Euro. Jnl of Applied Mathematics (2017), *vol.* 28, *pp.* 435–469. © Cambridge University Press 2016 435
doi:10.1017/S0956792516000371

Asymptotic behaviour of solutions of free boundary problems for Fisher-KPP equation

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(*Received 31 January 2016; revised 27 July 2016; accepted 2 August 2016; first published online 30 August 2016*)

We study a free boundary problem for the Fisher-KPP equation: $u_t = u_{xx} + f(u)$ $(g(t) < x < h(t))$ with free boundary conditions $h'(t) = -u_x(t, h(t)) - \alpha$ and $g'(t) = -u_x(t, g(t)) + \beta$ for $0 < \beta < \alpha$. This problem can model the spreading of a biological or chemical species, where free boundaries represent the spreading fronts of the species. We investigate the asymptotic behaviour of bounded solutions. There are two parameters α_0 and α^* with $0 < \alpha_0 < \alpha^*$ which play key roles in the dynamics. More precisely, (i) in case $0 < \beta < \alpha_0$ and $0 < \alpha < \alpha^*$, we obtain a trichotomy result: (i-1) spreading, i.e., $h(t) - g(t) \to +\infty$ and $u(t, \cdot + ct) \to 1$ with $c \in (c_L, c_R)$, where c_L and c_R are the asymptotic spreading speed of $g(t)$ and $h(t)$, respectively, ($c_R > 0 > c_L$ when $0 < \beta < \alpha < \alpha_0$; $c_R = 0 > c_L$ when $0 < \beta < \alpha = \alpha_0$; $0 > c_R > c_L$ when $\alpha_0 < \alpha < \alpha^*$ and $0 < \beta < \alpha_0$); (i-2) vanishing, i.e., $\lim_{t\to T} h(t) = \lim_{t\to T} g(t)$ and $\lim_{t\to T} u(t, x) = 0$, where T is some positive constant; (i-3) transition, i.e., $g(t) \to -\infty$, $h(t) \to -\infty$, $0 < \lim_{t\to\infty}[h(t) - g(t)] < +\infty$ and $u(t, x) \to V^*(x - c^*t)$ with $c^* < 0$, where $V^*(x - c^*t)$ is a travelling wave with compact support and which satisfies the free boundary conditions. (ii) in case $\beta \geqslant \alpha_0$ or $\alpha \geqslant \alpha^*$, vanishing happens for any solution.

Key words: Fisher-KPP equation, free boundary problem, zero number principle, compactly supported travelling wave

1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t = u_{xx} + f(u), & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -u_x(t, g(t)) + \beta, & t > 0, \\ h'(t) = -u_x(t, h(t)) - \alpha, & t > 0, \\ -g(0) = h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \leqslant x \leqslant h_0, \end{cases} \tag{1.1}$$

where $x = g(t)$ and $x = h(t)$ are moving boundaries to be determined together with $u(t, x)$, α and β are given positive constants with $0 < \beta < \alpha$, $f : [0, \infty) \to \mathbb{R}$ is a C^1 function satisfying

$$\begin{cases} f(0) = f(1) = 0,\ (1-u)f(u) > 0 \text{ for } u > 0 \text{ and } u \neq 1, \\ f'(0) > 0,\ f'(1) < 0 \text{ and } f(u) \leqslant f'(0)u \text{ for } u \geqslant 0. \end{cases} \tag{1.2}$$

One example is $f(u) = u(1-u)$. The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \left\{\phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0,\ \phi(x) \geqslant (\not\equiv) 0 \text{ in } (-h_0, h_0)\right\}. \quad (1.3)$$

The problem (1.1) may be used to model the spreading of a new or invasive species whose density is represented by $u(t,x)$, with the free boundaries $x = h(t)$ and $x = g(t)$ representing the expanding fronts of the species. When $\alpha = \beta = 0$, the problem (1.1) was studied by [7, 8], etc., they obtained a spreading-vanishing dichotomy result, namely the species either spreads to the whole environment and stabilizes at the positive state 1 (i.e., $u(t,\cdot) \to 1$ and $-g(t), h(t) \to +\infty$), or it vanishes (i.e., $u(t,\cdot) \to 0$, $g(t) \to g_\infty$ and $h(t) \to h_\infty$ for some $-g_\infty, h_\infty \in (0, +\infty)$. Such a result shows that free boundary problem has advantages in explaining the spreading of species compared with Cauchy problems. (The Cauchy problems have a hair-trigger effect: any positive solution converges to a positive constant no matter how small the positive initial data is, cf. [1, 2].) In [7, 8], an estimate of asymptotic spreading speed was obtained:

$$k^* := \lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{t} > 0.$$

Recently, [11] improved this result with sharper estimates. [6, 10, 12] also studied the corresponding problem of (1.1) in higher-dimensional spaces with $\alpha = \beta = 0$.

In this paper, we will study the problem (1.1) with $0 < \beta < \alpha$. We use these parameters to denote the decay rates at the boundary since there is a force resistant to spreading at the front for some species. Intuitively, the presence of $\alpha > 0$ makes the solution more difficult to spread than the case $\alpha = 0$. Indeed, $h'(t) > 0$ only if $u_x(t, h(t)) < -\alpha$.

The boundary condition we used can be derived from the following problems (cf. [15, 16]):

$$\begin{cases} u_t = \Delta u + f(u) - \dfrac{s_1 uv}{\varepsilon} - \dfrac{k_1(1-w)u}{\varepsilon}, & x \in \Omega,\ t > 0, \\ v_t = \Delta v + g(v) - \dfrac{s_2 uv}{\varepsilon} - \dfrac{k_2 wv}{\varepsilon}, & x \in \Omega,\ t > 0, \\ w_t = \dfrac{(1-w)u}{\varepsilon} - \dfrac{wv}{\varepsilon}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \quad (1.4)$$

where $k_i = \lambda s_i (i = 1, 2)$ for some $\lambda > 0$, ν denotes the outward unit normal vector to $\partial\Omega$, f and g are the growth terms, s_1/ε and s_2/ε are the interspecific competition rates between u and v. Denote the solution of (1.4) by $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$. From singular limit analysis, the authors proved that, as $\varepsilon \to 0^+$,

$$u^{\varepsilon} \to u,\ v^{\varepsilon} \to v \text{ in } L^2(\Omega \times T),$$

$$w^{\varepsilon} \to w \quad \text{weakly in } L^2(\Omega \times T),$$

and

$$\Omega_u(t) \cap \Omega_v(t) = \emptyset, \quad w(\cdot, t) = \begin{cases} 1 & in\ \Omega_u(t),\ t \in [0, T], \\ 0 & in\ \Omega_v(t),\ t \in [0, T], \end{cases}$$

where $\Omega_u(t) := \{x \in \Omega | u(x,t) > 0\}$ and $\Omega_v(t) := \{x \in \Omega | v(x,t) > 0\}$. Write $\Gamma := \bigcup_{0 \leqslant t \leqslant T} \Gamma(t)$, with $\Gamma(t) := \Omega \setminus [\Omega_u(t) \cup \Omega_v(t)]$. Then, (Γ, u, v) is the unique solution of the following free boundary problem:

$$\begin{cases} u_t = \Delta u + f(u), & (x,t) \in \Omega_u(t) \times (0, T], \\ v_t = \Delta v + g(v), & (x,t) \in \Omega_v(t) \times (0, T], \\ u(x,t) = v(x,t) = 0, & (x,t) \in \Gamma(t) \times (0, T], \\ V_n = -\dfrac{1}{\lambda s_1} \dfrac{\partial u}{\partial n} - \dfrac{1}{\lambda s_2} \dfrac{\partial v}{\partial n}, & (x,t) \in \Gamma(t), \end{cases} \tag{1.5}$$

where n is the unit normal vector on $\Gamma(t)$. [17] also studied a more general competition-diffusion system and obtained a similar result. When one of them (without loss of generality, we assume it is v) spreads and reaches a balanced state, then the profile of u has a shape like the travelling semi-wave (cf. [8]), so $\frac{\partial v}{\partial n} \approx constant$. Therefore, the last equation in (1.5) is our free boundary condition in the problem (1.1).

Moreover, such a free boundary condition is widely used in many biological models. For example, in [5, 13, 19–21], the authors studied some protocell models which mimic the biological process of growth and dissolution of an organism under external nutrient supply. When the nutrient is metabolized, it is transformed into building material of liquid phase which diffuses within the protocell. The liquid building material is polymerized into solid phase (or, more precisely, plastic phase) which builds the protocell. Besides polymerization, the liquid building material also undergoes disintegration due to factors such as aging, which causes the cell to shrink. On the other hand, the flux of building material at the boundary causes the cell to grow. The total result of these two effects is

$$V_n = -\frac{\partial C}{\partial n} - \gamma,$$

where C is the concentration of building materials of the cell, $\gamma > 0$ is the disintegration rate at the boundary, n is the exterior normal, V_n is the velocity of the boundary points in the direction n.

In the special case $\alpha = \beta > 0$, the problem (1.1) was considered recently in [3, 4]. They studied the asymptotic behaviour of solutions when $\alpha > 0$ is not large $\big($i.e., $\alpha < \alpha_0 := \sqrt{2 \int_0^1 f(s) ds}\big)$, and obtained a trichotomy result:

(i) spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ and

$$\lim_{t \to \infty} u(t, \cdot) = 1 \text{ locally uniformly in } \mathbb{R}^1;$$

https://doi.org/10.1017/S0956792516000371 Published online by Cambridge University Press

(ii) vanishing: $\lim_{t\to T} g(t) = \lim_{t\to T} h(t) \in [-2h_0, 2h_0]$ with $0 < T < +\infty$ and

$$\lim_{t\to T} \max_{g(t)\leqslant x\leqslant h(t)} u(t,x) = 0;$$

(iii) transition: (g_∞, h_∞) is a bounded interval and

$$\lim_{t\to\infty} u(t,\cdot) = v(\cdot) \text{ locally uniformly in } (g_\infty, h_\infty),$$

where v is the solution of

$$\begin{cases} v'' + f(v) = 0, & x \in (g_\infty, h_\infty), \\ v(g_\infty) = v(h_\infty) = 0, \ v'(g_\infty) = -v'(h_\infty) = \alpha. \end{cases}$$

In contrast with spreading-vanishing dichotomy result in [7, 8], the authors in [3, 4] obtained the third possibility, namely transition case (both $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} h(t)$ are *finite* numbers, and the solution $u(t,x)$ tends to a *stationary solution* of the equation $(1.1)_1$). Also, vanishing is a different case, since the free boundaries converge to the *same* point within a *finite* time.

Our main purpose in this paper is to study the influence of α, β on the asymptotic behaviour of solutions. As we will see below, the phenomena are much more complicated and more interesting than the case $\alpha = \beta$. To sketch the influence of α, β, we need to introduce two important travelling waves. We consider the following problem:

$$\begin{cases} q'' + cq' + f(q) = 0, & z \in (0, +\infty), \\ q(0) = 0, \ q(+\infty) = 1, \ q'(0) = -c + \beta, \ q'(z) > 0 \text{ for } z \in [0, +\infty). \end{cases} \tag{1.6}$$

Problem (1.6) has a unique solution (c_L, q_L) (see details in Lemma 3.4). Thus, $u(t,x) = q_L(x - c_L t)$ is a solution of $u_t = u_{xx} + f(u)$ with $u(t, c_L t) = 0$ and $c_L = -u_x(t, c_L t) + \beta$. It is called a *travelling semi-wave* since it is only defined for $x \geqslant c_L t$.

Similarly, one can obtain that the following problem,

$$\begin{cases} q'' + cq' + f(q) = 0, & z \in (-\infty, 0), \\ q(0) = 0, \ q(-\infty) = 1, \ -q'(0) = c + \alpha, \ q'(z) < 0 \text{ for } z \in (-\infty, 0], \end{cases} \tag{1.7}$$

also has a unique solution (c_R, q_R) (cf. Lemma 3.4). Moreover, the travelling semi-wave $u(t,x) = q_R(x - c_R t)$ also satisfies $u_t = u_{xx} + f(u)$ with $u(t, c_R t) = 0$ and $c_R = -u_x(t, c_R t) - \alpha$.

As we will see below, the asymptotic spreading speeds of the free boundaries $g(t)$ and $h(t)$ are c_L and c_R, respectively, when spreading happens, so we need consider to the relationship between c_L and c_R, which can determine the asymptotic locations of two free boundaries. We also write c_L as $c_L(\beta)$ and c_R as $c_R(\alpha)$ to emphasize the dependence of c_L, c_R on α, β, respectively. From phase plane analysis (see details in Proposition 3.5), we can derive that the following equation:

$$c_L(\beta) = c_R(\alpha) \tag{1.8}$$

has a unique root $\alpha = \alpha^*$. Obviously, α^* is dependent on β, so we also write α^* as $\alpha^*(\beta)$. In other words, there is $\alpha^* > \alpha_0$ such that the travelling semi-waves $q_L(x - c_L t)$ and $q_R(x - c_R t)$ have the same travelling speed.

The precise relationship between c_L and c_R can be seen in Corollary 3.6 below. According to their relations, we have the following situations. *Case 1.* When $0 < \beta < \alpha < \alpha_0$, the solution u with large initial data can grow up and converges to 1. Its right front (the part of the solution near $h(t)$) has a shape like the travelling semi-wave $q_R(\cdot)$ and moves rightwards at a speed $\approx c_R$. Its left front (the part of the solution near $g(t)$) has a shape like $q_L(\cdot)$ and moves leftwards at a speed $\approx c_L$. Moreover, in this case, we can derive from the phase plane analysis that $c_L < 0 < c_R$ (see details in Corollary 3.6), so the free boundary $h(t)$ moves to $+\infty$ and $g(t)$ moves to $-\infty$ as $t \to \infty$. *Case 2.* Assume that $0 < \beta < \alpha_0 < \alpha < \alpha^*$. If the initial data is sufficiently large, the solution $u(t, x)$ can also grow up between the left front and the right front, and the speeds of its left front and right front are determined by (1.6) and (1.7), respectively. In this case, $c_L < c_R < 0$ (cf. Corollary 3.6), so the free boundary $g(t)$ travels leftwards faster than $h(t)$. Hence, both $g(t)$ and $h(t)$ tend to $-\infty$, but the distance between $g(t)$ and $h(t)$ becomes larger and larger and tends to ∞ as time increases. *Case 3.* When $0 < \beta < \alpha_0 < \alpha^* < \alpha$, we have $c_R < c_L < 0$ (cf. Corollary 3.6), which means that both the left front and the right front move leftwards, and the right front moves faster than the left front. So the solution does not have enough space to grow up, and then only vanishing happens. *Case 4.* When $\alpha_0 < \beta < \alpha$, it follows from Corollary 3.6 that $c_R < 0 < c_L$, so two free boundaries move relative to one another and they move to the same point within a finite time. Hence, vanishing happens for any solution. This paper is organized as follows. In Section 2, we present the main results. In Section 3, we give some basic results including the comparison principles, several types of travelling waves, zero number results and converges results. These are fundamental for this research. In Section 4, we give some sufficient conditions for vanishing and spreading. In Section 5, we prove the main theorems.

2 Main results

In this section, we give the main results. By a similar argument as in [4, 7, 8], one can show that (1.1) has a unique solution (u, g, h) defined on the maximal interval $[0, T)$ (for some $T \in (0, +\infty]$) with $(u, g, h) \in C^{1+\gamma/2, 2+\gamma}(\overline{D}_T) \times C^{1+\gamma/2}([0, T]) \times C^{1+\gamma/2}([0, T])$ for any $\gamma \in (0, 1)$, where $D_T := \{(t, x) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in (0, T]\}$. Moreover, as in the proof of [4, Lemma 2.9], one can show that

$$h_T := \lim_{t \to T} h(t) \in [-\infty, +\infty] \quad \text{and} \quad g_T := \lim_{t \to T} g(t) \in [-\infty, +\infty]$$

exist. In particular, when $T = +\infty$, we also write h_T and g_T as h_∞ and g_∞, respectively.

The following main theorems (Theorem 2.1, Theorem 2.5) give a fairly complete description for the asymptotic behaviour of the solution (u, g, h) of the problem (1.1).

We first consider the case where β and α are not large: $0 < \beta < \alpha_0$ and $0 < \alpha < \alpha^*$, where $\alpha^* = \alpha^*(\beta)$ is the unique root of (1.8).

Theorem 2.1 *Assume $0 < \beta < \alpha_0$, $0 < \alpha < \alpha^*$ and $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$. Let (u, g, h) be a solution of the problem* (1.1) *defined on some maximal interval* $[0, T)$. *Then, there exists $\sigma^* = \sigma^*(h_0, \phi) \in (0, +\infty]$ such that*

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(i) **spreading** *happens when* $\sigma > \sigma^*$, *i.e.,* $T = +\infty$, *and*

(i-a) *when* $0 < \alpha < \alpha_0$, $(g_\infty, h_\infty) = \mathbb{R}$ *and*

$$\lim_{t\to\infty} u(t,\cdot) = 1 \quad \text{locally uniformly in } \mathbb{R}; \tag{2.1}$$

(i-b) *when* $\alpha_0 < \alpha < \alpha^*$,

$$h_\infty = g_\infty = -\infty, \quad \lim_{t\to\infty}[h(t) - g(t)] = +\infty$$

and

$$\lim_{t\to\infty} u(t, \cdot + ct) = 1 \quad \text{locally uniformly in } \mathbb{R}, \quad \text{for any } c \in (c_L, c_R), \tag{2.2}$$

where c_L *and* c_R *are the speeds of travelling semi-waves in* (1.6) *and* (1.7), *respectively, with* $c_L < c_R < 0$;

(i-c) *when* $\alpha = \alpha_0$, $g_\infty = -\infty$, $-\infty < h_\infty < +\infty$, *and*

$$\lim_{t\to\infty} u(t,x) = q_0(\cdot - h_\infty) \quad \text{locally uniformly in } (-\infty, h_\infty], \tag{2.3}$$

where $q_0(x)$ *is the unique solution of* (1.7) *with* $c = 0$,

(ii) **vanishing** *happens when* $\sigma < \sigma^*$, *i.e.,* $T < +\infty$,

$$\lim_{t\to T} g(t) = \lim_{t\to T} h(t) \in (-\infty, +\infty) \quad \text{and} \quad \lim_{t\to T} \max_{g(t)\leqslant x\leqslant h(t)} u(t,x) = 0,$$

(iii) *in the* **transition** *case when* $\sigma = \sigma^*$, *i.e.,* $T = +\infty$, $g_\infty = h_\infty = -\infty$,

$$g(t) = c^*t + x_1 + \varrho_1(t), \quad h(t) = c^*t + x_1 + \varrho_2(t), \quad \lim_{t\to\infty}[h(t) - g(t)] = L^*,$$

for some $x_1 \in \mathbb{R}$, $\varrho_1(t) \to 0$ *and* $\varrho_2(t) \to 0$ *as* $t \to \infty$, *and*

$$\lim_{t\to\infty} \|u(t,\cdot) - V^*(\cdot - c^*t - x_1)\|_{L^\infty(J(t))} = 0, \tag{2.4}$$

where $J(t) := \left[\max\{g(t), -L^* + c^*t + x_1\}, \min\{h(t), c^*t + x_1\}\right]$, *and* $(L, c, q) = (L^*, c^*, V^*)$ *is the unique solution of*

$$\begin{cases} q'' + cq' + f(q) = 0, & z \in (-L, 0), \\ q(-L) = q(0) = 0, \ q(z) > 0 \text{ for } z \in (-L, 0), \\ q'(-L) = -c + \beta, \ -q'(0) = c + \alpha \end{cases} \tag{2.5}$$

with $c^* \in (c_L, c_R)$, $L^* > 0$.

Remark 2.2 *For some biological and chemical species (such as the protocell), the process of growth is usually accompanied and counteracted by a process of dissolution or disintegration in a balanced manner so that the species finally forms a stationary configuration. When the disintegration rate at the boundary is not large (i.e.,* $0 < \beta < \alpha_0$, $0 < \alpha < \alpha^*$*), there*

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are three cases for the growth of the species. In the spreading case, the species expands to a stationary configuration (1, or q_0) when its initial density is sufficiently large. In the vanishing case, the species disappears in a finite time when the initial density is small. In the last case, the species converges to the stationary configuration with a certain radius.

Remark 2.3 *Compared with the results in [3, 4, 7, 8, 14], results (i-b) and (i-c) in the above theorem are new. The previous results showed that when spreading happens, two free boundaries move to $+\infty$ and $-\infty$, respectively, and the solution converges to 1 locally uniformly in $\mathbb{R}$. However, in our spreading case, for medium-sized α ($\alpha_0 \leqslant \alpha < \alpha^*$), the right free boundary $h(t)$ cannot spread to $+\infty$ no matter how large the initial data is, since we can find an upper solution that blocks the rightward expanding of the free boundary $h(t)$. More precisely, when $\alpha_0 < \alpha < \alpha^*$, both free boundaries move to $-\infty$, and the solution converges to 1 in some moving frame. When $\alpha = \alpha_0$, the free boundary $g(t) \to -\infty$ while $h(t)$ tends to a fixed point and the solution converges to a decreasing stationary solution defined on the half-line. This is also different from the phenomenon "virtual spreading" (the solution converges to 0 locally uniformly in $(g_\infty, +\infty)$ with $-\infty < g_\infty < 0$, but converges to 1 in some moving frame) in [14].*

Remark 2.4 *The transition case in the above theorem is also a new one. The authors in [3,4] showed that the solution converges to a stationary solution with compact support in their transition case. In our transition case, the solution $u(t,x)$ converges to $V^*(x-c^*t-x_1)$, where V^* is a unique solution of (2.5) with positive compact support, and it travels leftwards at a speed $c^* \in (c_L, c_R)$. We call such a solution the compactly supported travelling wave (see also [14]). In particular, if $\alpha = \beta + \epsilon$ for sufficiently small $\epsilon > 0$, then the asymptotic speed c^* of the transition solution depends on ϵ continuously, and satisfies $c^* \to 0$ as $\epsilon \to 0$. When $\epsilon \to 0$, the limit of $V^*(x-c^*t-x_1)$ is a stationary solution of $(1.1)_1$. Thus, the limiting case of Theorem 2.1 (iii) is the transition phenomenon of [3,4].*

We now show that only vanishing happens when β is large, or when α is large.

Theorem 2.5 *Assume that $\beta < \alpha_0$ and $\alpha \geqslant \alpha^*$, or $\beta \geqslant \alpha_0$. Let (u, g, h) be a solution of (1.1) with initial data $u_0 \in \mathscr{X}(h_0)$. Then, vanishing happens.*

In this case, α and β are too large for the free boundaries to expand outwards and their separation becomes smaller and smaller. In fact, we can construct two upper solutions on two sides of the solution moving relative to one another. So the solution is forced by these two upper solutions to tend to 0 and the two free boundaries come to one point at a finite time.

Finally, we show that when spreading happens, as in Theorem 2.1, the spreading speed and the asymptotic profile of the fronts are characterized by (1.6) and (1.7).

Theorem 2.6 *Assume spreading happens for a solution $u(t,x)$ of the problem (1.1) as in Theorem 2.1. Let (c_L, q_L) and (c_R, q_R) be the solutions of (1.6) and (1.7), respectively. Then, there exist H, $G \in \mathbb{R}$ such that*

$$\lim_{t\to\infty}[h(t) - c_R t - H] = 0, \quad \lim_{t\to\infty} h'(t) = c_R,$$

$$\lim_{t\to\infty}[g(t) - c_L t - G] = 0, \quad \lim_{t\to\infty} g'(t) = c_L,$$

and

$$\lim_{t\to\infty} \sup_{x\in[c^*t,h(t)]} |u(t,x) - q_R(x - h(t))| = 0, \tag{2.6}$$

$$\lim_{t\to\infty} \sup_{x\in[g(t),c^*t]} |u(t,x) - q_L(x - g(t))| = 0. \tag{2.7}$$

3 Some basic results

In this section, we first give the comparison principle and the definitions of upper solutions and lower solutions, then present the existence of two types of travelling waves, and the zero number arguments which will play key roles in our proofs. We finally prove the convergence results.

3.1 The comparison principle

We first give the following comparison theorems which can be proved similarly to [4, Lemma 2.3 and 2.4] and [7, Lemma 5.7].

Lemma 3.1 *Suppose that* $T \in (0,\infty)$, $\overline{g}, \overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leqslant T,\ \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \geqslant \overline{u}_{xx} + f(\overline{u}), & 0 < t \leqslant T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{g}'(t) \leqslant -\overline{u}_x + \beta, & 0 < t \leqslant T,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \geqslant -\overline{u}_x - \alpha, & 0 < t \leqslant T,\ x = \overline{h}(t). \end{cases}$$

If $[-h_0, h_0] \subseteq [\overline{g}(0), \overline{h}(0)]$, $u_0(x) \leqslant \overline{u}(0,x)$ *in* $[-h_0, h_0]$, *and if* (u, g, h) *is a solution of* (1.1) *with initial data* $u_0(x)$, *then*

$$g(t) \geqslant \overline{g}(t),\ h(t) \leqslant \overline{h}(t),\ u(x,t) \leqslant \overline{u}(x,t) \quad \textit{for } t \in (0,T] \textit{ and } x \in (g(t), h(t)).$$

Lemma 3.2 *Suppose that* $T \in (0,\infty)$, $\overline{g}, \overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leqslant T,\ \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \geqslant \overline{u}_{xx} + f(\overline{u}), & 0 < t \leqslant T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} \geqslant u, & 0 < t \leqslant T,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \geqslant -\overline{u}_x - \alpha, & 0 < t \leqslant T,\ x = \overline{h}(t), \end{cases}$$

with $h(t) > \overline{g}(t) \geqslant g(t)$ *in* $[0,T]$, $h_0 \leqslant \overline{h}(0)$, $u_0(x) \leqslant \overline{u}(0,x)$ *in* $[\overline{g}(0), h_0]$, *where* (u, g, h) *is a solution of* (1.1). *Then*

$$h(t) \leqslant \overline{h}(t),\ u(x,t) \leqslant \overline{u}(x,t) \ \textit{ for } t \in (0,T] \ \textit{ and } \ \overline{g}(t) < x < h(t).$$

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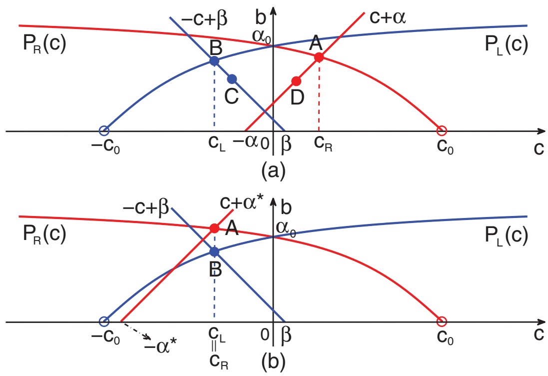

FIGURE 1. The c–b plane about travelling waves. Point A corresponds to a strictly decreasing travelling semi-wave, and point B corresponds to a strictly increasing travelling semi-wave, both of which satisfy the equation and free boundary condition. Point C (resp. point D) corresponds to a compactly supported travelling wave, which satisfies the equation and the free boundary condition on its left (resp. right) endpoint. (a) The case $\beta \in (0, \alpha_0)$ and $\alpha \in (0, \alpha^*)$; (b) the case $\beta \in (0, \alpha_0)$ and $\alpha = \alpha^*$.

Remark 3.3 *The function $\overline{u}$, or the triple $(\overline{u}, \overline{g}, \overline{h})$ in Lemmas 3.1 and 3.2, is usually called an upper solution of* (1.1) *and one can define a lower solution by reversing all the inequalities. There is a symmetric version of Lemma 3.2, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.*

3.2 Travelling semi-wave and compactly supported travelling wave

If $u(t, x) = q(x - ct)$ is a travelling wave of $u_t = u_{xx} + f(u)$, then (c, q) solves

$$q''(z) + cq'(z) + f(q) = 0. \tag{3.1}$$

It is well known that (3.1) has a unique positive decreasing solution $q(z)$ in $\mathbb{R}$ with $q(-\infty) = 1$, $q(+\infty) = 0$ and $q(0) = 1/2$ if and only if $c \geqslant c_0 := 2\sqrt{f'(0)}$.

Using phase plane analysis, section 3.2 in [14] have given a rather complete description of all the types of solutions for (3.1). We only concentrate on strictly monotonous solutions on the half-line and the compactly supported solutions, namely, types (ii), (iii) and (iv) in [14, section 3.2].

In this paper, we will use the following two types of travelling waves.

(I) Travelling semi-waves $q_L(x - c_L t)$ and $q_R(x - c_R t)$, where $c_L = c_L(\beta)$, $c_R = c_R(\alpha)$, q_L and q_R are given in the following lemma.

Lemma 3.4 *Assume $\alpha > \beta > 0$. Then*

(i) *there exists $c_L = c_L(\beta) \in (-c_0, \beta)$ such that, only for $c = c_L$, the equation* (3.1) *and*

$$q(0) = 0, \;\; q(+\infty) = 1, \;\; q'(0) = -c + \beta, \;\; q'(z) > 0 \textit{ for } z \in [0, \infty) \tag{3.2}$$

has a solution (cf. point B in Figure 1). The solution is unique and denoted by q_L;

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(ii) *there exists* $c_R = c_R(\alpha) \in (-\alpha, c_0)$ *such that, only for* $c = c_R$*, the equation* (3.1) *and*

$$q(0) = 0,\ q(-\infty) = 1,\ -q'(0) = c + \alpha,\ q'(z) < 0 \text{ for } z \in (-\infty, 0] \tag{3.3}$$

has a solution (cf. point A in Figure 1). The solution is unique and denoted by q_R*.*

Proof By phase plane analysis as in [14, section 3.2], one can show that

(1) for each $c > -c_0$, the equation (3.1) has a unique solution $U_L(\cdot; c) \in C^2([0,\infty))$, with $U_L(0; c) = 0$, $U_L(\infty; c) = 1$ and $U_L'(\cdot; c) > 0$ in $[0, \infty)$;
(2) for each $c < c_0$, the equation (3.1) has a unique solution $U_R(\cdot; c) \in C^2((-\infty, 0])$, with $U_R(0; c) = 0$, $U_R(-\infty; c) = 1$ and $U_R'(\cdot; c) < 0$ in $(-\infty, 0]$.

Define

$$P_L(c) := U_L'(0; c) > 0\ (c > -c_0) \quad \text{and} \quad P_R(c) := -U_R'(0; c) > 0\ (c < c_0).$$

One can easily obtain that

$$P_R(-c) = P_L(c) \quad \text{for all } c > c_0.$$

Using standard ODE theory for (3.1) as in the proof of [8, Lemma 6.1], we see that $P_L(c)$ (resp. $P_R(c)$) is continuous and strictly increasing (resp. decreasing) in $(-c_0, \infty)$ (resp. $(-\infty, c_0)$), $P_L(-c_0 + 0) = 0$ (resp. $P_R(c_0 - 0) = 0$) (see Figure 1). Because

$$\left[P_L(c) - (-c + \beta)\right]\Big|_{c=-c_0+0} < 0 \quad \text{and} \quad \left[P_L(c) - (-c + \beta)\right]\Big|_{c=\beta} > 0,$$

the equation $P_L(c) = -c + \beta$ has a unique root $c = c_L \in (-c_0, \beta)$. Writing $q_L(\cdot) := U_L(\cdot; c_L)$, the conclusion (i) holds. By similar arguments, one can prove that the equation $P_R(c) = c + \alpha$ has a unique root $c = c_R \in (-\alpha, c_0)$, which means that (3.1) and (3.3) have a unique solution $U_R(\cdot; c_R)$ denoted by $q_R(\cdot)$. □

Proposition 3.5 *Assume* $\alpha > \beta > 0$*. Then*

(i) $c_L(\beta)$ *is continuous and strictly increasing in* β*, and* $c_R(\alpha)$ *is continuous and strictly decreasing in* α*;*
(ii) $\alpha_0 := \sqrt{2\int_0^1 f(s)ds}$ *is the unique zero point of* $c_L(\beta)$ *and* $c_R(\alpha)$*;*
(iii) *for any* $\beta \in (0, \alpha_0)$*, the equation* $c_R(\alpha) = c_L(\beta)$ *has a unique root* $\alpha = \alpha^*(\beta) > \alpha_0$*.*

Proof

(i) By the proof of Lemma 3.4, we know $c = c_L(\beta)$ (resp. $c = c_R(\alpha)$) is the unique root of $P_L(c) = -c + \beta$ (resp. $P_R(c) = c + \alpha$). Combining with the continuity and monotonicity of $P_L(c)$ (resp. $P_R(c)$), one can get the conclusion (i) (cf. Figure 1).
(ii) Suppose $c = 0$, then $U_L(\cdot; 0)$ satisfies $U_L''(\cdot; 0) + f(U_L(\cdot; 0)) = 0$. Multiplying this equation by $U_L'(\cdot; 0)$ and integrate in $(0, \infty)$, we have $U_L'(0; 0) = \sqrt{2\int_0^1 f(s)ds} =: \alpha_0$, i.e., $P_L(0) = \alpha_0$. It follows from the definition of $c_L(\beta)$ that $c_L(\alpha_0) = 0$. By (i), one can

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then get α_0 is the unique zero point of $c_L(\beta)$. Using similar arguments, α_0 is also the unique zero point of $c_R(\alpha)$.

(iii) By the monotonicity of $c_L(\beta)$ and $c_L(\alpha_0) = 0$, we have $c_L(\beta) < 0$ when $\beta \in (0, \alpha_0)$. Hence

$$c_R(\alpha_0) = 0 > c_L(\beta), \qquad \text{for any fixed } \beta \in (0, \alpha_0). \tag{3.4}$$

On the other hand, because

$$P_R(-c_0) = -c_0 + \alpha \qquad \text{when } \alpha = P_R(-c_0) + c_0, \tag{3.5}$$

combining (3.5) with the definition of c_R, we have

$$c_R(P_R(-c_0) + c_0) = -c_0 < c_L(\beta). \tag{3.6}$$

The last inequality follows from Lemma 3.4 (i). Combining (3.4) and (3.6) with the continuity and monotonicity of $c_R(\alpha)$, one can derive that there exists a unique value $\alpha^* \in (\alpha_0, P_R(-c_0) + c_0)$ $(P_R(-c_0) > P_R(0) = P_L(0) = \alpha_0)$ such that $c_R(\alpha^*) = c_L(\beta)$. □

A direct consequence of Proposition 3.5 is as follows.

Corollary 3.6 *Assume* $\alpha > \beta > 0$. *Then*

(i) *when* $\beta \geqslant \alpha_0$, *we have* $c_L(\beta) \geqslant 0 > c_R(\alpha)$;

(ii) *when* $0 < \beta < \alpha_0$, *we have*

$$\begin{cases} c_L(\beta) < 0 < c_R(\alpha), \ \alpha \in (\beta, \alpha_0), \\ c_L(\beta) < 0 = c_R(\alpha), \ \alpha = \alpha_0, \\ c_L(\beta) < c_R(\alpha) < 0, \ \alpha \in (\alpha_0, \alpha^*(\beta)), \\ c_L(\beta) = c_R(\alpha) < 0, \ \alpha = \alpha^*(\beta), \\ c_R(\alpha) < c_L(\beta) < 0, \ \alpha \in (\alpha^*(\beta), \infty). \end{cases} \tag{3.7}$$

(II) Compactly supported travelling wave $V^*(x - c^*t)$, where c^* and V^* are given as below.

Lemma 3.7 *Assume* $0 < \beta < \alpha_0$ *and* $\beta < \alpha < \alpha^*$. *Then, there exits* $c^* \in \big(\max\{c_L, -\alpha\}, \min\{c_R, 0\}\big)$ *such that, only for* $c = c^*$, *the equation* (3.1) *and*

$$q(-L) = q(0) = 0, \ q'(-L) = -c + \beta, \ -q'(0) = c + \alpha, \ q(z) > 0 \text{ for } z \in (-L, 0), \tag{3.8}$$

has a solution, where $L = L(c)$ *is some positive constant. The solution is unique and denoted by* V^*. *We denote the width of the support of* V^* *by* L^*.

Proof By the phase plane analysis [14, section 3.2], we have

(1) for each $c \in (-c_0, c_0)$ and $b_R \in (0, P_R(c))$, the equation (3.1) has a unique solution $V_R(\cdot; c, b_R) \in C^2([-\widetilde{L}_2, L_1])$, with $V_R(-\widetilde{L}_2; c, b_R) = V_R(L_1; c, b_R) = 0$, $V_R'(L_1; c, b_R) =$

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$-b_R$, $V_R'(\cdot;c,b_R) > 0$ in $(-\widetilde{L}_2, 0)$ and $V_R'(\cdot;c,b_R) < 0$ in $(0, L_1)$ (from the last two inequalities we can deduce that $V_R'(0;c,b_R) = 0$);

(2) for each $c \in (-c_0, c_0)$ and $b_L \in (0, P_L(c))$, the equation (3.1) has a unique solution $V_L(\cdot;c,b_L) \in C^2([-L_2, \widetilde{L}_1])$, with $V_L(-L_2;c,b_L) = V_R(\widetilde{L}_1;c,b_L) = 0$, $V_L'(-L_2;c,b_L) = b_L$, $V_L'(\cdot;c,b_L) > 0$ in $(-L_2, 0)$ and $V_L'(\cdot;c,b_L) < 0$ in $(0, \widetilde{L}_1)$ (from the last two inequalities we can deduce that $V_L'(0;c,b_L) = 0$), where $L_1 = L_1(c)$, $\widetilde{L}_1 = \widetilde{L}_1(c)$, $L_2 = L_2(c)$ and $\widetilde{L}_2 = \widetilde{L}_2(c)$ are some positive constants.

For any $c \in \big(\max\{c_L, -\alpha\}, \min\{c_R, \beta\}\big)$ (this interval is well defined by Lemma 3.4 and Corollary 3.6), we have $c + \alpha \in (0, P_R(c))$ and $-c + \beta \in (0, P_L(c))$ (see Figure 1). Then, define

$$\widetilde{V}_R(z;c) := \big[V_R(z;c,c+\alpha)\big]\Big|_{z\in[0,L_1]} \quad \text{for} \quad c \in (-\alpha, c_R)$$

and

$$\widetilde{V}_L(z;c) := \big[V_L(z;c,-c+\beta)\big]\Big|_{z\in[-L_2,0]} \quad \text{for} \quad c \in (c_L, \beta).$$

$V_R(z;c,c+\alpha)$ and $V_L(z;c,-c+\beta)$ correspond to the points D and C in Figure 1, respectively. It follows from standard ODE theory that $\widetilde{V}_R(0;c)$ (resp. $\widetilde{V}_L(0;c)$) is continuous and strictly increasing (resp. decreasing) in c. Moreover, if $-\alpha \geqslant c_L$, then

$$\widetilde{V}_R(0;c) \to 0 \quad \text{as} \quad c \to -\alpha \quad \text{(point } D \text{ moves to point } (-\alpha, 0) \text{ in Figure 1)}$$

and

$$\widetilde{V}_L(0;c) \to m_L \in (0,1] \quad \text{as} \quad c \to -\alpha.$$

If $-\alpha < c_L$, then

$$\widetilde{V}_R(0;c) \to m_R \in (0,1) \quad \text{as} \quad c \to c_L$$

and

$$\widetilde{V}_L(0;c) \to 1 \quad \text{as} \quad c \to c_L \quad \text{(point } C \text{ moves to Point } B \text{ in Figure 1)}.$$

In summary,

$$\lim_{c\to\max\{c_L,-\alpha\}} \widetilde{V}_R(0;c) < \lim_{c\to\max\{c_L,-\alpha\}} \widetilde{V}_L(0;c). \tag{3.9}$$

Similarly,

$$\lim_{c\to\min\{c_R,\beta\}} \widetilde{V}_R(0;c) > \lim_{c\to\min\{c_R,\beta\}} \widetilde{V}_L(0;c). \tag{3.10}$$

Combining (3.9), (3.10) with the continuity and monotonicity of $\widetilde{V}_R(0;c)$ and $\widetilde{V}_L(0;c)$, one can derive that there exists a unique value $c^* \in \big(\max\{c_L, -\alpha\}, \min\{c_R, \beta\}\big)$ such that

$$\widetilde{V}_R(0;c^*) = \widetilde{V}_L(0;c^*) =: m^* \in (0,1),$$

which means that $V_R(z + L_1(c^*);c^*, c^* + \alpha)$ and $V_L(z + L_1(c^*);c^*, -c^* + \beta)$ are the same solution of (3.1) and (3.8) defined on $[-L_2(c^*) - L_1(c^*), 0]$ with $c = c^*$. We write $L^* := -L_2(c^*) - L_1(c^*)$ and $V^*(z) := V_R(z + L_1(c^*);c^*, c^* + \alpha)$.

Next, we prove that $c^* < 0$. Suppose $c^* \geqslant 0$. Multiply the equation $(V^*)'' + c^*(V^*)' +$

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$f(V^*) = 0$ by $(V^*)'$ and integrate in $(-L^*, 0)$. We have

$$\frac{(c^*+\alpha)^2}{2} - \frac{(c^*-\beta)^2}{2} + c^* \int_{-L^*}^{0} (V^*(x))^2 dx = 0,$$

that is,

$$2c^* + \alpha - \beta = -\frac{2c^*}{\alpha+\beta} \int_{-L^*}^{0} (V^*(x))^2 dx \leqslant 0,$$

which contradicts $2c^* + \alpha - \beta > 0$. Hence, $c^* \in \big(\max\{c_L, -\alpha\}, \min\{c_R, 0\}\big)$.

□

Corollary 3.8 *Assume $0 < \beta < \alpha_0$ and $\beta < \alpha < \alpha^*$. Then*

(i) *the following problem has a unique solution $q =: V_R^{\delta_1}$ (cf. point D in Figure 1) for any $\delta_1 \in (-\alpha - c^*, c_R - c^*)$:*

$$\begin{cases} q'' + (c^* + \delta_1)q' + f(q) = 0, & z \in (-l_1, 0), \\ q(-l_1) = q(0) = 0, \ -q'(0) = c^* + \delta_1 + \alpha, \ q(z) > 0 \ \textit{for} \ z \in (-l_1, 0), \end{cases}$$

where $l_1 = l_1(\delta_1)$ is some positive constant. We denote the width of the support of $V_R^{\delta_1}$ by $L_R^{\delta_1}$;

(ii) *the following problem exists a unique solution $q =: V_L^{\delta_2}$ (cf. point C in Figure 1) for any $\delta_2 \in (c^* - \beta, c^* - c_L)$:*

$$\begin{cases} q'' + (c^* - \delta_2)q' + f(q) = 0, & z \in (-l_2, 0), \\ q(-l_2) = q(0) = 0, \ q'(-l_2) = -c^* + \delta_2 + \beta, \ q(z) > 0 \ \textit{for} \ z \in (-l_2, 0), \end{cases}$$

where $l_2 = l_2(\delta_2)$ is some positive constant. We denote the width of the support of $V_L^{\delta_2}$ by $L_L^{\delta_2}$.

Here, we omit the proof. Actually, by the proof of Lemma 3.7, we can get

$$L_R^{\delta_1} = L_1(c^* + \delta_1) + \widetilde{L}_2(c^* + \delta_1), \quad L_L^{\delta_2} = \widetilde{L}_1(c^* - \delta_2) + L_2(c^* - \delta_2),$$

$$V_R^{\delta_1}(\cdot) = V_R(\cdot + L_1; c^* + \delta_1, c^* + \delta_1 + \alpha), \quad V_L^{\delta_2}(\cdot) = V_L(\cdot + \widetilde{L}_1; c^* - \delta_2, -c^* + \delta_2 + \beta)$$

(the definitions of L_1, $\widetilde{L}_1$, L_2, $\widetilde{L}_2$, V_R and V_L can be seen in the proof of Lemma 3.7).

By continuous dependence of the solution on the parameters, we have the following proposition.

Proposition 3.9 *Assume $0 < \beta < \alpha_0$ and $\beta < \alpha < \alpha^*$. Then*

$$V_R^{\delta}(\cdot) \to V^*(\cdot) \quad \textit{uniformly in } [-L^*, 0] \quad \textit{as } \delta \to 0,$$

and

$$V_L^{\delta}(\cdot) \to V^*(\cdot) \quad \textit{uniformly in } [-L^*, 0] \quad \textit{as } \delta \to 0.$$

3.3 Zero number arguments

For the reader's convenience, we next give two zero number arguments (see also [4,8,14]) which will be used to prove the main theorems and some lemmas. We use $Z_I(w)$ to denote the number of zeros of a continuous function $w(x)$ defined on $I \subset \mathbb{R}$.

Lemma 3.10 *Let $\xi_1(t) < \xi_2(t)$ be two continuous functions for $t \in (t_1, t_2)$. If $u(t,x)$ satisfies*

$$\begin{cases} u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u & \text{for } x \in (\xi_1(t), \xi_2(t)),\ t \in (t_1, t_2), \\ u(t, \xi_1(t)) \neq 0,\ u(t, \xi_2(t)) \neq 0 & \text{for } t \in (t_1, t_2), \end{cases} \tag{3.11}$$

where $a, 1/a, a_t, a_x, a_{xx}, b, b_t, b_x, c \in L^\infty$, then for each $t \in (t_1, t_2)$, the number of zeros of $u(t,\cdot)$ in $[\xi_1(t), \xi_2(t)]$ (which is denoted by $Z_{[\xi_1(t),\xi_2(t)]}[u(t,\cdot)]$) is finite. Moreover, $Z_{[\xi_1(t),\xi_2(t)]}[u(t,\cdot)]$ is non-increasing in t, and, if for some $s \in (t_1, t_2)$ the function $u(s,\cdot)$ has a degenerate zero $x_0 \in (\xi_1(s), \xi_2(s))$, then $Z_{[\xi_1(s_1),\xi_2(s_1)]}[u(s_1,\cdot)] > Z_{[\xi_1(s_2),\xi_2(s_2)]}[u(s_2,\cdot)]$ for all s_1, s_2 satisfying $t_1 < s_1 < s < s_2 < t_2$.

In the following proofs, we need to compare the the solution u of (1.1) with the compactly supported travelling wave V^* by studying the number of their intersection points. We set

$$l(t) := \max\{g(t), c^*t + z_0 - L^*\}, \quad r(t) := \min\{h(t), c^*t + z_0\},$$

$$L(t) := \min\{g(t), c^*t + z_0 - L^*\}, \quad R(t) := \max\{h(t), c^*t + z_0\},$$

$$\eta(t,x) := u(t,x) - V^*(x - c^*t - z_0) \quad \text{for } x \in J(t) := [l(t), r(t)],\ t \in (t_1, t_2)$$

(here we only consider the case $J(t) \neq \emptyset$ for $t \in (t_1, t_2)$, otherwise, u and V^* have no common domain and so there is no need to compare them), where z_0 is some constant. One can calculate that η satisfies

$$\eta_t = \eta_{xx} + c(t,x)\eta \quad \text{for } x \in (l(t), r(t)),\ t \in (t_1, t_2)$$

with $c(t,x) := [f(u(t,x)) - f(V^*(x - c^*t - z_0))]/\eta(t,x)$ when $\eta(t,x) \neq 0$, and $c(t,x) = 0$ otherwise.

Using similar arguments to the proofs of [9, Lemma 2.3] and [14, Lemma 3.10], one can get the following result on the number of zeros of $\eta(t,\cdot)$.

Lemma 3.11 *For any given $z_0 \in \mathbb{R}$, let $l(t)$, $L(t)$, $r(t)$, $R(t)$ and η be defined as above. Then*

(i) *$Z_{J(t)}[\eta(t,\cdot)]$ is finite and non-increasing in $t \in (t_1, t_2)$;*

(ii) *$Z_{J(\tau_1)}[\eta(\tau_1,\cdot)] \geqslant Z_{J(\tau_2)}[\eta(\tau_2,\cdot)] - 1$ when any one of the following conditions holds; $Z_{J(\tau_1)}[\eta(\tau_1,\cdot)] \geqslant Z_{J(\tau_2)}[\eta(\tau_2,\cdot)] - 2$ when any two of the following conditions hold; $Z_{J(\tau_1)}[\eta(\tau_1,\cdot)] \geqslant Z_{J(\tau_2)}[\eta(\tau_2,\cdot)] - 3$ when all of the following conditions hold for some $t_0 \in (t_1, t_2)$:*

  (1) *$l(t_0) = L(t_0)$;*

  (2) *$r(t_0) = R(t_0)$;*

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(3) *$\eta(t_0,\cdot)$ has a degenerate zero in the interior of $J(t_0)$,*

where τ_1 and τ_2 are any times satisfying $t_1 < \tau_1 < t_0 < \tau_2 < t_2$.

3.4 Convergence result

In this subsection, we present a locally uniformly convergence result. Before that, we give an estimate for $g(t) + h(t)$.

Lemma 3.12 *Suppose $0 < \beta < \alpha$. Let (u, g, h) be the solution of (1.1) defined on the maximal interval $[0, T)$, then*

$$g(t) + h(t) < 2h_0 \quad \textit{for all } t \in [0, T).$$

The proof of the above lemma is similar to the proof of [8, Lemma 2.8], so we omit it here. A direct consequence of Lemma 3.12 is $g_T + h_T \leqslant 2h_0$. So we have

Corollary 3.13 *There are only four possible situations for (g_T, h_T): (i) $g_T = h_T = -\infty$; (ii) $(g_T, h_T) = \mathbb{R}$; (iii) $-\infty = g_T < h_T < +\infty$; (iv) $-\infty < g_T \leqslant h_T < +\infty$.*

The stationary problem for (1.1) is written as

$$v'' + f(v) = 0, \quad \text{for } J \subset \mathbb{R}. \tag{3.12}$$

The solutions of (3.12) can be classified into the following categories:

(1) constant solutions: 0, 1;

(2) compactly supported solution v_γ when $\gamma \in (0, \alpha_0)$, where $(L, v) = (L_\gamma, v_\gamma)$ is the unique solution of

$$\begin{cases} v'' + f(v) = 0, \quad -L < x < 0, \\ v(0) = v(-L) = 0, \ v(x) > 0, \ x \in (-L, 0), \\ -v'(0) = v'(-L) = \gamma; \end{cases} \tag{3.13}$$

(3) the strictly decreasing solution q_0 with $-q_0'(0) = \alpha_0$, where q_0 is the solution of (1.7) with $c = 0$;

(4) the strictly increasing solution $\widetilde{q}_0$ with $\widetilde{q}_0'(0) = \alpha_0$, where $\widetilde{q}_0$ is the solution of (1.6) with $c = 0$.

Next, we consider the convergence of the time-global solution of (1.1) (i.e., $T = +\infty$).

Theorem 3.14 *Assume (u, g, h) is the time-global solution of (1.1), and $-\infty = g_\infty < h_\infty \leqslant +\infty$. Then $u(t,\cdot)$ converges to $v(\cdot - x_1)$ locally uniformly in (g_∞, h_∞), where x_1 is some constant and v is the solution of (3.12). Moreover,*

(i) *$v = 0$ or 1 when $0 < \beta < \alpha$ and $\alpha \neq \alpha_0$;*

(ii) *$v = 0$ or 1 or q_0 when $0 < \beta < \alpha = \alpha_0$. In particular, $v(x) = q_0(x)$ if $-\infty = g_\infty < h_\infty < +\infty$; in this situation $x_1 = h_\infty$.*

Sketch of the proof. Using a similar argument to that used in proving [8, Theorem 1.1], one can show that $u(t,\cdot)$ converges, as $t \to \infty$, to a stationary solution of $(1.1)_1$, that is, a solution v of $v'' + f(v) = 0$, locally uniformly in (g_∞, h_∞). Moreover, when $h_\infty < \infty$, making a change of the variable x to reduce $[g(t), h(t)]$ to the fixed finite interval $[-h_0, h_0]$ and applying the L^p estimates (as well as Sobolev embeddings) on the reduced equation with Dirichlet boundary conditions, we have

$$||u(t,\cdot) - v(\cdot)||_{C^{1+\nu}([h(t)-M, h(t)])} \to 0 \quad \text{as } t \to \infty,$$

for any $M > 0$ and $\nu \in (0,1)$. Hence, $v(h_\infty) = 0$ and $h'(t) = -u_x(t, h(t)) - \alpha \to -v_x(h_\infty) - \alpha$ as $t \to \infty$. On the other hand, $h'(t) \to 0$ when $h_\infty < +\infty$ since $h'(t)$ is Hölder continuous. Therefore, we have $v'(h_\infty) = -\alpha$ (cf. the proof of [4, Theorem 1.3]). So when $\alpha = \alpha_0$, v must be q_0 if $-\infty = g_\infty < h_\infty < +\infty$ since q_0 is the unique stationary solution defined on the half-line with $-q_0'(0) = \alpha_0$. When $\alpha \neq \alpha_0$, $h_\infty = +\infty$ since there is no stationary solution v defined on the half-line satisfying $-v'(0) = \alpha$.

So it follows by the categories of the solution of (3.12) that when $0 < \beta < \alpha$ and $\alpha \neq \alpha_0$, the only possible choice for the ω-limit of u in the topology of $L^\infty_{loc}((g_\infty, h_\infty))$ is 0 or 1; when $0 < \beta < \alpha = \alpha_0$, $\omega(u)$ consists of 0, 1 or $q_0(x - x_1)$. □

4 Vanishing and spreading phenomena

In this section, we give some necessary and sufficient conditions for vanishing and spreading phenomena.

4.1 Vanishing phenomena

We first give a sufficient condition for vanishing as follows.

Lemma 4.1 *Let $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, then u vanishes if $\|u_0\|_{L^\infty([-h_0,h_0])}$ is sufficiently small.*

Proof Consider the following problem:

$$\begin{cases} \overline{u}_t = \overline{u}_{xx} + f(\overline{u}), & \overline{g}(t) < x < \overline{h}(t),\ t > 0, \\ \overline{u}(t, \overline{g}(t)) = \overline{u}(t, \overline{h}(t)) = 0, & t > 0, \\ \overline{g}'(t) = -\overline{u}_x(t, \overline{g}(t)) + \beta, & t > 0, \\ \overline{h}'(t) = -\overline{u}_x(t, \overline{h}(t)) - \beta, & t > 0, \\ -\overline{g}(0) = \overline{h}(0) = h_0, \quad \overline{u}(0,x) = u_0(x), & -h_0 \leqslant x \leqslant h_0. \end{cases} \tag{4.1}$$

By [4, Proposition 5.2] one can show that vanishing happens for $\overline{u}(t,x)$ if the initial data $\|u_0\|_{L^\infty([-h_0,h_0])}$ is sufficiently small. On the other hand, by Lemma 3.1 we have $u(t,x) \leqslant \overline{u}(t,x)$ for $x \in [g(t), h(t)]$ and $t > 0$. Hence, vanishing happens for $u(t,x)$ if initial data $u_0(x)$ is sufficiently small. □

We now use (L_β, v_β) (which is the unique solution of (3.13) with $\gamma = \beta$) to give the following necessary and sufficient condition for vanishing.

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Lemma 4.2 *Assume $0 < \beta < \alpha_0$ and $\beta < \alpha < \alpha^*$. Let (u, g, h) be the solution of (1.1) on some maximal interval $[0, T)$ with $T \in (0, +\infty]$. Then, vanishing happens if and only if there exists $t_1 > 0$ and $x_1 \in \mathbb{R}$ such that*

$$u(t_1, x) \leqslant v_\beta(x - x_1) \quad \textit{for } x \in [g(t_1), h(t_1)] \subset [-L_\beta + x_1, x_1].$$

Proof By the strong maximum principle and Lemma 3.1, we have

$$u(t + t_1, x) < v_\beta(x - x_1) \text{ for } x \in [g(t + t_1), h(t + t_1)],\ t > 0, \tag{4.2}$$

and

$$-L_\beta + x_1 < g(t + t_1) < h(t + t_1) < x_1 \quad \text{for all } t > 0. \tag{4.3}$$

On the other hand, by Corollary 3.8, the following problem

$$\begin{cases} q'' + cq' + f(q) = 0, & z \in (-L, 0), \\ q(-L) = q(0) = 0,\ q'(-L) = -c + \beta,\ q(z) > 0,\ z \in (-L, 0) \end{cases} \tag{4.4}$$

has a unique solution (L_L^δ, V_L^δ) with $\delta := c^* - c$ for $c \in (c_L, \beta)$. Since V_L^δ depends on c continuously, we can choose a sufficiently small $c > 0$ such that

$$-L_L^\delta + x_1 < g(1 + t_1) < h(1 + t_1) < x_1 \tag{4.5}$$

and

$$u(1 + t_1, x) < V_L^\delta(x - x_1) \text{ for } x \in [g(1 + t_1), h(1 + t_1)]. \tag{4.6}$$

Write

$$\overline{u}(t, x) := V_L^\delta(x - ct - x_1),\ \overline{h}(t) := ct + x_1,\ \overline{g}(t) := ct - L_L^\delta + x_1,\ t > 0.$$

To compare $\overline{u}(t, \cdot)$ and $u(t + 1 + t_1, \cdot)$ on $[g(t + 1 + t_1), h(t + 1 + t_1)]$, we check that $(\overline{u}, \overline{g}, \overline{h})$ is an upper solution of (1.1) for $t > 0$. Clearly,

$$\overline{g}'(t) = c = -\overline{u}_x(t, \overline{g}(t)) + \beta. \tag{4.7}$$

Moreover, $\overline{u}(t, x)$ satisfies $(1.1)_1$ and $\overline{u}(t, \overline{g}(t)) = 0$. By (4.3), the definition of $\overline{h}(t)$ and $c > 0$, we have $h(t + 1 + t_1) < x_1 < \overline{h}(t)$, hence

$$\overline{u}(t, h(t + 1 + t_1)) > 0 = u(t + 1 + t_1, h(t + 1 + t_1)), \tag{4.8}$$

and it follows from the definition of $\overline{g}(t)$ and (4.5) that

$$\overline{g}(0) = -L_L^\delta + x_1 < g(1 + t_1). \tag{4.9}$$

Hence, by (4.6)–(4.9), one can show that $(\overline{u}, \overline{g}, \overline{h})$ is an upper solution of (1.1) for $t > 0$. By Lemma 3.2, for any $t > 0$, we have

$$u(t + 1 + t_1, x) < V_L^\delta(x - ct - x_1) \quad \text{for } x \in [g(t + 1 + t_1), h(t + 1 + t_1)], \tag{4.10}$$

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and

$$\overline{g}(t) < g(t+1+t_1) \text{ and } h(t+1+t_1) < \overline{h}(t) \text{ for all } t > 0. \tag{4.11}$$

Thus, by the definition of $\overline{g}(t)$, (4.3) and (4.11), we obtain

$$0 < h(t+1+t_1) - g(t+1+t_1) < x_1 - (ct - L_L^\delta + x_1). \tag{4.12}$$

We conclude from (4.12) that there is a $0 < T_0 \leqslant \frac{L_L^\delta}{c}$ such that

$$h(t+1+t_1) - g(t+1+t_1) \to 0 \text{ as } t \to T_0. \tag{4.13}$$

Hence

$$\lim_{t\to T}[h(t) - g(t)] = 0, \tag{4.14}$$

where $T := T_0 + t_1 + 1$.

On the other hand, by (1.2), one can show that there is a constant $C > 0$ such that $u(t,x) \leqslant C$ for $t \in [0,T)$ and $x \in [g(t), h(t)]$. Construct a function

$$U(t,x) = C[2M(h(t)-x) - M^2(h(t)-x)^2]$$

over the region $Q := \{(t,x) : t > 0, \max\{h(t) - M^{-1}, g(t)\} < x < h(t)\}$. When $M > 0$ is large, $U(t,x)$ is an upper solution of (1.1) on $\overline{Q}$. Since $\lim_{t\to T}[h(t) - g(t)] = 0$, there is $t_0 \in (0,T)$ such that $g(t) > h(t) - M^{-1}$. Hence, $u(t,x) \leqslant U(t,x)$ for $g(t) < x < h(t)$ and $t_0 < t < T$. According to $U(t,x) \to 0$ $(x \to h(t))$ and (4.14), we have

$$\lim_{t\to T} \max_{g(t)\leqslant x\leqslant h(t)} u(t,x) \leqslant \lim_{t\to T} \max_{g(t)\leqslant x\leqslant h(t)} U(t,x) = 0.$$

□

Next, we give another necessary and sufficient condition for vanishing in the following lemma.

Lemma 4.3 *Assume that* $0 < \beta < \alpha_0$ *and* $0 < \alpha < \alpha^*$. *Let* (u, g, h) *be a solution of* (1.1) *on* $[0,T)$. *Then, vanishing happens if and only if there exist* $t_2 < T$ *and* $x_2 \in \mathbb{R}$ *such that*

$$u(t_2, x) \leqslant (\not\equiv) V^*(x - x_2) \textit{ for } x \in [g(t_2), h(t_2)]. \tag{4.15}$$

This lemma can be proved by similar arguments to those for Lemma 4.2. By the strong maximum principle and (4.15) we have

$$u(t+t_2, x) < V^*(x - c^*t - x_2) \quad \text{for } x \in [g(t+t_2), h(t+t_2)],\ t > 0.$$

On the other hand, by Proposition 3.9, there is a small $\delta_1 < 0$ such that

$$u(1+t_2, x) < V_L^{\delta_1}(x - c^* - x_2) \quad \text{for } x \in [g(1+t_2), h(1+t_2)].$$

Then, replacing v_β and V_L^δ by V^* and $V_L^{\delta_1}$, respectively, in Lemma 4.2, we can prove the above lemma holds.

Remark 4.4 *When $0 < \beta < \alpha_0$ and $\alpha \geqslant \alpha^*$, or $\beta \geqslant \alpha_0$, only vanishing happens for any solution of* (1.1)*, this will be proved in Section 5 (i.e., the proof of Theorem 2.5).*

To prove vanishing happens in Lemma 4.2 and Lemma 4.3, we first prove that $h(t) - g(t) \to 0$ (see (4.14)), then use this to prove $u \to 0$. In fact, the following three arguments are equivalent, as can be proved as in [4, Theorem 1.2].

Lemma 4.5 *Let (u, g, h) be the solution of* (1.1) *in some maximal interval $[0, T)$ with $T \in (0, +\infty]$. Then, the following arguments are equivalent.*

(i) $\lim_{t\to T} \max_{g(t)<x<h(t)} u(t,x) = 0$;

(ii) $T < +\infty$;

(iii) $\lim_{t\to T} h(t) = \lim_{t\to T} g(t)$.

Proof For the reader's convenience, we give a sketch of the proof. First, the conclusion of (i) implies (ii) and (iii). Consider the problem (3.13) with $\gamma < \min\{\beta, \alpha_0\}$ and take its solution $v_\gamma(x)$ on some short interval $[-X, 0]$. When (i) holds, there exists a large time T_0 such that

$$u(t,x) \leqslant m := v_\gamma(-X),\ x \in [g(t), h(t)],\ t > T_0.$$

Choose a large $b_1 > 0$, then the function $v_\gamma(x - b_1)$ is an upper solution of the problem (1.1) on $[-X + b_1, b_1]$ and it blocks the extension of $h(t)$. Therefore, $h(t) < b_1$ for all $t > T_0$. Similarly, there is a $b_2 \in \mathbb{R}$ such that $g(t) > b_2$ for all $t > T_0$.

Set $L := 2(1 + b_1 + |b_2|)$ and

$$\eta_0(x) := \frac{2\varepsilon}{L^2}(L^2 - x^2),\ x \in [-L, L],$$

where $K_1 := \max_{0\leqslant u\leqslant 1} |f'(u)|$, and $\varepsilon > 0$ is small such that

$$8(\alpha + \sqrt{\alpha^2 + 4K_1})\varepsilon \leqslant \alpha, \quad 32\varepsilon \leqslant \alpha.$$

Consider the following problem:

$$\begin{cases} \eta_t = \eta_{xx} + \bar{f}(\eta), & \bar{g}(t) < x < \bar{h}(t),\ t > 0, \\ \eta(t, \bar{g}(t)) = \eta(t, \bar{h}(t)) = 0, & t > 0, \\ \bar{g}'(t) = -\eta_x(t, \bar{g}(t)) + \beta, & t > 0, \\ \bar{h}'(t) = -\eta_x(t, \bar{h}(t)) - \alpha, & t > 0, \\ -\bar{g}(0) = \bar{h}(0) = L, \quad \eta(0,x) = \eta_0(x), & -L \leqslant x \leqslant L, \end{cases} \tag{4.16}$$

where

$$\bar{f}(\eta) := 2K_1\eta\left(1 - \frac{\eta}{2\varepsilon}\right) \quad (\geqslant f(\eta) \text{ for } 0 \leqslant \eta \leqslant \varepsilon\ ).$$

Clearly, $\eta(t,x) \leqslant 2\varepsilon$ for $t \geqslant 0$. By the definition of $\bar{f}$ and η_0, we see that

$$U^\varepsilon(t,x) := 2\varepsilon[2M(\bar{h}(t) - x) - M^2(\bar{h}(t) - x)^2]$$

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is an upper solution of (4.16) over $\overline{Q} := \{(t,x) : t > 0, \max\{\overline{g}(t), \overline{h}(t) - M^{-1}\} \leqslant x \leqslant \overline{h}(t)\}$ with $M := \max\{\alpha + \sqrt{\alpha^2 + 4K_1}, 4\}$. Hence,

$$-\eta_x(t, \overline{h}(t)) \leqslant -U_x^{\varepsilon}(t, \overline{h}(t)) \leqslant 4M\varepsilon \leqslant \frac{\alpha}{2}.$$

Therefore, $\overline{h}'(t) \leqslant -\frac{\alpha}{2}$. Similarly, $\overline{g}'(t) \geqslant \frac{\beta}{2}$. Thus, $\overline{h}(t) - \overline{g}(t) \to 0$ as $t \to T_1 \leqslant \frac{4L}{\alpha+\beta}$. On the other hand, by $\lim_{t\to T} \max_{g(t)<x<h(t)} u(t,x) = 0$, there is a large time $T^* > 0$ such that $u(t,x) \leqslant \varepsilon$ for $t > T^*$, then $\eta(t,x)$ is an upper solution of (1.1) for $t > T^*$. Hence, $h(t+T^*) - g(t+T^*) \leqslant \overline{h}(t) - \overline{g}(t) \to 0$ as $t \to T_1$, which implies $T < +\infty$ and (iii) holds.

Secondly, (ii) implies (iii). Assume on the contrary that $\inf_{0<t<T}[h(t) - g(t)] > 0$, then by standard L^p estimates, the Sobolev embedding theorem and the Hölder estimates for parabolic equations, we can extend the solution to some interval $(0, \overline{T})$ with $\overline{T} > T$ as long as $\inf_{0<t<T}[h(t) - g(t)] > 0$, this is a contradiction since $(0, T)$ is the maximal existence interval of the solution $u(t,x)$.

Thirdly, (iii) implies (i). Using the same arguments as in the last proof of Lemma 4.2, one can show that (iii) implies (i). □

4.2 Spreading phenomena

To give some necessary and sufficient conditions for spreading, we first prove the following lemma for $0 < \alpha < \alpha^*$.

Lemma 4.6 *Assume $0 < \beta < \alpha_0$ and $0 < \alpha < \alpha^*$. Let (u, g, h) be a solution of (1.1). Suppose that there exist t^* and $x^* \in \mathbb{R}$ such that*

$$u(t^*, x) \geqslant (\not\equiv) V^*(x - x^*) \textit{ for } x \in [-L^* + x^*, x^*]. \tag{4.17}$$

Then

$$\lim_{t\to\infty} u(t, x + c^* t) = 1 \quad \textit{locally uniformly in } \mathbb{R} \tag{4.18}$$

and

$$g_\infty = -\infty, \quad \lim_{t\to\infty} [h(t) - g(t)] = +\infty. \tag{4.19}$$

Proof By Lemma 3.7, we know that $V^*(x - c^*t - x^*)$ satisfies $(1.1)_1$ and free boundary conditions at $x = l(t) := c^*t + x^* - L^*$ and $x = r(t) := c^*t + x^*$. By the strong maximum principle, we have

$$u(t + t^*, x) > V^*(x - c^*t - x^*) \text{ for } x \in [l(t), r(t)],\ t > 0 \tag{4.20}$$

and

$$g(t + t^*) < l(t), \quad h(t + t^*) > r(t) \quad \text{for all } t > 0. \tag{4.21}$$

In particular, (4.20) is true at $t = 1$. We now find a compactly supported travelling wave moving leftward slower than V^* to be a lower solution. By Proposition 3.9, there exists a sufficiently small $\delta_1 > 0$ such that $V_R^{\delta_1}$ is also below $u(t^* + 1, x)$ and the left endpoints of

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the compact support of $V_R^{\delta_1}$ and V^* are the same point $x = x^* + c^* - L^*$, that is,

$$u(t^*+1, x) > V_R^{\delta_1}\left(x - x^* - c^* + L^* - L_R^{\delta_1}\right) \text{ for } x \in \left[x^* + c^* - L^*, x^* + c^* - L^* + L_R^{\delta_1}\right] \tag{4.22}$$

and

$$\left[x^* + c^* - L^*, x^* + c^* - L^* + L_R^{\delta_1}\right] \subset (g(t^*+1), h(t^*+1)). \tag{4.23}$$

By (4.21), $c^* < 0$ (cf. Lemma 3.7) and $c^* + \delta_1 > c^*$, we have, for all $t > 0$,

$$g(t + t^* + 1) < l(t+1) = c^*(t+1) + x^* - L^* < (c^* + \delta_1)t + c^* + x^* - L^*. \tag{4.24}$$

Define $\underline{g}(t) := (c^* + \delta_1)t + x^* + c^* - L^*$, $\underline{h}(t) := (c^* + \delta_1)t + x^* + c^* - L^* + L_R^{\delta_1}$ and

$$\underline{u}(t,x) := V_R^{\delta_1}(x - (c^* + \delta_1)t - x^* - c^* + L^* - L_R^{\delta_1}) \quad \text{for } x \in [\underline{g}(t), \underline{h}(t)],\ t > 0.$$

Then, it follows from (4.23) and (4.24) that

$$\underline{h}(0) < h(t^*+1) \quad \text{and} \quad g(t + t^* + 1) < \underline{g}(t) \quad \text{for all } t \geqslant 0. \tag{4.25}$$

Note that $V_R^{\delta_1}(x - (c^* + \delta_1)t - x^* - c^* + L^* - L_R^{\delta_1})$ satisfies $(1.1)_1$ and the free boundary condition at $x = \underline{h}(t)$. Combining these, (4.22) with (4.24), one can check that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of the problem (1.1). Hence, by Lemma 3.2, we have

$$h(t + t^* + 1) > \underline{h}(t) \quad \text{for all } t > 0 \tag{4.26}$$

and

$$u(t + t^* + 1, x) > V_R^{\delta_1}(x - (c^* + \delta_1)t - x^* - c^* + L^* - L_R^{\delta_1}) \tag{4.27}$$

for $x \in [\underline{g}(t), \underline{h}(t)]$ and $t > 0$.

We next choose a compactly supported travelling wave moving leftwards faster than V^* to be also a lower solution. A similar discussion shows that there is a sufficiently small $\delta_2 > 0$ such that $(V_L^{\delta_2}(x - (c^* - \delta_2)t - x^* - c^*), \underline{g}_1(t), \underline{h}_1(t))$ is also a lower solution for $x \in [\underline{g}_1(t), \underline{h}_1(t)]$ and $t > 0$, where $\underline{g}_1(t) := (c^* - \delta_2)t + x^* + c^* - L_L^{\delta_2}$, $\underline{h}_1(t) := (c^* - \delta_2)t + x^* + c^*$. Hence,

$$g(t + t^* + 1) < \underline{g}_1(t),\ t > 0. \tag{4.28}$$

Consequently, combining (4.26) with (4.28), we have

$$H(t) := h(t + t^* + 1) - c^* t \geqslant \delta_1 t + c^* + x^* - L^* + L_R^{\delta_1} \to +\infty \text{ as } t \to \infty \tag{4.29}$$

and

$$G(t) := g(t + t^* + 1) - c^* t \leqslant -\delta_2 t + c^* + x^* - L_L^{\delta_2} \to -\infty \text{ as } t \to \infty. \tag{4.30}$$

Set

$$w(t,x) := u(t + t^* + 1, x + c^* t) \text{ for } G(t) \leqslant x \leqslant H(t),\ t \geqslant 0.$$

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It follows from (4.20) that w satisfies

$$w(t,x) > V^*(x - c^* - x^*) \text{ for } x \in [c^* + x^* - L^*, c^* + x^*],\ t \geqslant 0 \tag{4.31}$$

and

$$\begin{cases} w_t = w_{xx} + c^* w_x + f(w), \quad G(t) < x < H(t),\ t > 0, \\ w(t, G(t)) = 0,\ G'(t) = -w_x(t, G(t)) + \beta - c^*, \quad t > 0, \\ w(t, H(t)) = 0, H'(t) = -w_x(t, H(t)) - \alpha - c^*, \quad t > 0, \\ G(0) = g(t^* + 1),\ H(0) = h(t^* + 1),\ w(0, x) = u(t^* + 1, x),\ G(0) \leqslant x \leqslant H(0). \end{cases} \tag{4.32}$$

Using similar arguments to those applied in proving [14, Theorem 4.2] and [4, Theorem 1.3], one can show that $w(t, \cdot)$ converges, as $t \to \infty$, to a stationary solution of $(4.32)_1$, locally uniformly in $\mathbb{R}$. On the other hand, by Lemma 3.4 and Lemma 3.7, we have $c^* \in (-c_0, c_0)$. Hence, such a stationary solution must be 1 by (4.31). We conclude from this that spreading happens for w, this implies that (4.18) holds. Moreover, (4.19) follows from (4.26) and (4.28). □

Now, we are ready to give a necessary and sufficient condition for spreading.

Lemma 4.7 *Assume* $0 < \beta < \alpha_0$ *and* $0 < \alpha < \alpha^*$. *Let* (u, g, h) *be a solution of* (1.1). *Then, spreading happens if and only if there exist* t^* *and* $x^* \in \mathbb{R}$ *such that* (4.17) *holds.*

Proof The inequality (4.17) follows from the definition of spreading (cf. Theorem 2.1) immediately. We only need to prove that (4.17) is a sufficient condition for spreading.

(1) The case $0 < \alpha < \alpha_0$. We show that $h_\infty = +\infty$, $g_\infty = -\infty$ and (2.1) holds. It follows from (4.19) that $g_\infty = -\infty$, we now prove that $h_\infty = +\infty$. It follows from (3.6) that $c_R > 0$ when $0 < \beta < \alpha < \alpha_0$. Combining this and Corollary 3.8, there exists a constant $\delta \in (-\alpha - c^*, c_R - c^*)$ such that $c^* + \delta > 0$. By (4.18) and the definition of V_R^δ, there exist large $T_0 > 0$ and $x_1 \in \mathbb{R}$ such that

$$[-L_R^\delta + x_1, x_1] \subset [g(T_0) - c^* T_0, h(T_0) - c^* T_0],\ u(T_0, \cdot + c^* T_0) > V_R^\delta(\cdot - x_1) \text{ in } [-L_R^\delta + x_1, x_1], \tag{4.33}$$

that is,

$$[c^* T_0 + x_1 - L_R^\delta, c^* T_0 + x_1] \subset [g(T_0), h(T_0)],\ u(T_0, \cdot) > V_R^\delta(\cdot - c^* T_0 - x_1) \text{ in } [c^* T_0 + x_1 - L_R^\delta, c^* T_0 + x_1].$$

By (4.30), $g_\infty = -\infty$ and $c^* + \delta > 0$, we can choose the above T_0 large such that

$$g(t + T_0) < (c^* + \delta)t + c^* T_0 + x_1 - L_R^\delta \quad \text{for all } \ t > 0. \tag{4.34}$$

Set

$$\underline{g}(t) := (c^* + \delta)t + c^* T_0 + x_1 - L_R^\delta, \quad \underline{h}(t) := (c^* + \delta)t + c^* T_0 + x_1$$

and

$$\underline{u}(t, x) := V_R^\delta(x - (c^* + \delta)t - c^* T_0 - x_1) \quad \text{for } x \in [\underline{g}(t), \underline{h}(t)],\ t > 0.$$

Then, $\underline{u}$ satisfies the free boundary condition at $\underline{h}(t)$. Moreover, (4.34) implies that $g(t+T_0) < \underline{g}(t)$ for all $t > 0$. Using the definition of the lower solution, one can show that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of (1.1) in the domain $[\underline{g}(t), \underline{h}(t)]$ for $t > 0$. By Lemma 3.2, we have

$$u(t+T_0, x) > \underline{u}(t, x) \text{ for } x \in [\underline{g}(t), \underline{h}(t)],\ t > 0.$$

Hence

$$h(t+T_0) > \underline{h}(t) \quad \text{for all} \quad t > 0. \tag{4.35}$$

Since $c^* + \delta > 0$, (4.35) and the definition of $\underline{h}(t)$ imply that $h_\infty = +\infty$. On the other hand, (4.18) implies that there are a $\overline{T}_0 > 0$ and $x_2 \in \mathbb{R}$ such that

$$(-L_\alpha + x_2, x_2) \subset [g(\overline{T}_0) - c^*\overline{T}_0, h(\overline{T}_0) - c^*\overline{T}_0],\ u(\overline{T}_0, x + c^*\overline{T}_0) > v_\alpha(x - x_2)$$
$$\text{for } x \in (-L_\alpha + x_2, x_2),$$

where v_α is the unique solution of (3.13) with $\gamma = \alpha$. Then, by $\beta < \alpha$ and the comparison principle Lemma 3.1 one can get that, for all $t > 0$,

$$(-L_\alpha + x_2, x_2) \subset [g(t+\overline{T}_0) - c^*\overline{T}_0, h(t+\overline{T}_0) - c^*\overline{T}_0],\ u(t+\overline{T}_0, \cdot + c^*\overline{T}_0) > v_\alpha(\cdot - x_2)$$
$$\text{in } (-L_\alpha + x_2, x_2), \tag{4.36}$$

that is,

$$u(t+\overline{T}_0, x) > v_\alpha(x - c^*\overline{T}_0 - x_2) \quad \text{for all } x \in (c^*\overline{T}_0 - L_\alpha + x_2, c^*\overline{T}_0 + x_2),\ t > 0. \tag{4.37}$$

On the other hand, by Theorem 3.14 and $(g_\infty, h_\infty) = \mathbb{R}$, u converges to the stationary solution of $(1.1)_1$ on $\mathbb{R}$, i.e., 0 or 1. Combining this with (4.37), we derive that u converges to 1, that is,

$$\lim_{t\to\infty} u(t, \cdot) = 1 \quad \text{locally uniformly in } \mathbb{R}.$$

(2) The case $\alpha_0 < \alpha < \alpha^*$ and $\beta \in (0, \alpha_0)$. By Lemma 4.6, we only need to prove that $h_\infty = -\infty$ and (2.2) holds. Set $A := 2\max\{1, ||u_0||_{L^\infty([-h_0,h_0])}\}$. For some $\delta \in (0, -f'(1))$, define

$$\overline{g}(t) := g(t),\ \overline{h}(t) := c_R t - Me^{-\delta t} + H,\ \overline{u}(t, x) := (1 + Ae^{-\delta t})q_R(x - \overline{h}(t)),$$

for $\overline{g}(t) < x < \overline{h}(t)$, $t > 0$, where q_R is the travelling semi-wave (the solution of (1.7)). Then, a direct calculation as in the proof of [11, Lemma 3.2] shows that $(\overline{u}, \overline{g}, \overline{h})$ is an upper solution of (1.1) provided H, M are large. Therefore, we have

$$u(t, x) < \overline{u}(t, x) \ \text{ for } x \in [g(t), h(t)],\ t > 0$$

and

$$h(t) < \overline{h}(t) \quad \text{for large } t > 0. \tag{4.38}$$

On the other hand, by Lemma 4.6 we see that vanishing cannot happen when (4.17) holds. Hence (4.38) and $c_R < 0$ imply that $h_\infty = -\infty$ and $g_\infty = -\infty$. Next, we prove that, for any $c \in (c_L, c_R)$, (2.2) holds. We only consider the case $c \in (c^*, c_R)$ since the other case

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$c \in (c_L, c^*)$ can be considered similarly (when $c \in (c_L, c^*)$, one can use $V_L^{\delta_2}$ and $V_L^{\delta_2+\varepsilon}$ to replace $V_R^{\delta_1}$ and $V_R^{\delta_1+\varepsilon}$, respectively).
By (4.29) and (4.30), we have

$$h(t) - c^*t \to +\infty, \; g(t) - c^*t \to -\infty \quad \text{as } t \to \infty. \tag{4.39}$$

Combining these and (4.18), for any $c \in (c^*, c_R)$, writing $\delta_1 := c - c^* > 0$, there exist T_1, X_1 and B_1 such that

$$[-L_R^{\delta_1}+X_1, X_1] \subset [g(T_1)-c^*T_1, h(T_1)-c^*T_1], \; u(T_1, \cdot+c^*T_1) > V_R^{\delta_1}(\cdot-X_1) \text{ in } [-L_R^{\delta_1}+X_1, X_1] \tag{4.40}$$

and

$$[-L^*+B_1, B_1] \subset [g(T_1)-c^*T_1, h(T_1)-c^*T_1], \; u(T_1, \cdot+c^*T_1) > V^*(\cdot-B_1) \text{ in } [-L^*+B_1, B_1] \tag{4.41}$$

with $-L^* + B_1 < -L_R^{\delta_1} + X_1$, then $c^*t - L^* + B_1 < ct - L_R^{\delta_1} + X_1$. By the comparison principle Lemma 3.1, we have, for all $t \geqslant 0$,

$$u(t + T_1, x + c^*T_1) > V^*(x - c^*t - B_1) \quad \text{for } \; x \in [c^*t - L^* + B_1, c^*t + B_1] \tag{4.42}$$

and $[c^*t - L^* + B_1, c^*t + B_1] \subset [g(t + T_1) - c^*T_1, h(t + T_1) - c^*T_1]$. Hence

$$g(t + T_1) - c^*T_1 < c^*t - L^* + B_1 < ct - L_R^{\delta_1} + X_1, \; t > 0. \tag{4.43}$$

Define a lower solution of the problem (1.1) for $t > 0$ as follows:

$$\underline{u}(t, x) := V_R^{\delta_1}(x - ct - X_1), \; \underline{g}(t) := ct + X_1 - L_R^{\delta_1}, \; \underline{h}(t) := ct + X_1.$$

Then, (4.43) implies that $g(t+T_1)-c^*T_1 < \underline{g}(t)$ for all $t > 0$. By (4.40) and the comparison principle Lemma 3.1, we have

$$h(t+T_1)-c^*T_1 > \underline{h}(t), \; u(t+T_1, x+c^*T_1) > V_R^{\delta_1}(x-ct-X_1) \text{ for } x \in [\underline{g}(t), \underline{h}(t)], \; t > 0. \tag{4.44}$$

Moreover, by (4.40) and the continuous dependence of the solution on the parameters, there exists a sufficiently small $\varepsilon > 0$ such that $c + \varepsilon \in (c^*, c_R)$ and

$$u(T_1, x+c^*T_1) > V_R^{\delta_1+\varepsilon}(x - X_1 + L_R^{\delta_1} - L_R^{\delta_1+\varepsilon}) \quad \text{for } x \in [X_1 - L_R^{\delta_1}, X_1 - L_R^{\delta_1} + L_R^{\delta_1+\varepsilon}] \tag{4.45}$$

with $[X_1 - L_R^{\delta_1}, X_1 - L_R^{\delta_1} + L_R^{\delta_1+\varepsilon}] \subset [g(T_1) - c^*T_1, h(T_1) - c^*T_1]$. Define

$$\underline{g}_1(t) := (c + \varepsilon)t + X_1 - L_R^{\delta_1}, \; \underline{h}_1(t) := (c + \varepsilon)t + X_1 - L_R^{\delta_1} + L_R^{\delta_1+\varepsilon}$$

and

$$\underline{u}_1(t) := V_R^{\delta_1+\varepsilon}(x - (c + \varepsilon)t - X_1 + L_R^{\delta_1} - L_R^{\delta_1+\varepsilon}) \quad \text{for } x \in [\underline{g}_1(t), \underline{h}_1(t)], \; t > 0. \tag{4.46}$$

By $\underline{g}(t) < \underline{g}_1(t)$ and the comparison principle Lemma 3.2, we have

$$h(t + T_1) - c^*T_1 > \underline{h}_1(t) \quad \text{for } t > 0. \tag{4.47}$$

Now set

$w(t,x) := u(t+T_1, x+c(t+T_1)),\ H(t) := h(t+T_1) - c(t+T_1),\ G(t) := g(t+T_1) - c(t+T_1).$

Then it follows from (4.43), (4.47) and $c^* < c$ that $G(t) \to -\infty$ and $H(t) \to +\infty$ as $t \to \infty$. Combining this with (4.44), and using the similar arguments to those in the last part of the proof of Lemma 4.6, one can prove that

$$\lim_{t\to\infty} w(t,\cdot) = 1 \quad \text{locally uniformly in } \mathbb{R},$$

that is,

$$\lim_{t\to\infty} u(t,\cdot + ct) = 1 \quad \text{locally uniformly in } \mathbb{R}.$$

(3) The case $\alpha = \alpha_0$. By Lemma 4.6, it suffices to prove that $-\infty < h_\infty < +\infty$ and (2.3) holds.

By (4.18) and [18, Proposition A], there exists $\delta \in (0, -f'(1))$, $M > 0$ and $T > 0$ such that

$$u(t, x + c^* t) \geqslant 1 - Me^{-\delta t},\ x \in [-l, l] \subset (g(t) - c^* t, h(t) - c^* t),\ t \geqslant T \tag{4.48}$$

for any fixed $l > 0$, and

$$u(t, x + c^* t) \leqslant 1 + Me^{-\delta t},\ x \in [g(t) - c^* t, h(t) - c^* t],\ t \geqslant T.$$

Define

$$\underline{g}(t) := c^* t, \quad \underline{h}(t) := \sigma e^{-\delta t} + c^* T,$$
$$\underline{u}(t,x) = (1 - Me^{-\delta t}) q_0(x - \underline{h}(t)),$$

where σ is some positive constant to be determined later, and q_0 is the solution of (1.7) with $c = 0$. Using the similar arguments to those in [11, Lemma 3.3], one can derive that

$$\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \leqslant 0 \quad \text{for } x \in (\underline{g}(t), \underline{h}(t)),\ t > T, \tag{4.49}$$

when T and $\sigma > 0$ (independent of T) are sufficiently large.

Next, we choose $l > \sigma$ in (4.48), then for sufficiently large $T > 0$, we have

$$u(t, x + c^* t) \geqslant 1 - Me^{-\delta t} \quad \text{for } x \in [0, \sigma e^{-\delta t}] \subset [-l, l],\ t \geqslant T.$$

In particular, when $t = T$,

$$u(T, x) \geqslant 1 - Me^{-\delta T} \quad \text{for } x \in [\underline{g}(T), \underline{h}(T)].$$

Moreover, by the definition of $\underline{g}(t)$, we have, for $t > T$,

$$g(t) < \underline{g}(t),\ \underline{u}(t, \underline{g}(t)) = (1 - Me^{-\delta t}) q_0(\underline{g}(t) - \underline{h}(t)) < 1 - Me^{-\delta t} \leqslant u(t, \underline{g}(t)). \tag{4.50}$$

It is obvious that $\underline{u}(t, \underline{h}(t)) = 0$, and a direct calculation shows that

$$\underline{h}'(t) \leqslant -\underline{u}_x(t, \underline{h}(t)) - \alpha \quad \text{for } t > T, \tag{4.51}$$

provided $\sigma > \frac{M\alpha_0}{\delta}$.

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Consequently, $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of (1.1). Therefore, $h(t) \geqslant \underline{h}(t)$ for $t > T$, and this implies that

$$h_\infty > -\infty \quad \text{and} \quad \liminf_{t\to\infty} u(t,x) \geqslant q_0(x - c^*T). \tag{4.52}$$

Next, we show that $h_\infty < +\infty$. We need to construct an upper solution. Define, for some $\sigma_1 > 0$ and $H_0 > 0$,

$$\overline{g}(t) := g(t),\ \overline{h}(t) := H_0 + \sigma_1 M(1 - e^{-\delta t}),\ \overline{u}(t,x) := (1 + Me^{-\delta t})q_0(x - \overline{h}(t)).$$

One can calculate directly as in [11, Lemma 3.2] to prove that, when σ_1 and H_0 are large, $(\overline{u}, \overline{g}, \overline{h})$ is an upper solution of (1.1) for large t. Therefore, $h(t) \leqslant \overline{h}(t)$ for large t, which implies that $h_\infty < +\infty$. Hence, by Theorem 3.14, we have

$$\lim_{t\to\infty} u(t,x) = q_0(x - h_\infty) \quad \text{locally uniformly in } (-\infty, h_\infty].$$

□

5 Proof of main theorems

In this section, we will study the influence of α, β on the asymptotic behaviour of solutions and prove the main theorems.

For any given $h_0 > 0$ and $\phi \in \mathscr{X}(h_0)$, we write the solution (u, g, h) of (1.1) also as $(u(t,x;\sigma\phi), g(t;\sigma\phi), h(t;\sigma\phi))$ to emphasize the dependence on the initial data $u_0 = \sigma\phi$. Set

$$\sigma_* = \sigma_*(h_0, \phi) := \sup\{\sigma \geqslant 0 :\ \text{vanishing happens for } (u, g, h)\}$$

and

$$\sigma^* = \sigma^*(h_0, \phi) := \inf\{\sigma > 0 :\ \text{spreading happens for } (u, g, h)\}.$$

By the comparison principle, we have $\sigma_* \leqslant \sigma^* \leqslant +\infty$. Moreover, by Lemma 4.1, the solution $u(t,x;\sigma\phi)$ vanishes provided $\sigma > 0$ is small. Therefore, $\sigma_* \in (0, +\infty]$. We next prove Theorem 2.1.

5.1 Proof of Theorem 2.1

If $\sigma_* = \infty$, then there is nothing left to prove. So we assume that σ_* is a finite positive number below. We divide the proof into four steps.

Step 1. Vanishing happens for $\sigma \in (0, \sigma_*)$. This can be proved directly by the definition of σ_* and the comparison principle.

Step 2. Transition happens when $\sigma \in [\sigma_*, \sigma^*]$. We first prove that vanishing and spreading cannot happen for any $\sigma \in [\sigma_*, \sigma^*]$. In fact, suppose that vanishing happens for some $\sigma \in [\sigma_*, \sigma^*]$, then there exists t_0 and $x_0 \in \mathbb{R}$ such that

$$u(t_0, x;\sigma\phi) < v_\beta(x - x_0) \quad \text{for } x \in (g(t_0;\sigma\phi), h(t_0;\sigma\phi)),$$

where v_β is the unique solution of (3.13) with $\gamma = \beta$. By the continuous dependence of the solution on the initial data value, we can find $\varepsilon > 0$ sufficiently small such that the solution $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ of (1.1) with initial data $u_0 = (\sigma + \varepsilon)\phi$ satisfies

$$u_\varepsilon(t_0, x) < v_\beta(x - x_0) \quad \text{for } x \in (g_\varepsilon(t_0), h_\varepsilon(t_0)).$$

Hence, it follows from Lemma 4.2 that vanishing happens for $u(t, x; (\sigma + \varepsilon)\phi)$, a contradiction to the definition of σ_*.

On the other hand, spreading cannot happen for any $\sigma \in [\sigma_*, \sigma^*]$. Suppose on the contrary that spreading happens for $u(t, x; \sigma\phi)$, then we can find some $t_1 > 0$ large and $x_1 \in \mathbb{R}$ such that

$$u(t_1, x; \sigma\phi) > V^*(x - x_1) \text{ for } x \in [-L^* + x_1, x_1] \subset [g(t_1; \sigma\phi), h(t_1; \sigma\phi)]. \tag{5.1}$$

Due to the continuous dependence of the solution on the initial data again, we can find a small $\varepsilon > 0$ such that the solution $(u^\varepsilon, g^\varepsilon, h^\varepsilon)$ of (1.1) with initial data $u_0 = (\sigma - \varepsilon)\phi$ satisfies (5.1). Hence, Lemma 4.7 implies spreading happens for $(u^\varepsilon, g^\varepsilon, h^\varepsilon)$, this is a contradiction to the definition of σ^*.

Therefore, vanishing and spreading cannot happen when $\sigma \in [\sigma_*, \sigma^*]$. We next prove that $u(t, x; \sigma\phi)$ is in the transition case: $h_\infty = -\infty$, $g_\infty = -\infty$, $h_\infty - g_\infty = L^*$ and (2.4) holds.

Claim 1 The limits $\lim_{t\to\infty}[h(t) - c^*t] \in (-\infty, +\infty)$ and $\lim_{t\to\infty}[g(t) - c^*t] \in (-\infty, +\infty)$ exist. Define

$$H(t) := h(t) - c^*t,\ G(t) := g(t) - c^*t,\ t > 0$$

and

$$w(t, x) := u(t, x + c^*t) \quad \text{for} \ \ G(t) < x < H(t),\ t > 0.$$

Then, $w(t, x)$ satisfies

$$\begin{cases} w_t = w_{xx} + c^* w_x + f(w), \quad G(t) < x < H(t),\ t > 0, \\ w(t, G(t)) = 0,\ G'(t) = -w_x(t, G(t)) + \beta_1, \quad t > 0, \\ w(t, H(t)) = 0, H'(t) = -w_x(t, H(t)) - \alpha_1, \quad t > 0, \\ -G(0) = H(0) = h_0,\ w(0, x) = u(0, x),\ -h_0 \leqslant x \leqslant h_0, \end{cases} \tag{5.2}$$

where $\beta_1 := \beta - c^*$, $\alpha_1 := \alpha + c^*$. Comparing with $V^*(x)$, one can prove that $H(t)$, as well as $G(t)$, does not move across any fixed point for infinitely many times (cf. [4, Lemma 2.9]), that is, $H_\infty := \lim\limits_{t\to\infty} H(t) \in [-\infty, +\infty]$ and $G_\infty = \lim\limits_{t\to\infty} G(t) \in [-\infty, +\infty]$ exist.

Now, we only prove $\lim_{t\to\infty}[h(t) - c^*t] \in (-\infty, +\infty)$ since the other one can be proved similarly. Suppose on the contrary that

$$h(t) - c^*t \to +\infty \text{ as } t \to \infty. \tag{5.3}$$

The other case ($h(t) - c^*t \to -\infty$ as $t \to \infty$) can be considered similarly. We will derive a contradiction by considering the number of the intersection points of $u(t, x)$ ($x \in [g(t), h(t)]$) and $V^*(x - c^*t - L^* - h_0)$ ($x \in [l_1(t), r_1(t)]$, where $l_1(t) := c^*t + h_0$, $r_1(t) := c^*t + L^* + h_0$).

By $h(0) = h_0$, we have $h(0) = l_1(0)$. Combine this and (5.3), there is $T_1 \geqslant 0$ such that

$$h(T_1) = l_1(T_1) \quad \text{and} \quad h(t) > l_1(t) \quad \text{for } t > T_1. \tag{5.4}$$

Moreover, by (5.3) again, we can choose $\hat{T} > T_1$ such that

$$h(\hat{T}) = r_1(\hat{T}) \quad \text{and} \quad h(t) < r_1(t) \quad \text{for all } t \in [T_1, \hat{T}]. \tag{5.5}$$

Set $J(t) = [\max\{g(t), l_1(t)\}, h(t)]$, then it is not empty for $t \in [T_1, \hat{T}]$. Define

$$\eta(t,x) := u(t,x) - V^*(x - c^*t - L^* - h_0) \text{ for } x \in J(t),\ t \in [T_1, \hat{T}].$$

Since $h(T_1) = l_1(T_1)$, then $Z_{J(T_1)}[\eta(T_1,\cdot)] = 1$. To analyse the zero number of η at $t = \hat{T}$, we need consider the position of $g(\hat{T})$. By Lemma 3.11, only two cases may happen as follows.

Case 1. $g(\hat{T}) < l_1(\hat{T})$. In this case, by Lemma 3.11 and (5.5), we have $Z_{J(t)}[\eta(t,\cdot)] = 1$ for $t \in [T_1, \hat{T}]$. In particular, when $t = \hat{T}$, the unique zero point of η moves to $x = h(\hat{T}) = r(\hat{T})$, so

$$u(\hat{T}, x) \geqslant (\not\equiv) V^*(x - c^*\hat{T} - L^* - h_0), \quad x \in [l_1(\hat{T}), r_1(\hat{T})].$$

This implies that spreading happens for u by Lemma 4.7, a contradiction.

Case 2. $g(\hat{T}) > l_1(\hat{T})$. Then, there exists $t_0 \in (T_1, \hat{T})$ such that

$$g(t_0) = l_1(t_0) \text{ and } h(t_0) < r_1(t_0).$$

By Lemma 3.11, we have $Z_{J(t)}[\eta(t,\cdot)] = 1$ for $t \in [T_1, t_0]$. In particular, when $t = t_0$, the unique zero point of η moves to $x = g(t_0) = l_1(t_0)$ (the left endpoint of u and V^*), so

$$u(t_0, x) \leqslant (\not\equiv) V^*(x - c^*t_0 - L^* - h_0) \quad \text{for } x \in [l_1(t_0), h(t_0)].$$

This implies that vanishing happens for u by Lemma 4.3, a contradiction.

We conclude from Case 1 and Case 2 that (5.3) is impossible. Similarly, the case $h(t) - c^*t \to -\infty$ as $t \to \infty$ is also impossible by considering the zero numbers of

$$\zeta(t,x) := u(t,x) - V^*(x - c^*t + h_0) \text{ for } x \in J_1(t),\ t > 0,$$

where

$$J_1(t) := [\max\{g(t), -L^* + c^*t - h_0\}, \min\{c^*t - h_0, h(t)\}].$$

Therefore, claim 1 holds.

Claim 2 $h_\infty = -\infty$, $g_\infty = -\infty$ and $\lim_{t\to\infty}[h(t) - g(t)] \in (0, +\infty)$. These can be deduced from claim 1 directly.

Claim 3 $\lim_{t\to\infty}[h(t) - g(t)] = L^*$ and (2.4) holds. Consider the problem (5.2) again. It follows from claim 1 and the definition of H_∞ and G_∞ that $H_\infty, G_\infty \in (-\infty, +\infty)$. By the proof of [8, Theorem 1.1], along a sequence $t_n \to +\infty$, $w(t_n, \cdot)$ converges to the stationary solution $v(\cdot)$ of $(5.2)_1$ in $C^{1+\gamma}_{\text{loc}}((G_\infty, H_\infty))$ for some $\gamma \in (0,1)$, where v satisfies

$$\begin{cases} v'' + c^*v' + f(v) = 0, & x \in (G_\infty, H_\infty), \\ v(G_\infty) = v(H_\infty) = 0, & t > 0. \end{cases} \tag{5.6}$$

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Moreover, by making a change of the variable x to reduce $[G(t), H(t)]$ to the fixed finite interval $[-h_0, h_0]$ and applying the L^p estimates (as well as Sobolev embeddings) on the reduced equation with Dirichlet boundary conditions, we have, by passing to a subsequence,

$$\|w(t_n,\cdot)-v(\cdot)\|_{C^{1+\frac{\gamma}{2}}([G(t_n),H(t_n)])} \to 0 \quad \text{as } n\to\infty. \tag{5.7}$$

From this, we have $H'(t_n) = -w_x(t_n, H(t_n)) - \alpha_1 \to -v'(H_\infty) - \alpha_1$ and $G'(t_n) = -w_x(t_n, G(t_n)) + \beta_1 \to -v'(G_\infty) + \beta_1$ as $n \to \infty$. On the other hand, $H(t)$ and $G(t)$ are Hölder continuous. Combining this and $G_\infty, H_\infty \in (-\infty, +\infty)$, we have $H'(t) \to 0$ and $G'(t) \to 0$ (cf. the proof of [4, Theorem 1.3]). Therefore, $-v'(H_\infty) = \alpha_1$, $v'(G_\infty) = \beta_1$. Since (3.1) with (3.8) has a unique positive solution (cf. Lemma 3.7), we can derive that

$$H_\infty - G_\infty = L^* \quad \text{and} \quad v(x) \equiv V^*(x - H_\infty) \quad \text{for all } x \in [G_\infty, H_\infty]. \tag{5.8}$$

Moreover, (2.4) follows from (5.7) and (5.8).

Step 3. $\sigma_* = \sigma^*$. Otherwise, there exist $\sigma_1, \sigma_2 \in [\sigma_*, \sigma^*]$ such that $\sigma_1 < \sigma_2$. Denote the solution of (1.1) with $\sigma = \sigma_1$ by (u_1, g_1, h_1), and let (u_2, g_2, h_2) be the solution of (1.1) with $\sigma = \sigma_2$. By the comparison principle Lemma 3.1, we have

$$(g_1(1), h_1(1)) \subset (g_2(1), h_2(1)) \quad \text{and} \quad u_1(1,x) < u_2(1,x) \quad \text{for} \quad x \in (g_1(1), h_1(1)).$$

Hence, we can find $\epsilon_0 > 0$ small such that for all $\varepsilon \in [0, \varepsilon_0]$,

$$(g_1(1)-\varepsilon, h_1(1)-\varepsilon) \subset (g_2(1), h_2(1)) \quad \text{and} \quad u_1(1, x+\varepsilon) < u_2(1,x) \text{ for } x \in (g_1(1)-\varepsilon, h_1(1)-\varepsilon).$$

Define

$$u_1^\varepsilon(t,x) := u_1(t+1, x+\varepsilon),\ g_1^\varepsilon(t) := g_1(t+1) - \varepsilon,\ h_1^\varepsilon(t) := h_1(t+1) - \varepsilon.$$

Clearly, $(u_1^\varepsilon, g_1^\varepsilon, h_1^\varepsilon)$ is a solution of (1.1) with $u_0(x) = u_1(1, x+\varepsilon)$. By the comparison principle Lemma 3.1, we have, for all $t > 0$ and $\varepsilon \in (0, \varepsilon_0]$,

$$[g_1^\varepsilon(t), h_1^\varepsilon(t)] \subset (g_2(t+1), h_2(t+1)),\ u_1^\varepsilon(t,x) \leqslant u_2(t+1, x) \text{ in } [g_1^\varepsilon(t), h_1^\varepsilon(t)]. \tag{5.9}$$

By Step 2, we have $u_1(t,x) \to V^*(x - c^*t - b_1)$ uniformly for some $b_1 \in \mathbb{R}$ and $u_2(t,x) \to V^*(x - c^*t - b_2)$ uniformly for some $b_2 \in \mathbb{R}$ as $t \to \infty$. Then it follows from (5.9) that

$$V^*(x - c^*t - b_1 + \varepsilon) \leqslant V^*(x - c^*t - b_2),\ x \in [c^*t + b_1 - \varepsilon - L^*, c^*t + b_1 - \varepsilon],$$

for any $\varepsilon \in (0, \varepsilon_0]$. This is impossible by the definition of V^*. Hence, $\sigma_* = \sigma^*$.

Step 4. Spreading happens for $\sigma \in (\sigma^*, \infty)$. By step 3 and our assumption, we have $\sigma^* = \sigma_* \in (0, \infty)$. It follows by the definition of σ^* and the comparison principle that spreading happens when $\sigma > \sigma^*$. □

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5.2 Proof of Theorem 2.5

We prove this theorem by constructing two upper solutions. Define, for some M_1, X_1, $\sigma_1 > 0$ and $\delta_1 \in (0, -f'(1))$,

$$\overline{g}_1(t) := g(t),\ \overline{h}_1(t) := c_R t - \sigma_1 M_1 e^{-\delta_1 t} + X_1,\ \overline{u}_1(t,x) := (1 + M_1 e^{-\delta_1 t}) q_R(x - \overline{h}_1(t)),$$

and, for some M_2, $X_2 \ll X_1$, $\sigma_2 > 0$, $\delta_2 \in (0, -f'(1))$,

$$\overline{h}_2(t) := h(t),\ \overline{g}_2(t) := c_L t + \sigma_2 M_2 e^{-\delta_2 t} + X_2,\ \overline{u}_2(t,x) := (1 + M_2 e^{-\delta_2 t}) q_L(x - \overline{g}_2(t)).$$

One can calculate directly as in [11, Lemma 3.2] to show that, when σ_1, σ_2 are large, $(\overline{u}_1, \overline{g}_1, \overline{h}_1)$ and $(\overline{u}_2, \overline{g}_2, \overline{h}_2)$ are upper solutions of (1.1) for large $t > 0$. Therefore,

$$h(t) \leqslant \overline{h}_1(t) \quad \text{and} \quad g(t) \geqslant \overline{g}_2(t) \tag{5.10}$$

for large t. By Corollary 3.6, $c_R < c_L$ when $\alpha > \alpha^*$ and $\beta < \alpha_0$, or $\beta \geqslant \alpha_0$. Hence, (5.10) implies that $h(t) - g(t) \to 0$ as $t \to T^*$ for some $T^* < +\infty$, that is, vanishing happens by Lemma 4.5.

Now, we consider the case $\alpha = \alpha^*$. In this case, $c_L = c_R$. It follows by (5.10) and Lemma 4.5 that $0 < h_\infty - g_\infty < +\infty$ or $h(t) - g(t) \to 0$ as $t \to T_0$ for some $T_0 < +\infty$. The latter implies vanishing. We next prove that the former case is impossible. Otherwise, define

$$\widehat{H}(t) := h(t) - c_R t \quad \text{and} \quad \widehat{G}(t) := g(t) - c_R t \quad \text{for } t > 0$$

and

$$\widehat{w}(t,x) := u(t, x + c_R t) \quad \text{for } \widehat{G}(t) < x < \widehat{H}(t),\ t > 0.$$

Then, $\widehat{w}(t,x)$ satisfies

$$\begin{cases} \widehat{w}_t = \widehat{w}_{xx} + c_R \widehat{w}_x + f(\widehat{w}), & \widehat{G}(t) < x < \widehat{H}(t),\ t > 0, \\ \widehat{w}(t, \widehat{G}(t)) = 0,\ \widehat{G}'(t) = -\widehat{w}_x(t, G_1(t)) + \beta - c_R, & t > 0, \\ \widehat{w}(t, \widehat{H}(t)) = 0, \widehat{H}'(t) = -\widehat{w}_x(t, \widehat{H}(t)) - (\alpha + c_R), & t > 0, \\ -\widehat{G}(0) = \widehat{H}(0) = h_0,\ \widehat{w}(0,x) = u(0,x),\ -h_0 \leqslant x \leqslant h_0. \end{cases} \tag{5.11}$$

Comparing with $q_R(x)$ and $q_L(x)$, one can prove that $\widehat{H}(t)$, as well as $\widehat{G}(t)$, does not move across any fixed point for infinitely many times (cf. [4, Lemma 2.9]), that is, $\widehat{H}_\infty := \lim_{t\to\infty} \widehat{H}(t) \in [-\infty, +\infty]$ and $\widehat{G}_\infty := \lim_{t\to\infty} \widehat{G}(t) \in [-\infty, +\infty]$ exist. Moreover, it follows by (5.10) that $\overline{g}_2(t) \leqslant g(t) < h(t) \leqslant \overline{h}_1(t)$ for large t. Therefore, $\widehat{H}_\infty, \widehat{G}_\infty \in (-\infty, +\infty)$. Using similar arguments to those for claim 3 in the above proof, one can show that $\widehat{w}(t,x)$ converges to the stationary solution V of $(5.11)_1$ uniformly in $[\widehat{G}_\infty, \widehat{H}_\infty]$, where V satisfies:

$$\begin{cases} V'' + c_R V' + f(V) = 0, & x \in (\widehat{G}_\infty, \widehat{H}_\infty), \\ V(\widehat{G}_\infty) = V(\widehat{H}_\infty) = 0,\ t > 0. \\ V'(\widehat{G}_\infty) = \beta - c_R,\ -V'(\widehat{H}_\infty) = \alpha + c_R. \end{cases} \tag{5.12}$$

This is a contradiction, since (5.12) has no positive solution by Lemma 3.4 and the uniqueness of the solution for ODE. □

5.3 Proof of Theorem 2.6

We divide the proof into several steps.

Step 1. Boundedness of $h(t)-c_Rt$ and $g(t)-c_Lt$. By Theorem 2.1, using similar arguments to those for [18, Proposition A], there exist $\delta \in (0, -f'(1))$, $T > 0$ and $M > 0$ such that for $t \geqslant T$, $c^*t \subset [g(t), h(t)]$ and

$$u(t, c^*t) \geqslant 1 - Me^{-\delta t}, \quad u(t,x) \leqslant 1 + Me^{-\delta t} \text{ for } x \in [g(t), h(t)]. \tag{5.13}$$

Here, we have used (4.18) in the first inequality. Define, for some $N_1, L_1(T) > 0$,

$$\overline{g}(t) := g(t), \quad \overline{h}(t) := c_Rt - \sigma N_1 e^{-\delta t} + L_1(T), \quad \overline{u}(t,x) := (1 + N_1 e^{-\delta t})q_R(x - \overline{h}(t)),$$

and for some $N_2, L_2(T) > 0$,

$$\underline{g}(t) := c^*t, \quad \underline{h}(t) := c_Rt + \sigma N_2 e^{-\delta t} + L_2(T), \quad \underline{u}(t,x) := (1 - N_2 e^{-\delta t})q_R(x - \underline{h}(t)).$$

By (5.13), one can calculate directly as in [11] to prove that $(\bar{u}, \bar{g}, \bar{h})$ is an upper solution and $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of (1.1) provided that $\sigma > 0$ is sufficiently large. Therefore, $\underline{h}(t) \leqslant h(t) \leqslant \bar{h}(t)$ for large t. This implies the boundedness of $h(t) - c_Rt$. The boundedness of $g(t) - c_Lt$ is proved similarly.

Step 2. Convergence of $h(t) - c_Rt$, $g(t) - c_Lt$, $h'(t)$, $g'(t)$. Define, for $t > 0$,

$$H(t) := h(t) - c_Rt, \; G(t) := g(t) - c_Rt, \; w(t,z) := u(t, z + c_Rt), \; z \in [G(t), H(t)].$$

By Step 1 and $c_L < c_R$ when $\alpha < \alpha^*$, we have $G(t) \to -\infty$ as $t \to \infty$. Moreover, w satisfies (5.11) and $H := \lim_{t\to\infty} H(t)$ exists and it is finite by the boundedness of $H(t)$. By the limit of $H(t)$ and the uniform Hölder estimate for $H'(t)$: $\|H'(t)\|_{C^{\nu/2}([1,\infty))} \leqslant C$ for C independent of t (cf. the proof of Theorem 1.6 in [4]), it is easy to show that $\lim_{t\to\infty} H'(t) = 0$, that is, $\lim_{t\to\infty} h'(t) = c_R$. Similarly, the limit $G := \lim_{t\to\infty}[g(t) - c_Lt]$ exists and $\lim_{t\to\infty} g'(t) = c_L$.

Step 3. (2.6) *and* (2.7) *hold.* Define, for any small $\varepsilon > 0$, and some $H_1, N, B, K > 0$, $T > 0$, $\delta \in (0, -f'(1))$,

$$\begin{aligned}
&\overline{g}_1(t) = g(t),\\
&\overline{h}_1(t) = c_Rt + H_1 + N\varepsilon + N\varepsilon B(1 - e^{-\delta(t-T)}),\\
&\overline{u}_1(t,x) = (1 + K\varepsilon e^{-\delta(t-T)})q_R(x - \overline{h}_1(t)).
\end{aligned}$$

One can calculate directly as in [11, section 3.3] to show that, when N (with $N > 1$), T, K and B are sufficiently large, $(\overline{u}_1, \overline{g}_1, \overline{h}_1)$ is an upper solution. Then, by the monotonicity of q_R, we obtain

$$u(t,x) \leqslant q_R(x - c_Rt - H_1 - N\varepsilon(1 + B)) + \varepsilon K e^{-\delta(t-T)}. \tag{5.14}$$

Similarly, define a lower solution as follows:

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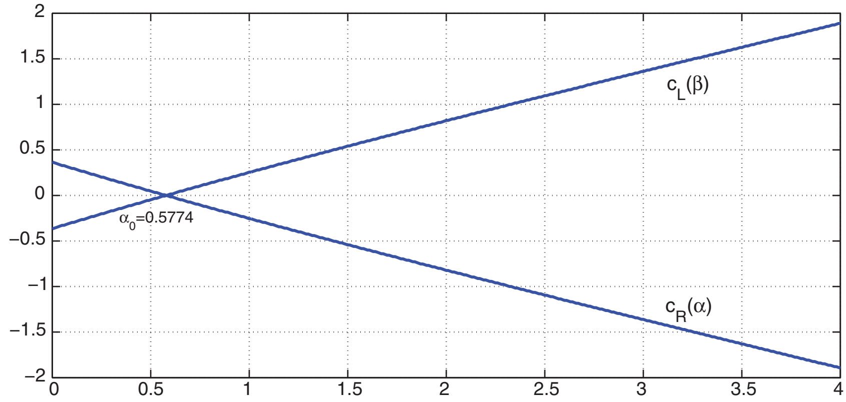

FIGURE 2. Graphs of $c_R(\alpha)$ and $c_L(\beta)$ when $f(u) = u(1-u)$.

$$
\begin{aligned}
&\underline{g}_1(t) = c^* t,\\
&\underline{h}_1(t) = c_R t + H_1 - N\varepsilon - N\varepsilon B(1 - e^{-\delta(t-T)}),\\
&\underline{u}_1(t,x) = (1 - K\varepsilon e^{-\delta(t-T)}) q_R(x - \underline{h}_1(t)).
\end{aligned}
$$

We may apply the comparison principle to obtain $\underline{u}(t,x) \leqslant u(t,x)$ for $x \in [\underline{g}(t), \underline{h}(t)]$ and $t > 0$. More precisely,

$$q_R(x - c_R t - H_1 + N\varepsilon(1+B)) - \varepsilon K e^{-\delta(t-T)} \leqslant u(t,x). \tag{5.15}$$

By (5.14), (5.15), the mean value theorem and the monotonicity of q_R, we have

$$\limsup_{t\to\infty} \sup_{x\in[c^* t, h(t)]} |u(t,x) - q_R(x - h(t))| \leqslant C\varepsilon,$$

where C is dependent of $||q_R'||_\infty$ but independent of ε. Letting $\varepsilon \to 0$, we deduce

$$\limsup_{t\to\infty} \sup_{x\in[c^* t, h(t)]} |u(t,x) - q_R(x - h(t))| = 0.$$

One can similarly show that

$$\limsup_{t\to\infty} \sup_{x\in[g(t), c^* t]} |u(t,x) - q_L(x - h(t))| = 0.$$

□

6 Numerical simulation and discussion

In this section, we give the numerical results of the asymptotic speeds of free boundaries when spreading and transition happen, that is the numerical simulations for $c_R(\alpha)$, $c_L(\beta)$

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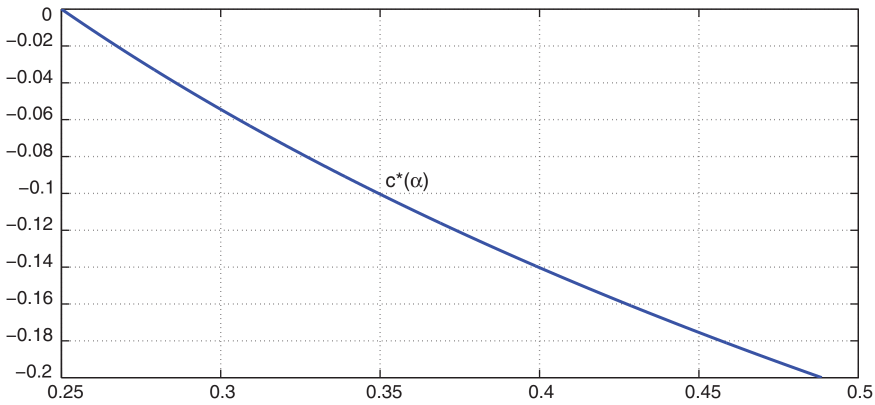

FIGURE 3. Graph of $c^*(\alpha)$ when $f(u) = u(1-u)$ and $\beta = 0.25$.

and c^*. For simplicity, we choose $f(u) = u(1-u)$. In this case,

$$\alpha_0 = \sqrt{2\int_0^1 f(s)ds} = \frac{\sqrt{3}}{3} \approx 0.5774.$$

Our computational results show that $c_R(0) = -c_L(0) \approx 0.3640$. In Figure 2, we plot the graph of $c_R(\alpha)$ for $\alpha \geqslant 0$ (which has a decreasing slope), and the graph of $c_L(\beta)$ for $\beta \geqslant 0$ (which has an increasing slope). Then fix $\beta = 0.25 < \alpha_0$, it is easy to see from Figure 3 that $c^*(\alpha)$ is decreasing in $\alpha \geqslant 0.25$ and $c^*(0.25) = 0$.

Figure 2 shows that the asymptotic spreading speed of the right (resp. left) free boundary is decreasing (resp. increasing) in $\alpha > 0$ (resp. $\beta > 0$), and it is positive (resp. negative) when α (resp. β) is small while it is negative (resp. positive) when α (resp. β) is large. The biological significance of the numerical and main results can be read as:

(a) when the resistant force at the boundary is small (i.e., $0 < \beta < \alpha < \alpha_0$), the free boundaries can expand outwards and tend to $+\infty$ and $-\infty$, respectively. The species can successfully spread into infinity eventually and stabilize as a positive equilibrium state, and the occupied space $(g(t), h(t))$ of the species tends to the entire space;

(b) when the resistant force is small at the left boundary and medium at the right boundary (i.e., $0 < \beta < \alpha_0 < \alpha < \alpha^*$, where $\alpha = \alpha^*$ is the unique value such that $c_R(\alpha) = c_L(\beta)$), both the left and the right free boundaries expand leftwards with $c_L < c_R < 0$. This means that the environment at the right is more hostile than the left. So the species moves leftwards and its occupied space become larger and larger;

(c) when the resistant force is small at the left boundary and large at the right boundary (i.e., $0 < \beta < \alpha_0$ and $\alpha > \alpha^*$), the right free boundary moves leftwards faster than the left free boundary. In this case, the species moves leftwards, but its occupied space become smaller and smaller, and the species will die out eventually;

(d) when the resistant force at the boundaries is large (i.e., $\alpha > \beta > \alpha_0$), the right front moves leftwards while the left front moves rightwards, and the occupied space of the species tend to a point, the species can not spread into infinity and instead vanishes in a finite time.

Additionally, we obtain a transition case: The solution converges to a compactly supported travelling wave moving leftwards with a certain speed. We also analyse the influence of α on the speed when β is fixed. Figure 3 shows that the speed of the travelling wave is decreasing in α, which means that in the transition case, the species moves leftwards faster when the resistant force at the right boundary is larger.

The spreading case (when $0 < \beta < \alpha_0 < \alpha < \alpha^*$) and the transition case in our problem are new phenomena compared with that when $\alpha = \beta = 0$. The condition $\alpha > \beta > 0$ is such that the species prefers moving leftwards than to moving rightwards, since the resistant force at the right boundary is larger than that at the left boundary. Hence for some sufficiently large initial density, the species moves leftwards and its occupied space becomes larger and larger (see Theorem 2.1 (i-b)), and for a particular initial density, the species moves leftwards with a certain speed and its occupied space tend to a constant (see Theorem 2.1 (iii)).

Acknowledgements

The authors would like to thank Doctor Junfan Lu for valuable discussion. This research was sponsored by NSFC (No. 11526134) and Natural Science Foundation of Shanghai (No. 13ZR1454900).

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