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WEYL SPECTRAL IDENTITY AND BIQUASITRIANGULARITY

C. S. KUBRUSLY[1] AND B. P. DUGGAL[2]

[1] *Catholic University of Rio de Janeiro, 22453-900, Rio de Janeiro, Brazil* (carlos@ele.puc-rio.br)
[2] *University of Niš, Department of Mathematics, PO Box 224, Niš, Serbia* (bpduggal@yahoo.co.uk)

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Abstract Let A and B be operators acting on infinite-dimensional complex Banach spaces. We say that the Weyl spectral identity holds for the tensor product $A\otimes B$ if $\sigma_{\mathrm{w}}(A\otimes B) = \sigma_{\mathrm{w}}(A)\cdot\sigma(B)\cup\sigma(A)\cdot\sigma_{\mathrm{w}}(B)$, where $\sigma(\cdot)$ and $\sigma_{\mathrm{w}}(\cdot)$ stand for the spectrum and the Weyl spectrum, respectively. Conditions on A and B for which the Weyl spectral identity holds are investigated. Especially, it is shown that if A and B are biquasitriangular (in particular, if the spectra of A and B have empty interior), then the Weyl spectral identity holds. It is also proved that if A and B are biquasitriangular, then the tensor product $A\otimes B$ is biquasitriangular.

Keywords: tensor product; Weyl spectral identity; biquasitriangular operators

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1. Introduction

Given a pair of operators A and B, it is relevant to enquire whether the Weyl spectral identity (WSI) holds. An example where this condition plays an important role is the result that says that *if A and B are isoloid, satisfy Weyl's theorem, and the WSI holds, then the tensor product $A\otimes B$ satisfies Weyl's theorem* [**20**, Theorem 1] (see also [**24**, Proof of Theorem 1] and [**19**, Corollary 4]). The aim of this paper is to investigate the conditions on A and B for which the WSI holds. The main result (Theorem 5.1) says that *if A and B are biquasitriangular operators (in particular, if the spectra of A and B have empty interior), then the WSI holds, and this implies that biquasitriangularity is transferred from the operators to their tensor product $A\otimes B$.*

2. Notation and terminology

Notation in this area is not standard. Thus, to begin with, we introduce the notation and terminology that will be used throughout the text. By an *operator* we mean a *bounded* linear transformation of a normed space into itself. Throughout this paper, T will denote an arbitrary operator acting on a complex infinite-dimensional Banach space $\mathcal{X}$, and

I will denote the identity operator on $\mathcal{X}$. Let the kernel and range of T be denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, let $\mathcal{X}/\mathcal{R}(T)^-$ be the quotient space of $\mathcal{X}$ modulo the closure of $\mathcal{R}(T)$ (which in a Hilbert space is identified with $\mathcal{N}(T^*)$), let $\sigma(T)$ and $\sigma_{\mathrm{P}}(T)$ stand for the spectrum and point spectrum (i.e. the set of all eigenvalues) of T, and let $\sigma_{\mathrm{AP}}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not bounded below}\}$ be the approximate point spectrum of T. Recall that $\sigma(T)$ is compact and non-empty. Set

$$\sigma_{\mathrm{le}}(T) = \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim \mathcal{N}(\lambda I - T) = \infty\},$$
$$\sigma_{\mathrm{re}}(T) = \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed or } \dim \mathcal{X}/\mathcal{R}(\lambda I - T) = \infty\},$$

the left and right essential spectra in a Hilbert space setting, or the upper and lower semi-Fredholm spectra in a Banach setting; and let

$$\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{le}}(T) \cup \sigma_{\mathrm{re}}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not a Fredholm operator}\}$$

be the essential spectrum (also called the Fredholm spectrum) of T. Let

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not a Fredholm operator of index zero}\}$$

be the Weyl spectrum of T. Set

$$\sigma_0(T) = \sigma(T) \backslash \sigma_{\mathrm{w}}(T),$$

the complement of the Weyl spectrum $\sigma_{\mathrm{w}}(T)$ in the whole spectrum $\sigma(T)$. The pair of sets $\{\sigma_{\mathrm{w}}(T), \sigma_0(T)\}$ forms a partition of the spectrum $\sigma(T)$. Observe that in a Hilbert space, where T^* denotes the adjoint of T, we obtain

$$\sigma_0(T) = \{\lambda \in \sigma_{\mathrm{P}}(T) \colon \mathcal{R}(\lambda I - T) \text{ is closed and } \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\bar{\lambda} I - T^*) < \infty\}$$

(see, for example, [**17**, § 5.3]: in a Banach space the same result still holds with $\mathcal{N}(\bar{\lambda} I - T^*)$ replaced with the quotient space $\mathcal{X}/\mathcal{R}(\lambda I - T)^-$, the upper bar standing for closure). Consider the set $\sigma_{\mathrm{PF}}(T)$ of all eigenvalues of T of finite multiplicity,

$$\sigma_{\mathrm{PF}}(T) = \{\lambda \in \sigma_{\mathrm{P}}(T) \colon \dim \mathcal{N}(\lambda I - T) < \infty\},$$

so that $\sigma_0(T) \subseteq \sigma_{\mathrm{PF}}(T)$, and set

$$\pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{\mathrm{PF}}(T),$$

where $\sigma_{\mathrm{iso}}(T)$ denotes the set of all isolated points of the spectrum $\sigma(T)$. Its complement $\sigma_{\mathrm{acc}}(T)$ in $\sigma(T)$ is the set of all accumulation points of the spectrum: $\sigma_{\mathrm{acc}}(T) = \sigma(T) \backslash \sigma_{\mathrm{iso}}(T)$ (these are sometimes also denoted by $\operatorname{iso} \sigma(T)$ and $\operatorname{acc} \sigma(T)$, respectively). One says that an operator T satisfies Weyl's theorem if

$$\sigma_0(T) = \pi_{00}(T),$$

and it is said to satisfy Browder's theorem if

$$\sigma_0(T) \subseteq \pi_{00}(T).$$

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Set

$$\pi_0(T) = \sigma_{\rm iso}(T) \cap \sigma_0(T).$$

The set

$$\sigma_{\rm b}(T) = \sigma(T) \backslash \pi_0(T)$$

is referred to as the Browder spectrum of T, and so $\{\sigma_{\rm b}(T), \pi_0(T)\}$ forms another partition of the spectrum $\sigma(T)$. Recall that

$$\sigma_{\rm e}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T) \subseteq \sigma(T)$$

and

$$\sigma_{\rm w}(T) = \sigma_{\rm b}(T) \text{ if and only if } T \text{ satisfies Browder's theorem.}$$

These are well-known results (see, for example, [**17**, Chapter 5]). An operator T is isoloid if $\sigma_{\rm iso}(T) \subseteq \sigma_{\rm P}(T)$ (i.e. if every isolated point of the spectrum is an eigenvalue).

3. Auxiliary results

In this section we explore the relationship among biquasitriangular operators, operators without isolated points in their spectra and, at the other end, operators whose spectra have empty interior, which will be needed in what follows. Let $\sigma(T)^\circ$ denote the interior of the spectrum $\sigma(T)$ of T.

An operator T on a complex infinite-dimensional separable Hilbert space is quasitriangular if there is a sequence $\{P_n\}$ of finite-rank projections that converges strongly to the identity operator I and $\{(I - P_n)TP_n\}$ converges uniformly to the null operator [**10**, § 2]. If both T and T^* are quasitriangular, then T is biquasitriangular ($\mathcal{BQT}$). Biquasitriangular operators are equivalently described as

$$T \text{ is } \mathcal{BQT} \text{ if and only if } \sigma_{\rm le}(T) = \sigma_{\rm re}(T) = \sigma_{\rm e}(T) = \sigma_{\rm w}(T)$$

(see [**2**, Theorem 5.4], [**3**, Theorem 2.1] and also [**21**, p. 37]), which means that [**18**, § 4]

$$T \text{ is } \mathcal{BQT} \text{ if and only if } \sigma_{\rm e}(T) \text{ has no holes and no pseudo-holes}$$

(see Lemma 3.1 (e)). Thus, we take the above equivalent statement as the definition of a $\mathcal{BQT}$ operator on a Banach space. By the above characterization, if there is an operator on a complex infinite-dimensional separable Hilbert space without a non-trivial invariant subspace, then it must be biquasitriangular [**21**, p. 47]. Indeed, since $\sigma(T)\backslash\sigma_{\rm e}(T) \subseteq \sigma_{\rm PF}(T) \cup \sigma_{\rm PF}(T^*)^*$, $\sigma_{\rm le}(T)\backslash\sigma_{\rm re}(T) \subseteq \sigma_{\rm P}(T)\backslash\sigma_{\rm PF}(T)$, $\sigma_{\rm re}(T)\backslash\sigma_{\rm le}(T) \subseteq \sigma_{\rm P}(T^*)^*\backslash\sigma_{\rm PF}(T^*)^*$ (see, for example, [**17**, Theorem 5.16 and Corollary 5.18]), and recalling that $\sigma_{\rm le}(T) \cup \sigma_{\rm re}(T) = \sigma_{\rm e}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T) \subseteq \sigma(T)$, we obtain

$$\sigma_{\rm le}(T) = \sigma_{\rm re}(T) = \sigma_{\rm e}(T) = \sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma(T)$$

whenever $\sigma_{\rm P}(T) = \sigma_{\rm P}(T^*) = \varnothing$. In particular,

$$\sigma_{\rm P}(T) = \sigma_{\rm P}(T^*) = \varnothing \text{ implies that } T \text{ is } \mathcal{BQT}.$$

Equivalently,

$$\sigma(T) = \sigma_{\mathrm{C}}(T) \text{ implies that } T \text{ is } \mathcal{BQT},$$

where $\sigma_{\mathrm{C}}(T) = \sigma(T)\backslash(\sigma_{\mathrm{P}}(T) \cup \sigma_{\mathrm{P}}(T^*)^*)$ stands for the continuous spectrum of T. Recalling that if T has no non-trivial invariant subspace, then $\sigma(T) = \sigma_{\mathrm{C}}(T)$, it follows that if T has no non-trivial invariant subspace, then T is $\mathcal{BQT}$.

Lemma 3.1. *Let T be an arbitrary operator.*

(a) *If $\sigma_{\mathrm{iso}}(T) = \varnothing$, then $\sigma_{\mathrm{b}}(T) = \sigma(T)$.*

(b) *If $\sigma_0(T) = \varnothing$, then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$.*

(c) *If $\sigma_{\mathrm{iso}}(T) = \sigma_0(T) = \varnothing$, then T satisfies Weyl's theorem.*

(d) *If $\sigma_0(T) = \varnothing$ and $\sigma_{\mathrm{e}}(T)$ has no holes, then $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$.*

(e) *$\sigma_{\mathrm{e}}(T)$ has no holes and no pseudo-holes if and only if T is $\mathcal{BQT}$.*

Proof. If $\sigma_{\mathrm{iso}}(T) = \varnothing$, then $\pi_0(T) = \varnothing$, and hence (a) $\sigma_{\mathrm{b}}(T) = \sigma(T)$. Recall that $\sigma_{\mathrm{e}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T) \subseteq \sigma(T)$. If $\sigma_0(T) = \sigma(T)\backslash\sigma_{\mathrm{w}}(T) = \varnothing$, then (b) $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$ (which implies that T satisfies Browder's theorem). If, in addition, $\sigma_{\mathrm{iso}}(T) = \varnothing$, then (c) T satisfies Weyl's theorem as well (since $\pi_{00} = \varnothing$ trivially). Since $\sigma_{\mathrm{w}}(T)$ is the union of $\sigma_{\mathrm{e}}(T)$ and its holes (the Schechter theorem: see, for example, [**17**, Theorem 5.24]), we get (d) from (b). Consequently, $\sigma_{\mathrm{e}}(T)$ has no holes (which means that $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T)$) and the pseudo-holes $\sigma_{+\infty}(T) = \sigma_{\mathrm{le}}(T)\backslash\sigma_{\mathrm{re}}(T)$ and $\sigma_{-\infty}(T) = \sigma_{\mathrm{re}}(T)\backslash\sigma_{\mathrm{le}}(T)$ (see, for example, [**17**, Theorem 5.16]) of $\sigma_{\mathrm{e}}(T)$ are empty (so that $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T)$) if and only if $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T)$, which means that T is $\mathcal{BQT}$. □

Lemma 3.2. *Suppose that $\sigma(T)^\circ = \varnothing$. Then*

(a) *$\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ (i.e. T is $\mathcal{BQT}$ and satisfies Browder's theorem),*

(b) *$\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$ if and only if $\sigma_0(T) = \varnothing$.*

Proof. Recall that $\sigma_0(T) = \tau_0(T) \cup \pi_0(T)$, where $\tau_0(T) = \sigma_0(T)\backslash\pi_0(T)$ is an open set in $\mathbb{C}$ (see [**17**, Corollary 5.20]). Suppose that $\sigma(T)^\circ = \varnothing$. Then $\sigma_0(T) = \pi_0(T)$, which is equivalent to saying that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ (i.e. T satisfies Browder's theorem: see, for example, [**17**, Corollary 5.41]). Moreover, since the holes of $\sigma_{\mathrm{e}}(T)$ are open sets, the Schechter theorem (see, for example, [**17**, Theorem 5.24]) ensures that $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T)$. Furthermore, the pseudo-holes of $\sigma_{\mathrm{e}}(T)$ are also open sets, so that $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T)$ (see, for example, [**17**, Theorem 5.16]). This concludes the proof of (a). From (a), $\sigma_0(T) = \varnothing$ (i.e. $\sigma_{\mathrm{w}}(T) = \sigma(T)$) if and only if $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma(T)$ (since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T) \subseteq \sigma(T)$), and so we get (b). □

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Remark 3.3.

(a) Let T act on $\mathcal{X}$. Recall that

$$\sigma_{\rm e}(T) = \sigma_{\rm le}(T) \cup \sigma_{\rm re}(T) = (\sigma_{\rm le}(T) \cap \sigma_{\rm re}(T)) \cup \sigma_{+\infty}(T) \cup \sigma_{-\infty}(T);$$

that $\sigma_{\rm le}(T)$, $\sigma_{\rm re}(T)$, $\sigma_{\rm e}(T)$, $\sigma_{\rm w}(T)$ and $\sigma_{\rm b}(T)$ are subsets of $\sigma(T)$ that are all closed in $\mathbb{C}$; and also that the pseudo-holes of $\sigma_{\rm e}(T)$, namely, $\sigma_{+\infty}(T) = \sigma_{\rm le}(T)\backslash\sigma_{\rm re}(T)$ and $\sigma_{-\infty}(T) = \sigma_{\rm re}(T)\backslash\sigma_{\rm le}(T)$, which are holes of $\sigma_{\rm re}(T)$ and $\sigma_{\rm le}(T)$, are open in $\mathbb{C}$. Therefore, it can be verified (see [**17**, Theorem 5.16, Corollary 5.18, Remarks 5.15 (a), 5.27 (a) and 5.40 (a)]) that assertions (i)–(iii) below are pairwise equivalent.

  (i) $\dim \mathcal{X} = \infty$.

  (ii) One of the sets $\sigma_{\rm le}(T)$, $\sigma_{\rm re}(T)$, $\sigma_{\rm e}(T)$, $\sigma_{\rm w}(T)$ or $\sigma_{\rm b}(T)$ is not empty.

  (iii) All the sets $\sigma_{\rm le}(T)$, $\sigma_{\rm re}(T)$, $\sigma_{\rm e}(T)$, $\sigma_{\rm w}(T)$ and $\sigma_{\rm b}(T)$ are not empty.

(b) Thus, the sets in Lemma 3.2 (a) are non-empty. If $\sigma_{\rm iso}(T) = \sigma(T)$, then $\#\sigma(T) < \infty$ (# means cardinality), and so $\sigma(T)^\circ = \varnothing$, implying the assumption of Lemma 3.2.

(c) The result in Lemma 3.2 (b) also holds if T is $\mathcal{BQT}$. That is, *if T is $\mathcal{BQT}$, then $\sigma_{\rm le}(T) = \sigma_{\rm re}(T) = \sigma_{\rm e}(T) = \sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma(T)$ if and only if $\sigma_0(T) = \varnothing$.*

(d) The converse of Lemma 3.2 (a) fails: $\sigma_{\rm le}(T) = \sigma_{\rm re}(T) = \sigma_{\rm e}(T) = \sigma_{\rm w}(T) = \sigma_{\rm b}(T)$ does not imply $\sigma(T)^\circ = \varnothing$. Example 6.H (Part 2) of [**16**] exhibited a Hilbert space operator such that $\sigma(T) = \sigma_{\rm C}(T) = \mathbb{D}$, the closed unit disc, and so $\sigma_{\rm P}(T) = \sigma_{\rm P}(T^*) = \varnothing$, which implies that $\sigma_{\rm le}(T) = \sigma_{\rm re}(T) = \sigma_{\rm e}(T) = \sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma(T)$.

4. Weyl spectral identity

By a tensor product space $\mathcal{X} \otimes \mathcal{Y}$ of (complex infinite-dimensional) Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we mean the completion endowed with a reasonable uniform cross norm [**22**, § 6.1] of the algebraic tensor product of $\mathcal{X}$ and $\mathcal{Y}$ [**5**, pp. 22–25], [**25**, § 3.4]. Let the (bounded linear) operator $A \otimes B$ on $\mathcal{X} \otimes \mathcal{Y}$ denote the tensor product of (bounded linear) operators A on $\mathcal{X}$ and B on $\mathcal{Y}$. (A and B will always stand for operators on Banach spaces.) As far as tensor product properties used in this paper are concerned, there will be no fuss in considering tensor products either in a Banach or in a Hilbert space setting [**15**, § 2]. For instance, that the spectrum of a tensor product coincides with the product of the spectra of the factors

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B)$$

was proved in a Hilbert space setting [**4**, § 1], but it is well known (see [**23**, Theorem 2.1], [**11**, Theorem 4.13] and [**9**, Theorem 3.2]) that this has a natural extension to a Banach space setting (reasonably uniformly cross normed). For the essential and Browder spectra it was proved in [**12**, Theorem 4.2 (a) and (c)] that

$$\sigma_{\mathrm{e}}(A \otimes B) = \sigma_{\mathrm{e}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{e}}(B),$$
$$\sigma_{\mathrm{b}}(A \otimes B) = \sigma_{\mathrm{b}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{b}}(B),$$

while for the Weyl spectrum it was also proved in [**12**, Theorem 4.2 (f)] that

$$\sigma_{\mathrm{w}}(A \otimes B) \subseteq \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B).$$

Until quite recently it remained an open question as to whether the above inclusion might be an identity. This question was solved by using the counterexample from [**14**, § 3] (which exhibits a pair of operators that satisfy Weyl's theorem but whose tensor product does not satisfy Browder's theorem) together with a result from [**19**, Corollary 6] (which says that Browder's theorem is transferred from a pair of operators to their tensor product if and only if the above inclusion is an identity). This ensures the existence of pairs of operators for which the above inclusion is proper. If a pair of operators $\{A, B\}$ is such that

$$\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B),$$

then we say that the Weyl spectral identity (WSI) holds for $A \otimes B$.

It is important to enquire whether the WSI holds. An example where this condition plays a crucial role is the result that says that *if A and B are isoloid, satisfy Weyl's theorem, and the WSI holds, then $A \otimes B$ satisfies Weyl's theorem* [**20**, Theorem 1] (also see [**24**, Proof of Theorem 1] and [**19**, Corollary 4]).

In the next sections we investigate conditions on A and B for which the WSI holds. We begin with a collection of intermediate results, exhibiting conditions that imply the WSI or are implied by it, which are closely linked with Browder's theorem.

Lemma 4.1. *Let A and B be operators acting on infinite-dimensional spaces.*

(a) *If $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B)$, then the WSI holds (i.e. if $A \otimes B$ satisfies Browder's theorem, then the WSI holds).*

(b) *In particular, if $\sigma_0(A \otimes B) = \varnothing$, then the WSI holds with $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B) = \sigma(A \otimes B)$.*

(c) *If $\sigma_{\mathrm{e}}(A)\backslash\{0\} = \sigma_{\mathrm{w}}(A)\backslash\{0\}$ and $\sigma_{\mathrm{e}}(B)\backslash\{0\} = \sigma_{\mathrm{w}}(B)\backslash\{0\}$, then the WSI holds with $\sigma_{\mathrm{e}}(A \otimes B) = \sigma_{\mathrm{w}}(A \otimes B)$.*

(d) *If $\sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{w}}(B) = \{0\}$, then the WSI holds with $\sigma_0(A \otimes B) = \sigma_0(A) \cdot \sigma_0(B)$.*

(e) *If $\sigma_{\mathrm{w}}(A) = \sigma(A)$ or $\sigma_{\mathrm{w}}(B) = \sigma(B)$, and if the WSI holds, then $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B) = \sigma(A \otimes B)$ and $\sigma_0(A \otimes B) = \sigma_0(A) \cdot \sigma_0(B) = \varnothing$.*

(f) *If $\sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{b}}(A)$, $\sigma_{\mathrm{w}}(B) = \sigma_{\mathrm{b}}(B)$ and the WSI holds, then $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B)$ (i.e. if A and B satisfy Browder's theorem, and if the WSI holds, then the tensor product $A \otimes B$ satisfies Browder's theorem).*

Proof. (a) Since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$, if $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B)$, we have

$$\begin{aligned}\sigma_{\mathrm{w}}(A \otimes B) &\subseteq \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B) \\ &\subseteq \sigma_{\mathrm{b}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{b}}(B) = \sigma_{\mathrm{b}}(A \otimes B) = \sigma_{\mathrm{w}}(A \otimes B)\end{aligned}$$

(see [**12**, Theorem 4.2 (a) and (f)]).

(b) Since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T) \subseteq \sigma(T)$, if $\sigma_0(A \otimes B) = \varnothing$, or equivalently, if $\sigma_{\mathrm{w}}(A \otimes B) = \sigma(A \otimes B)$, then $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{b}}(A \otimes B)$, and we get back to item (a).

(c) Suppose that $\sigma_{\mathrm{e}}(A)\backslash\{0\} = \sigma_{\mathrm{w}}(A)\backslash\{0\}$ and $\sigma_{\mathrm{e}}(B)\backslash\{0\} = \sigma_{\mathrm{w}}(B)\backslash\{0\}$, and recall that $\sigma_{\mathrm{e}}(T) \subseteq \sigma_{\mathrm{w}}(T)$. Therefore, since $\sigma_{\mathrm{w}}(A \otimes B) \subseteq \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B)$ and $\sigma_{\mathrm{e}}(A \otimes B) = \sigma_{\mathrm{e}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{e}}(B)$ [**12**, Theorem 4.2 (c) and (f)], we obtain

$$\begin{aligned}\sigma_{\mathrm{w}}(A \otimes B)\backslash\{0\} &\subseteq (\sigma_{\mathrm{w}}(A) \cdot \sigma(B))\backslash\{0\} \cup (\sigma(A) \cdot \sigma_{\mathrm{w}}(B))\backslash\{0\} \\ &= \sigma_{\mathrm{w}}(A)\backslash\{0\} \cdot \sigma(B)\backslash\{0\} \cup \sigma(A)\backslash\{0\} \cdot \sigma_{\mathrm{w}}(B)\backslash\{0\} \\ &= \sigma_{\mathrm{e}}(A)\backslash\{0\} \cdot \sigma(B)\backslash\{0\} \cup \sigma(A)\backslash\{0\} \cdot \sigma_{\mathrm{e}}(B)\backslash\{0\} \\ &= (\sigma_{\mathrm{e}}(A) \cdot \sigma(B))\backslash\{0\} \cup (\sigma(A) \cdot \sigma_{\mathrm{e}}(B))\backslash\{0\} \\ &= \sigma_{\mathrm{e}}(A \otimes B)\backslash\{0\} \subseteq \sigma_{\mathrm{w}}(A \otimes B)\backslash\{0\},\end{aligned}$$

and so

$$\sigma_{\mathrm{w}}(A \otimes B)\backslash\{0\} = \sigma_{\mathrm{w}}(A)\backslash\{0\} \cdot \sigma(B)\backslash\{0\} \cup \sigma(A)\backslash\{0\} \cdot \sigma_{\mathrm{w}}(B)\backslash\{0\},$$

which in turn implies that the WSI holds, since $0 \in \sigma(A) \cup \sigma(B)$ if and only if $0 \in \sigma_{\mathrm{w}}(A \otimes B)$, because $0 \notin \sigma_0(A \otimes B)$ [**19**, Proposition 5 (a)].

(d) Recall that the Weyl spectrum is non-empty on infinite-dimensional spaces. Since $\sigma_{\mathrm{w}}(A \otimes B) \subseteq \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B)$, it follows that if $\sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{w}}(B) = \{0\}$, then $\sigma_{\mathrm{w}}(A \otimes B) = \{0\}$ (because $0 \notin \sigma_0(A \otimes B)$), and so the WSI holds. Moreover, since $\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B)$ and $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{w}}(B) = \{0\}$, it follows that $\sigma(A \otimes B)\backslash\sigma_{\mathrm{w}}(A \otimes B) = \sigma(A)\backslash\sigma_{\mathrm{w}}(A) \cdot \sigma(B)\backslash\sigma_{\mathrm{w}}(B)$.

(e) If $\sigma_{\mathrm{w}}(A) = \sigma(A)$ (equivalently, if $\sigma_0(A) = \varnothing$) and if the WSI holds, then

$$\begin{aligned}\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B) &= \sigma(A) \cdot \sigma(B) \cup \sigma_{\mathrm{w}}(A) \cdot \sigma_{\mathrm{w}}(B) \\ &= \sigma(A) \cdot \sigma(B) \\ &= \sigma(A \otimes B),\end{aligned}$$

and hence $\sigma_0(A \otimes B) = \varnothing$, which leads to the claimed identities. Clearly, the assumption $\sigma_{\mathrm{w}}(A) = \sigma(A)$ can be replaced with $\sigma_{\mathrm{w}}(B) = \sigma(B)$.

(f) If $\sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{b}}(A)$ and $\sigma_{\mathrm{w}}(B) = \sigma_{\mathrm{b}}(B)$, and if the WSI holds, then

$$\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{w}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{w}}(B) = \sigma_{\mathrm{b}}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\mathrm{b}}(B) = \sigma_{\mathrm{b}}(A \otimes B).$$

That is, if A and B satisfy Browder's theorem, and if the WSI holds, then $A \otimes B$ satisfies Browder's theorem. □

Remark 4.2. Lemma 4.1 (f) provides a way to look for operators for which the WSI does not hold:

> if A and B satisfy Browder's theorem, and if $A \otimes B$ does not satisfy Browder's theorem, then the WSI does not hold.

The results in Lemma 4.1 (a) and Lemma 4.1 (f) were originally presented in [**19**, Propositions 6 (a) and 7 (a)], leading to the following equivalences (see [**19**, Corollary 6]).

(a) A and B satisfy Browder's theorem
$$\Longrightarrow \{A \otimes B \text{ satisfies Browder's theorem} \Longleftrightarrow \text{the WSI holds}\}.$$

Equivalently,

(b) $\{A$ and B satisfy Browder's theorem $\Longrightarrow$ $A \otimes B$ satisfies Browder's theorem$\}$
$$\Longleftrightarrow \text{ the WSI holds.}$$

Moreover, the following implication was shown in [**19**, Proposition 5].

(c) If the WSI holds, then $\sigma_0(A \otimes B) \subseteq \sigma_0(A) \cdot \sigma_0(B)$,

and the inclusion may be proper even if the WSI holds, with A, B and $A \otimes B$ being isoloid operators satisfying Weyl's theorem (see [**19**, Remark 2]). Lemma 4.1 (e) is a particular case of (c) giving another way to verify whether the WSI does not hold:

> if $\sigma_0(A) = \varnothing$ and if $\sigma_0(A \otimes B) \neq \varnothing$ for some B, then the WSI does not hold.

5. Biquasitriangular

The class of all biquasitriangular operators is quite a large class. For instance, let $\mathcal{N}$, $\mathcal{K}$, $\mathcal{A}$lg, $\mathcal{N}$il and $\mathcal{QN}$il denote the classes of normal, compact, algebraic, nilpotent and quasinilpotent operators, respectively. Let $\mathcal{N} + \mathcal{K}$ be the class of all sums of normal plus compact, which trivially includes $\mathcal{N}$ and $\mathcal{K}$ individually. These classes are included in the class of biquasitriangular operators, and are related as follows (see, for example, [**21**, pp. 37–40, 48]), where the upper bar stands for closure:

$$\mathcal{N} + \mathcal{K} \subset \mathcal{BQT}, \qquad \mathcal{N}\mathrm{il} \subset \mathcal{A}\mathrm{lg} \subset \mathcal{A}\mathrm{lg}^- = \mathcal{BQT}, \qquad \mathcal{N}\mathrm{il} \subset \mathcal{QN}\mathrm{il} \subset \mathcal{N}\mathrm{il}^- \subset \mathcal{BQT}.$$

Theorem 5.1. *Let A and B be operators acting on infinite-dimensional spaces.*

(a) *If A and B are $\mathcal{BQT}$, then the WSI holds.*

(b) *If A and B are $\mathcal{BQT}$, then the tensor product $A \otimes B$ is $\mathcal{BQT}$ as well.*

Proof. (a) If A and B are $\mathcal{BQT}$, then $\sigma_e(A) = \sigma_w(A)$ and $\sigma_e(B) = \sigma_w(B)$, so that $\sigma_e(A)\backslash\{0\} = \sigma_w(A)\backslash\{0\}$ and $\sigma_e(B)\backslash\{0\} = \sigma_w(B)\backslash\{0\}$, and therefore Lemma 4.1 (c) ensures that the WSI holds.

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(b) Suppose A and B are $\mathcal{BQT}$. This means that

$$\sigma_{\rm le}(A) = \sigma_{\rm re}(A) = \sigma_{\rm e}(A) = \sigma_{\rm w}(A) \quad \text{and} \quad \sigma_{\rm le}(B) = \sigma_{\rm re}(B) = \sigma_{\rm e}(B) = \sigma_{\rm w}(B).$$

Item (a) says that the WSI holds (the spectral identity holds for the Weyl spectrum). Since the spectral identity always holds for the essential spectrum (and since A and B are $\mathcal{BQT}$, which implies that $\sigma_{\rm e}(A) = \sigma_{\rm w}(B)$ and $\sigma_{\rm e}(A) = \sigma_{\rm w}(B)$), we obtain

$$\sigma_{\rm e}(A \otimes B) = \sigma_{\rm w}(A \otimes B).$$

Claim 5.2. *If T is $\mathcal{BQT}$, then $\sigma_{\rm AP}(T) = \sigma(T)$.*

Proof. Recall that, for any operator T (see, for example, [**17**, p. 148]),

$$\sigma_{\rm AP}(T) = \sigma_{\rm le}(T) \cup \sigma_{\rm PF}(T).$$

Therefore, if T is $\mathcal{BQT}$, then

$$\sigma_{\rm AP}(T) = \sigma_{\rm w}(T) \cup \sigma_{\rm PF}(T).$$

However, since $\sigma(T)\backslash\sigma_{\rm w}(T) = \sigma_0(T) \subseteq \sigma_{\rm PF}(T)$, it follows that

$$\sigma(T) = \sigma_{\rm w}(T) \cup \sigma_0(T) \subseteq \sigma_{\rm w}(T) \cup \sigma_{\rm PF}(T) = \sigma_{\rm AP}(T) \subseteq \sigma(T),$$

which concludes the proof of Claim 5.2. □

Now recall from [**13**, Theorem 4.4 (a) and (b)] that $\sigma_{\rm AP}(A \otimes B) = \sigma_{\rm AP}(A) \cdot \sigma_{\rm AP}(B)$ and

$$\sigma_{\rm le}(A \otimes B) = \sigma_{\rm le}(A) \cdot \sigma_{\rm AP}(B) \cup \sigma_{\rm AP}(A) \cdot \sigma_{\rm le}(B).$$

Thus, if A and B are $\mathcal{BQT}$, then Claim 5.2 ensures that

$$\sigma_{\rm le}(A \otimes B) = \sigma_{\rm e}(A) \cdot \sigma(B) \cup \sigma(A) \cdot \sigma_{\rm e}(B) = \sigma_{\rm e}(A \otimes B).$$

Dually, the above identity implies that if A and B are $\mathcal{BQT}$, then

$$\sigma_{\rm re}(A \otimes B) = \sigma_{\rm e}(A \otimes B).$$

(The duality holds in any appropriate Banach space setting, and has a nice proof in a Hilbert space setting. Indeed, since $\sigma_{\rm e}(T) = \sigma_{\rm e}(T^*)^*$, $\sigma_{\rm re}(T) = \sigma_{\rm le}(T^*)^*$, T is $\mathcal{BQT}$ if and only if T^* is $\mathcal{BQT}$ and $(A \otimes B)^* = A^* \otimes B^*$; it follows that $\sigma_{\rm re}(A \otimes B) = \sigma_{\rm le}((A \otimes B)^*)^* = \sigma_{\rm le}((A^* \otimes B^*))^* = \sigma_{\rm e}((A^* \otimes B^*))^* = \sigma_{\rm e}((A \otimes B)^*)^* = \sigma_{\rm e}(A \otimes B)$.) The outcome, then, is that

$$\sigma_{\rm le}(A \otimes B) = \sigma_{\rm re}(A \otimes B) = \sigma_{\rm e}(A \otimes B) = \sigma_{\rm w}(A \otimes B),$$

which means that $A \otimes B$ is $\mathcal{BQT}$. □

A note on the approximate WSI (a-WSI). The notion of the a-WSI is an extension of the WSI. Take the set

$$\sigma_{\mathrm{aw}}(T) = \{\lambda \in \sigma_{AP}(T) \colon \text{either } \lambda \in \sigma_{\mathrm{le}}(T) \text{ or } \lambda I - T \text{ has positive index}\},$$

which we refer to as the approximate Weyl spectrum of an operator T. We say that the a-WSI holds for a tensor product $A \otimes B$ if the original WSI holds with spectra replaced with approximate point spectra and Weyl spectra replaced with approximate Weyl spectra. In other words, the a-WSI holds for $A \otimes B$ if

$$\sigma_{\mathrm{aw}}(A \otimes B) = \sigma_{\mathrm{aw}}(A) \cdot \sigma_{\mathrm{AP}}(B) \cup \sigma_{\mathrm{AP}}(A) \cdot \sigma_{\mathrm{aw}}(B).$$

(See [**7**, Theorem 1] for the approximate version of Remark 4.2 (a).) Now observe that

$$T \in \mathcal{BQT} \quad \text{implies} \quad \sigma_{\mathrm{AP}}(T) = \sigma(T) \text{ and } \sigma_{\mathrm{aw}}(T) = \sigma_{\mathrm{w}}(T).$$

(Reason: Claim 5.2 in the proof of Theorem 5.1 ensures that $\sigma_{\mathrm{AP}}(T) = \sigma(T)$; moreover, $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{w}}(T)$, and $\lambda I - T$ has index zero for $\lambda \in \sigma_0(T) = \sigma(T) \backslash \sigma_{\mathrm{w}}(T)$.) Hence, for $\mathcal{BQT}$ operators, the a-WSI holds if and only if the WSI holds, and, in this case, the results in Remark 4.2 (a) and those in [**7**, Theorem 1] are equivalent.

Corollary 5.3. *Let A and B be operators acting on infinite-dimensional spaces.*

(a) *If $\sigma(A)^\circ = \sigma(B)^\circ = \varnothing$, then the WSI holds, and A, B and $A \otimes B$ are $\mathcal{BQT}$ and satisfy Browder's theorem.*

(b) *If $\sigma(A)^\circ = \sigma_{\mathrm{iso}}(A) = \varnothing$ and $\sigma(B)^\circ = \sigma_{\mathrm{iso}}(B) = \varnothing$, then (in addition) A, B and $A \otimes B$ satisfy Weyl's theorem with*

$$\sigma_0(A) = \sigma_0(B) = \sigma_0(A \otimes B) = \sigma_{\mathrm{iso}}(A \otimes B) = \varnothing.$$

Proof. (a) If $\sigma(A)^\circ = \sigma(B)^\circ = \varnothing$, then A and B are $\mathcal{BQT}$ and satisfy Browder's theorem by Lemma 3.2 (a). Thus, Theorem 5.1 (a) ensures that the WSI holds, and Theorem 5.1 (b) ensures that $A \otimes B$ is $\mathcal{BQT}$. Since A and B satisfy Browder's theorem and the WSI holds, it follows that $A \otimes B$ satisfies Browder's theorem by Lemma 4.1 (f).

(b) Recall that $\sigma_0(T) \subseteq \tau_0(T) \cup \sigma_{\mathrm{iso}}(T)$, where $\tau_0(T) = \sigma_0(T) \backslash \pi_0(T) \subseteq \sigma(T)$ is an open subset of $\mathbb{C}$ (see [**17**, Corollary 5.20]). Hence,

$$\sigma(T)^\circ = \sigma_{\mathrm{iso}}(T) = \varnothing \quad \text{implies} \quad \sigma_0(T) = \varnothing,$$

and so $\sigma(T)^\circ = \sigma_{\mathrm{iso}}(T) = \varnothing$ implies $\sigma_{\mathrm{iso}}(T) = \sigma_0(T) = \varnothing$, which in turn implies that T satisfies Weyl's theorem by Lemma 3.1 (c). Therefore, if $\sigma(A)^\circ = \sigma_{\mathrm{iso}}(A) = \varnothing$ and $\sigma(B)^\circ = \sigma_{\mathrm{iso}}(B) = \varnothing$, then (i) both A and B satisfy Weyl's theorem, and (ii) A and B are isoloid (there are no isolated points in their spectra, and so no isolated point that is not an eigenvalue). Since (a) ensures that the WSI holds, it follows by [**20**, Theorem 1] that $A \otimes B$ satisfies Weyl's theorem. Moreover, $\sigma_{\mathrm{iso}}(A) = \sigma_{\mathrm{iso}}(B) = \varnothing$ implies that $\sigma_{\mathrm{iso}}(A \otimes B) = \varnothing$ (see, for example, [**19**, Proposition 3 (c)]) and, since the WSI holds, $\sigma_0(A \otimes B) \subseteq \sigma_0(A) \cdot \sigma_0(B)$ (see Remark 4.2 (c)), so that $\sigma_0(A \otimes B) = \varnothing$. □

The following are immediate consequences of Corollary 5.3 (a).

Corollary 5.4. *Let A and B be operators acting on infinite-dimensional spaces.*

(a) $\#\sigma_{\mathrm{acc}}(A) < \infty$ *and* $\#\sigma_{\mathrm{acc}}(B) < \infty$ *imply that the WSI holds.*

In particular,

(b) $\sigma(A) = \sigma_{\mathrm{iso}}(A)$ *and* $\sigma(B) = \sigma_{\mathrm{iso}}(B)$ *imply that the WSI holds.*

Remark 5.5. (a) If the WSI holds, then $\sigma_0(A \otimes B) \subseteq \sigma_0(A) \cdot \sigma_0(B)$, as we saw in Remark 4.2 (c). Does the reverse inclusion imply that A and B are isoloid when the WSI holds? In other words, *is it true that if the WSI holds, and if* $\sigma_0(A \otimes B) = \sigma_0(A) \cdot \sigma_0(B)$, *then A and B must be isoloid*? Corollary 5.3 offers a negative answer to this question. Indeed, take a pair of compact weighted bilateral shifts A and B on ℓ^2, so that they are quasinilpotent and their spectra coincide with their continuous spectra, $\sigma(A) = \sigma_{\mathrm{C}}(A) = \sigma(B) = \sigma_{\mathrm{C}}(B) = \{0\}$, and hence they are not isoloid. Moreover, $\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) = \{0\}$. Since A and B act on infinite-dimensional spaces, their Weyl spectra are not empty, and so $\sigma_{\mathrm{w}}(A \otimes B) = \sigma_{\mathrm{w}}(A) = \sigma_{\mathrm{w}}(B) = \{0\}$, which implies that $\sigma_0(A \otimes B) = \sigma_0(A) \cdot \sigma_0(B) = \varnothing$. Furthermore, Corollary 5.3 (a) ensures that the WSI holds.

(b) The preceding question suggests the next one. *Is it true that if A and B are isoloid, then the WSI holds*? Again, the very same setup discussed in § 4 offers a negative answer to this question, too: the pair of operators that satisfy Weyl's theorem but whose tensor product does not satisfy Browder's theorem, given in [**14**, § 3], are isoloid, and the WSI does not hold for their tensor product according to [**19**, Corollary 6] (see Remark 4.2 (b)). Such a pair of operators exhibited in [**14**, § 3] is constructed as follows. Let S be the canonical unilateral shift on $\mathcal{X} = \ell^2_+$ and consider the following operators A and B on the (orthogonal) direct sum $\mathcal{X} \oplus \mathcal{X} = \ell^2_+ \oplus \ell^2_+$,

$$A = (I - SS^*) \oplus (\tfrac{1}{2}S - I), \qquad B = -(I - SS^*) \oplus (\tfrac{1}{2}S^* + I),$$

whose spectra are given by

$$\sigma(A) = \{0, 1\} \cup (\tfrac{1}{2}\mathbb{D} - 1), \qquad \sigma(B) = \{0, -1\} \cup (\tfrac{1}{2}\mathbb{D} + 1),$$

where $\mathbb{D}$ is the closed unit disc centred at the origin of the complex plane, and take their tensor product $A \otimes B$ acting on $(\mathcal{X} \oplus \mathcal{X}) \otimes (\mathcal{X} \oplus \mathcal{X})$, so that $\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) = (\{0, 1\} \cup (\tfrac{1}{2}\mathbb{D} - 1)) \cdot (\{0, -1\} \cup (\tfrac{1}{2}\mathbb{D} + 1))$. The operators A and B satisfy Weyl's theorem, while their tensor product $A \otimes B$ does not satisfy Browder's theorem [**14**, § 3]. Hence, according to [**19**, Corollary 6] (see Remark 4.2 (b)), the WSI does not hold. However, observe that the isolated points 0 and 1 of $\sigma(A)$ and -1 and 0 of $\sigma(B)$, are eigenvalues of A and B, and so A and B are isoloid.

Remark 5.6. An operator T is said to have the single-valued extension property (SVEP) at a point $\mu \in \mathbb{C}$ if, for every open neighbourhood $\Lambda_\mu \subseteq \mathbb{C}$ of μ, the only analytic solution $f \colon \Lambda_\mu \to \mathcal{X}$ to the equation

$$(\lambda I - T) f(\lambda) = 0 \quad \text{for every } \lambda \in \Lambda_\mu$$

is the null function ($f = 0$). Clearly, every operator T has the SVEP at every point of the resolvent set $\rho(T) = \mathbb{C}\backslash\sigma(T)$, and at every point in the boundary of $\sigma(T)$, and so at every isolated point of the spectrum $\sigma(T)$. An operator T is said to have the SVEP if it has the SVEP at every $\mu \in \sigma(T)$. A countable spectrum implies the SVEP, and, if T has no eigenvalues (i.e. if $\sigma_{\mathrm{P}}(T) = \varnothing$), then T has the SVEP; dually, if T^* has no eigenvalues (i.e. if $\sigma_{\mathrm{P}}(T^*) = \varnothing$), then T^* has the SVEP. It can be verified (see, for example, [**1**, Theorems 3.16 and 3.17, Corollary 3.19]) that the following result holds true.

> If T and T^* have the SVEP at every point in $\sigma_0(T) = \sigma(T)\backslash\sigma_{\mathrm{w}}(T)$, then T is $\mathcal{BQT}$ and satisfies Browder's theorem (i.e. $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$).

In such a case, $\sigma_{\mathrm{b}}(T) = \sigma(T)$ if and only if T has no isolated eigenvalues of finite multiplicity (see Lemma 3.2 (b)). Observe that for a decomposable operator T, both T and T^* have the SVEP [**1**, Theorem 6.21], and so T is $\mathcal{BQT}$. According to the above displayed result we obtain the following consequence of Theorem 5.1 and Lemma 4.1 (f).

> If A and A^* have the SVEP on $\sigma_0(A)$, and if B and B^* have the SVEP on $\sigma_0(B)$, then the WSI holds, and A, B and $A \otimes B$ are $\mathcal{BQT}$ and satisfy Browder's theorem.

In particular, if A, B, A^* and B^* have the SVEP, then the WSI holds, and A, B and $A \otimes B$ are $\mathcal{BQT}$ and satisfy Browder's theorem (compare with Corollary 5.3 (a)).

Remark 5.7. Let A and B be arbitrary operators and consider the (bounded linear) transformers L_A and R_B, the left and right multiplication, defined by $L_A(X) = AX$ and $R_B(X) = XB$ for every operator X. Recall that the transformers $\delta_{A,B}$ (the generalized derivation) and $\varDelta_{A,B}$ (the elementary operator) are given by $\delta_{A,B} = L_A - R_B$ and $\varDelta_{A,B} = L_A R_B - I$, where I stands for the identity: $I(X) = X$. The following corollary of Theorem 5.1 was verified in [**6**, Corollary 1].

(a) If A and B are $\mathcal{BQT}$, then $\sigma_{\mathrm{w}}(L_A R_B) = \sigma(A) \cdot \sigma_{\mathrm{w}}(B) \cup \sigma_{\mathrm{w}}(A) \cdot \sigma(B)$.

(b) If A and B are $\mathcal{BQT}$, then $L_A R_B$ is $\mathcal{BQT}$.

Note that the identity $\sigma_{\mathrm{w}}(L_A R_B) = \sigma(A) \cdot \sigma_{\mathrm{w}}(B) \cup \sigma_{\mathrm{w}}(A) \cdot \sigma(B)$ is the analogue of the WSI (which was defined for tensor products), replacing $A \otimes B$ with $L_A R_B$. The above result implies, in particular, that if A is $\mathcal{BQT}$, then L_A and R_A are $\mathcal{BQT}$. It was also shown in [**6**] that if A and B are $\mathcal{BQT}$, then $\delta_{A,B} = L_A - R_B$ is $\mathcal{BQT}$, and $\sigma_{\mathrm{w}}(\delta_{A,B}) \subseteq (\sigma(A) - \sigma_{\mathrm{w}}(B)) \cup (\sigma_{\mathrm{w}}(A) - \sigma(B))$. (Also see [**8**].)

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