rightarrow B\}) \rightarrow \bigvee \Delta$ and ξ is a finite proof of the formula $F \rightarrow (G \rightarrow H)$ in GL.

Case 2. If π has the form

$$\rightarrow_{\mathsf{R}} \frac{\begin{matrix}\pi' \\ \vdots \\ \Sigma; \Gamma, A \Rightarrow B, \Delta\end{matrix}}{\Sigma; \Gamma \Rightarrow A \rightarrow B, \Delta},$$

then we define $f(\pi)$ as

$$\mathsf{mp}\ \frac{\begin{matrix}f(\pi') \\ \vdots \\ F\end{matrix} \qquad \begin{matrix}\xi \\ \vdots \\ F \rightarrow G\end{matrix}}{G},$$

where $F = \bigwedge(\Gamma \cup \{A\}) \rightarrow \bigvee(\{B\} \cup \Delta)$, $G = \bigwedge \Gamma \rightarrow \bigvee(\{A \rightarrow B\} \cup \Delta)$ and ξ is a finite proof of the formula $F \rightarrow G$ in GL.

Case 3. Now consider the final case when π has the form

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$$\Box\,\frac{\begin{array}{c}\pi'\\ \vdots\\ \Sigma;\,\Sigma_0,\Gamma,\Box\Gamma \Rightarrow A\end{array}}{\Sigma;\,\Pi,\Box\Gamma \Rightarrow \Box A,\,\Delta}\;(\Sigma_0 \text{ is a finite subset of } \Sigma).$$

We define $f(\pi)$ as

$$\mathsf{mp}\,\dfrac{\mathsf{mp}\,\dfrac{\begin{array}{c}\eta\\ \vdots\\ \bigwedge\Sigma_0\end{array}\qquad \mathsf{mp}\,\dfrac{\begin{array}{c}f(\pi')\\ \vdots\\ F\end{array}\qquad \begin{array}{c}\zeta\\ \vdots\\ F\to(\bigwedge\Sigma_0\to G)\end{array}}{\bigwedge\Sigma_0\to G}}{\mathsf{nec}\,\dfrac{G}{\Box G}}\qquad \begin{array}{c}\xi\\ \vdots\\ \Box G\to H\end{array}}{H},$$

where $F = \bigwedge(\Sigma_0 \cup \Gamma \cup \Box\Gamma) \to A$, $G = \bigwedge(\Gamma \cup \Box\Gamma) \to A$, $H = \bigwedge(\Pi \cup \Box\Gamma) \to \bigvee(\{\Box A\} \cup \Delta)$ and η is a finite derivation of the formula $\bigwedge \Sigma_0$ in GL from the set of assumptions Σ_0. In addition, ζ and ξ are finite proofs in GL of the corresponding formulas $F \to (\bigwedge \Sigma_0 \to G)$ and $\Box G \to H$.

It is not hard to prove that every infinite branch in $f(\pi)$ contains infinitely many applications of the rule (nec) and, in addition, any assumption leaf of $f(\pi)$ is boxed and is marked by an element of Σ. Hence $f(\pi)$ is the required ∞-derivation. □

COROLLARY 8.2. *For any sets of modal formulas Σ and Γ, and for any modal formula A, we have*

$$\Sigma;\Gamma \vdash A \iff \Sigma;\Gamma \Vdash A \iff \Sigma;\Gamma \Vdash^* A \iff \Sigma;\Gamma \vDash A \iff \Sigma;\Gamma \vDash^* A.$$

Proof. The left-to-right implications follow from Proposition 3.6, the definition of the consequence relations $\Vdash^*$, Proposition 4.10, and the definition of the consequence relations $\vDash^*$. Assume $\Sigma;\Gamma \vDash^* A$. Applying Theorem 5.5, we have $\Sigma;\Gamma \vDash A$. By Theorem 6.2, there is a finite subset Γ_0 of Γ such that $\Sigma;\Gamma_0 \vDash A$. By Corollary 7.4, the sequent $\Sigma;\Gamma_0 \Rightarrow A$ is provable in S. From Theorem 8.1, we have $\Sigma;\varnothing \vdash \bigwedge\Gamma_0 \to A$. Notice that $\varnothing;\Gamma \vdash \bigwedge\Gamma_0$. Thus, $\Sigma;\Gamma \vdash \bigwedge\Gamma_0$ and $\Sigma;\Gamma \vdash \bigwedge\Gamma_0 \to A$. Applying an inference rule (mp), we obtain $\Sigma;\Gamma \vdash A$. □

REMARK. *In this corollary, we do not mention the corresponding semantic consequence relation over Kripke GL-frames (transitive conversely well-founded relational frames), because the set $\{\Box^n(\Box p_{n+1} \to p_n) \mid n \in \mathbb{N}\}$ globally entails p_0 over such frames. However, $\{\Box^n(\Box p_{n+1} \to p_n) \mid n \in \mathbb{N}\} \nvDash_g p_0$. Here, $\Box^n$ denotes n consequent applications of $\Box$ and $\Box^0(\Box p_1 \to p_0) = \Box p_1 \to p_0$.*

Let $\mathcal{O}$ be the Esakia frame corresponding to the first uncountable ordinal with the interval topology. The logic GL enriched with non–well-founded derivations is complete for its neighbourhood interpretation over $\mathcal{O}$ in the following sense.

COROLLARY 8.3. *For any sets of modal formulas Σ and Γ, and for any modal formula A, if $\Sigma;\Gamma \nVdash A$, then there is a valuation θ over $\mathcal{O}$, a world x of $\mathcal{O}$ and a neighbourhood U of x such that*

$$\forall C \in \Sigma\ \forall y \in U\setminus\{x\}\ \ (\mathcal{O},\theta),y \vDash C, \quad \forall B \in \Gamma\ \ (\mathcal{O},\theta),x \vDash B \quad \textit{and} \quad (\mathcal{O},\theta),x \nvDash A.$$

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Proof. Any countable ordinal with the interval topology is an open subframe of the first uncountable ordinal. Thus, this statement follows from the previous corollary, Corollary 5.3 and Lemma 4.4 immediately. □

REMARK. *By Proposition 4.12, the logic* GL *with non–well-founded derivations is not complete with respect to any countable ordinal.*

Acknowledgements. The development of main ideas of this article took place during my stay in Tash-Bulak village, the Kyrgyz Republic, in 2015. I heartily thank my uncle Kengebek Shamkanov and his family for their hospitality. In addition, I am grateful to Tadeusz Litak. Proposition 3.3 has been inspired by his constructive comments. Also, I thank Andrey Kudinov for helpful discussions concerning §5 and an anonymous reviewer for stimulating recommendations. I express my deep gratitude to my grandmother Anastasia Shelutinskaya. SDG.

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STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES
GUBKINA STR. 8, 119991, MOSCOW, RUSSIA
and
NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS
FACULTY OF MATHEMATICS
USACHEVA STR. 6, 119048, MOSCOW, RUSSIA
E-mail: daniyar.shamkanov@gmail.com