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THE UNDECIDABILITY OF THE DEFINABILITY OF PRINCIPAL SUBCONGRUENCES

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Abstract. For each Turing machine $\mathcal{T}$, we construct an algebra $\mathbb{A}'(\mathcal{T})$ such that the variety generated by $\mathbb{A}'(\mathcal{T})$ has definable principal subcongruences if and only if $\mathcal{T}$ halts, thus proving that the property of having definable principal subcongruences is undecidable for a finite algebra. A consequence of this is that there is no algorithm that takes as input a finite algebra and decides whether that algebra is finitely based.

§1. Introduction. Given a variety $\mathcal{V}$, the *residual bound* of $\mathcal{V}$ is the least cardinal λ that is strictly larger than the cardinality of every subdirectly irreducible (=SI) member of $\mathcal{V}$. If such a λ exists we write $\kappa(\mathcal{V}) = \lambda$, and if no such λ exists, then we write $\kappa(\mathcal{V}) = \infty$. If $\mathcal{V} = \mathcal{V}(\mathbb{A})$ is the variety generated by the algebra $\mathbb{A}$, we define $\kappa(\mathbb{A}) = \kappa(\mathcal{V}(\mathbb{A}))$. The *RS-conjecture* (see [4]) is the conjecture that $\kappa(\mathbb{A}) \geq \omega$ implies $\kappa(\mathbb{A}) = \infty$ for finite $\mathbb{A}$. McKenzie [7] disproves this conjecture by exhibiting an algebra with residual bound precisely ω. This algebra is used by McKenzie as a basis for his groundbreaking paper [6], in which to each Turing machine $\mathcal{T}$ an algebra $\mathbb{A}(\mathcal{T})$ is associated such that $\kappa(\mathbb{A}(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts, thus proving that the property of having a finite residual bound is an undecidable property of a finite algebra.

The problem of algorithmically determining whether $\kappa(\mathbb{A}) < \omega$ is closely related to a problem due to Alfred Tarski, called *Tarski's finite basis problem*. An algebra $\mathbb{A}$ is said to be *finitely based* if the infinite set of identities which are true in $\mathbb{A}$ can be derived from a finite subset of them. Tarski's problem is the question: is there an algorithm that takes as input a finite algebra and determines whether it is finitely based? McKenzie [8] uses a construction similar to his $\mathbb{A}(\mathcal{T})$ construction to provide a negative answer to this question, and Willard [10] shows that, in fact, the original $\mathbb{A}(\mathcal{T})$ is finitely based if and only if $\mathcal{T}$ halts. Thus, there is no algorithm that decides whether an algebra is finitely based for every finite algebra.

An algebra $\mathbb{A}$ is finitely based if and only if $\mathcal{V}(\mathbb{A})$ (the variety generated by $\mathbb{A}$) is finitely axiomatizable. One approach to proving that $\mathcal{V}(\mathbb{A})$ is finitely axiomatizable is to first show that $\kappa(\mathbb{A}) < \omega$, and then to show that $\mathcal{V}(\mathbb{A})$ has definable principal congruences. These two features are sufficient to imply that $\mathcal{V}(\mathbb{A})$ is finitely axiomatizable. A formula $\psi(w, x, y, z)$ defined in an algebraic first-order language L is

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said to be a *congruence formula* if ψ is existential positive and for every model $\mathbb{A}$ of L and for all $a, b, c, d \in A$, $\mathbb{A} \models \psi(a, b, c, d)$ implies that (a, b) belongs to the congruence generated by the pair (c, d). A class $\mathcal{C}$ of algebras of the same type is said to have *definable principal congruences (DPC)* if there is a congruence formula ψ such that for every $\mathbb{A} \in \mathcal{C}$ and every $a, b \in A$ the formula $\psi(x, y, a, b)$ defines the relation "$(x, y) \in \mathrm{Con}^{\mathbb{B}}(a, b)$". McKenzie [5] shows that if a variety $\mathcal{V}$ has definable principal congruences and $\kappa(\mathcal{V}) < \omega$, then $\mathcal{V}$ is finitely based.

Baker and Wang [2] generalize DPC by saying that a class of algebras $\mathcal{C}$ (all of the same type) has *definable principal subcongruences (DPSC)* if there are congruence formulas Γ and ψ such that for all $\mathbb{A} \in \mathcal{C}$ and all $a, b \in A$, if $a \neq b$ then there exist $c, d \in A$ such that $c \neq d$, $\mathbb{A} \models \Gamma(c, d, a, b)$, and $\psi(-, -, c, d)$ defines $\mathrm{Cg}^{\mathbb{A}}(c, d)$. It is convenient to observe that if the type of $\mathcal{C}$ is finite, then there is a first-order formula, $\Pi_\psi(y, z)$ so that $\mathbb{A} \models \Pi_\psi(c, d)$ (where $\mathbb{A}$ is any algebra of this type) if and only if $\{(a, b) \mid \mathbb{A} \models \psi(a, b, c, d)\}$ is the congruence generated by (c, d). In symbols, a class $\mathcal{C}$ of algebras of the same finite type has DPSC if there are congruence formulas $\Gamma(w, x, y, z)$ and $\psi(w, x, y, z)$ such that for all $\mathbb{A} \in \mathcal{C}$,

$$\mathbb{A} \models \forall a, b \left[a \neq b \rightarrow \exists c, d \left[c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_\psi(c, d)\right]\right].$$

See Figure 1. Baker and Wang [2] use the fact that congruence distributive varieties have definable principal subcongruences to give a new proof of K. Baker's Finite Basis Theorem [1]: if $\mathbb{A}$ is a finite algebra of finite type and $\mathcal{V}(\mathbb{A})$ is congruence distributive, then $\mathbb{A}$ is finitely based. Willard [11] extends Baker's theorem by showing that if the variety has finite type, is residually finite, and is congruence $\wedge$-semidistributive ($\mathcal{V}(\mathbb{A}(\mathcal{T}))$ has these features if $\mathcal{T}$ halts), then the variety is finitely based. Since $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ is finitely axiomatizable if and only if $\mathcal{T}$ halts, and finitely axiomatizability is so closely related to DPC and DPSC, it is natural to consider whether the failure of finite axiomatizability when $\mathcal{T}$ does not halt is related to a failure of DPC or DPSC in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$.

The main result of this paper is to construct an algebra $\mathbb{A}'(\mathcal{T})$ based on McKenzie's $\mathbb{A}(\mathcal{T})$ and to show that $\mathbb{A}'(\mathcal{T})$ has definable principal subcongruences

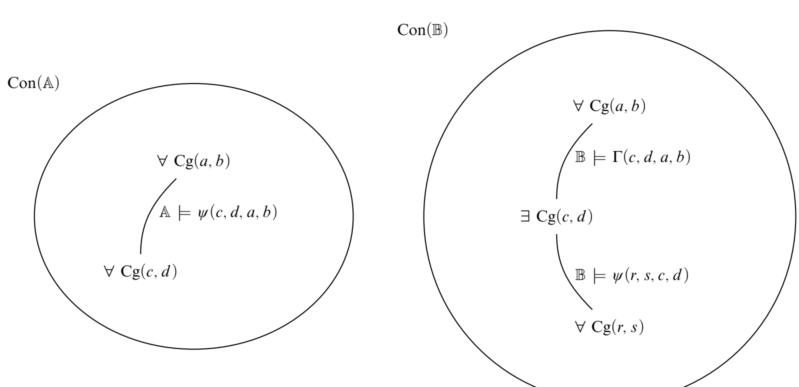

FIGURE 1. $\mathbb{A}$ has DPC via ψ, and $\mathbb{B}$ has DPSC via Γ and ψ.

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if and only if $\mathcal{T}$ halts. Since the halting problem is undecidable, this proves that the property of having DPSC is undecidable (i.e., there is no algorithm that takes as input a finite algebra and giving as output the correct answer to the question: "does the variety generated by this algebra have DPSC?"). The proof of this involves many cases, an exploration of "$\mathbb{A}'(\mathcal{T})$-arithmetic", and a fine analysis of the polynomials of $\mathbb{A}'(\mathcal{T})$. We begin in Section 2 with a description of the algebra $\mathbb{A}'(\mathcal{T})$. Section 3 details the modifications to McKenzie's original argument that are necessary to show that $\kappa(\mathbb{A}'(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts. Section 4 then gives a detailed description of the subdirectly irreducible algebras of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ that will be needed throughout. The proof that DPSC is undecidable is broken into two cases, depending on whether $\mathcal{T}$ does or does not halt. The case where $\mathcal{T}$ halts is addressed in Section 5, and is quite complicated, involving many subcases. The case where $\mathcal{T}$ does not halt is addressed in Section 6, and a short negative answer to Tarski's problem using the undecidability of definable principal subcongruences is given in this section as well.

The results in this paper originated with the examination of properties of $\mathbb{A}(\mathcal{T})$. Finite axiomatizability is closely related to the properties of definable principal congruences and definable principal subcongruences, and there was a natural question of whether McKenzie's negative answer to Tarski's finite basis problem was the consequence of a more primitive result concerning either DPC or DPSC. Although it is true that this is the situation for $\mathbb{A}'(\mathcal{T})$, it was recently shown by the author in [9] that the original $\mathbb{A}(\mathcal{T})$ does *not* have DPSC. This is the first known example of a congruence $\wedge$-semidistributive variety with finite residual bound that does not have DPSC. The methods used to prove the undecidability of definable principal subcongruences do not appear to be amenable to proving the undecidability of definable principal congruences, but the overall structure of the argument and the fine analysis of polynomials in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ may provide a foundation for proving the undecidability of DPC as well.

§2. Defining $\mathbb{A}'(\mathcal{T})$.

We define a *Turing machine* $\mathcal{T}$ to be a finite list of 5-tuples (s, r, w, d, t), called the *instructions* of $\mathcal{T}$, and interpreted as "if in state s and reading r, then write w, move d, and enter state t." The set of states is finite, $r, w \in \{0, 1\}$, and $d \in \{\mathrm{L}, \mathrm{R}\}$. A Turing machine takes as input an infinite bidirectional tape $\tau : \mathbb{Z} \to \{0, 1\}$ that has finite support (i.e., $\tau^{-1}(\{1\})$ is finite). If $\mathcal{T}$ stops computation on some input, then $\mathcal{T}$ is said to have *halted* on that input. For this reason, we say that the Turing machine halts (without specifying the input) if it halts on the empty tape $\tau(x) = 0$. We enumerate the states of $\mathcal{T}$ as $\{\mu_0, \dots, \mu_n\}$, where μ_1 is the initial (starting) state, and μ_0 is the halting state (which might not ever be reached). $\mathcal{T}$ has no instruction of the form (μ_0, r, w, d, t) but for every pair (i, r) with $1 \le i \le n$ and $r \in \{0, 1\}$ does have precisely one instruction of the form (μ_i, r, w, d, t).

Given a Turing machine $\mathcal{T}$ with states $\{\mu_0, \dots, \mu_n\}$, we associate to $\mathcal{T}$ an algebra $\mathbb{A}'(\mathcal{T})$. We will now describe the algebra $\mathbb{A}'(\mathcal{T})$. Let

$$U = \{1, 2, \mathrm{H}\}, \qquad W = \{C, D, \partial C, \partial D\}, \qquad A = \{0\} \cup U \cup W,$$
$$V_{ir}^{s} = \{C_{ir}^{s}, D_{ir}^{s}, M_i^r, \partial C_{ir}^{s}, \partial D_{ir}^{s}, \partial M_i^r\} \quad \text{for} \quad 0 \le i \le n \text{ and } \{r, s\} \subseteq \{0, 1\},$$
$$V_{ir} = V_{ir}^{0} \cup V_{ir}^{1}, \qquad V_i = V_{i0} \cup V_{i1}, \qquad V = \bigcup\{V_i \mid 0 \le i \le n\}.$$

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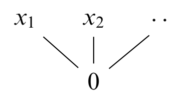
The underlying set of $\mathbb{A}'(\mathcal{T})$ is $A'(\mathcal{T}) = A \cup V$. In the operations defined below, the "∂" is taken to be a permutation of order 2 with domain $V \cup W$ (e.g. $\partial\partial C = C$), and is referred to as "bar". It should be mentioned that ∂ is *not* an operation of $\mathbb{A}'(\mathcal{T})$. We now describe the fundamental operations of $\mathbb{A}'(\mathcal{T})$. The algebra $\mathbb{A}'(\mathcal{T})$ is a height 1 $\wedge$-semilattice (i.e., it is "flat") with bottom element 0:

$$x \wedge y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$



This semilattice structure induces an order, $\leq$, on algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. There is a binary nonassociative "multiplication", defined by

$$2 \cdot D = H \cdot C = D, \qquad 1 \cdot C = C,$$
$$2 \cdot \partial D = H \cdot \partial C = \partial D, \qquad 1 \cdot \partial C = \partial C,$$

and $x \cdot y = 0$, otherwise. The next operations play the role of controlling the production of large SI's (i.e., those SI's not contained in $\mathbf{HS}(\mathbb{A}'(\mathcal{T}))$) in McKenzie's original argument. Such SI's are fully described in Section 4. Define

$$J(x, y, z) = (x \wedge \partial y \wedge z) \vee (x \wedge y) = \begin{cases} x & \text{if } x = y, \\ x \wedge z & \text{if } x = \partial y \in V \cup W, \\ 0 & \text{otherwise,} \end{cases}$$

$$J'(x, y, z) = (x \wedge y \wedge z) \vee (x \wedge \partial y) = \begin{cases} x \wedge z & \text{if } x = y, \\ x & \text{if } x = \partial y \in V \cup W, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} K(x, y, z) &= (\partial x \wedge y) \vee (\partial x \wedge \partial y \wedge z) \vee (x \wedge y \wedge z) \\ &= \begin{cases} y & \text{if } x = \partial y \in V \cup W, \\ z & \text{if } x = y = \partial z \in V \cup W, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases} \end{aligned}$$

(In expressions like $x \wedge \partial y \wedge z$, if y does not lie in the domain of ∂, then we take ∂y to be 0.) As we will see, the J and J' operations force a certain easy-to-analyze structure on the SI's of the variety, and the K operation allows us to simplify certain kinds of polynomials in the SI's in $\mathbf{HS}(\mathbb{A}'(\mathcal{T}))$ (i.e., the small SI's). This simplification of polynomials will be the key to showing that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has DPSC when $\mathcal{T}$ halts. Define

$$S_0(u, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u \in V_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_1(u, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u \in \{1, 2\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_2(u, v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u = \partial v \in V \cup W, \\ 0 & \text{otherwise.} \end{cases}$$

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The operation of left multiplication by y, $\lambda_y(x) = y \cdot x$, is, in general, not injective. The next operation will allow us to produce "barred" elements (i.e., produce ∂x from x) in cases when λ_y is not injective. Let

$$T(w,x,y,z) = \begin{cases} w \cdot x & \text{if } w \cdot x = y \cdot z \text{ and } (w,x) = (y,z), \\ \partial(w \cdot x) & \text{if } w \cdot x = y \cdot z \neq 0 \text{ and } (w,x) \neq (y,z), \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define operations that emulate the computation of the Turing machine on some tape. First, we define an operation that when applied to certain elements of $A'(\mathcal{T})^{\mathbb{Z}}$ will produce something that represents a "blank tape":

$$I(x) = \begin{cases} C_{10}^0 & \text{if } x = 1, \\ M_1^0 & \text{if } x = \mathrm{H}, \\ D_{10}^0 & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For each instruction of $\mathcal{T}$ of the form $(\mu_i, r, s, \mathrm{L}, \mu_j)$ and each $t \in \{0,1\}$ define an operation

$$L_{irt}(x,y,u) = \begin{cases} C_{jt}^{s'} & \text{if } x = y = 1 \text{ and } u = C_{ir}^{s'} \text{ for some } s', \\ M_j^t & \text{if } x = \mathrm{H},\ y = 1, \text{ and } u = C_{ir}^t, \\ D_{jt}^s & \text{if } x = 2,\ y = \mathrm{H}, \text{ and } u = M_i^r, \\ D_{jt}^{s'} & \text{if } x = y = 2 \text{ and } u = D_{ir}^{s'} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } L_{irt}(x,y,\partial u) = v \in V \text{ by the above lines}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the set of all such operations. Similarly, for each instruction of $\mathcal{T}$ of the form $(\mu_i, r, s, \mathrm{R}, \mu_j)$ and each $t \in \{0,1\}$ define an operation

$$R_{irt}(x,y,u) = \begin{cases} C_{jt}^{s'} & \text{if } x = y = 1 \text{ and } u = C_{ir}^{s'} \text{ for some } s', \\ C_{jt}^s & \text{if } x = \mathrm{H},\ y = 1, \text{ and } u = M_i^r, \\ M_j^t & \text{if } x = 2,\ y = \mathrm{H}, \text{ and } u = D_{ir}^t, \\ D_{jt}^{s'} & \text{if } x = y = 2 \text{ and } u = D_{ir}^{s'} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } R_{irt}(x,y,\partial u) = v \in V \text{ by the above lines}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{R}$ be the set of all such operations.

When applied to certain elements from $A'(\mathcal{T})^{\mathbb{Z}}$, these operations simulate the computation of the Turing machine $\mathcal{T}$ on different input tapes. Certain other elements of $\{1,2,H\}^{\mathbb{Z}}$ serve to track the position of the Turing machine's head when operations from $\mathcal{L} \cup \mathcal{R}$ are applied to elements of $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$ that encode the contents of the tape. Define a binary relation $\prec$ on $\{1,2,\mathrm{H}\}$ by $x \prec y$ if and only if $(x,y) = (2,2)$, $(x,y) = (2,\mathrm{H})$, or $(x,y) = (1,1)$. For $F \in \mathcal{L} \cup \mathcal{R}$ note that $F(x,y,z) = 0$ except when $x \prec y$. As with the multiplication operation, operations of the form $m(x) = F(u,v,x)$ with $F \in \mathcal{L} \cup \mathcal{R}$ are not injective. The next two operations are very much like the T operation in that they allow us to produce

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barred elements in some situations, where $m(x)$ fails to be injective. For $F \in \mathcal{L} \cup \mathcal{R}$ define operations

$$U_F^0(x, y, z, u) = \begin{cases} \partial F(y, z, u) & \text{if } x \prec z,\ x \neq y, \text{ and } F(y, z, u) \neq 0, \\ F(y, z, u) & \text{if } x \prec z,\ x = y, \text{ and } F(y, z, u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$U_F^1(x, y, z, u) = \begin{cases} \partial F(x, y, u) & \text{if } x \prec z,\ y \neq z, \text{ and } F(x, y, u) \neq 0, \\ F(x, y, u) & \text{if } x \prec z,\ y = z, \text{ and } F(x, y, u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operations of $\mathbb{A}'(\mathcal{T})$ are

$$\{0, \wedge, (\cdot), J, J', K, S_0, S_1, S_2, T, I\} \cup \mathcal{L} \cup \mathcal{R} \cup \{U_F^0, U_F^1 \mid F \in \mathcal{L} \cup \mathcal{R}\}.$$

Observe that all operations of $\mathbb{A}'(\mathcal{T})$ are monotonic with respect to the order induced by the semilattice structure. The $\mathbb{A}(\mathcal{T})$ algebra from [6] has the same underlying set as $\mathbb{A}'(T)$, and all of the same operations except for K. McKenzie proved the following theorem.

THEOREM 2.1 (McKenzie [6]). *$\kappa(\mathbb{A}(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts.*

The fact that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has only finitely many subdirectly irreducible algebras, all finite, if $\mathcal{T}$ halts is needed to prove that this variety has definable principal subcongruences. Since we have modified the algebra that this theorem refers to, we must show that this theorem as well as other important properties of $\mathbb{A}(\mathcal{T})$ still hold.

§3. Modifying McKenzie's Argument.

McKenzie's argument is quite detailed and long, and fortunately only needs to be added to, not changed. In this section we will detail the specific additions to the arguments in papers [7] and [6] that are needed, in order to prove that the large subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ satisfy $K(x, y, z) = x \wedge y \wedge z$ and are otherwise precisely the same as the large subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ as described in [6].

LEMMA 3.1. *Suppose that $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is flat and $\mathbb{B} \models S_2(u, v, x, y, z) \approx 0$. Then*

(1) $\mathbb{B} \models K(x, y, z) \approx x \wedge y \wedge z$, *and*
(2) $\mathbb{B} \models J'(x, y, z) \approx x \wedge y \wedge z$.

PROOF. We begin with item (1). $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$, so say $\mathbb{B} = \mathbb{C}/\theta$, where $\mathbb{C} \leq \mathbb{A}'(\mathcal{T})^L$ and $\theta \in \mathrm{Con}(\mathbb{C})$. Suppose that $B \not\models K(x, y, z) \approx x \wedge y \wedge z$. Then there are $a, b, c \in C$ with $K(a, b, c) \not\theta (a \wedge b \wedge c)$. In particular, from the definition of K this means that at least 2 of a, b, c lie in distinct θ-classes, and by flatness, $(a \wedge b \wedge c)\ \theta\ 0$. We therefore have that $K(a, b, c) \not\theta\ 0$.

Let $\alpha = K(a, b, c)$. From the definition of K and since $K(a, b, c) \neq (a \wedge b \wedge c)$, for each $l \in \mathrm{Supp}(\alpha)$ we have $a(l) \in \{\alpha(l), \partial\alpha(l)\}$. If $\alpha\ \theta\ a$, then since $\mathbb{A}'(\mathcal{T}) \models K(x, y, z) \wedge x \approx K(x, y, z) \wedge x \wedge y \wedge z$,

$$K(a, b, c) = \alpha\ \theta\ \alpha \wedge a = K(a, b, c) \wedge a = K(a, b, c) \wedge a \wedge b \wedge c.$$

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By the flatness of $\mathbb{B}$, this implies $K(a,b,c)\ \theta\ (a \wedge b \wedge c)$, which contradicts observations in the first paragraph. It follows that $\alpha\ \not\theta\ a$, so $(\alpha \wedge a)\ \theta\ 0$. From the definition of K, the hypothesis that $\mathbb{B} \models S_2(u,v,x,y,z) \approx 0$, and these observations,

$$0\ \theta\ S_2(a,\alpha,a,a,a) = K(\alpha,a,0)\ \theta\ K(\alpha,a,\alpha \wedge a) = \begin{cases} a(l) & \text{for } l \in \operatorname{Supp}(\alpha), \\ 0 & \text{for } l \notin \operatorname{Supp}(\alpha). \end{cases}$$

Let $a' = K(\alpha,a,\alpha \wedge a)$. Then $K(a,b,c) = K(a',b,c)\ \theta\ K(0,b,c) = 0$, a contradiction.

We will now prove item (2) from item (1). Assume to the contrary that $\mathbb{B} \not\models J'(x,y,z) \approx x \wedge y \wedge z$. Then there are $d,e,f \in B$ such that $J'(d,e,f) \neq d \wedge e \wedge f$. If $d \wedge e \wedge f \neq 0$, then by the flatness of $\mathbb{B}$ it follows that $d = e = f$. In this case $J'(d,e,f) = d \wedge e \wedge f$ by definition, so it must be that $d \wedge e \wedge f = 0$. Thus, we are assuming that $J'(d,e,f) \neq 0$.

Since $\mathbb{A}'(\mathcal{T}) \models K(x,y,x) \approx J'(x,y,x) \geq J'(x,y,z)$, for any $d,e,f \in B$ by the proof of item (1) above we have

$$d \wedge e = K(d,e,d) = J'(d,e,d) \geq J'(d,e,f).$$

$\mathbb{B}$ is flat, and by the previous paragraph $J'(d,e,f) \neq 0$. This means that $J(d,e,f) = d \wedge e \neq 0$. Therefore $d = e$, and

$$J'(d,e,f) = J'(d,d,f) = d \wedge d \wedge f = d \wedge e \wedge f,$$

a contradiction. ⊣

Many additions to McKenzie's argument occur where induction on polynomial complexity is used, and the following lemma is the crux of the additional argumentation in most of these instances.

Lemma 3.2. *Let L be an index set and suppose that $\mathbb{B} \leq \mathbb{A}'(\mathcal{T})^L$ and $C \subseteq B$ are such that*

(1) *if $c \in C$ then $c(l) \neq 0$ for all $l \in L$ (we will say that c is nowhere 0), and*
(2) *if $c \in C$ and $a \in B$ are such that $c(l) \in \{a(l), \partial a(l)\}$ for all $l \in L$, then $c = a$.*

If $f_1(x), f_2(x), f_3(x)$ are polynomials of $\mathbb{B}$ such that for all i either $f_i(x)$ is constant or $f_i^{-1}(C) \subseteq C$, then the polynomial $f(x) = K(f_1(x), f_2(x), f_3(x))$ is also either constant in $\mathbb{B}$ or $f^{-1}(C) \subseteq C$.

Proof. Let $f_1(x), f_2(x), f_3(x)$ be as in the statement of the lemma, and let $f(x) = K(f_1(x), f_2(x), f_3(x))$. We will show that $f(x)$ is either constant or $f^{-1}(C) \subseteq C$. Suppose that $a \in B$ and $f(a) \in C$. Since

$$\begin{aligned} f(a) &= K(f_1(a), f_2(a), f_3(a)) \\ &= (\partial f_1(a) \wedge f_2(a)) \vee (\partial f_1(a) \wedge \partial f_2(a) \wedge f_3(a)) \vee (f_1(a) \wedge f_2(a) \wedge f_3(a)), \end{aligned}$$

and $f(a)$ is nowhere 0 and $\mathbb{A}'(\mathcal{T})$ is flat, for each $l \in L$

- $\partial f_1(a)(l) = f_2(a)(l) = f(a)(l)$, or
- $\partial f_1(a)(l) = \partial f_2(a)(l) = f_3(a)(l) = f(a)(l)$, or
- $f_1(a)(l) = f_2(a)(l) = f_3(a)(l) = f(a)(l)$.

For $b \in \{f_1(a), f_2(a)\}$ it follows from these formulas that $b(l) \in \{f(a)(l), \partial f(a)(l)\}$ for all $l \in L$. Hypothesis (2) implies that $f(x) = f_1(a) = f_2(a)$, and then $f(a) \leq f_3(a)$. Since $f(a)$ is nowhere 0, we have $f(a) = f_3(a)$ as well. Thus, $f_i(a) \in C$ for $i \in \{0, 1, 2\}$. Thus, finally, if $a \notin C$, then each f_i is constant on B, and then so is f. ⊣

Definition 3.3. Let $\mathcal{C}$ be a class of algebras of the same type whose reduct to $\{0, \wedge\}$ is a meet semilattice. $\mathcal{C}$ is said to be 0-*absorbing* if for every fundamental operation $F(x_1, \ldots, x_n)$, every $\mathbb{A} \in \mathcal{C}$, and every $a_1, \ldots, a_n \in A$,

$$0 \in \{a_1, \ldots, a_n\} \quad \text{implies} \quad F(a_1, \ldots, a_n) = 0.$$

$\mathcal{C}$ is said to *commute with* $\wedge$ if for every fundamental operation F of some arity n,

$$\mathcal{C} \models F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \approx F(x_1 \wedge y_1, \ldots, x_n \wedge y_n).$$

We now enumerate the needed additions to McKenzie's proofs in papers [7] and [6]. To avoid needlessly long definitions and discussions, the additions will be presented assuming that the reader has the appropriate paper on hand to reference. Overall, we will proceed through the main argument in [6] and divert to [7] when the main argument makes reference to it.

(1) In general, we note that K is monotonic, and if $\mathbb{A} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is flat and $\mathbb{A} \models S_2(u, v, x, y, z) \approx 0$, then $\mathbb{A} \models J'(x, y, z) \approx K(x, y, z) \approx x \wedge y \wedge z$. This is Lemma 3.1.

(2) In [6] in the proof of Lemma 4.1, elements α_n and β_n of $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$ are defined as

$$\alpha_n(k) = \begin{cases} 1 & \text{if } k < n, \\ H & \text{if } k = n, \\ 2 & \text{if } k > n, \end{cases} \quad \text{and} \quad \beta_n(k) = \begin{cases} C & \text{if } k < n, \\ D & \text{if } k \geq n. \end{cases}$$

Let Γ be the subuniverse of the algebra generated by these elements, Σ the set of all configuration elements generated by the α_n (that is, the set of all nowhere 0 outputs of $\mathcal{L} \cup \mathcal{R} \cup \{I\}$), and Γ_0 the subset of Γ consisting of elements that are 0 at some coordinate. It is necessary to prove that the set

$$\Gamma' = \Gamma_0 \cup \Sigma \cup \{\alpha_n, \beta_n \mid n \in \mathbb{Z}\}$$

is closed under the operation K. By construction, if $u \in \Gamma' \setminus \Gamma_0$, then for each $l \in L$, $u(l)$ cannot be a barred element (e.g. ∂C, ∂D, ∂C^s_{ir}, etc.). From the definition of K, we have that if $a, b, c \in \Gamma' \setminus \Gamma_0$, then $K(a, b, c) = a \wedge b \wedge c \in \Gamma$. Thus $K(\Gamma' \setminus \Gamma_0, \Gamma' \setminus \Gamma_0, \Gamma' \setminus \Gamma_0) \subseteq \Gamma'$. The set Γ_0 contains elements that have a value of 0 at some coordinate. Since K is 0-absorbing in its first and second coordinates, $K(\Gamma_0, \Gamma', \Gamma') \cup K(\Gamma', \Gamma_0, \Gamma') \subseteq \Gamma_0$. Furthermore, if $a, b \in \Gamma' \setminus \Gamma_0$ and $c \in \Gamma_0$ then since $a(l) \neq \partial b(l)$ for any $l \in \mathbb{Z}$ ($\Gamma' \setminus \Gamma_0$ contains no barred elements), we have that $K(a, b, c) = a \wedge b \wedge c$, so in this case $K(a, b, c) \in \Gamma_0$ since $w \in \Gamma_0$. Therefore $K(\Gamma' \setminus \Gamma_0, \Gamma' \setminus \Gamma_0, \Gamma_0) \subseteq \Gamma_0$. We have now show that $K(\Gamma', \Gamma', \Gamma') \subseteq \Gamma'$, so we are done.

(3) The proof of Lemma 5.3 in [6] needs only a few added words at the end of the first full paragraph on page 41 to demonstrate that our new operation K can be dealt with the same way as the operations J and J' are handled in this proof.

(4) Prior to the statement of Lemma 5.5 in [6], it is written that the lemma is a restatement of Lemmas 6.7-6.9 of [7]. All of these lemmas go through

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without modification, except for Lemma 6.8. Lemma 6.8 concerns itself with a subalgebra $\mathbb{B}$ of $\mathbb{A}'(\mathcal{T})^L$ and a subset B_1 of B defined by

$$B_1 = \{u \in B \mid u = p \text{ or } x_0 x_1 \cdots x_n = p \text{ and } u \in \{x_0 \ldots, x_n\}\}$$

(p is a fixed element of $\mathbb{B}$ that has the property, amongst many others, of being nowhere 0). The product in $x_0 x_1 \cdots x_n$ in the definition of B_1 associates to the right. At the very start of the proof of Lemma 6.8, induction on the complexity of polynomials is used to prove that if $u \in B$ and $f(u) \in B_1$ then $f(x)$ is either constant or $u \in B_1$. Lemma 6.6 in [7] states that B_1 consists of elements that are nowhere 0 and having the property that if $u \in B_1$ and $v \in B$ are such that $u(l) \in \{v(l), \partial v(l)\}$ for all $l \in L$, then $u = v$. Taking $C = B_1$ in Lemma 3.2 above, the inductive step of the proof for the K operation follows.

(5) In [6] in the proof of Lemma 5.7 part (iii), induction on the complexity of polynomials is used to prove that if $f(x)$ is a nonconstant polynomial of $\mathbb{B}$ and $f(u) \in B_1$ for some $u \in B$, then $u \in B_1$. This is the same argument that appears in the previous item above.

(6) A consequence of these lemmas is that every large SI of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is flat and models $S_i(\overline{u}, x, y, z) \approx 0$ for every $i \in \{0, 1, 2\}$. By Lemma 3.1, we have that $K(x, y, z) \approx x \wedge y \wedge z$ in large SI's (in fact, in large SI's $K(x, y, z) \approx x \wedge y \wedge z \approx J(x, y, z)$). Therefore, the addition of the K operation does not change the structure of the large SI's of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$.

This completes the changes that are needed to adapt McKenzie's description of large SI algebras in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. We will now give an explicit description of exactly what these algebras look like.

§4. Subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Define terms e_0, e_1, e_2 in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ by

$$\begin{aligned} &e_0(m, x) = S_0(m, x, x, x), \quad e_2(m, n, x) = S_2(m, n, x, x, x), \\ &e_1(m, x) = S_1(m, x, x, x). \end{aligned} \tag{4.1}$$

The argument in the previous section shows that large SI's model $e_i(\overline{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. The small subdirectly irreducible algebras break into two categories: those that satisfy $\exists \overline{n}[e_i(\overline{n}, x) \approx x]$ for some $i \in \{0, 1, 2\}$, and those that do not (in which case they satisfy $e_i(\overline{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$). As we will see in Lemma 5.4, all SI algebras that do model $\exists \overline{n}[e_i(\overline{n}, x) \approx x]$ for some $i \in \{0, 1, 2\}$ have Jónsson polynomials and are thus congruence distributive, and by an argument due to Baker and Wang [2], these algebras will also have DPSC.

There are only three different isomorphism types for small SI algebras satisfying $e_i(\overline{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. Two of these small SI's are subalgebras of $\mathbb{A}'(\mathcal{T})$), and the remaining one is the 4-element quotient

$$\mathbb{W} = \langle H, C \rangle / \mathrm{Cg}(M_1^0, 0) = \{0, H, C, D, M_1^0\} / \mathrm{Cg}(M_1^0, 0) \tag{4.2}$$

(this will be proved in Lemma 4.1). The fundamental operations of $\mathbb{A}'(\mathcal{T})$ are all identically 0 in $\mathbb{W}$ except for $\wedge$, which makes $\langle W; \wedge \rangle$ a flat semilattice, and the following operations:

$$x \cdot y = 0 \quad \text{except for} \quad H \cdot C = D,$$
$$T(w,x,y,z) = 0 \quad \text{except for} \quad T(H,C,H,C) = D,$$
$$J(x,y,z) = x \wedge y, \quad \text{and} \quad J'(x,y,z) = K(x,y,z) = x \wedge y \wedge z.$$

Lemma 4.1. *Let $\mathbb{B} \in \boldsymbol{HS}(\mathbb{A}'(T))$ be nontrivial and subdirectly irreducible and such that $\mathbb{B} \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0,1,2\}$. Then $\mathbb{B}$ is isomorphic to the two element subalgebra $\{0, C\} \leq \mathbb{A}'(\mathcal{T})$, the three element subalgebra $\{0, H, M_1^0\} \leq \mathbb{A}'(\mathcal{T})$, or to the four element quotient $\mathbb{W}$.*

Proof. We will first consider subalgebras. Suppose that $\mathbb{B} \leq \mathbb{A}'(\mathcal{T})$ is SI. Since $\mathbb{B} \models S_i(\overline{n}, x, x, x) \approx 0$ for all i, we have $(\{1,2\} \cup V_0) \cap B = \emptyset$ and $\{x, \partial x\} \not\subseteq B$ for $x \in W \cup V$ (i.e., the "bar-able" elements of $A'(\mathcal{T})$). It follows that all fundamental operations are identically 0 except for $\wedge$ and

$$\begin{aligned} I(x) = 0 &\quad \text{except for} \quad I(H) = M_1^0, \\ x \cdot y = 0 &\quad \text{except for} \quad H \cdot C = D \text{ and } H \cdot \partial C = \partial D, \\ &J(x,y,z) = x \wedge y, \\ K(x,y,z) &= J'(x,y,z) = x \wedge y \wedge z, \text{ and} \\ &T(w,x,y,z) = (w \wedge y) \cdot (x \wedge z). \end{aligned} \tag{4.3}$$

There are two cases depending upon whether or not H is an element of B. For the first case, suppose that $H \notin B$. Then $x \cdot y = T(w,x,y,z) = I(x) = 0$, so $\mathbb{B}$ is a flat semilattice. It follows that if $x, y \in B$ are distinct and nonzero, then $\mathrm{Cg}^{\mathbb{B}}(x,0)$ and $\mathrm{Cg}^{\mathbb{B}}(y,0)$ are distinct and cover $\mathbf{0}$ in $\mathrm{Con}(\mathbb{B})$, and hence $\mathbb{B}$ is not subdirectly irreducible. Therefore $B = \{0, x\}$, so $\mathbb{B}$ is isomorphic to the subalgebra $\{0, C\}$.

For the second case, suppose that $H \in B$. Then $I(H) = M_1^0 \in B$ as well. If $F(x)$ is a fundamental translation of $\mathbb{B}$, then $F(M_1^0) = M_1^0$ or $F(M_1^0) = 0$ (see the description of the fundamental operations above). Two consequences of this are that $\mathrm{Cg}^{\mathbb{B}}(M_1^0, 0)$ is the monolith of $\mathbb{B}$ and that if $\mathrm{Cg}^{\mathbb{B}}(a,0) = \mathrm{Cg}^{\mathbb{B}}(M_1^0, 0)$ then $a = M_1^0$.

We will now show that $B = \{0, H, M_1^0\}$. Suppose that $x \in B \setminus \{0, H, M_1^0\}$. If $x = C$, then $H \cdot C = D$, so $D \in B$ as well. An argument similar to the one in the previous paragraph will show that $\mathrm{Cg}^{\mathbb{B}}(D,0)$ covers $\mathbf{0}$. Likewise, if $x = \partial C$, then $\mathrm{Cg}^{\mathbb{B}}(\partial D, 0)$ covers $\mathbf{0}$. Both of these possibilities are contradictions. If $x \notin \{C, \partial C\}$ then $\mathrm{Cg}^{\mathbb{B}}(x,0)$ also covers $\mathbf{0}$, again contradicting $\mathbb{B}$ being subdirectly irreducible. Therefore it must be that $B \setminus \{0, H, M_1^0\} = \emptyset$. It follows that the only subdirectly irreducible subalgebras of $\mathbb{A}'(\mathcal{T})$ are isomorphic to either $\{0, C\}$ or $\{0, H, M_1^0\}$.

We now examine the situation when $\mathbb{B}$ is a proper quotient of a subalgebra of $\mathbb{A}'(\mathcal{T})$. Suppose that $\mathbb{B} = \mathbb{B}_1/\theta \in \mathbf{HS}(\mathbb{A}'(\mathcal{T}))$ is subdirectly irreducible. In the quotient $\mathbb{B}$, the equations (4.3) hold by the same argument appearing at the start of the proof. Since $\mathbb{A}'(\mathcal{T})$ is a flat semilattice, the only possibly nontrivial class of θ is the one containing 0. As before, we have two cases to consider, depending on whether B contains a nonzero H/θ. If $H \notin B_1$ or $(H, 0) \in \theta$, then $\mathbb{B}$ is a subdirectly irreducible flat semilattice (i.e., all the operations are 0 except for $\wedge$), so $\mathbb{B}$ must be isomorphic to the 2-element subalgebra $\{0, C\}$ of $\mathbb{A}'(\mathcal{T})$.

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Suppose now that $H \in B_1$ and $(H, 0) \notin \theta$. There are three cases to consider:

- $(M_1^0, 0) \notin \theta$,
- $(M_1^0, 0) \in \theta$ and [$C \notin B_1$ or $(C, 0) \in \theta$], or
- $(M_1^0, 0) \in \theta$ and [$C \in B_1$ and $(C, 0) \notin \theta$].

If $(M_1^0, 0) \notin \theta$ then $\mathrm{Cg}^{\mathbb{B}}(M_1^0, 0)$ is the monolith of $\mathbb{B}$, so by the last paragraph $\mathbb{B}$ is isomorphic to the 3-element subalgebra $\{0, H, M_1^0\}$ of $\mathbb{A}'(\mathcal{T})$. If instead $(M_1^0, 0) \in \theta$ and [$C \notin B_1$ or $(C, 0) \in \theta$], then $\mathrm{Cg}^{\mathbb{B}}(H, 0)$ must cover $\mathbf{0}$, so $\mathbb{B}$ is isomorphic to the 2-element subalgebra $\{0, C\}$. Suppose now that $(M_1^0, 0) \in \theta$ and [$C \in B_1$ and $(C, 0) \notin \theta$]. If $(D, 0) \in \theta$, then both $\mathrm{Cg}^{\mathbb{B}}(H, 0)$ and $\mathrm{Cg}^{\mathbb{B}}(C, 0)$ cover $\mathbf{0}$, contradicting the subdirect irreducibility. If $(D, 0) \notin \theta$, then $\mathrm{Cg}^{\mathbb{B}}(D, 0)$ covers $\mathbf{0}$. An argument similar to when $\mathbb{B}$ is a subalgebra shows that $x \notin B_1$ or $(x, 0) \in \theta$ for all $x \in B_1 \setminus \{0, H, C, D\}$. When this happens $\mathbb{B}$ is isomorphic to the algebra $\mathbb{W}$ described in (4.2) above. ⊣

Large subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ come in two types: sequential and machine. Both of these types of algebras model the identities $S_i(\overline{n}, x, y, z) \approx 0$, $J(x, y, z) \approx x$, and $J'(x, y, z) \approx K(x, y, z) \approx x \wedge y \wedge z$. Sequential algebras are distinguished as additionally modeling the identities $I(x) \approx F(x, y, z) \approx U_F^{\varepsilon}(w, x, y, z) \approx 0$ for all $F \in \mathcal{L} \cup \mathcal{R}$ and all $\varepsilon \in \{0, 1\}$. Machine algebras are distinguished as modeling the identities $x \cdot y \approx T(w, x, y, z) \approx 0$.

We begin the description of the sequential algebras by describing an algebra $\mathbb{S}_{\mathbb{Z}}$ in which every sequential algebra is embeddable (but which may not belong to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$). The algebra $\mathbb{S}_{\mathbb{Z}}$ has underlying set $S_{\mathbb{Z}} = \{0, a_i, b_i \mid i \in \mathbb{Z}\}$ and fundamental operations of $S_{\mathbb{Z}}$ are the same as $\mathbb{A}'(\mathcal{T})$, but are all identically 0 except for $\wedge$, $(\cdot)$, T, J, J', and K. The operation $\wedge$ is defined so that $\langle S_{\mathbb{Z}}; \wedge \rangle$ is a flat meet semilattice with bottom element 0. The operation $(\cdot)$ is defined so that $a_n \cdot b_{n+1} = b_n$, and 0 otherwise. The operations T, J, J', and K are defined

$$J(x, y, z) = x \wedge y, \qquad J'(x, y, z) = K(x, y, z) = x \wedge y \wedge z,$$
$$T(w, x, y, z) = (w \cdot x) \wedge (y \cdot z).$$

Define $\mathbb{S}_{\omega}$ to be the subalgebra of $\mathbb{S}_{\mathbb{Z}}$ with universe $\{0, a_i, b_i \mid i \geq 0\}$, and define $\mathbb{S}_n$ to be the subalgebra of $\mathbb{S}_{\mathbb{Z}}$ with universe $\{0, a_0, b_0, \ldots, a_n, b_n\}$. The algebras $\mathbb{S}_{\omega}$ and $\mathbb{S}_n$ are subdirectly irreducible, with monoliths $\mathrm{Cg}(b_0, 0)$. With the additions described earlier in Section 3, McKenzie's argument in [6] proves that $\mathbb{S}_{\mathbb{Z}} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ if and only if $\mathcal{T}$ does not halt, and that $\mathcal{T}$ halts if and only if there is some maximum $N \in \mathbb{N}$ such that $\mathbb{S}_N \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$. $\mathbb{S}_{\mathbb{Z}}$, $\mathbb{S}_{\omega}$, and $\mathbb{S}_n$ for $n \in \mathbb{N}$ are the sequential algebras, but only S_n and $\mathbb{S}_{\omega}$ are subdirectly irreducible.

Next, we restate the description of machine algebras given by McKenzie [6]. We begin the description of machine algebras by defining an algebra (possibly not in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$) that will have a quotient isomorphic to our hypothetical machine algebra. Let $N \subseteq \mathbb{Z}$ be a nonempty interval and let $\mathcal{Q} = \langle \tau, j, \gamma \rangle$ be any configuration of the Turing machine $\mathcal{T}$ (here τ is the tape function, $j \in N$ is the head position, and γ is the state of the machine). We say that $\mathcal{Q}$ is an initial configuration if τ is the blank tape (the tape consisting of all 0's, written as τ_0 below) and $\gamma = \mu_1$ (the starting state). We say that $\mathcal{Q}$ is a halting configuration if $\gamma = \mu_0$ (the halting state). Let Ω_N denote the set of all configurations $\langle \tau, j, \gamma \rangle$ with $j \in N$ and $\tau(\mathbb{Z} \setminus N) = \{0\}$. Write

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$\mathcal{P} \leq_N \mathcal{Q}$ if there is a finite sequence $\mathcal{Q} = \mathcal{Q}_0, \ldots, \mathcal{Q}_m = \mathcal{P}$ with $\mathcal{Q}_i \in \Omega_N$ and such that $\mathcal{Q}_{i+1} = \mathcal{T}(\mathcal{Q}_i)$.

Let $\Sigma_N = \{a_n \mid n \in N\}$, and assume that Σ_N, Ω_N, and $\{0\}$ are pairwise disjoint. Let $\mathbb{P}_N$ be the algebra where

- the universe is $P_N = \{0\} \cup \Sigma_N \cup \Omega_N$,
- the operations $(\cdot), S_0, S_1, S_2, T$ are identically 0,
- $\wedge$ makes $\langle P_N, \wedge \rangle$ a flat semilattice,
- $J(x, y, z) = x \wedge y$ and $J'(x, y, z) = K(x, y, z) = x \wedge y \wedge z$,
- $I(a_n) = \langle \tau_0, n, \mu_1 \rangle \in \Omega_N$ and $I(x) = 0$ otherwise (here τ_0 is the tape consisting of all 0's),
- if $F = L_{ir\varepsilon} \in \mathcal{L}$ where $\mu_i r s L \mu_j$ is an instruction of $\mathcal{T}$ and $\mathcal{Q} = \langle \tau, n+1, \mu_i \rangle$ is a configuration in Ω_N, then $F(a_n, a_{n+1}, \mathcal{Q}) = \mathcal{T}(\mathcal{Q})$ provided that $n \in N$, $\mathcal{T}(\mathcal{Q}) \in \Omega_N$, $\tau(n+1) = r$, and $\tau(n) = \varepsilon$. In all other cases $F(x, y, z) = 0$. The case when $F = R_{ir\varepsilon} \in \mathcal{R}$ is defined analogously,
- if $F \in \mathcal{L} \cup \mathcal{R}$ and $n, n+1 \in N$, we have

$$U_F^0(a_n, a_{n+1}, a_{n+1}, x) = F(a_n, a_{n+1}, x) = U_F^1(a_n, a_n, a_{n+1}, x),$$

  and $U_F^j(w, x, y, z) = 0$ otherwise.

Next, we describe the congruence of $\mathbb{P}_N$ which we will quotient by. Assume the set $\Phi \subseteq \Omega_N$ and the element $\mathcal{P} \in \Phi$ satisfy the following conditions.

- For all $\mathcal{Q} \in \Phi$ we have $\mathcal{P} \leq_N \mathcal{Q}$.
- If $\mathcal{Q} \in \Phi$ and $\mathcal{P} \leq_N \mathcal{T}(\mathcal{Q})$ then $\mathcal{T}(\mathcal{Q}) \in \Phi$.
- If $\mathcal{Q} \in \Omega_N$ is an initial configuration and $\mathcal{P} \leq_N \mathcal{Q}$ then $\mathcal{Q} \in \Phi$.
- If $\mathcal{Q}, \mathcal{Q}' \in \Omega_N$, $\mathcal{Q}'$ is a halting configuration, and $\mathcal{Q}' \leq_N \mathcal{Q}$ then $\mathcal{Q} \notin \Phi$.
- $|N| > 1$ and for every $n \in N$, there is some $\langle \tau, n, \gamma \rangle \in \Phi$.

Define Γ to be $(\Omega_N \setminus \Phi) \cup \{0\}$ and let $\Theta_{(\Phi)}$ be the congruence of P_N whose only nontrivial class is Γ. McKenzie gives the following theorem at the end of [6], which with the addition of the arguments above still holds for the modified $\mathbb{A}'(\mathcal{T})$.

Theorem 4.2 (McKenzie [6]). *$\Theta_{(\Phi)}$ is a congruence relation of $\mathbb{P}_N$ and the algebra $\mathbb{P}_N/\Theta_{(\Phi)}$ is a subdirectly irreducible algebra that belongs to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Every large SI in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is either embeddable in $\mathbb{S}_\omega$ or is isomorphic to $\mathbb{P}_N/\Theta_{(\Phi)}$ for some N and Φ as above.*

The above description of the SI algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ extends to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ the result that $\kappa(\mathbb{A}'(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts to $\mathbb{A}'(\mathcal{T})$.

Theorem 4.3. *$\kappa(\mathbb{A}'(\mathcal{T})) < \omega$ if and only if $\mathcal{T}$ halts.*

§5. If $\mathcal{T}$ halts. The argument to show that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences if $\mathcal{T}$ halts is quite long and intricate, so we will begin by giving an description of the different cases.

Definition 5.1. Let $F(x_1, \ldots, x_n)$ be a fundamental operation of $\mathbb{A}'(\mathcal{T})$, $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$, and $b_1, \ldots b_n \in B$. The polynomials

$$F^{(i)}_{b_1,\ldots,b_n}(x) = F(b_1, \ldots, \overset{i}{\hat{x}}, \ldots, b_n) \quad \text{for } i \in \{1, \ldots, n\}$$

are called *fundamental translations of F*.

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If $h(x)$ is a polynomial of $\mathbb{B}$ that is generated under composition by fundamental translations, we will say that $h(x)$ is a *primitive polynomial*. The set of all primitive polynomials of $\mathbb{B}$ will be denoted

$$\mathcal{P}(\mathbb{B}) = \{h(x) \in \mathrm{Pol}_1(\mathbb{B}) \mid h(x) \text{ is generated by fundamental translations}\}.$$

When the algebra $\mathbb{B}$ is clear from the context which $h(x)$ is mentioned in, $\mathcal{P}(\mathbb{B})$ will be shortened to just $\mathcal{P}$.

If $\mathbb{B}$ is an algebra, then by Maltsev's Lemma, $(c,d) \in \mathrm{Cg}^{\mathbb{B}}(a,b)$ if and only if there is a sequence of elements, $c = k_1, k_2, \dots, k_n = d$, terms $f_1, \dots, f_{n-1}$, and constants $\overline{e} \in B^m$ such that $\{f_i(\overline{e}, a), f_i(\overline{e}, b)\} = \{k_i, k_{i+1}\}$ for all i. Equivalently, we can take the polynomials $f_i(\overline{e}, x)$ to be generated by fundamental translations. A *congruence scheme*, as in [3], is a first-order formula, $\varphi(w,x,y,z)$, that asserts the existence of such elements $k_1, \dots, k_n$ and constants $\overline{e}$ for some fixed sequence of terms. A disjunction of congruence schemes is a *congruence formula*, and every $(c,d) \in \mathrm{Cg}^{\mathbb{B}}(a,b)$ satisfies some congruence scheme. Thus, showing that a principal congruence is definable can be reduced to finding a finite number of schemes that fully describe the congruence, and showing that a variety has definable principal congruences can be reduced to show that there is a finite number of congruence schemes that fully describe every principal congruence in every algebra in the variety.

Begin with an arbitrary $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ with subdirect representation $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$, where each $\mathbb{C}_l$ is subdirectly irreducible. Recall from the previous section that $e_i(\overline{n}, x) = S_i(\overline{n}, x, x, x)$, where $\overline{n} = n_1$ if $i \in \{0,1\}$ and $\overline{n} = (n_1, n_2)$ if $i = 2$. The isomorphism types of the $\mathbb{C}_l$ come in 4 different flavors. If $\mathbb{C}_l$ is subdirectly irreducible, then exactly one of the following holds:

(a) $\mathbb{C}_l \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$ for some $i \in \{0,1,2\}$. Any such $\mathbb{C}_l$ is necessarily small (i.e., contained in $\mathbf{HS}(\mathbb{A}'(\mathcal{T}))$ (see Lemma 5.2 in [6]). For fixed $i \in \{0,1,2\}$ and $\overline{m} \in B^2 \cup B$, every model of $e_i(\overline{m}, x) \approx x$ is congruence distributive (see the proof of Lemma 5.4), and the class of these models (for a single fixed i) has definable principal subcongruences (see Lemma 5.4).

(b) $\mathbb{C}_l$ is small and $\mathbb{C}_l \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0,1,2\}$. In this case there are just 3 isomorphism types (see Lemma 4.1).

(c) $\mathbb{C}_l$ is large (i.e., not contained in $\mathbf{HS}(\mathbb{A}'(\mathcal{T}))$) and $\mathbb{C}_l \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0,1,2\}$ and $\mathbb{C} \models I(x) \approx F(x,y,z) \approx 0$ for all $F \in \mathcal{L} \cup \mathcal{R}$. In this case, $\mathbb{C}$ is *sequential*. SI's of this type were fully described in Section 4.

(d) $\mathbb{C}_l$ is large and $\mathbb{C} \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0,1,2\}$ and $\mathbb{C} \models x \cdot y \approx T(w,x,y,z) \approx 0$, In this case, $\mathbb{C}$ is *machine*. SI's of this type were fully described in Section 4.

In order to show that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences, we will produce congruence formulas Γ and ψ such that for any $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and any $a', b' \in B$ there is $(c,d) \in \mathrm{Cg}^{\mathbb{B}}(a',b')$ witnessed by $\Gamma(c,d,a',b')$ and such that the relation "$(x,y) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$" is defined by $\psi(x,y,c,d)$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. The way in which (c,d) is produced depends on the isomorphism types of the $\mathbb{C}_l$ with $l \in L$ such that $a'(l) \neq b'(l)$. Our first step is to assume without loss of generality that $a' \not\leq b'$ and to take $a = a'$ and $b = a' \wedge b'$ so that $b < a$. Let $K = \{l \in L \mid a(l) \neq b(l)\}$. The case distinctions follow.

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(1) There is $k \in K$ such that $\mathbb{C}_k \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$. These are the SI's described in item (a) above.
(2) The previous case does not hold, and there is $k \in K$ such that $\mathbb{C}_k$ is sequential. In this case either a translation of the operation $(\cdot)$ will distinguish a and b, or $(a(l), b(l))$ lies in the monolith of $\mathbb{C}_l$ for all $l \in L$. These are the SI's described in item (c) above.
(3) The previous cases do not hold, and there is $k \in K$ such that $\mathbb{C}_k$ is machine, and either a translation of one of the operations from $\mathcal{L} \cup \mathcal{R} \cup \{I\}$ will separate a and b, or $(a(l), b(l))$ lies in the monolith of $\mathbb{C}_l$ for all $l \in L$. These are the SI's described in item (d) above.
(4) The previous cases do not hold, so it must be that the only $k \in K$ are such that $\mathbb{C}_k$ is one of the three small SI's that satisfy $e_i(\overline{n}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. These are the SI's described in item (b) above.

We begin the proof for Case 1 with a slightly specialized version of a theorem from Baker and Wang [2].

LEMMA 5.2. *Let $\mathcal{V}$ be a locally finite variety and let*

$$P(\overline{c}) = \{p_j(\overline{c}, x_1, x_2, x_3) \mid 1 \leq j \leq K\}$$

be terms in $\mathcal{V}$ with (a fixed number of) constant symbols $\overline{c}$. Suppose that $J(\overline{c})$ is the set consisting of the Jónsson identities for the polynomials $P(\overline{c})$ in the variables $\{x_1, x_2, x_3\}$. Then the class

$$\mathcal{M} = Mod_{\mathcal{V}}(\exists \overline{c}\ J(\overline{c})) = \{\mathbb{B} \in \mathcal{V} \mid \mathbb{B} \models \exists \overline{c}\ J(\overline{c})\}$$

has definable principal subcongruences if $\kappa(\mathcal{V}) = N < \omega$.

PROOF. The notable modification of the proof given in [2] is at (5.1) below.

Let $\mathbb{B} \in \mathcal{M}$, let $a, b \in B$ be distinct, and fix $\overline{c} \in B^n$ witnessing $\mathbb{B} \models J(\overline{c})$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras such that whenever $k, l \in L$ and $\mathbb{C}_k \equiv \mathbb{C}_l$ then $\mathbb{C}_k = \mathbb{C}_l$. Since $\kappa(\mathcal{V}) < \omega$, each $\mathbb{C}_l$ is finite and there are only finitely many distinct ones. We will construct a finite subalgebra $\mathbb{C} \leq \mathbb{B}$, and then find a pair $(c, d) \in \mathrm{Cg}^{\mathbb{C}}(a, b)$ such that $c \neq d$ and $\mathrm{Cg}^{\mathbb{B}}(c, d)$ is uniformly definable (i.e., definable in a way that depends only on $\mathcal{V}$, and not on $\mathbb{B}$, c, or d).

Choose $k \in L$ such that $a(k) \neq b(k)$ and $|C_k|$ is maximal with this property. Choose preimage representatives $s_1, \dots s_M \in B$ of $\mathbb{C}_k$ and let

$$\mathbb{C} = \langle \{a, b, \overline{c}\} \cup \{s_1, \dots, s_M\} \rangle. \tag{5.1}$$

Since $\kappa(\mathcal{V}) = N < \omega$ and $\mathcal{V}$ is locally finite, any such $\mathbb{C}$ has size bounded by a number depending only on N and the number of constants $\overline{c}$. Since $\mathbb{C}$ has bounded size, congruences are defined by a finite number of congruence schemes. By construction, $\pi_k(\mathbb{C}) = \mathbb{C}_k$ and since any subalgebra of $\mathbb{B}$ containing $\overline{c}$ is congruence distributive (any such subalgebra has Jónsson polynomials), $\mathbb{C}$ is congruence distributive.

$\mathbb{C}_k$ is subdirectly irreducible, so $\ker(\pi_k|_{\mathbb{C}})$ is completely meet irreducible in $\mathrm{Con}(\mathbb{C})$ (the congruence lattice of $\mathbb{C}$). Since $\mathbb{C}$ is congruence distributive, the interval $[0, \ker(\pi_k|_{\mathbb{C}})]$ is a prime ideal and therefore the complement is a filter with a least element, call it α, which is join-prime. Therefore α is a principal congruence, say $\alpha = \mathrm{Cg}^{\mathbb{C}}(c, d)$, and α is the least congruence not below $\ker(\pi_k|_{\mathbb{C}})$. Since

$\mathrm{Cg}^{\mathbb{C}}(a,b) \not\leq \ker(\pi_k|_{\mathbb{C}})$, by minimality of α we have $\alpha = \mathrm{Cg}^{\mathbb{C}}(c,d) \leq \mathrm{Cg}^{\mathbb{C}}(a,b)$. By the previous paragraph, $|C|$ is bounded by a number depending only on N and the number of constants $\overline{c}$. It follows that there is a congruence formula determined entirely by this bound that witnesses $(c,d) \in \mathrm{Cg}^{\mathbb{C}}(a,b)$.

Let $l \in L$ and suppose that $c(l) \neq d(l)$. Then $\mathrm{Cg}^{\mathbb{C}}(c,d) \not\leq \ker(\pi_l|_{\mathbb{C}})$ and $a(l) \neq b(l)$. By the minimality of $\alpha = \mathrm{Cg}^{\mathbb{C}}(c,d)$, it must be that $\ker(\pi_l|_{\mathbb{C}}) \leq \ker(\pi_k|_{\mathbb{C}})$. Hence there is a surjective mapping

$$\pi_l(\mathbb{C}) \cong \mathbb{C}/\ker(\pi_l|_{\mathbb{C}}) \twoheadrightarrow \mathbb{C}/\ker(\pi_k|_{\mathbb{C}}) \cong \pi_k(\mathbb{C}) = \mathbb{C}_k.$$

Now, $\mathbb{C}_k$ was chosen to be maximal such that $a(k) \neq b(k)$, so the mapping must also be injective since $\mathbb{C}_l$ is finite. Thus $\pi_l(\mathbb{C}) = \mathbb{C}_k$.

Let $r, s \in B$ be distinct with $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$. We shall construct a finite $\mathbb{D}$ such that $(r,s) \in \mathrm{Cg}^{\mathbb{D}}(c,d)$. Let $\mathbb{D} = \langle C \cup \{r,s\}\rangle$. As with $\mathbb{C}$, any such $\mathbb{D}$ has size bounded by a number depending only on N and the number of constants $\overline{c}$, and so congruences in $\mathbb{D}$ are defined by a congruence formula determined entirely by this bound. Since $\overline{c} \in D^n$, we also have that $\mathbb{D}$ is congruence distributive. Let $l \in L$. If $c(l) \neq d(l)$ then by the above paragraph $\pi_l(\mathbb{D}) = \pi_l(\mathbb{C}) = \mathbb{C}_k$, so

$$(r(l), s(l)) \in \mathrm{Cg}^{\pi_l(\mathbb{C})}(c(l), d(l)) = \mathrm{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l)).$$

On the other hand, if $c(l) = d(l)$ then $r(l) = s(l)$, so it follows that $(r(l), s(l)) \in \mathrm{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l)) = \mathbf{0}_{\pi_l(\mathbb{D})}$. In either case, $(r(l), s(l)) \in \mathrm{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l))$ for all $l \in L$. To complete the proof we need only prove the following claim.

Claim. *Let $\mathbb{D}$ be finite and congruence distributive and let $\mathbb{D} \leq \prod_{i \in I} \mathbb{C}_i$. Then $(r,s) \in Cg^{\mathbb{D}}(c,d)$ if and only if $(r(i), s(i)) \in Cg^{\pi_i(\mathbb{D})}(c(i), d(i))$ for all $i \in I$.*

Proof of claim. One direction is clear, since the i-th projection map is a homomorphism. For the other direction, we have

$$(r,s) \in \mathrm{Cg}^{\mathbb{D}}(c,d) \vee \ker(\pi_i) \quad \text{for each } i.$$

The set $\Gamma = \{\ker(\pi_i) \mid i \in I\}$ of congruences of $\mathbb{D}$ is finite since D is finite. Let $\Gamma = \{\ker(\pi_j) \mid j \in J\}$, where J is a finite subset of I. Then by the congruence distributivity of $\mathbb{D}$,

$$\begin{aligned}(r,s) \in \bigwedge_{j \in J} \left(\mathrm{Cg}^{\mathbb{D}}(c,d) \vee \ker(\pi_j)\right) &= \mathrm{Cg}^{\mathbb{D}}(c,d) \vee \bigwedge_{j \in J} \ker(\pi_j),\\ &= \mathrm{Cg}^{\mathbb{D}}(c,d) \vee \bigwedge_{i \in I} \ker(\pi_i) = \mathrm{Cg}^{\mathbb{D}}(c,d) \vee \mathbf{0}_{\mathbb{D}},\\ &= \mathrm{Cg}^{\mathbb{D}}(c,d),\end{aligned}$$

as claimed. ⊣

⊣

Let $\mathcal{V} = \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and define subclasses of $\mathcal{V}$,

$$\mathcal{M}_i = \mathrm{Mod}_{\mathcal{V}}\left(\exists \overline{m}\ \left[e_i(\overline{m}, x) \approx x\right]\right) \quad \text{for} \quad i \in \{0,1,2\}.$$

We will make use of the fact that if $\mathbb{C}$ is subdirectly irreducible, then either $\mathbb{C} \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$ for some $i \in \{0,1,2\}$ or $\mathbb{C} \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0,1,2\}$. This fact follows from the description of SI's in Section 4 and from the definition of the S_i and the e_i. In the case, where $\mathbb{C} \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$, McKenzie [6] proves that since $\mathbb{C}$ is subdirectly irreducible it is necessarily small. The condition

$$\text{“}x \in e_i(\overline{u}, B) \text{ for some } i \in \{0, 1, 2\} \text{ and some } \overline{u} \in B^2 \cup B\text{”}$$

and it's negation will be referred to quite often in the upcoming argument, so we now define an easier way to reference it.

Definition 5.3. We will say that $S \subseteq B$ is *unhappy* if

$$\forall i \in \{0, 1, 2\}\ \forall \overline{u} \in B^2 \cup B \left[S \not\subseteq e_i(\overline{u}, B)\right],$$

and S is *happy* otherwise. The element $s \in B$ is *unhappy* (resp. *happy*) if $\{s\}$ is unhappy (resp. happy). The function $h : B^n \to B$ is *unhappy* (resp. *happy*) if Range(h) is unhappy (resp. happy). Note that these definitions depend on the algebra $\mathbb{B}$, but the particular algebra something is happy or unhappy with respect to will always be clear from the context.

Here are some useful (and straightforward) observations about happiness with respect to an algebra $\mathbb{B}$.

- If a set S only contains unhappy elements then this is a stronger property than S being unhappy.
- The operations S_i for $i \in \{0, 1, 2\}$ are happy.
- If the function $h : B^n \to B$ is 0-absorbing in the ith variable position, then it *preserves happiness* in the sense that if $a_1, \dots a_n \in B$ and a_i is happy, then $h(a_1, \dots, a_n)$ is happy as well.
- If $c, d \in B$, $d \leq c$, and $\{d, c\}$ is unhappy, then c must be unhappy.

Lemma 5.4. *If $\mathcal{T}$ halts then each $\mathcal{M}_i$ has definable principal subcongruences.*

Proof. Let $i \in \{0, 1, 2\}$. We will show that $\mathcal{M}_i$ satisfies the hypotheses of Lemma 5.2 and thus has definable principal subcongruences. Let $\mathbb{B} \in \mathcal{M}_i$. Choose $\overline{m} \in B^2 \cup B$ witnessing $\mathbb{B} \models e_i(\overline{m}, x) \approx x$. Now, $\mathbb{B} \models e_i(\overline{m}, x) \approx x$ if and only if $\mathbb{B} \models S_i(\overline{m}, x, y, z) \approx (x \wedge y) \vee (x \wedge z)$. Therefore there exists $\overline{m} \in B^2 \cup B$ such that the following

$$\begin{aligned} &p_0(x, y, z) = x, && p_1(x, y, z) = S_i(\overline{m}, x, y, z) = (x \wedge y) \vee (x \wedge z),\\ &p_2(x, y, z) = x \wedge z, && p_3(x, y, z) = S_i(\overline{m}, z, y, x) = (y \wedge z) \vee (x \wedge z),\\ &p_4(x, y, z) = z, \end{aligned}$$

are polynomials of $\mathbb{B}$ and satisfy the Jónsson identities. If $J_i(\overline{m})$ is the set of Jónsson identities for these polynomials, then $\mathcal{M}_i \subseteq \text{Mod}_{\mathcal{V}}(\exists \overline{m}\ J_i(\overline{m}))$. Since $\mathcal{T}$ halts, $\kappa(\mathcal{V}(\mathcal{M}_i)) \leq \kappa(\mathbb{A}'(\mathcal{T})) < \omega$. By Lemma 5.2, it follows that $\mathcal{M}_i$ has definable principal subcongruences. ⊣

Let $\Gamma_0^i(w, x, y, z)$ and $\psi_0^i(w, x, y, z)$ be the congruence formulas witnessing definable principal subcongruences for $\mathcal{M}_i$. Define

$$\psi_0(w, x, y, z) = \bigvee_{i=0}^{2} \psi_0^i(w, x, y, z) \quad \text{and} \quad \Gamma_0(w, x, y, z) = \bigvee_{i=0}^{2} \Gamma_0^i(w, x, y, z). \quad (5.2)$$

Theorem 5.5. *The class $\mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$ has definable principal subcongruences witnessed by the congruence formulas Γ_0 and ψ_0.*

Proof. Since Γ_0^i and ψ_0^i are congruence formulas, so are Γ_0 and ψ_0. Let $\Pi_{\psi_0}(x, y)$ be the formula expressing that the pair (x, y) generates a congruence that is defined by $\psi_0(-,-,x,y)$ in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ (i.e., the formula asserting that $\psi_0(-,-,x,y)$ is an equivalence relation, is invariant under fundamental translations, and that $\psi_0(x,y,x,y)$ holds). Since each $\mathcal{M}_i$ has definable principal subcongruences and Γ_0 and ψ_0 are the disjunctions of the formulas witnessing DPSC, Γ_0 and ψ_0 witness definable principal subcongruences for the class $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. ⊣

In symbols, Theorem 5.5 says

$$\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \models \forall a, b \left[a \neq b \to \exists c, d \left[c \neq d \wedge \Gamma_0(c,d,a,b) \wedge \Pi_{\psi_0}(c,d)\right]\right].$$

In terms of happiness, Theorem 5.5 means that if $\mathbb{B}$ is a happy algebra, then $\mathbb{B}$ has DPSC witnessed by Γ_0 and ψ_0.

The next 5 lemmas provide the groundwork for analyzing the polynomials that make up a hypothetical Maltsev chain. Specifically, they describe the extent to which the non-0-absorbing operations commute with the other operations.

Lemma 5.6. *Each of the following hold for every algebra $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$.*

(1) *If $f(x)$ is 0-absorbing, then $g(x) = f(S_j(\overline{n}, p, q, x))$ is happy for all $j \in \{0,1,2\}$, $\overline{n} \in B^2 \cup B$, and $p, q \in B$.*
(2) *If $f(x)$ is happy, then there is $j \in \{0,1,2\}$ and $\overline{n} \in B^2 \cup B$ such that*

$$f(x) = S_j(\overline{n}, f(x), f(x), f(x)).$$

(3) *If $f(x)$ is a polynomial, $c, d \in B$, $d \leq c$, and $\{f(c), f(d)\}$ is happy, then there is $j \in \{0,1,2\}$ and $\overline{n} \in B^2 \cup B$ such that the polynomial*

$$g(x) = S_j(\overline{n}, f(c), f(d), f(x))$$

satisfies $(g(c), g(d)) = (f(c), f(d))$.

Proof. We begin with (1). Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras and define

$$I = \{l \in L \mid \pi_l(S_j(\overline{n}, p, q, B)) \neq \{0\}\} \quad \text{and} \quad J = L \setminus I.$$

Write a typical $y \in B$ as $y = (y_I, y_J)$, where $y_I = \pi_I(y)$ and $y_J = \pi_J(y)$. Therefore $S_j(\overline{n}, y, y, y) = e_j(\overline{n}, y) = (y_I, 0)$, so

$$g(x) = f(S_j(\overline{n}, p, q, x)) = f\begin{pmatrix} (p_I \wedge q_I) \vee (p_I \wedge x_I) \\ 0_J \end{pmatrix}$$

$$= \begin{pmatrix} f((p_I \wedge q_I) \vee (p_I \wedge x_I)) \\ f(0_J) \end{pmatrix} = \begin{pmatrix} f((p_I \wedge q_I) \vee (p_I \wedge x_I)) \\ 0_J \end{pmatrix} \in e_j(B).$$

For (2), say that $f(x)$ is happy because $\mathrm{Range}(f(x)) \subseteq e_j(\overline{n}, B)$ for some $j \in \{0,1,2\}$ and $\overline{n} \in B^2 \cup B$. Observing that $e_j(\overline{n}, \mathbb{B}) \models e_j(\overline{n}, x) \approx x$, the conclusion follows.

For (3), say (as in (2)) that $\{f(c), f(d)\}$ is happy because $\{f(c), f(d)\} \subseteq e_j(\overline{n}, B)$ for some $j \in \{0,1,2\}$ and $\overline{n} \in B^2 \cup B$. The operations of $\mathbb{A}'(\mathcal{T})$ are monotonic, so $f(d) \leq f(c)$. Therefore

$$g(c) = S_j(\overline{n}, f(c), f(d), f(c)) = (f(c) \wedge f(d)) \vee (f(c) \wedge f(c)) = f(c),$$

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and likewise

$$g(d) = S_j(\overline{n}, f(c), f(d), f(d)) = (f(c) \wedge f(d)) \vee (f(c) \wedge f(d)) = f(d). \quad \dashv$$

LEMMA 5.7. *Let $\{c, d\} \subseteq B$ be happy with $d \leq c$ and let $g(x) \in \mathcal{P}$. Then there are constants $p, q \in B$ such that*

(1) *if $\{g(c), g(d)\}$ is happy, then there is $j \in \{0, 1, 2\}$, $\overline{n} \in B^2 \cup B$, and a happy polynomial $h(x) \in \mathcal{P}$ such that*

$$g'(x) = S_j(\overline{n}, p, q, h(x))$$

has $(g(c), g(d)) = (g'(c), g'(d))$.

(2) *if $\{g(c), g(d)\}$ is unhappy, then there are some fundamental translations $F_1, \ldots, F_M$ and a happy polynomial $h(x) \in \mathcal{P}$ such that for some choice of operation $G \in \{J, J', K\}$, the polynomial*

$$g'(x) = F_M \circ \cdots \circ F_1 \circ G(p, q, h(x))$$

has $(g(c), g(d)) = (g'(c), g'(d))$ and the set

$$\{F_k \circ \cdots \circ F_1 \circ G(p, q, h(c)) \mid 1 \leq k \leq M\} \cup \{G(p, q, h(c))\}$$

contains only unhappy elements.

PROOF. Item (1) is a restatement of Lemma 5.6.

Suppose that $\{g(c), g(d)\}$ is unhappy. The polynomial $g(x)$ is primitive, so there are fundamental translations $F_1, \ldots, F_N$ such that $g(x) = F_N \circ \cdots \circ F_1(x)$. Define a sequence of polynomials $g_l(x)$ and elements c_l, d_l by $g_l(x) = F_l \circ \cdots \circ F_1(x)$ and $(c_l, d_l) = (g_l(c), g_l(d))$ with $(c_0, d_0) = (c, d)$. Choose L maximal such that $\{c_L, d_L\}$ is unhappy but $\{c_{L-1}, d_{L-1}\}$ is happy. Since $c_L = F_L(c_{L-1})$ and $d_L = F_L(c_{L-1})$, the translation F_L must map some happy elements to unhappy elements (i.e., it does not preserve happiness). The only way this can happen is if F_L is non-0-absorbing (by Lemma 5.6, 0-absorbing functions preserve happiness).

The only fundamental translations that are not 0-absorbing are the S_j in the last 2 variables, and J, J', and K in the last variable. If F_L is such a translation of an S_j operation, then it is happy, which contradicts the unhappiness of $\{c_L, d_L\}$. Therefore

$$F_L(x) \in \{J(p, q, x), J'(p, q, x), K(p, q, x)\}$$

for some $p, q \in B$. Let the happiness of the set $\{c_{L-1}, d_{L-1}\}$ be witnessed by $\{c_{L-1}, d_{L-1}\} \subseteq e_j(\overline{n}, B)$ for some $j \in \{0, 1, 2\}$ and $\overline{n} \in B^2 \cup B$, and define

$$g'(x) = F_M \circ \cdots \circ F_L \circ h(x) \quad \text{where} \quad h(x) = S_j(\overline{n}, c_{L-1}, d_{L-1}, F_{L-1} \circ \cdots \circ F_1(x)).$$

The polynomial $h(x)$ is clearly happy and primitive, and by the maximality of L the set

$$\{F_k \circ \cdots \circ F_{L+1} \circ F_L(c) \mid L + 1 \leq k \leq N\} = \{c_k \mid L + 1 \leq k \leq N\}$$

contains only unhappy elements.

The only assertion left to verify is $(g'(c), g'(d)) = (g(c), g(d))$. Since all operations are monotone and $d \leq c$, we have that $d_{L-1} \leq c_{L-1}$. Since $\{c_{L-1}, d_{L-1}\} \subseteq e_j(\overline{n}, B)$ and $d_{L-1} \leq c_{L-1}$,

$$\begin{aligned} F_M \circ \cdots \circ F_L \circ h \begin{pmatrix} c \\ d \end{pmatrix} &= F_M \circ \cdots \circ F_L \circ S_j \left(\overline{n}, c_{L-1}, d_{L-1}, F_{L-1} \circ \cdots \circ F_1 \begin{pmatrix} c \\ d \end{pmatrix} \right) \\ &= F_M \circ \cdots \circ F_L \circ S_j \left(\overline{n}, c_{L-1}, d_{L-1}, \begin{pmatrix} c_{L-1} \\ d_{L-1} \end{pmatrix} \right) \\ &= F_M \circ \cdots \circ F_L \circ \begin{pmatrix} c_{L-1} \\ d_{L-1} \end{pmatrix} \\ &= F_M \circ \cdots \circ F_L \circ F_{L-1} \circ \cdots \circ F_1 \begin{pmatrix} c \\ d \end{pmatrix} = g \begin{pmatrix} c \\ d \end{pmatrix}. \end{aligned}$$

Therefore, $(g'(c), g'(d)) = (g(c), g(d))$. Reindexing $F_M, \ldots F_{L+1}$ completes the proof of (2). ⊣

The next 3 lemmas quantify the extent to which the unhappy operations J, J', and K in the polynomial $g'(x)$ from the conclusion of Lemma 5.7 "commute" with other unhappy fundamental operations.

Lemma 5.8. *Let $F_1, \ldots, F_M$ be fundamental translations, $h(x)$ a happy primitive polynomial, and $p, q, c, d \in B$ with $d \leq c$ such that the set*

$$\{F_k \circ \cdots \circ F_1(J(p, q, h(c))) \mid 1 \leq k \leq M\} \cup \{J(p, q, h(c))\}$$

contains only unhappy elements. If $g(x) = F_M \circ \cdots \circ F_1 \circ J(p, q, h(x))$ then there are constants $p', q' \in B$ and a happy $h'(x) \in \mathcal{P}$ (actually having Range$(h') \subseteq e_2(p', q', B)$) such that the polynomial

$$g'(x) = J(p', q', h'(x))$$

satisfies $(g(c), g(d)) = (g'(c), g'(d))$.

Proof. For convenience, let $r = g(c)$ and $s = g(d)$. We begin by noting that

$$J(x, y, z) = (x \wedge y) \vee (x \wedge \partial y \wedge z) = (x \wedge y) \vee (x \wedge \partial y \wedge e_2(x, y, z)),$$

from the definition of S_2 (recall $e_2(x, y, z) = S_2(x, y, z, z, z)$) and J. Thus, it will be sufficient to prove that the polynomial $g'(x)$ in the statement of the lemma satisfies $g'(c) = g(c)$ and $g'(d) = g(d)$ without any restrictions on the happiness of $h'(x)$.

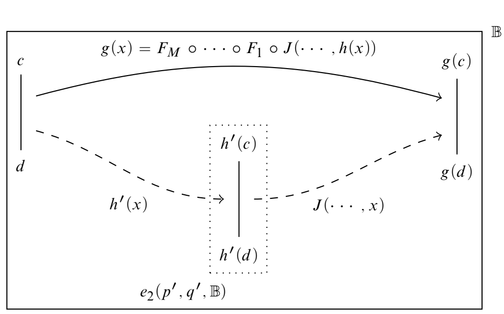

Figure 2. Lemma 5.8 illustration.

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We will prove the lemma in the restricted setting of $M = 1$ (i.e., when $g(x) = F \circ J(p, q, h(x))$). Repeated applications of the restricted proof will then prove the lemma for general M. Say that $g(x) = F \circ J(p, q, h(x))$, where F is a fundamental translation and $g(c) = r$ is unhappy (and thus $\{r, s\}$ is unhappy). Note that F must be unhappy since $r = F \circ J(p, q, h(c))$ is unhappy. In particular, this means that F is not a translation of an S_i operation.

Composing the polynomial $J(p, q, h(x))$ with translations of operations from $\{(\cdot), I\} \cup \mathcal{L} \cup \mathcal{R}$ produce either constant polynomials or the composition is commutative (i.e., $F(J(p, q, h(x))) = J(F(p), F(q), F \circ h(x))$). Thus the claim holds for these operations.

CASE $\wedge$: We have that $u \wedge J(p, q, h(x)) = J(p, q, h(x)) \wedge u = J(p \wedge u, q, h(x))$.

CASE J: The first translation is easy since $J(x, y, z) \wedge w = J(x \wedge w, y, z)$ and $J(x, y, z) \leq J(x, y, x)$. We have

$$\begin{aligned} J(J(p, q, h(x)), u, v) &= J(p, q, h(x)) \wedge J(J(p, q, p), u, v) \\ &= J(p \wedge J(J(p, q, p), u, v), q, h(x)). \end{aligned}$$

For $g(x) = J(u, J(p, q, h(x)), v)$, let

$$g'(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))),$$

where $g(d) = s \leq r = g(c)$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. We will show that $g'(c) = r$ and $g'(d) = s$ componentwise. We have that

$$\begin{aligned} g(x) = (u \wedge p \wedge q) \vee (u \wedge p \wedge \partial q \wedge h(x)) \\ \vee (u \wedge \partial p \wedge \partial q \wedge v) \vee (u \wedge \partial p \wedge \partial\partial q \wedge \partial h(x)). \end{aligned}$$

The argument at this point breaks down into many subcases, depending on whether $r(l)$ is equal to $p(l)$, $q(l)$, $\partial p(l)$, or $\partial q(l)$ (if $r(l) \neq 0$, then by the flatness of $\mathbb{C}_l$ it must take on one of these values). The easiest way to keep track of everything is with a table. Since $r(l) = 0$ implies $g'(x)(l) = 0$ and $s(l) = 0$, we will assume that $r(l) \neq 0$. For ease of reading, in the table below we will omit the coordinate when giving values of functions (i.e., "(l)" will be omitted from $r(l)$). Additionally, those coordinates which permit $r(l) \neq s(l)$ have been indicated.

r	$K(r, p, q)$	$S_2(r, K(r, p, q), r, s, g(x))$	$g'(x)$	$r \neq s$
$p = q$	$p = r$	0	$r \wedge r$	N
$p = \partial q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	Y
$\partial p = \partial q$	$p = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	N
$\partial p = q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	Y

Since $r = g(c)$ and $s = g(d)$, we see that $g'(c)(l) = r(l)$ and $g'(d)(l) = s(l)$, except for possibly when $r(l) = p(l) = q(l)$. In this subcase, however, from the description of $g(x)$ in terms of $\wedge$ and $\vee$ above we see that $g(x)(l)$ is constant, so it must be that

$$r(l) = g(c)(l) = g(d)(l) = s(l).$$

Therefore, $g'(c) = r$ and $g'(d) = s$, as claimed.

In the subcase where $g(x) = J(u, v, J(p, q, h(x)))$, let

$$g'(x) = J(u, v, S_2(u, v, r, s, g(x))).$$

The note in the first paragraph of the proof shows that $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Case J': The first translation is similar to the case for $\wedge$, and the second translation reduces to the J case. We have

$$\begin{aligned} J'(J(p,q,h(x)),u,v) &= J(p,q,h(x)) \wedge J'(J(p,q,p),u,v) \\ &= J(J'(J(p,q,p),u,v),q,h(x)), \quad \text{and} \\ J'(u,J(p,q,h(x)),v) &= J(J'(u,J(p,q,p),v),J(p,q,h(x)),J'(u,J(p,q,p),v))^{\dagger} \end{aligned}$$

[†: see Case J above]. For $g(x) = J'(u,v,J(p,q,h(x)))$, let

$$g'(x) = J(r,K(r,v,q),S_2(r,K(r,v,q),r,s,g(x))).$$

An argument similar to the one requiring the table above will show that $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Case S_i: Since $\{r,s\}$ is unhappy, we can exclude these translations.

Case K: For $g(x) = K(J(p,q,h(x)),u,v)$, let

$$g'(x) = J(r,K(r,p,q),S_2(r,K(r,p,q),r,s,g(x))).$$

The approach is to take a subdirect representation of $\mathbb{B}$ and show that $g'(c) = r$ and $g'(d) = s$ componentwise, as in Case J above. Since

$$\begin{aligned} g(x) = (\partial p \wedge \partial q \wedge u) &\vee (\partial p \wedge \partial\partial q \wedge \partial h(x) \wedge u) \vee (\partial p \wedge \partial q \wedge \partial u \wedge v) \\ &\vee (\partial p \wedge \partial\partial q \wedge h(x) \wedge \partial u \wedge v) \vee (p \wedge q \wedge u \wedge v) \\ &\vee (p \wedge \partial q \wedge h(x) \wedge u \wedge v), \end{aligned}$$

the l-th projection of the polynomial $S_2(r,K(r,p,q),r,s,g(x))$ maps $c(l)$ to $r(l)$ and $d(l)$ to $s(l)$ unless $r(l) = (p \wedge q \wedge u \wedge v)(l)$. From the definition of J, it therefore follows that $g'(c) = r$ and $g'(d) = s$.

For the two remaining subcases where either $g(x) = K(u,J(p,q,h(x)),v)$ or $g(x) = K(u,v,J(p,q,h(x)))$, let

$$g'(x) = J(r,K(r,u,q),S_2(r,K(r,u,q),r,s,g(x))).$$

An argument similar to the previous subcase shows that $g'(c) = r$ and $g'(d) = s$.

Case T: Since $T(w,x,y,z) = T(y,z,w,x)$, we need only consider translations through the first two coordinates. The equation

$$T(J(p,q,h(x)),u,v,w) = T(p \wedge q,u,v,w)$$

holds in our variety, so we move on to the subcase $g(x) = T(u,J(p,q,h(x)),v,w)$. If $g(x) = T(u,J(p,q,h(x)),v,w)$ then

$$r = \big[r \wedge (u \cdot J(p,q,h(x)))\big] \vee \big[r \wedge \partial(u \cdot J(p,q,h(x)))\big]$$

($x \vee y$ is not a polynomial in our variety, but since $\mathbb{A}'(\mathcal{T})$ is a height 1 semilattice, if $x,y \leq z$ then the quantity $x \vee y$ is uniquely defined). From the above equation,

$$g'(x) = J(r,J(u \cdot p,u \cdot q,u \cdot h(x)),v \cdot w)$$

has $g'(c) = r$ and $g'(d) = s$ and Case J applies again.

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Case $F \in \{U^0_M, U^1_M \mid M \in \mathcal{L} \cup \mathcal{R}\}$: The only difficulty in this case arises when $g(x) = U^i_M(u, v, w, J(p, q, h(x))$. In this subcase, let $t = K(r, M(v, w, p), M(v, w, q))$ and $g'(x) = J(r, t, S_2(r, t, r, s, g(x)))$.

This completes the proof of the restricted setting of $M = 1$. By repeatedly applying this argument, the lemma is proved for general M. ⊣

The Lemma 5.7 shows that for 2 fixed inputs, the J operation can be taken to commute in a very specific way with the other fundamental operations. The situation for J' is similar, but much more complicated, requiring a sequence of inputs and a mix of the J and J' operations.

Lemma 5.9. *Let $F_1, \dots, F_M$ be fundamental translations, $h(x)$ a happy primitive polynomial, and $p, q, c, d \in B$ with $d \leq c$ such that the set*

$$\{F_k \circ \cdots \circ F_1(J'(p, q, h(c))) \mid 1 \leq k \leq M\} \cup \{J'(p, q, h(c))\}$$

contains only unhappy elements. If $g(x) = F_M \circ \cdots \circ F_1 \circ J'(p, q, h(x))$ and $(r, s) = (g(c), g(d))$, then there is a decreasing Maltsev chain $g(c) = r_1, r_2, \dots, r_n = g(d)$ connecting $g(c)$ to $g(d)$ with associated polynomials $g_1(x), \dots, g_{n-1}(x)$ of the form

$$g_k(x) = G_k(p_k, q_k, h_k(x)), \quad \textit{where} \quad G_k \in \{J, J'\},\ p_k, q_k \in B,\ h_k(x) \in \mathcal{P} \textit{ happy}.$$

Proof. For convenience, let $r = g(c)$ and $s = g(d)$. As in Lemma 5.8, we will prove the claim in the restricted setting of $M = 1$ (i.e., when $g(x) = F \circ J'(p, q, h(x))$). Repeated applications on the proof in the restricted setting and of Lemma 5.8 will prove the lemma for general M. Say that $g(x) = F \circ J'(p, q, h(x)))$, where F is a fundamental translation and $g(c) = r$ is unhappy (and thus $\{r, s\}$ is unhappy). Note that F must be unhappy since $r = F \circ J'(p, q, h(c)))$ is unhappy. In particular, this means that F is not a translation of an S_i operation.

Composing the polynomial $J'(p, q, h(x))$ with translations of operations from $\{(\cdot), I\} \cup \mathcal{L} \cup \mathcal{R}$ produce either constant polynomials or the composition is commutative (i.e., $F \circ J'(p, q, h(x))) = J'(F(p), F(q), F \circ h(x)))$. Since these operations are 0-absorbing, they are happiness preserving, and the claim holds for them.

Case $\wedge$: We have that $u \wedge J'(p, q, h(x)) = J'(p, q, h(x)) \wedge u = J'(p \wedge u, q, h(x))$.

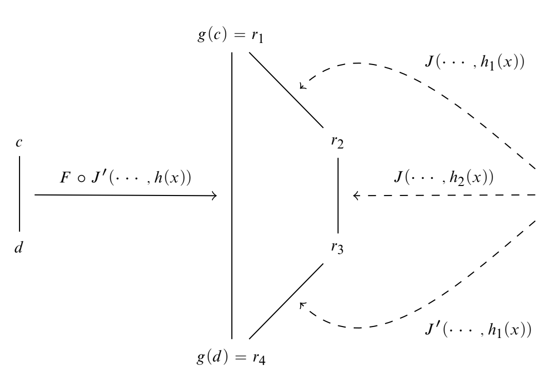

Figure 3. Lemma 5.9 illustration.

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Case J: The first translation is easy since $J(x,y,z)\wedge w = J(x\wedge w,y,z)$ and $J'(x,y,z)\leq J'(x,y,x)$. We have

$$\begin{aligned}J(J'(p,q,h(x)),u,v) &= J'(p,q,h(x))\wedge J(J'(p,q,p),u,v)\\ &= J'(p\wedge J(J'(p,q,p),u,v),q,h(x)).\end{aligned}$$

For $g(x) = J(u,J'(p,q,h(x)),v)$ we must introduce a new "link" in our Maltsev chain. Let

$$\begin{aligned}g_1(x) &= J(r,K(r,p,q),S_2(r,K(r,p,q),r,s,g(x))) &&\text{and}\\ g_2(x) &= J'(t_1,K(t_1,p,q),h(x)), &&\text{where } t_1 = g_1(d)\end{aligned}$$

(recall that $r = g(c)$ and $s = g(d)$). We have that

$$\begin{aligned}g(x) = (u\wedge p\wedge q\wedge h(x)\wedge v)\vee(u\wedge p\wedge \partial q\wedge v)\\ \vee(u\wedge \partial p\wedge \partial q\wedge \partial h(x))\vee(u\wedge\partial p\wedge\wedge q).\end{aligned}$$

The argument at this point breaks down into many subcases, depending on whether $r(l)$ is equal to $p(l)$, $q(l)$, $\partial p(l)$, or $\partial q(l)$ (if $r(l)\neq 0$, then by the flatness of $\mathbb{C}_l$ it must take on one of these values). The easiest way to keep track of everything is with a table. Since $r(l)=0$ implies $g'(x)(l)=0$ and $s(l)=0$, we will assume that $r(l)\neq 0$. For ease of reading, in the table below we will omit the coordinate when giving values of functions (i.e., "(l)" will be omitted from $r(l)$). Additionally, those coordinates which permit $r(l)\neq s(l)$ have been indicated.

r	$K(r,p,q)$	$S_2(r,K(r,p,q),r,s,g(x))$	$g_1(x)$	$r\neq s$
$p=q$	$p=r$	0	$r\wedge r$	Y
$p=\partial q$	$q=\partial r$	$(r\wedge s)\vee(r\wedge g(x))$	$s\vee(r\wedge g(x))$	N
$\partial p=\partial q$	$p=\partial r$	$(r\wedge s)\vee(r\wedge g(x))$	$s\vee(r\wedge g(x))$	Y
$\partial p=q$	$q=\partial r$	$(r\wedge s)\vee(r\wedge g(x))$	$s\vee(r\wedge g(x))$	N

Since $r=g(c)$ and $s=g(d)$, we have that $g_1(c)=r$, and $t_1(l)=g_1(d)(l)=s(l)$ except for possibly when $r(l)=p(l)=q(l)$. It follows (by flatness) that $s\leq t_1\leq r$. We will now show (with another similar table) that $g_2(c)=t_1$ and $g_2(d)=s$. The first column of the table below corresponds to the 2nd-to-last column of the table above evaluated at $x=d$.

$t_1=g_1(d)$	$K(t_1,p,q)$	$g_2(x)$	$t_1\neq s$
$r=p=q$	$p=t_1$	$t_1\wedge h(x)$	Y
$r=s=p=\partial q$	$q=\partial t_1$	t_1	N
s	$(p\wedge\partial s)=\partial t_1$	t_1	N
$r=s=\partial p=q$	$q=\partial t_1$	t_1	N

From the table we can see that $g_2(c)(l)=t_1(l)$ in all subcases except for possibly when $r(l)=p(l)=q(l)$. In this event, from the definition of $g(x)$ we have that $r(l)=h(c)(l)$, so $g_2(c)(l)=t_1(l)$ (the previous table indicates that $t_1(l)=r(l)$ when $r(l)=p(l)=q(l)$). Therefore, $g_2(c)=t_1$. Since $t_1(l)$ differs from $s(l)$ only when $r(l)=p(l)=q(l)$, and since in this subcase $h(d)(l)=s(l)$ (from the definition of $g(x)$ at the start of the case), it follows that $g_2(d)=s$.

In the case where $g(x)=J(u,v,J'(p,q,h(x)))$, let

$$g_1(x)=J(u,v,S_2(u,v,r,s,h(x))).$$

An argument similar to the case for $J(\cdots, J(\cdots, h(x)))$ in Lemma 5.8 will show that $g_1(c) = g(c) = r$ and $g_1(d) = g(d) = s$.

CASE J': We have

$$J'(J'(p, q, h(x)), u, v) = J'(p, q, h(x)) \wedge J'(J(p, q, p), u, v)$$
$$= J'(J'(J(p, q, p), u, v), q, h(x)).$$

For $g(x) = J'(u, J'(p, q, h(x)), v)$ we must again introduce a new "link" in our Maltsev chain. Let

$$g_1(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))) \quad \text{and}$$
$$g_2(x) = J'(t_1, K(t_1, p, q), h(x)), \quad \text{where } t_1 = g_1(d).$$

An argument similar to the corresponding subcase of Case J will show that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$. For $g(x) = J'(u, v, J'(p, q, h(x)))$, let

$$g_1(x) = J'(r, K(r, v, q), h(x)).$$

An argument similar to the one at the start of Case J above will show that $g_1(c) = r$ and $g_2(d) = s$.

CASE S_i: Since $\{r, s\}$ is unhappy, we can exclude these translations.

CASE K: For $g(x) = K(J'(p, q, h(x)), u, v)$, let

$$g_1(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))) \quad \text{and}$$
$$g_2(x) = J'(t_1, K(t_1, p, q), h(x)), \quad \text{where } t_1 = g_1(d).$$

An argument similar to the one in Case J requiring the tables shows that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$. For the two remaining subcases where we have either $g(x) = K(u, J'(p, q, h(x)), v)$ or $g(x) = K(u, v, J'(p, q, h(x)))$, let

$$g_1(x) = J(r, K(r, u, p), S_2(r, K(r, u, q), r, s, g(x))) \quad \text{and}$$
$$g_2(x) = J'(t_1, K(t_1, u, q), h(x)), \quad \text{where } t_1 = g_1(d).$$

An argument similar to the one using the tables above will show that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$.

CASE T: Since $T(w, x, y, z) = T(y, z, w, x)$, we need only consider translations through the first two coordinates. If $g(x) = T(J'(p, q, h(x)), u, v, w)$, then

$$r = \left[r \wedge (J'(p, q, h(x)) \cdot u)\right] \vee \left[r \wedge \partial(J'(p, q, h(x)) \cdot u)\right].$$

Therefore

$$g'(x) = J'(r, J'(p \cdot u, q \cdot u, h(x) \cdot u), v \cdot w)$$

has $g(c) = r$ and $g(d) = s$ and Case J' applies. Similarly if we have $g(x) = T(u, J'(p, q, h(x)), v, w)$, then

$$r = \left[r \wedge (u \cdot J'(p, q, h(x)))\right] \vee \left[r \wedge \partial(u \cdot J'(p, q, h(x)))\right].$$

Therefore

$$g'(x) = J'(r, J'(u \cdot p, u \cdot q, u \cdot h(x)), v \cdot w)$$

has $g(c) = r$ and $g(d) = s$ and Case J' applies again.

Case $F \in \{U^0_M, U^1_M \mid M \in \mathcal{L} \cup \mathcal{R}\}$: If $g(x) = U^i_F(u, v, w, J'(p, q, h(x)))$, then let

$$g'(x) = J'(r, U^i_F(u, v, w, q), U^i_F(u, v, w, h(x)));$$

if $g(x) = U^i_F(u, v, J'(p, q, h(x)), w)$, then let

$$g'(x) = J'(r, U^i_F(u, v, p, w), U^i_F(u, v, h(x), w));$$

if $g(x) = U^i_F(u, J'(p, q, h(x)), v, w)$, then let

$$g'(x) = J'(r, U^i_F(u, p, v, w), U^i_F(u, h(x), v, w));$$

if $g(x) = U^i_F(J'(p, q, h(x)), u, v, w)$, then let

$$g'(x) = J'(r, U^i_F(p, u, v, w), U^i_F(h(x), u, v, w)).$$

This completes the proof of the restricted setting of $M = 1$. By repeatedly applying this argument and Lemma 5.8, the lemma is proved for general M. ⊣

In the Lemma 5.10, we see that the K operation behaves essentially the same as the J' operation.

Lemma 5.10. *Let $F_1, \dots, F_M$ be fundamental translations, $h(x)$ a happy primitive polynomial, and $p, q, c, d \in B$ with $d \leq c$ such that the set*

$$\{F_k \circ \cdots \circ F_1(K(p, q, h(c))) \mid 1 \leq k \leq M, \} \cup \{K(p, q, h(c))\}$$

contains only unhappy elements. If $g(x) = F_M \circ \cdots \circ F_1 \circ K(p, q, h(x))$, then there is a decreasing Maltsev chain $g(c) = r_1, r_2, \dots, r_n = g(d)$ connecting $g(c)$ to $g(d)$ with associated polynomials $g_1(x), \dots, g_{n-1}(x)$ of the form

$$g_k(x) = G_k(p_k, q_k, h_k(x)), \quad \text{where} \quad G_k \in \{J, J'\},\ p_k, q_k \in B,\ h_k(x) \in \mathcal{P} \text{ happy}.$$

Proof. Let $g'(x) = K(p, q, h(x))$, where p, q and $h(x)$ are as in the hypotheses of the lemma, and let $r = g'(c)$ and $s = g'(d)$. Define

$$f_1(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, h(x))) \qquad \text{and}$$
$$f_2(x) = J'(t_1, K(t_1, p, q), h(x)), \qquad \text{where } t_1 = f_1(d).$$

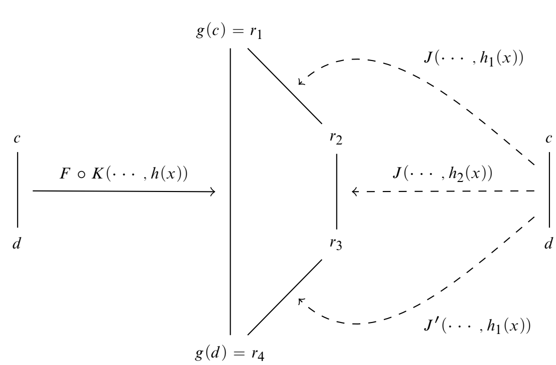

Figure 4. Lemma 5.10 illustration.

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We will show that $r = f_1(c)$, $t_1 = f_2(c)$, and $s = f_2(d)$. Since s is unhappy and $s \leq t_1 \leq r$, the elements s, t_1, and r are all unhappy. Using this fact and Lemmas 5.8 and 5.9, the conclusion will follow.

Let $l \in L$. The proof breaks into cases depending on whether $r(l) = 0$. Overall, it is useful to note that J and J' are 0-absorbing in the first and second variables, that $s \leq t_1 \leq r$, and if $r(l) \neq 0$, then $r(l) = h(c)(l)$ or $r(l) = \partial h(c)(l)$.

If $r(l) = 0$ then $s(l) = 0$ and $t_1(l) = 0$. Thus $r(l) = 0 = f_1(x)(l) = f_1(c)(l)$ and $t_1(l) = s(l) = 0 = f_2(x)(l) = f_2(c)(l) = f_2(d)(l)$. If $r(l) \neq 0$, then by the definition of K, either $p(l) = \partial q(l)$, $p(l) = q(l) = \partial r(l)$, or $p(l) = q(l) = r(l)$. In each of these cases, the equations $f_1(c)(l) = r(l)$, $t_1(l) = f_2(c)(l)$, and $s(l) = f_2(d)(l)$ are easily verified from the definitions. ⊣

Lemma 5.11. *Let $c, d \in B$ be such that $d \leq c$ and $\{c, d\}$ is happy. Suppose that $(r, s) \in Cg^{\mathbb{B}}(c, d)$ with $s \leq r$ is witnessed by the decreasing Maltsev sequence $r = u_1, \dots, u_n = s$ with associated primitive polynomials $\lambda_1(x), \dots, \lambda_{n-1}(x)$. Then there is another decreasing Maltsev sequence, $r = t_1, \dots, t_m = s$, with associated primitive polynomials $g_1(x), \dots, g_{m-1}(x)$ such that for each $k \in \{1, \dots, m-1\}$, one of*

1. *$g_k(x)$ is happy,*
2. *$g_k(x) = J(t_k, q_k, h_k(x))$ and $h_k(x) \in \mathcal{P}$ is happy, or*
3. *$g_k(x) = J'(t_k, q_k, h_k(x))$ and $h_k(x) \in \mathcal{P}$ is happy*

holds for some constants $q_k \in B$.

Proof. Select a consecutive pair, u_k and u_{k+1} from the Maltsev sequence. We will show that the claim holds for the pair, and by applying the argument to each consecutive pair, it therefore must hold for the entire sequence. By Lemma 5.7, we can assume that one of the following is true:

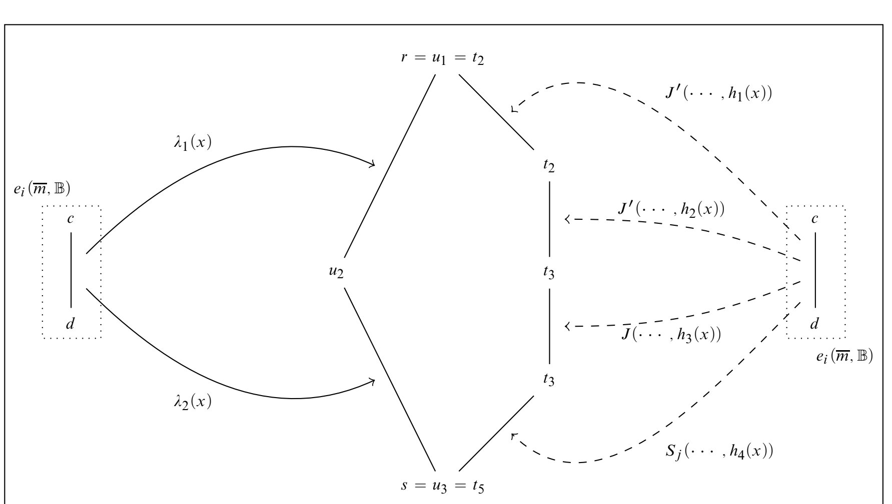

Figure 5. Lemma 5.11 illustration.

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(1) $\{u_k, u_{k+1}\}$ is happy, so there is a happy primitive $g_k(x)$ with $(u_k, u_{k+1}) = (g_k(c), g_k(d))$, or
(2) $\{u_k, u_{k+1}\}$ is unhappy, so there are fundamental translations $F_1, \dots, F_M$ and a happy polynomial $h(x) \in \mathcal{P}$ such that for some $G \in \{J, J', K\}$ and some $p, q \in B$ the polynomial

$$g_k'(x) = F_m \circ \dots \circ F_1 \circ G(p, q, h(x))$$

has $(u_k, u_{k+1}) = (g_k'(c), g_k'(d))$ and the set

$$\{F_k \circ \dots \circ F_1 \circ G(p, q, h(c)) \mid 1 \le k \le M\} \cup \{G(p, 1, h(c))\}$$

contains only unhappy elements.

In the first possibility, we are done. In the second possibility, apply Lemma 5.8 (if $G = J$), 5.9 (if $G = J'$), or 5.10 (if $G = K$) to get a decreasing Maltsev sequence $u_k = t_k, t_{k+1}, \dots, t_{k+m'} = u_{k+1}$ with associated primitive polynomials $g_k(x), \dots, g_{k+m'-1}(x)$ such that for all $l \in \{k, \dots, k+m'-1\}$,

$$g_l(x) = G_l(p_l, q_l, h_l(x)) \quad \text{where} \quad G_l \in \{J, J'\},\ p_l, q_l \in B,\ h_l(x) \in \mathcal{P} \text{ happy.}$$

This is almost the conclusion of the lemma. To finish, we observe that if $f(x) = G(p, q, h(x))$ is a polynomial with $G \in \{J, J'\}$ and $f(d) \le f(c)$, then

$$f(c) = G(f(c), q, h(c)) \quad \text{and} \quad f(d) = G(f(c), q, h(d)).$$

Applying this observation to the $g_l(x)$ and using the fact that $t_k, \dots, t_{k+m'}$ is a decreasing sequence completes the proof. ⊣

At this point, we have established the tools necessary to transform general decreasing Maltsev chains into longer chains whose associated polynomials are of a very specific form. Now, we move on to show that these longer chains can be shortened and come in just 7 types, and that these 7 different types of chains are definable. The following definition simplifies the discussion.

DEFINITION 5.12. Let $r_1, \dots, r_n \in B$ be a sequence of elements. We write

$$r_1 \xrightarrow{F_1} r_2 \xrightarrow{F_2} r_3 \cdots r_{n-1} \xrightarrow{F_{n-1}} r_n$$

for $F_i \in \{J, J', S_0, S_1, S_2\}$ if both of the following hold

(1) if $F_i \in \{J, J'\}$, then there exist constants $p_i, q_i \in B$ and $\overline{n}_i \in B^2 \cup B$ such that

$$r_i = F_i(p_i, q_i, e_{j_i}(\overline{n}_i, r_i)) \quad \text{and} \quad r_{i+1} = F_i(p_i, q_i, e_{j_i}(\overline{n}_i, r_{i+1}))$$

for some $j_i \in \{0, 1, 2\}$, and
(2) if $F_i \in \{S_0, S_1, S_2\}$, then there exists $\overline{n}_i \in B^2 \cup B$ such that

$$r_i = F_i(\overline{n}_i, r_i, r_i, r_i) \quad \text{and} \quad r_{i+1} = F_i(\overline{n}_i, r_{i+1}, r_{i+1}, r_{i+1}).$$

Such a sequence will be referred to as an F_1-F_2-$\cdots$-F_{n-1} *chain*. If it is the case that for all i, $(r_i, r_{i+1}) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ for some $c, d \in B$, then we will say that $(r_1, r_n) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ is *witnessed* by an F_1-…-F_{n-1} chain.

LEMMA 5.13. *Let $c, d \in e_i(\overline{m}, B)$ for some $i \in \{0, 1, 2\}$ and $\overline{m} \in B^2 \cup B$ and assume that the congruence formula $\psi(-, -, c, d)$ defines $Cg^{e_i(\overline{m}, \mathbb{B})}(c, d)$ in $e_i(\overline{m}, \mathbb{B})$.*

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Suppose that $r, s \in e_j(\overline{n}, B)$ *for some* $j \in \{0, 1, 2\}$ *and* $\overline{n} \in B^2 \cup B$ *with* $s \leq r$. *Then* $(r, s) \in Cg^{\mathbb{B}}(c, d)$ *if and only if*

$$\mathbb{B} \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d)$$

and $r = S_j(\overline{n}, r, s, e_i(\overline{m}, r))$ *and* $s = S_j(\overline{n}, r, s, e_i(\overline{m}, s))$.

PROOF. Suppose first that $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ and let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Define

$$I = \{l \in L \mid e_i(\overline{m}, B)(l) \neq \{0\}\} \quad \text{and} \quad J = L \setminus I,$$

and write a typical element $x \in B$ as $x = (x_I, x_J)$, where $x_I \in \pi_I(B)$ and $x_J \in \pi_J(B)$. Since $c, d \in e_i(\overline{m}, B)$, we have $c = (c_I, 0_J)$ and $d = (d_I, 0_J)$. Hence, if $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$, it must be that $r = (r_I, z_J)$ and $s = (s_I, z_J)$ (i.e., $\pi_J(r) = \pi_J(s)$).

From the definition of S_i, we have that $e_i(\overline{m}, -)$ is a homomorphism from $\mathbb{B}$ to $e_i(\overline{m}, \mathbb{B})$. Therefore $(e_i(\overline{m}, r), e_i(\overline{m}, s)) \in \mathrm{Cg}^{e_i(\overline{m}, \mathbb{B})}(c, d)$, and

$$\mathbb{B} \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d),$$

since ψ is existentially quantified (it is a congruence formula) and $e_i(\overline{m}, \mathbb{B}) \leq \mathbb{B}$. Since $r \in e_j(\overline{n}, B)$, if $t \leq r$ then $t \in e_j(\overline{n}, B)$. Therefore

$$\{e_i(\overline{m}, r), e_i(\overline{m}, s)\} \subseteq e_i(\overline{m}, e_j(\overline{n}, B)) \subseteq e_i(\overline{n}, B).$$

It follows that

$$\begin{aligned}
S_j(\overline{n}, r, s, e_i(\overline{m}, r)) &= S_j\left(\overline{n}, \begin{pmatrix} r_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ z_J \end{pmatrix}, \begin{pmatrix} r_I \\ 0 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ z_J \end{pmatrix}\right) \vee \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} r_I \\ 0 \end{pmatrix}\right) = \begin{pmatrix} s_I \\ z_J \end{pmatrix} \vee \begin{pmatrix} r_I \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} r_I \\ z_J \end{pmatrix} = r, \quad \text{and likewise} \\
S_j(\overline{n}, r, s, e_i(\overline{m}, s)) &= S_j\left(\overline{n}, \begin{pmatrix} r_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ 0 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ z_J \end{pmatrix}\right) \vee \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ 0 \end{pmatrix}\right) = \begin{pmatrix} s_I \\ z_J \end{pmatrix} \vee \begin{pmatrix} s_I \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} s_I \\ z_J \end{pmatrix} = s,
\end{aligned}$$

completing the forward direction.

Suppose now that $\mathbb{B} \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d)$, and $r = S_j(\overline{n}, r, s, e_i(\overline{m}, r))$ and $s = S_j(\overline{n}, r, s, e_i(\overline{m}, s))$. Since ψ is a congruence formula and $e_i(\overline{m}, -)$ is a homomorphism from $\mathbb{B}$ to $e_i(\overline{m}, \mathbb{B})$ and $c, d \in e_i(\overline{m}, B)$, we have

$$e_i(\overline{m}, \mathbb{B}) \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d).$$

Thus, $(e_i(\overline{m}, r), e_i(\overline{m}, s)) \in \mathrm{Cg}^{e_i(\overline{m}, \mathbb{B})}(c, d) \subseteq \mathrm{Cg}^{\mathbb{B}}(c, d)$. By hypothesis, we also have that $r = S_j(\overline{n}, r, s, e_i(\overline{m}, r))$ and $s = S_j(\overline{n}, r, s, e_i(\overline{m}, s))$, so it follows that $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$. ⊣

In light of the Lemma 5.12, define

$$\psi_S(w,x,y,z) = \bigvee_{i=0}^{2}\bigvee_{j=0}^{2} \exists \overline{m},\overline{n}\,\big[y = e_i(\overline{m},y) \wedge z = e_i(\overline{m},z)$$
$$\wedge \psi_0(e_i(\overline{m},w), e_i(\overline{m},x), y, z)$$
$$\wedge\, w = S_j(\overline{n}, w, x, e_i(\overline{m},w))$$
$$\wedge x = S_j(\overline{n}, w, x, e_i(\overline{m},x))\big] \tag{5.3}$$

(recall that ψ_0 was defined in (5.2)). If c, d, r, s satisfy the hypotheses of Lemma 5.13, then $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$ if and only if $\mathbb{B} \models \psi_S(r,s,c,d)$. That is, if $\{c,d\}$ and $\{r,s\}$ are happy with $d \le c$ and $s \le r$, then $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$ if and only if $\mathbb{B} \models \psi_S(r,s,c,d)$.

LEMMA 5.14. *Suppose that* $(r,s) \in Cg^{\mathbb{B}}(c,d)$ *for some* $c, d \in B$ *and that there is a decreasing sequence* $r = r_1 \ge r_2 \ge \cdots \ge r_n = s$ *and some constants* $p_1, q_1, \ldots, p_{n-1}, q_{n-1} \in B$ *such that*

$$r_i = J(p_i, q_i, r_i) \quad and \quad r_{i+1} = J(p_i, q_i, r_{i+1})$$

for $1 \le i \le n-1$. *Then there exists a constant* $\rho \in B$ *such that* $r = J(r, \rho, r')$ *and* $s = J(r,\rho,s')$, *where* $r' = e_2(r,\rho,r)$ *and* $s' = e_2(r,\rho,s)$.

Since $J(x,y,z) = J(x,y,e_2(x,y,z))$ (*from the definition of* J), *this is equivalent to the assertion that for each decreasing* J-J-…-J *chain* (*of any length*), *there is a* (*length* 1) J *chain with the same endpoints.*

PROOF. Note that $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$ and the presence of a semilattice operation implies $(r_i, r_{i+1}) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$. Next, observe that since the chain is decreasing and $s \le r$, if we replace q_i with $J(q_i, p_i, q_i)$, then we can replace each p_i with r. Thus, we may assume that

$$r_i = J(r, q_i, r_i) \quad \text{and} \quad r_{i+1} = J(r, q_i, r_{i+1}).$$

The proof shall be by induction on n (the length of the chain). If $n = 1$, then

$$r = J(r, q_1, r) \quad \text{and} \quad s = J(r, q_1, s).$$

Therefore

$$r = (r \wedge \partial q_1 \wedge r) \vee (r \wedge q_1) \quad \text{and} \quad s = (r \wedge \partial q_1 \wedge s) \vee (r \wedge q_1).$$

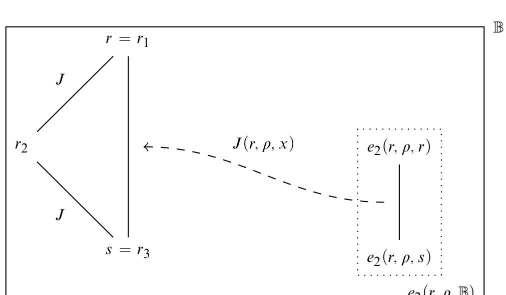

FIGURE 6. Lemma 5.14 illustration.

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Hence without loss of generality, we can replace the last occurrence of r in $r = J(r, q_1, r)$ with $r' = e_2(r, q_1, r)$, and the last occurrence of s in $s = J(r, q_1, s)$ with $s' = e_2(r, q_1, s)$. After making these replacements, the conclusion of the lemma follows with $\rho = q_1$.

Assume now that the lemma holds for all chains of length less than N, and consider a chain of length N: $r = r_1, \dots, r_N = s$. Applying the inductive hypothesis to the subchain $r = r_1, \dots, r_{N-1}$, there exists $\rho_1 \in B$ with $r = J(r, \rho_1, r'')$ and $r_{N-1} = J(r, \rho_1, r''_{N-1})$, where $r'' = e_2(r, \rho_1, r)$ and $r''_{N-1} = e_2(r, \rho_1, r_{N-1})$. We therefore have

$$\begin{aligned} &r = J(r, \rho_1, r''), \qquad r_{N-1} = J(r, \rho_1, r''_{N-1}) = J(r, q_{N-1}, r_{N-1}), \\ &s = J(r, q_{N-1}, s). \end{aligned} \tag{5.4}$$

Let $\rho = K(r, \rho_1, q_{N-1})$, $r' = e_2(r, \rho, r)$, and $s' = e_2(r, \rho, s)$. We will now show that $r = J(r, \rho, r')$ and $s = J(r, \rho, s')$, proving the lemma.

Let $\mathbb{B} = \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. We will analyze the polynomial $J(r, \rho, x)$ coordinatewise, and as usual it will be easiest to use a table. Before the table is constructed, however, we will determine which coordinates permit $r(l) \neq s(l)$. Since $s \leq r_{N-1} \leq r$, either $r(l) \neq r_{N-1}(l) = s(l) = 0$, or $r(l) = r_{N-1}(l) \neq s(l) = 0$. The equalities (5.4) give us

$$\begin{aligned} r &= (r \wedge \rho_1) \vee (r \wedge \partial\rho_1 \wedge r''), & r_{N-1} &= (r \wedge \rho_1) \vee (r \wedge \partial\rho_1 \wedge r''_{N-1}), \\ r_{N-1} &= (r \wedge q_{N-1}) \vee (r \wedge \partial q_{N-1} \wedge r_{N-1}), & s &= (r \wedge q_{N-1}) \vee (r \wedge \partial q_{N-1} \wedge s). \end{aligned}$$

Observe that $r(l) = \partial\rho_1(l)$ implies $r(l) = e_2(r, \partial\rho_1, r)(l) = r''(l)$. Assume first that $r(l) \neq r_{N-1}(l) = s(l) = 0$. Under this assumption, it must be that $r(l) = \partial\rho_1(l)$ and $r(l) = \partial q_{N-1}(l)$. Assume now that $r(l) = r_{N-1}(l) \neq s(l) = 0$. Under this assumption, it must be that $r(l) = r_{N-1}(l) \in \{\rho_1, \partial\rho_1\}$ and $r = \partial q_{N-1}$. We now assemble all of this in the table below. As usual, since $r(l) = 0$ implies $r_{N-1}(l) = s(l) = 0$, we assume that $r(l) \neq 0$. In particular, this means that $r(l) \in \{\rho_1(l), \partial\rho_1(l)\}$.

r	$\rho = K(r, \rho_1, q_{N-1})$	$J(r, \rho, e_2(r, \rho, x))$	$r \neq s$
$\rho_1 = q_{N-1}$	r	r	N
$\rho_1 = \partial q_{N-1}$	$q_{N-1} = \partial r$	$e_2(r, \partial r, x)$	Y
$\partial\rho_1$	$\rho_1 = \partial r$	$e_2(r, \partial r, x)$	Y

If $r(l) = \partial\rho(l)$, then $r'(l) = e_2(r, \rho, r)(l) = r(l)$ and $s'(l) = e_2(e, \rho, s)(l) = s(l)$. Therefore, the table above show that $J(r, \rho, r') = r$ and $J(r, \rho, s') = s$. ⊣

In light of the Lemma 5.13, define

$$\begin{aligned} \psi_J(w, x, y, z) = \exists b \, \big[&\psi_S(e_2(w, b, w), e_2(w, b, x), y, z) \\ &\wedge w = J(w, b, e_2(w, b, w)) \wedge x = J(w, b, e_2(w, b, x)) \big] \end{aligned} \tag{5.5}$$

(ψ_S was defined in (5.3)). From the Lemma 5.13, if $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$, $\{c, d\}$ is happy, and $s \leq r$, then $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ is witnessed by a decreasing J-$\cdots$-J, chain if and only if $\mathbb{B} \models \psi_J(r, s, c, d)$.

LEMMA 5.15. *Suppose that* $(r, s) \in Cg^{\mathbb{B}}(c, d)$ *for some* $c, d \in B$ *and that there is a decreasing sequence* $r = r_1 \geq r_2 \geq \cdots \geq r_n = s$ *and some constants* $p_1, q_1, \ldots, p_{n-1}, q_{n-1} \in B$ *such that*

$$r_i = J'(p_i, q_i, e_{j_i}(\overline{n}_i, r_i)) \quad \text{and} \quad r_{i+1} = J'(p_i, q_i, e_{j_i}(\overline{n}_i, r_{i+1}))$$

for some $j_i \in \{0, 1, 2\}$ *and* $\overline{n}_i \in B^2 \cup B$. *Then there exist constants,* $\rho, t \in B$ *such that*

$$r = J'(r, q_1, e_{j_1}(\overline{n}_1, r)), \qquad t = J'(r, q_1, e_{j_1}(\overline{n}_1, s)) = J(t, \rho, r'),$$
$$s = J(t, \rho, s'),$$

where $r' = e_2(r, \rho, r)$, *and* $s' = e_2(r, \rho, s)$.

That is, for every J'-$\cdots$-J' *chain (of arbitrary length) there is a* J'-J *chain with the same endpoints.*

PROOF. Note that $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ and the presence of a semilattice operation implies $(r_i, r_{i+1}) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$. The proof shall be by induction on n (the length of the sequence). If $n = 1$, the lemma is trivially true. Assume now that the lemma holds for all sequences of length less than N, and consider a sequence of length N: $r = r_1, \ldots, r_N = s$. Apply the inductive hypothesis to the subsequence $r_2, \ldots, r_N = s$ to get

$$r_2 = J'(r_2, q_2, e_{j_2}(\overline{n}_2, r_2)), \qquad t_1 = J'(r_2, q_2, e_{j_2}(\overline{n}_2, s)) = J(t_1, \rho_1, r_2')$$
$$s = J(t_1, \rho_1, s'), \qquad \text{where } r_2' = e_2(r_2, \rho, r_2) \text{ and } s' = e_2(r_2, \rho, s)$$

for some constants $\rho_1, t_1 \in B$. Since the sequence is decreasing, by replacing q_2 with $J'(q_2, r_2, q_2)$ we are free to replace r_2 with r. After doing this replacement we have

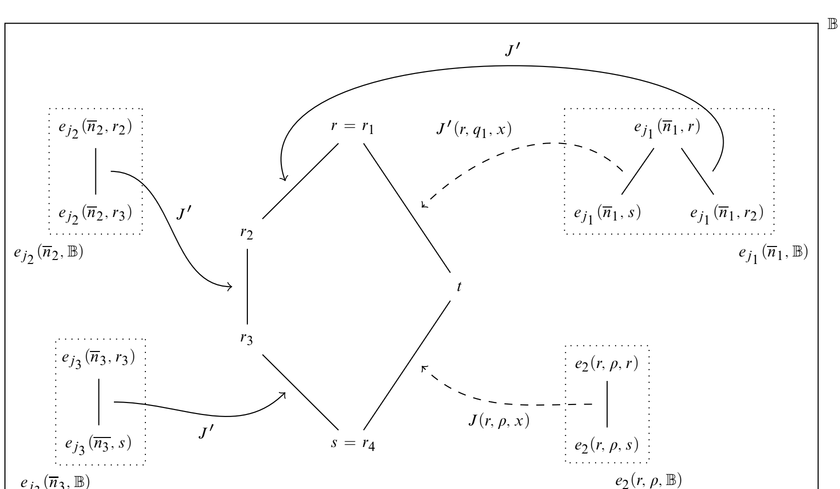

FIGURE 7. Lemma 5.15 illustration.

$$r = J'(r, q_1, e_{j_1}(\overline{n}_1, r)),$$
$$r_2 = J'(r, q_1, e_{j_1}(\overline{n}_1, r_2)) = J'(r, q_2, e_{j_2}(\overline{n}_2, r_2)),$$
$$t_1 = J'(r, q_2, e_{j_2}(\overline{n}_2, s)) = J(t_1, \rho_1, r_2'), \text{ and}$$
$$s = J(t_1, \rho_1, s').$$

We will analyze the subsequence r, r_2, t_1.

Let $t = J'(r, q_1, e_{j_1}(\overline{n}_1, t_1))$. We will show that

$$r = J'(r, q_1, e_{j_1}(\overline{n}_1, r)), \qquad t = J'(r, q_1, e_{j_1}(\overline{n}_1, t_1)) = J(t, q_1, r'),$$
$$t_1 = J(t, q_1, t_1'), \qquad \text{for } r' = e_2(r, q_1, r) \text{ and } t_1' = e_2(r, q_1, t_1)$$

(that is, $r \overset{J'}{\text{---}} r_2 \overset{J'}{\text{---}} t \overset{J}{\text{---}} s$ implies $r \overset{J'}{\text{---}} t \overset{J}{\text{---}} t_1 \overset{J}{\text{---}} s$). The only equalities that have not been shown already are $t = J(t, q_1, r')$ and $t_1 = J(t, q_1, t_1')$. As usual, let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. We will proceed componentwise.

We begin by showing that $t = J(t, q_1, r')$. Since $J(t, q_1, r') \leq t$, by the flatness of $\mathbb{C}_l$, it will be sufficient to show that $t(l) \neq 0$ implies $J(t, q_1, r')(l) \neq 0$. Suppose that $t(l) \neq 0$. Since $t = J'(r, q_1, e_{j_1}(\overline{n}_1, t_1))$, either $t(l) = q_1(l)$ or $t(l) = \partial q_1(l)$. If $t(l) = q_1(l)$, then $J(t, q_1, r')(l) = (t \wedge q_1)(l) = t(l)$. If $t(l) = \partial q_1(l)$, then $r(l) = t(l)$, since $\mathbb{C}_l$ is flat and $t \leq r$. Therefore, $r'(l) = e_2(r, q_1, r)(l) = r(l) = t(l)$, and so $J(t, q_1, r')(l) = t(l)$. Hence $J(t, q_1, r') = t$.

Next, we show that $t_1 = J(t, q_1, t_1')$. Again, we will assume that $t(l) \neq 0$, since $t(l) = 0$ implies that $t_1(l) = 0$ and $J(t, q_1, t_1')(l) = 0$. Since $\mathbb{C}_l$ is flat, if $t(l) \neq 0$, then $t_1(l) = t(l)$ and $t(l) \in \{q_1(l), \partial q_1(l)\}$. If $t(l) = q_1(l)$, then $J(t, q_1, t_1')(l) = t(l) = t_1(l)$. If $t(l) = \partial q_1(l)$, then $t_1'(l) = e_2(r, q_1, t_1) = t_1(l)$, so $J(t, q_1, t_1')(l) = t_1'(l) = t_1(l)$. Hence $J(t, q_1, t_1') = t_1$.

We now have

$$r = J'(r, q_1, e_{j_1}(\overline{n}_1, r)), \qquad t = J'(r, q_1, e_{j_1}(\overline{n}_1, t_1)) = J(t, q_1, r'),$$
$$t_1 = J(t, q_1, t_1') = J(t_1, \rho_1, r_2'), \qquad s = J(t_1, \rho_1, s'),$$

where $r' = e_2(r, q_1, r)$, $t_1' = e_2(r, q_1, t_1)$, $r_2' = e_2(t_1, \rho_1, t_1)$, and $s' = e_2(r_2, \rho_1, s)$. Apply Lemma 5.14 to the sequence t, t_1, s (the part of the sequence in the range of J) to get an element $\rho \in B$ such that $t = J(t, \rho, t'')$ and $s = J(t, \rho, s'')$ for $t'' = e_2(t, \rho, t)$ and $s'' = e_2(t, \rho, s)$. Since $t \leq r$, if $r' = e_2(r, \rho, r)$ and $s' = e_2(r, \rho, s)$, we have $t = J(t, \rho, r')$ and $s = J(t, \rho, s')$. Finally, we now have

$$r = J'(r, q_1, e_{j_1}(\overline{n}_1, r)), \qquad t = J'(r, q_1, e_{j_1}(\overline{n}_1, s)) = J(t, \rho, r')$$
$$s = J(t, \rho, s'), \qquad \text{for } r' = e_2(r, \rho, r) \text{ and } s' = e_2(r, \rho, s),$$

proving the lemma. ⊣

In light of the Lemma 5.14, define

$$\psi_{J'J}(w, x, y, z) = \exists t \left[\alpha(t, w, x, y, z) \wedge \beta(t, w, x, y, z)\right], \tag{5.6}$$

where

$$\alpha(t, w, x, y, z) = \bigvee_{i=0}^{2} \exists \overline{n}, a \left[\psi_S(e_i(\overline{n}, w), e_i(\overline{n}, x), y, z) \right.$$
$$\left. \wedge w = J'(w, a, e_i(\overline{n}, w)) \wedge t = J'(w, a, e_i(\overline{n}, x))\right]$$

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and

$$\beta(t, w, x, y, z) = \exists b \left[\psi_S(e_2(w, b, w), e_2(w, b, x), y, z) \right.$$
$$\left. \wedge t = J(t, b, e_2(w, b, w)) \wedge x = J(t, b, e_2(w, b, x))\right].$$

Recall that ψ_S was defined in (5.3). From the Lemma 5.14, if $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and $c, d, r, s \in B$ with $c < d$ and $s \leq r$, then $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ witnessed by a decreasing J'-$\cdots$-J' chain implies that $\mathbb{B} \models \psi_{J'J}(r, s, c, d)$. Conversely, $\mathbb{B} \models \psi_{J'J}(r, s, c, d)$ implies that $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ (although this is perhaps not witnessed by a J'-$\cdots$-J' chain).

At this point, we have the machinery necessary to change a general decreasing Maltsev chain into a longer chain whose associated polynomials all have J, J', or S_j as the outermost operations, and then to collapse repeated occurrences of J and J' to either a single occurrence of J or the chain J'-J. In order to fully collapse the chain, we still need to address what happens when the chain has alternating J and J' operations.

LEMMA 5.16. *Let $r, t, s \in B$ be such that $s \leq t \leq r$ and $(r, t), (t, s) \in Cg^{\mathbb{B}}(c, d)$ for some $c, d \in B$. Suppose that for constants $p_1, p_2, q_1, q_2 \in B$,*

$$r = J(p_1, q_1, r), \qquad t = J(p_1, q_1, t) = J'(p_2, q_2, t'),$$
$$s = J'(p_2, q_2, s'),$$

for $t' = e_i(\overline{n}, t)$ and $s' = e_i(\overline{n}, s)$ for some $i \in \{0, 1, 2\}$ and $\overline{n} \in B^2 \cup B$. Then there exist constants $\rho, u \in B$ such that

$$r = J'(r, \rho, r'), \qquad u = J'(r, \rho, s') = J(u, \rho, r''),$$
$$s = J(u, \rho, s''),$$

where $r' = e_i(\overline{n}, r)$, $r'' = e_2(r, \rho, r)$, and $s'' = e_2(r, \rho, s)$.

That is, for every J-J' chain there is a J'-J chain with the same endpoints.

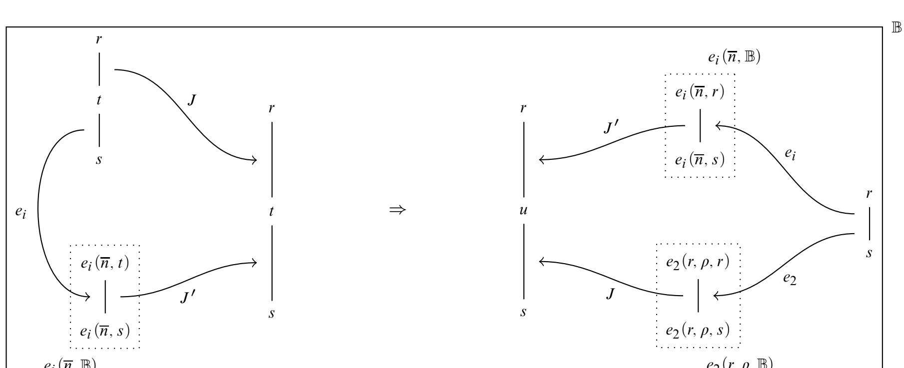

FIGURE 8. Lemma 5.16 illustration.

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PROOF. Since $s \leq t \leq r$, in the equations in the hypothesis by replacing q_1 with $J(p_1, q_1, p_1)$ and q_2 with $J'(p_2, q_2, p_2)$, we can replace p_1 and p_2 with r. Thus,

$$\begin{aligned} r &= J(r, q_1, r) = (r \wedge \partial q_1 \wedge r) \vee (r \wedge q_1), \\ t &= J(r, q_1, t) = (r \wedge \partial q_1 \wedge t) \vee (r \wedge q_1) \\ &= J'(r, q_2, t') = (r \wedge q_2 \wedge t') \vee (r \wedge \partial q_2), \text{ and} \\ s &= J'(r, q_2, s') = (r \wedge q_2 \wedge s') \vee (r \wedge \partial q_2). \end{aligned} \tag{5.7}$$

Let $\rho = K(r, q_1, q_2)$ and $u = J'(r, \rho, s')$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. Note that from the definition of K and the equations (5.7), for all $l \in L$, $r(l) \in \{\rho, \partial\rho, 0\}$. We will show that the equalities in the conclusion of the lemma hold componentwise.

We begin by showing that $r = J'(r, \rho, r')$. As usual, a table is the easiest way to organize the proof. Since $r(l) = 0$ implies $J'(r, \rho, r')(l) = 0$, assume that $r(l) \neq 0$.

r	$\rho = K(r, q_1, q_2)$	$J'(r, \rho, r')$	$r \neq t$
$q_1 = q_2$	$q_1 = q_2 = r$	$r \wedge r'$	N
$q_1 = \partial q_2$	$q_2 = \partial r$	r	N
$\partial q_1 = q_2$	$q_1 = \partial r$	r	Y
$\partial q_1 = \partial q_2$	$q_1 = \partial r$	r	N
$q_1 \notin \{q_2, \partial q_2\}$	0	0	N
$\partial q_1 \notin \{q_2, \partial q_2\}$	$q_1 = \partial r$	r	Y

The only possibly problematic cases are when $r(l) = q_1(l) = q_2(l)$ and when $r(l) = q_1(l) \notin \{q_2(l), \partial q_2(l)\}$.

CASE $r(l) = q_1(l) = q_2(l)$: If $r(l) = q_1(l)$, then $r(l) = t(l)$, by (5.7), so $r(l) = t(l) = t'(l)$. Since $t'(l) \leq r'(l)$ (because $e_i(\overline{n}, -)$ is monotonic and $t \leq r$), it follows that $r'(l) = r(l)$. Thus $J(r, \rho, r')(l) = r(l)$ in this case.

CASE $r(l) = q_1(l) \notin \{q_2(l), \partial q_2(l)\}$: If $r(l) = q_1(l)$, then $r(l) = t(l)$, but if $r(l) \notin \{q_2(l), \partial q_2(l)\}$ then $t(l) = 0$ by (5.7), contradicting our assumption that $r(l) \neq 0$. Therefore, in this case $J'(r, \rho, r')(l) = r(l)$ as well.

Next, we show that $u = J(u, \rho, r'')$. Since $J(u, \rho, r'') \leq u$ and each $\mathbb{C}_l$ is flat, it will be sufficient to show that when $u(l) \neq 0$, $J(u, \rho, r'')(l) \neq 0$ as well. When $u(l) \neq 0$, since $u \leq r$, it must be that $u(l) = r(l)$. From the construction of ρ, if $r(l) \neq 0$ either $r(l) \in \{\rho(l), \partial\rho(l)\}$. If $\rho(l) = r(l)$, then $J(u, \rho, r'')(l) = r(l) = u(l)$. Suppose now that $\rho(l) = \partial r(l)$. Then $r''(l) = e_2(r, \rho, r)(l) = r(l) = u(l)$, so $J(u, \rho, r'')(l) = r(l)$. Since $\rho(l) = r(l)$ or $\rho(l) = \partial r(l)$ for all $l \in L$, we have that $u = J(r, \rho, r'')$.

Finally, we show that $s = J(u, \rho, s'')$. There are three possibilities: $r(l) = u(l) = s(l)$, $r(l) = u(l)$ but $s(l) = 0$, or $r(l) \neq 0$ but $u(l) = s(l) = 0$.

CASE $r(l) = u(l) = s(l)$: If $\rho(l) = r(l)$, then $J(u, \rho, s'') = (u \wedge \rho)(l) = r(l) = s(l)$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s)(l) = s(l)$, so $J(u, \rho, s'') = (u \wedge \rho \wedge s'')(l) = s(l)$.

CASE $r(l) = u(l)$ **but** $s(l) = 0$: If $\rho(l) = r(l)$, then $r(l) = q_1(l) = q_2(l)$, so $s(l) = J'(r, q_2, s')(l) = J'(r, \rho, s')(l) = u(l)$, contradicting $u(l) \neq 0$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s) = s(l)$, so $J(u, \rho, s'')(l) = s''(l) = s(l)$.

CASE $r(l) \neq 0$ **and** $u(l) = s(l) = 0$: If $\rho(l) = r(l)$, then $J(u, \rho, r'')(l) = u(l) = J(u, \rho, s'')(l)$, so $s(l) = J(u, \rho, s'')$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s) = s(l)$, so $J(u, \rho, s'')(l) = s''(l) = s(l)$.

In all cases, we have $J(u, \rho, s'')(l) = s(l)$, so it must be that $J(u, \rho, s'') = s$, completing the proof. ⊣

Lemma 5.15 allows us to reduce a chain consisting of a string of J' operations to a J'-J chain. A chain of length 1 consisting of a single J' operation is an example of a J'-J chain since $J(x, x, x) \approx x$ in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$.

Since $e_i(\overline{n}, \mathbb{B})$ has the property that $a \in e_i(\overline{n}, B)$ and $b \leq a$ implies $b \in e_i(\overline{n}, B)$, for any decreasing Maltsev chain, if any one of the intermediate elements is happy, then all subsequent ones are happy as well. Thus, every Maltsev chain must terminate in a (possibly length 0) S_i chain, and S_i chains do not appear anywhere else in the chain except at the end. Lemma 5.13 allows us to collapse repeated S_i links to a single S_i. Hence, to the already defined ψ_S, ψ_J, and $\psi_{J'J}$ we add the following:

$$\psi_{JS}(w, x, y, z) = \exists t \left[\psi_J(w, t, y, z) \wedge \psi_S(t, x, y, z)\right], \tag{5.8}$$

$$\psi_{J'JS}(w, x, y, z) = \exists t \left[\psi_{J'J}(w, t, y, z) \wedge \psi_S(t, x, y, z)\right] \tag{5.9}$$

(see equations (5.3), (5.5), and (5.6) for definitions of ψ_S, ψ_J, and $\psi_{J'J}$, respectively).

LEMMA 5.17. *Let $\{c, d\}$ be happy. If $(r, s) \in Cg^{\mathbb{B}}(c, d)$ is witnessed by a decreasing Maltsev sequence whose associated polynomials are primitive, then $(r, s) \in Cg^{\mathbb{B}}(c, d)$ is witnessed by one of the following chains*:

(1) *S_j for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_S(r, s, c, d)$,*
(2) *J and $\mathbb{B} \models \psi_J(r, s, c, d)$,*
(3) *J'-J and $\mathbb{B} \models \psi_{J'J}(r, s, c, d)$,*
(4) *J-S_j for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_{JS}(r, s, c, d)$, or*
(5) *J'-J-S_j for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_{J'JS}(r, s, c, d)$.*

Moreover, if $\mathbb{B} \models \psi_G(r, s, c, d)$ for $G \in \{J, J', J'J, JS, J'S, J'JS\}$ then $(r, s) \in Cg^{\mathbb{B}}(c, d)$.

PROOF. In all of the cases, that $\mathbb{B}$ models the claimed first-order formula follows from the definition of the formula and the conclusion of the appropriate lemmas: 5.13 for formulas whose subscript ends in S, 5.14 for formulas whose subscript begins in J, and 5.15 and 5.11 for formulas whose subscript begin with J'. The "moreover" part of the lemma follows from the fact that each ψ_G is a congruence formula.

Let $r = r_1, r_2, \ldots, r_n = s$ be the decreasing Maltsev sequence witnessing $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ and let $\lambda_1(x), \ldots, \lambda_{n-1}(x)$ be the primitive polynomials associated to it. From Lemma 5.11, without loss of generality we may assume that for each $k \in \{1, \ldots, n-1\}$ one of the following holds

(1) $\lambda_k(x) = S_{j_k}(\overline{m}_k, r_k, r_{k+1}, h_k(x))$ for some $j_k \in \{0, 1, 2\}$ and $\overline{m}_k \in B^2 \cup B$ (i.e., λ_k is happy),
(2) $\lambda_k(x) = J(r_k, q_k, h_k(x))$ for some $q_k \in B$, or
(3) $\lambda_k(x) = J'(r_k, q_k, h_k(x))$ for some $q_k \in B$,

where the polynomials $h_k(x)$ are happy and primitive for all k. Since $e_i(\overline{n}, B)$ has the property that if $q \in e_i(\overline{n}, B)$ and $p \leq q$ then $p \in e_i(\overline{n}, B)$, if $r_k \in e_i(\overline{n}, B)$, then

$$r = r_1 \xrightarrow{F_1} r_2 \xrightarrow{F_2} r_3 \cdots r_{k-1} \xrightarrow{F_{k-1}} r_k \xrightarrow{S_i} r_{k+1} \xrightarrow{S_i} r_{k+2} \cdots r_{n-1} \xrightarrow{S_i} r_n = s, \quad F_i \in \{J, J'\}.$$

Lemma 5.13 can be applied to the pair $(r_k, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ to collapse the end of the chain, and produce a new shorter chain of the form F_1-F_2-$\cdots$-S_i.

From Lemmas 5.14 and 5.15, subchains consisting of entirely J or J' can be converted to subchains consisting of a single J or J'-J, respectively:

$$J - J - \cdots - J \Rightarrow J,$$
$$J' - J' - \cdots J' \Rightarrow J' - J.$$

Thus, we need only consider chains in which the J and J' are mixed. We will show that all such chains can be reduced to $J' - J$ chains. We have

$$\begin{aligned}
&J - J' - J' \Rightarrow J - J' - J \Rightarrow J' - J - J \Rightarrow J' - J,\\
&J' - J - J' \Rightarrow J' - J' - J \Rightarrow J' - J - J \Rightarrow J' - J,\\
&J' - J' - J \Rightarrow J' - J - J \Rightarrow J' - J,\\
&J' - J - J \Rightarrow J' - J,\\
&J - J' - J \Rightarrow J' - J - J \Rightarrow J' - J, \text{ and}\\
&J - J - J' \Rightarrow J - J' \Rightarrow J' - J
\end{aligned}$$

(using Lemmas 5.14, 5.15, and 5.16). It follows that all mixed chains of J and J' can be reduced to a J'-J chain. The conclusion of the lemma follows. ⊣

Given the Lemma 5.16, let

$$\psi_1(w, x, y, z) = \psi_S(w, x, y, z) \vee \psi_J(w, x, y, z) \vee \psi_{J'J}(w, x, y, z) \\ \vee \psi_{JS}(w, x, y, z) \vee \psi_{J'JS}(w, x, y, z).$$

From Lemma 5.16, if $\{c, d\}$ is happy and $s \leq r$, then $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_1(r, s, c, d)$.

All of the lemmas above required that $s \leq r$. Since $\mathbb{B}$ is a semilattice, if $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$, then there is an intermediate element, $t \leq r \wedge s$, such that $(r, t), (t, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ and there are decreasing Maltsev chains connecting r to t and s to t (this will be proved in detail in Theorem 5.18). Therefore, define

$$\psi_2(w, x, y, z) = \exists t \left[\psi_1(w, t, y, z) \wedge \psi_1(x, t, y, z)\right]. \tag{5.10}$$

Finally, we will now use the above lemmas to prove that there is a congruence formula Γ_1 (defined in the theorem below) such that if $a, b \in B$ are distinguished by a polynomial of the form $e_i(\overline{n}, x)$ for some $i \in \{0, 1, 2\}$ and some $\overline{n}$, then $\mathrm{Cg}^{\mathbb{B}}(a, b)$ has a subcongruence witnessed by $\Gamma_1(-, -, a, b)$ and that this subcongruence is defined by ψ_2.

THEOREM 5.18. *Let $a, b \in B$ and suppose that there is $i \in \{0, 1, 2\}$ and $\overline{m} \in B^2 \cup B$ such that $e_i(\overline{m}, a) \neq e_i(\overline{m}, b)$. Let*

$$\Gamma_1(w, x, y, z) = \bigvee_{j=0}^{2} \exists \overline{n}\ \Gamma_0(w, x, e_j(\overline{n}, y), e_j(\overline{n}, z))$$

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(Γ_0 *was defined in* (5.2)). *The congruence* $Cg^{\mathbb{B}}(a,b)$ *has a principal subcongruence witnessed by* $\Gamma_1(-,-,a,b)$ *and defined by* ψ_2. *That is,*

$$\mathbb{B} \models \exists c, d \left[\Gamma_1(c,d,a,b) \wedge \Pi_{\psi_2}(c,d)\right].$$

PROOF. From the definition of e_i (4.1) and Lemma 5.4, $e_i(\overline{m},\mathbb{B})$ is congruence distributive and has definable principal subcongruences witnessed by Γ_0 and ψ_0 (defined in (5.2)). Therefore there are $c, d \in e_i(\overline{m}, B)$ such that

$$(c,d) \in \mathrm{Cg}^{e_i(\overline{m},\mathbb{B})}(e_i(\overline{m},a), e_i(\overline{m},b))$$

is witnessed by Γ_0 and $\mathrm{Cg}^{e_i(\overline{m},\mathbb{B})}(c,d)$ is defined in by ψ_0. By Lemma, we have that 5.13, $e_i(\overline{m},\mathbb{B}) \models \Pi_{\psi_2}(c,d)$. Summarizing,

$$e_i(\overline{m},\mathbb{B}) \models \Gamma_0(c,d,e_i(\overline{m},a),e_i(\overline{m},b)) \qquad \text{and} \qquad e_i(\overline{m},\mathbb{B}) \models \Pi_{\psi_2}(c,d).$$

Since Γ_0 is existentially quantified (it is a congruence formula) and $e_i(\overline{m},\mathbb{B}) \leq \mathbb{B}$, $\mathbb{B} \models \Gamma_1(c,d,a,b)$. It remains to be shown that $\mathbb{B} \models \Pi_{\psi_2}(c,d)$ (that is, that $\mathrm{Cg}^{\mathbb{B}}(c,d)$ is defined in $\mathbb{B}$ by ψ_2).

Let $r, s \in B$ and $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$. To show that $\mathbb{B} \models \psi_2(r,s,c,d)$, by Lemma 5.17 we need only show that there are decreasing Maltsev sequences connecting r to some t and s to t and whose associated polynomials are primitive.

Let $r = r_1, \dots, r_n = s$ be a Maltsev sequence connecting r to s with associated primitive polynomials $\lambda_1(x), \dots, \lambda_{n_1}(x)$. Let

$$t_i = \begin{cases} r_1 \wedge r_2 \wedge \cdots \wedge r_i & \text{if } i \leq n, \\ t_{i-n} \wedge r_{i-n+1} \cdots \wedge r_n & \text{if } n \leq i \leq 2n, \end{cases}$$

and

$$\mu_i(x) = \begin{cases} \lambda_i(x) \wedge t_i & \text{if } i < n, \\ \lambda_{i-n} \wedge t_{i+1} & \text{if } n < i \leq 2n. \end{cases}$$

Then the sequences $r = t_1, t_2, \dots, t_n$ and $s = t_{2n}, \dots t_{n+1} = t_n$ are decreasing Maltsev sequences witnessed by the primitive polynomials $\mu_i(x)$. Thus,

$$\mathbb{B} \models \psi_1(r,t_n,c,d) \wedge \psi_1(s,t_n,c,d),$$

and hence $\mathbb{B} \models \psi_2(r,s,c,d)$. From the definition of ψ_2, it is a congruence formula, so if $\mathbb{B} \models \psi_2(u,v,c,d)$ then $(u,v) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$. Therefore, $\mathbb{B} \models \Pi_{\psi_2}(c,d)$. ⊣

Having completed the argument for the case when $a, b \in B$ are distinguished by a polynomial of the form $e_i(\overline{m},x)$ for some $i \in \{0,1,2\}$ and some $\overline{m} \in B^2 \cup B$, we move on to the case where a, b are distinguished by an operation from a sequential SI. The Lemma 5.19 is crucial for this case as well as the case for machine SI's.

LEMMA 5.19. *Let* $c, d \in B$ *be such that* $d \leq c$ *and* $e_i(\overline{m},c) = e_i(\overline{m},d)$ *for all* $i \in \{0,1,2\}$ *and all* $\overline{m} \in B^2 \cup B$. *Suppose that*

$$r = f_1(c), \qquad t = f_1(d) = f_2(c),$$
$$s = f_2(d),$$

for some polynomials $f_1(x)$ *and* $f_2(x)$. *Then* $r = t$ *or* $t = s$.

PROOF. Suppose that $t \neq s$. We will show that $r = t$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. Since each $\mathbb{C}_l$ is flat and $s < t \leq r$, there is $k \in L$ such that $r(k) = t(k) \neq 0$, and $s(k) = 0$.

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Claim. *if $s(k) = 0$ then $d(k) = 0$.*

Proof of claim. Suppose to the contrary that $d(k) \neq 0$ but $0 = s(k) = f_2(d)(k)$. Since $\mathbb{C}_k$ is flat, $d(k) \neq 0$ implies that $d(k) = c(k)$, so

$$t(k) = f_2(c)(k) = f_2(d)(k) = s(k) = 0.$$

This contradicts our choosing k such that $t(k) \neq s(k) = 0$, and proves the claim. ⊣

By the above claim, we have that $d(k) = 0$, but since the only $\mathbb{C}_l$ where $d(l) \neq c(l)$ are 0-absorbing (see the description of SI's in Section 4), this implies that either $f_1(d)(k) = 0$, contradicting $t(k) = f_1(d)(k) \neq s(k) = 0$, or that $f_1(c)(l) = f_1(d)(l)$ for all l such that $\mathbb{C}_l$ is 0-absorbing (i.e., $f_1(x)$ doesn't depend on x in the 0-absorbing $\mathbb{C}_l$). Since $c(l)$ and $d(l)$ can only differ when $\mathbb{C}_l$ is 0-absorbing, this means that $f_1(c) = f_1(d)$, implying that $r = t$. ⊣

Our last remaining task is to address the case where $a, b \in B$ differ at coordinate that is one of the three small SI's that model $e_i(\overline{y}, x) \approx 0$. These algebras are described in Section 4.

Lemma 5.20. *Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras, and suppose that $c, d \in B$ are such that*

(1) $d \leq c$,
(2) *$e_i(\overline{n}, c) = e_i(\overline{n}, d)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$, and*
(3) *for each $l \in L$, $Cg^{\mathbb{C}_l}(c(l), d(l))$ lies in the monolith of $\mathbb{C}_l$.*

Let

$$\mathcal{C} = \{id(x)\} \cup \{F_1(a_1, b_1, F_2(a_2, b_2, \cdots F_n(a_n, b_n, x) \cdots)) \mid n \in \mathbb{N}, F_i \in \mathcal{L} \cup \mathcal{R}, \text{and } a_i, b_i \in B\}.$$

If $g(x)$ is a primitive polynomial of $\mathbb{B}$ such that $g(c) \neq g(d)$, then there is some $\rho \in B$, some $F(x) \in \mathcal{C}$, and some polynomial $g'(x) = J'(g(c), \rho, F(x))$ such that $(g(c), g(d)) = (g'(c), g'(d))$.

Proof. Let $\mathbb{B} = \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras $\mathbb{C}_l$. We begin by proving a claim.

Claim. *If $h(x) = H(f(x))$ for a fundamental translation $H(x)$ and polynomial $f(x)$, then $h'(x) = J'(h(c), \rho, F(f(x)))$ satisfies $(h(c), h(d)) = (h'(c), h'(d))$ for some $\rho \in B$ and some $F(x) \in \mathcal{C}$.*

Proof of claim. For convenience, let $r = h(c)$ and $s = h(d)$. The proof shall be by cases, depending on which particular fundamental operation H is a translation of. As usual, we shall proceed componentwise. The only possible $l \in L$ with $c(l) \neq d(l)$ are such that $\mathbb{C}_l \models e_i(\overline{n}, x) \approx 0$, by the second hypothesis, and the third hypothesis implies that

$$\mathbb{B} \models [x \cdot c \approx x \cdot d] \wedge [c \cdot x \approx d \cdot x].$$

Therefore by Lemma 4.1 and from the hypotheses, the only fundamental translations that possibly do not collapse (c, d) are translations of the operations $\wedge$, J, J', K, E, U_E^0, and U_E^1, where $E \in \mathcal{L} \cup \mathcal{R}$. In all cases except for operations from $\mathcal{L} \cup \mathcal{R}$ we will take the $F(x) \in \mathcal{C}$ in the statement of the claim to be $\mathrm{id}(x)$.

Before beginning with the cases, note that if $h(x) = H(f(x)) \leq f(x)$, then since $\mathbb{C}_l$ is flat either $r(l) = f(c)(l)$ or $r(l) = 0$, and likewise for $s(l)$. The polynomial $h'(x) = J'(r, r, f(x)) = r \wedge f(x)$ therefore has $h'(c) = r(l)$ and $h'(d) = s(l)$. Therefore in cases where $h(x) \leq f(x)$, taking $\rho = r$ and $F = \text{id}$ is sufficient.

Case $\wedge$: If $h(x) = u \wedge f(x)$, then $h(x) \leq f(x)$, so by the above remarks, take $\rho = r$.

Case J: If $h(x) = J(f(x), u, v)$, then $h(x) \leq f(x)$, so by the remarks at the start of the cases let $\rho = r$. If $h(x) = J(u, f(x), v)$, then let $\rho = K(r, f(c), r)$. Since many of the later cases are similar to this, we will carefully prove that $h'(x) = J'(r, \rho, f(x))$ satisfies $(h'(c), h'(d)) = (r, s)$. We have

$$h(x) = J(u, f(x), v) = (u \wedge f(x)) \vee (u \wedge \partial f(x) \wedge v).$$

This yields the following table (assume that $r(l) \neq 0$, since $h'(x)(l) = 0$ otherwise).

r	$\rho = K(r, f(c), r)$	$h'(c)$	$h'(d)$
$f(c)$	r	$r \wedge f(c) = r$	$r \wedge f(d) = s$
$\partial f(c)$	$f(c) = \partial r$	r	r

The only possibly problematic case is when $r(l) = \partial f(c)(l)$, but in this case we have that $s(l) = \partial f(d)(l)$, so $r(l) = e_2(r, f(c), r)(l)$ and $s(l) = e_2(r, f(c), s)(l)$, contradicting hypothesis (2) in the statement of the lemma. It follows that $h'(c) = r$ and $h'(d) = s$.

If $h(x) = J(u, v, f(x))$, then $h(c)(l)$ and $h(d)(l)$ agree whenever $u(l) = v(l)$ and can only possibly differ when $u(l) = \partial v(l)$. Hence, if $h(c) \neq h(d)$, then $e_2(u, v, h(c)) \neq e_2(u, v, h(d))$, contradicting hypothesis (2) again.

Case J': If $h(x) = J'(f(x), u, v)$, then $h(x) \leq f(x)$, so by the remarks at the start of the cases let $\rho = r$. If $h(x) = J'(u, f(x), v)$ or $h(x) = J'(u, v, f(x))$, let $\rho = K(r, f(c), r)$. An argument similar to the one in Case J will work.

Case K: If $h(x) = K(f(x), u, v)$, then let $\rho = K(r, f(c), r)$. If we have $h(x) = K(u, f(x), v)$, then let $\rho = K(r, u, r)$. If $h(x) = K(u, v, f(x))$, then let $\rho = K(r, u, r)$. Arguments similar to the one in Case J will work.

Case $E \in \mathcal{L} \cup \mathcal{R}$: If $g(x) = E(f(x), u, v)$ or $g(x) = E(u, f(x), v)$ then hypothesis (3) implies that when $\mathbb{C}_l$ is of machine type and $f(c)(l) = g(c)(l)$ then $c(l)$ is a configuration element and therefore $g(c)(l) = 0$. Thus in this case, $g(c) = g(d)$, a contradiction. Thus we need only examine $g(x) = E(u, v, f(x))$. In this case, $g'(x) = J'(r, r, f(x))$ clearly works.

Case U_E^i **for** $E \in \mathcal{L} \cup \mathcal{R}$ **and** $i \in \{0, 1\}$: This case is quite similar to the previous one. Use the fact that $c(l)$ and $d(l)$ only differ on sequential and machine $\mathbb{C}_l$, that $U_E^i(w, x, y, z) \approx 0$ on sequential SI's, and that

$$U_E^i(u, v, w, x) = 0 \qquad \text{except for} \qquad U_E^0(v, u, v, x) = E(u, v, x) = U_E^1(u, u, v, x)$$

in the machine SI's. ⊣

The polynomial $g(x)$ is primitive and therefore generated by fundamental translations. It follows that there is some fundamental translation $G(x)$ such that $g(x) = G(f(x))$. Apply the above claim to $g(x) = G(f(x))$ to get that there is $\rho \in B$ and $F(x) \in \mathcal{C}$ such that the polynomial $g'(x) = J'(g(c), \rho, F(f(x)))$ satisfies $(g(c), g(d)) = (g'(c), g'(d))$.

Since $g(x) = G(f(x))$ is a primitive polynomial, $f(x)$ is also primitive. Therefore there is a fundamental translation $H(x)$ such that $f(x) = H(h(x))$. Apply the above claim to the $f(x)$ in $g'(x) = J'(g(c), \rho, F(f(x)))$ from the above paragraph to get that there is $\rho' \in B$ and $F'(x) \in \mathcal{C}$ such that the polynomial

$$g''(x) = J'(g(c), \rho, F(J'(f(c), \rho', F'(h(x)))))$$

satisfies $(g''(c), g''(d)) = (g'(c), g'(d)) = (g(c), g(d))$. The second claim will show how to reduce this polynomial to the form required by the conclusion of the lemma.

Claim. *If $h(x) = J'(u, v, F(J'(p, q, E(f(x)))))$ for constants $u, v, p, q \in B$ and $F(x), E(x) \in \mathcal{C}$ then there is some $\rho \in B$ and $E_1 \in \mathcal{C}$ such that*

$$h'(x) = J'(h(c), \rho, E_1(f(x)))$$

has $h'(c) = h(c)$ and $h'(d) = h(d)$.

Proof of claim. Our first task will be to find another polynomial that agrees with $F(J'(p, q, E(f(x))))$ on $\{c, d\}$ but has the form $J'(r_1, \rho_1, G(f(x)))$ for some $r_1, \rho_1 \in B$.

If $h_1(x) = G_1(J'(p, q, f_1(x)))$, where $G_1(x) = G_1'(a_1, b_1, x)$ for $G_1' \in \mathcal{L} \cup \mathcal{R}$, then there is $\rho_1 \in B$ such that $h_1'(x) = J'(h_1(c), \rho_1, G_1(f(x)))$ has $h_1'(c) = h_1(c)$ and $h_1'(d) = h_1(d)$. To see this, let $\rho_1 = G_1(q)$. We have

$$h_1(x) = G_1(J'(p, q, f_1(x))) = G_1((p \wedge \partial q) \vee (p \wedge q \wedge f_1(x))),$$

and $G_1(\partial x) = \partial G_1(x)$ (this last equation is true because $G_1' \in \mathcal{L} \cup \mathcal{R}$). Therefore, $h_1'(x) = J'(h(c), G_1(q), G_1(f_1(x)))$ agrees with $h_1(x)$ on $\{c, d\}$.

By repeatedly applying the result of the above paragraph to $F(J'(p, q, E(f(x))))$, (and using the fact that F is a composition of translations of the form $F_i(a_i, b_i, x)$ for $F_i \in \mathcal{R} \cup \mathcal{L}$), we obtain a polynomial of the form $J'(r_1, \rho_1, G(f(x)))$ that agrees with $F(J'(p, q, E(f(x))))$ on $\{c, d\}$.

At this point, we can take the polynomial $h(x) = J'(u, v, F(J'(p, q, E(f(x)))))$ in the statement of the claim and produce a polynomial

$$h_1(x) = J'(u, v, J'(r_1, \rho_1, G(f(x))))$$

for some $r_1, \rho_1 \in B$ and $G \in mathcalC$ such that $(h_1(c), h_1(d)) = (h(c), h(d))$.

Next, we will show that there is some $\rho \in B$ such that $h'(x) = J'(h(c), \rho, G(f(x)))$ satisfies $(h'(c), h'(d)) = (h_1(c), h_1(d)) = (h(c), h(d))$. Let $\rho = K(h(c), v, \rho_1)$. We have

$$\begin{aligned} h_1(x) &= J'(u, v, J'(r_1, \rho_1, G(f(x)))) \\ &= (u \wedge v \wedge r_1 \wedge \rho_1 \wedge G(f(x))) \vee (u \wedge v \wedge r_1 \wedge \partial \rho_1) \vee (u \wedge \partial v). \end{aligned}$$

This gives us the following table of cases (as usual, assume that $r(l) \neq 0$ since $h_1(x)(l) = 0$ otherwise).

$h_1(c) = h(c)$	$\rho = K(h(c), v, q)$	$h'(c)$	$h'(d)$	$h(c) \neq h(d)$
$v = \rho_1$	$h(c)$	$h(c) \wedge G(f(c))$	$h(c) \wedge G(f(d))$	Y
$v = \partial \rho_1$	$\rho_1 = \partial h(c)$	$h(c)$	$h(c)$	N
∂v	$v = \partial h(c)$	$h(c)$	$h(c)$	N

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In the case where $v(l) = \rho_1(l)$ we also have that $h(c)(l) = G(f(c))(l)$ and $h(d)(l) = G(f(d))(l)$, so the table above indicates that $h'(c) = h(c)$ and $h'(d) = h(d)$. ⊣

Applying this claim to the previously computed

$$g''(x) = J'(g(c), \rho, F(J'(f(c), \rho', F'(h(x)))))$$

produces a polynomial $g_1(x) = J'(g(c), \rho_1, F_1(h(x)))$ such that

$$(g_1(c), g_1(d)) = (g''(c), g''(d)) = (g(c), g(d)).$$

Repeating this argument with $g_1(x)$ proves the lemma. ⊣

If $a, b \in B$ differ at a coordinate that is sequential, then the Lemma 5.21 proves that there is some polynomial that maps (a, b) coordinatewise into the monoliths of the $\mathbb{C}_l$ (the subdirect factors of $\mathbb{B}$) and does not collapse (a, b).

LEMMA 5.21. *There is a finite set of terms P depending only on $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ such that if $a, b \in B$ are distinct, $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$, and there is $p \in B$ with $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, then there is $t(\overline{y}, x) \in P$ and $\overline{m} \in B^n$ with the property that if $c = t(\overline{m}, a)$ and $d = t(\overline{m}, b)$ then*

- $c \neq d$,
- $x \cdot c = x \cdot d$,
- $c \cdot x = d \cdot x$,
- $I(c) = I(d)$,
- $F(x, y, c) = F(x, y, d)$,
- $F(x, c, y) = F(x, d, y)$, *and*
- $F(c, x, y) = F(d, x, y)$,

for $F \in \mathcal{L} \cup \mathcal{R}$ and for all $x, y \in B$.

PROOF. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. The subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ can be divided into two groups: either $\mathbb{C}_l \models e_i(\overline{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and some $\overline{n} \in C_l^2 \cup C_l$ or $\mathbb{C}_l \models e_i(\overline{n}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. Since $e_i(\overline{n}, a) = e_i(\overline{n}, b)$, the projections $a(l)$ and $b(l)$ must agree on all factors that satisfy $\mathbb{C}_l \models e_i(\overline{n}, x) \approx x$ for some i and some $\overline{n}$, and can only possibly disagree on factors satisfying $\mathbb{C}_l \models e_i(\overline{n}, x) \approx 0$ for all i.

CLAIM. *There is a finite number $N \in \mathbb{N}$ such that for all $l \in L$, if $\mathbb{C}_l \models e_1(n, x) \approx 0$ then*

$$\mathbb{C}_l \models x_1 \cdot x_2 \cdots x_{N-1} \cdot x_N \approx 0.$$

PROOF OF CLAIM. First recall that since $\mathcal{T}$ halts, there are only finitely many subdirectly irreducible algebras, all finite. Therefore, if $\mathbb{C}_l$ does not model the identity in the claim, there must exist nonzero elements $r, s \in C_l$ such that $r \cdots r \cdot s = s$. Considering $\mathbb{C}_l$ as a quotient of a product of subalgebras of $\mathbb{A}'(\mathcal{T})$, this means that there is some coordinate of the preimages (under the quotient map) of r and s such that $(r(i), s(i)) \in \{(1, C), (2, D)\}$. Therefore, $e_1(r, s) \neq 0$, contradicting our assumption that $\mathbb{C}_l \models e_1(n, x) \approx 0$. Let S be a finite set containing a representative of each isomorphism type of the subdirectly irreducible algebras of $\mathcal{V}(\mathbb{A}'(T))$, and for $\mathbb{C} \in S$, let $n_{\mathbb{C}} \in \mathbb{N}$ be minimal such that $\mathbb{C} \models x_1 \cdots x_{n_{\mathbb{C}}} \approx 0$. Taking $N = \max\{n_{\mathbb{C}} \mid \mathbb{C} \in S\}$ completes the proof of the claim. ⊣

Since the product $(\cdot)$ associates to the left, every polynomial of the form $f(x) = y_1 \cdots y_m \cdot x \cdot y_{m+1} \cdots y_M$ can be rewritten as $f(x) = y_1 \cdots y_m \cdot x \cdot z$, where $z = y_{m+1} \cdots y_M$. Let

$$P = \{f(y_1, \dots, y_M, x) = y_1 \cdots y_M \cdot x,$$
$$g(y_1, \dots, y_M, x) = y_1 \cdots y_{M-1} \cdot x \cdot y_M \mid 0 \leq M < N\}.$$

Thus, there is a term $t(\overline{y}, x) \in P$ and constants $\overline{m} \in B^n$ such that $t(\overline{m}, a) \neq t(\overline{m}, b)$ and $x \cdot t(\overline{m}, a) = x \cdot t(\overline{m}, b)$ and $t(\overline{m}, a) \cdot x = t(\overline{m}, b) \cdot x$ for all $x \in B$. Furthermore, since $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, the term t is not the identity. Therefore, $t(\overline{m}, x)(l) \approx 0$ when $\mathbb{C}_l$ is machine (recall that machine $\mathbb{C}_l$ model $x \cdot y \approx 0$). Thus for all $x, y, z \in B$ and all $F \in \mathcal{L} \cup \mathcal{R}$,

$$F(t(\overline{m}, z), x, y) = F(x, t(\overline{m}, z), y) = F(x, y, t(\overline{m}, z)) = I(t(\overline{m}, x)) = 0. \qquad \dashv$$

Let the set P be as in Lemma 5.21 and define

$$\Gamma_{(\cdot)}(w, x, y, z) = \bigvee_{t \in P \cup \{\mathrm{id}(x)\}} \exists \overline{n} \left[w = t(\overline{n}, y) \wedge x = t(\overline{n}, z) \right]. \tag{5.11}$$

Given Lemmas 5.20 and 5.19, define

$$\psi_{(\cdot)}(w, x, y, z) = \exists t \left[w = J'(w, t, y) \wedge x = J'(w, t, z) \right]. \tag{5.12}$$

If $c, d \in B$ with $d < c$ satisfy the conclusion of Lemma 5.21, then c, d also satisfy the hypotheses of Lemma 5.20. In this situation, $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_{(\cdot)}(r, s, c, d)$. Since we will be employing a strategy similar to the proof of Theorem 5.18, where a general Maltsev sequence is divided into 2 strictly decreasing sequences, let

$$\psi_3(w, x, y, z) = \exists t \left[\psi_{(\cdot)}(w, t, y, z) \wedge \psi_{(\cdot)}(x, t, y, z) \right]. \tag{5.13}$$

THEOREM 5.22. *Let $a, b \in B$ be distinct and such that $b \leq a$ and $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$. If one of the following*

(1) *there is $p \in B$ such that $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, or*
(2) *for all $u, v \in B$ and $F \in \mathcal{L} \cup \mathcal{R}$ each of the translations $x \cdot u$, $u \cdot x$, $I(x)$, $F(u, v, x)$, $F(u, x, v)$, and $F(x, u, v)$ are constant for $x \in \{a, b\}$*

holds, then the congruence $Cg^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by the formula $\Gamma_{(\cdot)}(-, -, a, b)$ and defined by the formula ψ_3:

$$\mathbb{B} \models \exists c, d \left[c \neq d \wedge \Gamma_{(\cdot)}(c, d, a, b) \wedge \Pi_{\psi_3}(c, d) \right].$$

PROOF. Let $\mathbb{B} = \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras $\mathbb{C}_l$. If (1) holds, then from Lemma 5.21 the pair (a, b) differs at a coordinate that is sequential, and $(a, b)(l)$ lies outside of the monolith of some sequential $\mathbb{C}_l$. Find $t \in P$ and constants $\overline{m}$ such that if $c = t(\overline{m}, a)$ and $d = t(\overline{m}, b)$ then

- $c \neq d$,
- $x \cdot c = x \cdot d$,
- $I(c) = I(d)$,
- $c \cdot x = d \cdot x$,
- $F(x, y, c) = F(x, y, d)$,

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- $F(x,c,y) = F(x,d,y)$, and
- $F(c,x,y) = F(d,x,y)$.

If (2) holds, then the pair (a,b) differ at a coordinate that is sequential, but $(a,b)(l)$ lies in the monolith of each sequential $\mathbb{C}_l$. Let $c = a$ and $d = b$. In both (1) and (2), $\mathbb{B} \models \Gamma_{(\cdot)}(c,d,a,b)$ and c and d satisfy the hypotheses of Lemma 5.20.

Since $b \leq a$ and the operations of $\mathbb{B}$ are monotonic, $d \leq c$. Suppose now that $(r,s) \in \mathrm{Cg}^{\mathbb{B}}(c,d)$. Then using the same argument as in the proof of Theorem 5.18, there are decreasing Maltsev chains $r = r_1, \dots, r_m = t$ and $s = s_1, \dots, s_n = t$ with associated primitive polynomials. Using first Lemma 5.20 and the description of c and d in the preceding paragraph, and then applying Lemma 5.19, we have that there are constants ρ and ρ' such that

$$r = J'(r,\rho,c), \qquad t = J'(r,\rho,d) = J'(s,\rho',d),$$
$$s = J'(s,\rho',c).$$

Hence $\mathbb{B} \models \psi_{(\cdot)}(r,t,c,d) \wedge \psi_{\cdot}(s,t,c,d) = \psi_3(r,s,c,d)$, completing the proof. ⊣

Next, we move on to analyzing the case where $a, b \in B$ differ at a machine coordinate. We will employ a strategy similar to the sequential case, and produce from (a,b) a pair (c,d) such that $(c,d)(l)$ lies in the monolith of $\mathbb{C}_l$ for each $l \in L$.

Lemma 5.23. *There are finite sets of terms S and T depending only on $V(\mathbb{A}'(\mathcal{T}))$ such that if $a, b \in B$ are distinct such that $b \leq a$, $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0,1,2\}$ and all $\overline{n} \in B^2 \cup B$, one of*

(1) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u,v,a) \neq F(u,v,b)$, or*
(2) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u,v,I(a)) \neq F(u,v,I(b))$,*
(3) *for all $F \in \mathcal{L} \cup \mathcal{R}$ and all $u, v \in B$,*
    (a) *$I(a) = I(b)$,*
    (b) *$u \cdot a = u \cdot b$ and $a \cdot u = b \cdot u$, and*
    (c) *$F(u,v,a) = F(u,v,b)$,*

holds, then there is $t(\overline{y}, x) \in S$ and constants $\overline{m} \in B$ such that if $c = t(\overline{m}, a)$ and $d = t(\overline{m}, b)$ then for any $n \in \mathbb{N}$ and $F_1, \dots, F_n \in \mathcal{L} \cup \mathcal{R}$ and any $a_1, \dots, a_{2n} \in B$ there is $G(\overline{y}, x) \in T$ and $\overline{b} \in B^m$ such that

$$F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, c) \dots)) = G(\overline{b}, c) \qquad \textit{and}$$
$$F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, d) \dots)) = G(\overline{b}, d).$$

Furthermore, if $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ is a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras then $(c(l), d(l))$ lies in the monolith of $\mathbb{C}_l$ for each $l \in L$.

Proof. We will begin by examining algebras whose only subdirect factors are machine. If $\mathbb{D}$ is a machine SI, then using the notation from the discussion of large SI's in Section 4, the monolith of $\mathbb{D}$ is $\mathrm{Cg}^{\mathbb{D}}(\mathcal{P}, 0)$, and there are two possibilities for its structure: either

- $\mathcal{T}(\mathcal{P}) = 0$, in which case the only nontrivial class of the monolith is $\{\mathcal{P}, 0\}$, or
- there is $N \in \mathbb{N}$ such that $\mathcal{T}^N(\mathcal{P}) = \mathcal{T}(\mathcal{T}(\cdots\mathcal{T}(\mathcal{P})\cdots)) = \mathcal{P}$ (that is, the Turing machine enters a nonterminating loop), in which case the only nontrivial class of the monolith is $\{\mathcal{P}, \mathcal{T}(\mathcal{P}), \dots, \mathcal{T}^{N-1}(\mathcal{P}), 0\}$.

Let $(\mathbb{C}_k)_{k \in K}$ be a family of machine SI's and suppose that $\mathbb{C} \leq \prod_{k \in K} \mathbb{C}_k$ and that $c, d \in C$ are such that $c \geq d$ and $(c(k), d(k))$ lies in the monolith of $\mathbb{C}_k$ for

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each $k \in K$. We now make two straightforward observations that follow from the description of the monoliths of the machine SI's in the above paragraph and in Section 4 and from $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ having finite residual bound:

- if $F \in \mathcal{L} \cup \mathcal{R}$, $k \in K$, and $a_1, a_2 \in C_k$ then $F(a_1, a_2, c(k)) \in \{0, \mathcal{T}(c(k))\}$ (likewise for d in place of c), and
- the set $T' = \{\mathcal{T}^i(d), \mathcal{T}^i(c) \mid 0 \leq i < \infty\}$ is finite (we apply $\mathcal{T}$ coordinatewise).

We now define the set T. For each isomorphism type of a machine SI, $\mathbb{D}$, let $N_{\mathbb{D}}$ be minimal such that $\mathcal{T}^{N_{\mathbb{D}}}(\mathcal{P}) \in \{0, \mathcal{P}\}$, and let N be the least common multiple of the $N_{\mathbb{D}}$. Since $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has finite residual bound, N is finite. Define

$$T = \{G_1(y_1, y_2, G_2(y_3, y_4, \dots, G_m(y_{2m-1}, y_{2m}, x)\dots)) \mid m \leq N, G_i \in \mathcal{L} \cup \mathcal{R}\} \cup \{\mathrm{id}(x)\}.$$

A consequence of these observations and the definition of T is that for any $n \in \mathbb{N}$, $F_1, \dots, F_n \in \mathcal{L} \cup \mathcal{R}$, and $a_1, \dots, a_{2n} \in C$, there is $G(\overline{y}, x) \in T$ and $\overline{b} \in C^m$ such that

$$F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, c)\dots)) = G(\overline{b}, c) \qquad \text{and}$$
$$F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, d)\dots)) = G(\overline{b}, d).$$

Next, we move on to the set S. $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is residually finite, so there is a finite set of terms S depending only on $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ such that for all $\mathbb{C} \leq \prod_{k \in K} \mathbb{C}_k$ (recall $(\mathbb{C}_k)_{k \in K}$ is a family of machine SI's) and all $p, q \in C$ with $q < p$ there is a term $t \in S$ and constants $\overline{m} \in C^m$ such that $t(\overline{m}, p) \neq t(\overline{m}, q)$ and $(t(\overline{m}, p)(k), t(\overline{m}, q)(k))$ lies in the monolith of $\mathbb{C}_k$ for each $k \in K$. Note that the set of terms S can be taken to consist of the identity and a finite subset of terms generated by composing operations from $\mathcal{L} \cup \mathcal{R} \cup \{I(x)\}$. At this point we have produced S and T that will work for algebras $\mathbb{C}$ whose subdirect factors are all machine.

We now examine algebras whose subdirect factors contain nonmachine SI's. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. By the hypotheses, $a(l) \neq b(l)$ on some machine $\mathbb{C}_l$. From the paragraph above, it follows that there is a term $t \in S$ and constants $\overline{m} \in B^m$ such that $(t(\overline{m}, a), t(\overline{m}, b))(l)$ lies in the monolith of $\mathbb{C}_l$ for all machine $\mathbb{C}_l$. Let $(c, d) = (t(\overline{m}, a), t(\overline{m}, b))$. The term t is (if hypotheses (1) or (2) hold) a composition of operations from $\mathcal{L} \cup \mathcal{R} \cup \{I(x)\}$ or (if hypothesis (3) holds) the identity. Aside from the identity, terms from S are constant in SI's modeling $e_i(\overline{x}, y) \approx 0$ except for machine SI's and the 3-element small SI $\{0, H, M_1^0\}$. Therefore, $c(l) = d(l)$ on nonmachine $\mathbb{C}_l$ and $(c, d)(l)$ lies in the monolith of $\mathbb{C}_l$ for machine $\mathbb{C}_l$. Applying the observations from above about the set T now proves the lemma. ⊣

Let the sets S and T be as in Lemma 5.23 and define

$$\Gamma_{\mathcal{T}}(w, x, y, z) = \bigvee_{t \in S} \exists \overline{n} \left[w = t(\overline{n}, y) \wedge x = t(\overline{n}, z) \right]. \tag{5.14}$$

Given Lemmas 5.20 and 5.19, define

$$\psi_{\mathcal{T}}(w, x, y, z) = \exists t \left[\bigvee_{G \in T} \exists \overline{b} \left[w = J'(w, t, G(\overline{b}, y)) \wedge x = J'(w, t, G(\overline{b}, z)) \right] \right]. \tag{5.15}$$

If $c, d \in B$ satisfy the conclusion of Lemma 5.23, then c, d also satisfy the hypotheses of Lemma 5.20. Then $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_{\mathcal{T}}(r, s, c, d)$. Since we will be employing a strategy similar to the proof of Theorems 5.18 and 5.22, where a Maltsev chain is broken into 2 decreasing segments, let

$$\psi_4(w, x, y, z) = \exists t \left[\psi_{\mathcal{T}}(w, t, y, z) \wedge \psi_{\mathcal{T}}(x, t, y, z)\right]. \tag{5.16}$$

THEOREM 5.24. *Let $a, b \in B$ be distinct and such that $b \leq a$ and $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$. If*

(1) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u, v, a) \neq F(u, v, b)$, or*
(2) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u, v, I(a)) \neq F(u, v, I(b))$,*
(3) *for all $F \in \mathcal{L} \cup \mathcal{R}$ and all $u, v \in B$,*
   (a) *$I(a) = I(b)$,*
   (b) *$u \cdot a = u \cdot b$ and $a \cdot u = b \cdot u$, and*
   (c) *$F(u, v, a) = F(u, v, b)$,*

holds, then the congruence $Cg^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_{\mathcal{T}}(-, -, a, b)$ and defined by ψ_4. In symbols,

$$\mathbb{B} \models \exists c, d \left[c \neq d \wedge \Gamma_{\mathcal{T}}(c, d, a, b) \wedge \Pi_{\psi_4}(c, d)\right].$$

PROOF. By hypothesis, Lemma 5.23 holds. Let T and S be the finite sets of terms and $c, d \in B$ be the elements guaranteed by the conclusion of Lemma 5.23. Then there is $t \in S$ and $\overline{m} \in B^n$ such that $c = t(\overline{m}, a)$ and $d = t(\overline{m}, b)$. Furthermore, if

$$F(x) \in \{\mathrm{id(x)}\} \cup \{F_1(a_1, b_1, F_2(a_2, b_2, \ldots, F_n(a_n, b_n, x)\ldots)) \mid \\ n \in \mathbb{N}, F_i \in \mathcal{L} \cup \mathcal{R}, \text{ and } a_i, b_i \in B\},$$

then there is $G \in T$ and $\overline{b} \in B^n$ such that $F(c) = G(\overline{b}, c)$ and $F(d) = G(\overline{b}, d)$ (the existence of such elements is the conclusion of Lemma 5.23). Thus $\mathbb{B} \models \Gamma_{\mathcal{T}}(c, d, a, b)$ and c and d satisfy the hypotheses of Lemma 5.20.

Since $b \leq a$ and the operations of $\mathbb{B}$ are monotone, $d \leq c$. Suppose now that $(r, s) \in \mathrm{Cg}^{\mathbb{B}}(c, d)$. Using the same argument as in the proof of Theorem 5.18, there are decreasing Maltsev chains $r = r_1, \ldots, r_m = t$ and $s = s_1, \ldots, s_n = t$ with associated primitive polynomials. Using first Lemma 5.20, and then Lemma 5.19, we have that there are constants $\rho, \rho' \in B$ such that

$$r = J'(r, \rho, G(\overline{b}, c)), \qquad t = J'(r, \rho, G(\overline{b}, c)) = J'(s, \rho', G'(\overline{b}', d)),$$
$$s = J'(s, \rho', G'(\overline{b}', c))$$

for some $G, G' \in T$ and constants $\overline{b}, \overline{b}' \in B^n$. Hence $\mathbb{B} \models \psi_{\mathcal{T}}(r, t, c, d) \wedge \psi_{\mathcal{T}}(s, t, c, d)$, completing the proof. ⊣

The last case where $a, b \in B$ differ at a coordinate that is small but that does not satisfy $\exists \overline{n}[e_i(\overline{n}, x) \approx x]$ remains. From Lemma 4.1, we know that there are only 3 isomorphism types for such SI's. If the coordinate is isomorphic to $\{0, C\}$, then the lemmas used in the sequential case apply. We are therefore concerned with the remaining two isomorphism types. To this end, let

$$\Gamma_I(w, x, y, z) = \\ \exists u, v \left[\left(u = I(y) \wedge v = I(z) \wedge \Gamma_{(\cdot)}(w, x, u, v)\right) \vee \Gamma_{(\cdot)}(w, x, y, z)\right]. \tag{5.17}$$

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LEMMA 5.25. *Suppose that $a, b \in B$ are distinct, but that $I(x)$ is the only fundamental operation that distinguishes them. Then the congruence $Cg^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_I(-,-,a,b)$ and defined by ψ_3 (see* (5.13) *and* (5.17)*):*

$$\mathbb{B} \models \exists c, d \left[c \neq d \wedge \Gamma_I(c, d, a, b) \wedge \Pi_{\psi_3}(c, d)\right].$$

PROOF. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of $\mathbb{B}$ by subdirectly irreducible algebras. If $a, b \in B$ are distinct and only distinguished by $I(x)$, then $a(l) \neq b(l)$ if and only if $\mathbb{C}_l \cong \mathbb{W}$ or $\mathbb{C}_l \cong \{0, H, M_1^0\}$ (see (4.2) and Lemma 4.1).

If $a' = I(a)$ and $b' = I(b)$, then a' and b' satisfy the hypotheses of Theorem 5.22 and thus $\mathrm{Cg}^{\mathbb{B}}(a', b')$ has a principal subcongruence witnessed by $\Gamma_{(\cdot)}(-,-,a',b')$ and defined by ψ_3. Therefore, $\mathrm{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_I(-,-,a,b)$ and defined by ψ_3, as claimed. ⊣

THEOREM 5.26. *If $\mathcal{T}$ halts then $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences.*

PROOF. Let

$$\Gamma(w, x, y, z) = \Gamma_1(w, x, y, z) \vee \Gamma_{(\cdot)}(w, x, y, z) \vee \Gamma_{\mathcal{T}}(w, x, y, z) \vee \Gamma_I(w, x, y, z)$$

(see Theorem 5.18 and equations (5.11), (5.14), and (5.17) for definitions of these), and

$$\psi(w, x, y, z) = \psi_2(w, x, y, z) \vee \psi_3(w, x, y, z) \vee \psi_4(w, x, y, z)$$

(see equations (5.10), (5.13), and (5.16) for definitions of these). We claim that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal congruences witnessed by Γ and ψ. In symbols,

$$\mathcal{V}(\mathbb{A}'(\mathcal{T})) \models \forall a, b \left[a \neq b \rightarrow \exists c, d \left[c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_{\psi}(c, d)\right]\right].$$

Let $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ with $a, b \in B$ distinct and let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation by subdirectly irreducible algebras. Since a and b are distinct, there is some $l \in L$ such that $a(l) \neq b(l)$. Let

$$K = \{l \in L \mid a(l) \neq b(l)\}.$$

The case distinction breaks down as follows:

(1) There is some $k \in K$ such that $\mathbb{C}_k \models e_i(\overline{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and some $\overline{n} \in C_k^2 \cup C_k$. In this case, Theorem 5.18 applies.

(2) The previous case does not apply, but there is some $k \in K$ such that $\mathbb{C}_k$ is sequential. If this is the case, there is some $u \in B$ such that $u \cdot a \neq u \cdot b$ or $a \cdot u \neq b \cdot u$, or the machine operations $\mathcal{L} \cup \mathcal{R}$ cannot distinguish between a and b. In this case, Theorem 5.22 applies.

(3) The previous cases do not apply, but there is some $k \in K$ such that $\mathbb{C}_k$ is machine. If this is the case, there is some machine operation in $\mathcal{L} \cup \mathcal{R}$ that can distinguish between a and b. In this case, Theorem 5.24 applies.

(4) The previous cases do not apply, so there must be some $k \in K$ such that $\mathbb{C}_k$ is small and models $e_i(\overline{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$ (see Lemma 4.1). If $C_k = \{0, C\}$ or $\mathbb{C} \cong \mathbb{W}$, then Theorem 5.22 applies. If $C_k = \{0, H, M_1^0\}$ then either Theorem 5.22 applies (if $a(k) = M_1^0$) or Lemma 5.25 applies (if $a(k) = H$).

Since the SI's of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ either satisfy $e_i(\overline{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and $\overline{n} \in B^2 \cup B$, are sequential, are machine, or are isomorphic to one of the 3 small algebras given in Lemma 4.1, this completes the proof. ⊣

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One of the interesting applications of definable principal subcongruences is in defining the subdirectly irreducible members of some class of algebras. If $\mathcal{C}$ is a class of algebras with definable principal subcongruences witnessed by congruence formulas Γ and ψ, then

$$\mathcal{C} \models \forall a, b \left[a \neq b \rightarrow \exists c, d \left[c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_\psi(c, d)\right]\right],$$

and the sentence

$$\sigma = \exists r, s \left[r \neq s \wedge \forall a, b \left[a \neq b \rightarrow \exists c, d \left[\Gamma(c, d, a, b) \wedge \psi(r, s, c, d)\right]\right]\right]$$

defines the subdirectly irreducible algebras in $\mathcal{C}$. Baker and Wang [2] use this to prove the following theorem.

THEOREM 5.27 (Baker, Wang [2]). *A variety $\mathcal{V}$ with definable principal subcongruences is finitely based if and only if the class of subdirectly irreducible members of $\mathcal{V}$ is finitely axiomatizable.*

In particular, if $\kappa(\mathcal{V}) < \omega$ then the class of subdirectly irreducible members of $\mathcal{V}$ is finitely axiomatizable since there are only finitely many of them, all finite. This observation and the above theorem yields a corollary to Theorem 5.26.

COROLLARY 5.28. *If $\mathcal{T}$ halts, then $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is finitely based.*

§6. If $\mathcal{T}$ does not halt.

In the case where $\mathcal{T}$ halts, every sequential subdirectly irreducible algebra is finite and there are only finitely many of them. In the case where $\mathcal{T}$ does not halt, McKenzie [6] and the additions from Section 3 show that the algebra $\mathbb{S}_\mathbb{Z}$ (defined in Section 4) is a member of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. McKenzie [8] uses $\mathbb{S}_\mathbb{Z}$ to show that if $\mathcal{T}$ does not halt, then $\mathbb{A}(\mathcal{T})$ is inherently nonfinitely based. Although $\mathbb{S}_\mathbb{Z}$ is not subdirectly irreducible, it contains an infinite subalgebra $\mathbb{S}_\omega$ which is, and every finite sequentiable SI can be embedded in it. We will use the presence of $\mathbb{S}_\mathbb{Z}$ in $\mathcal{V}(\mathbb{A}'(T))$ to show that if $\mathcal{T}$ does not halt, then $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ doesn't have DPSC.

An algebra $\mathbb{C}$ is said to be *finitely subdirectly irreducible (FSI)* if for all $a, b, c, d \in C$ such that $a \neq b$ and $c \neq d$, $\mathrm{Cg}^\mathbb{C}(a, b) \cap \mathrm{Cg}^\mathbb{C}(c, d) \neq \mathbf{0}$ (i.e., $\mathbf{0}$ is meet irreducible). Every SI is FSI, but not every FSI is SI.

THEOREM 6.1. *The class of finitely subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is not axiomatizable if $\mathcal{T}$ does not halt.*

PROOF. We will use an ultrapower argument. Suppose to the contrary that the class of finitely subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is axiomatizable, say by Φ. $\mathcal{T}$ does not halt if and only if $\mathbb{S}_\mathbb{Z} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Let $\mathbb{S}$ be an ultrapower of $\mathbb{S}_\omega$, so that $\mathbb{S}$ satisfies all first-order properties of $\mathbb{S}_\omega$. In particular, since $\mathbb{S}_\omega \models \Phi$, we have that $\mathbb{S} \models \Phi$, so $\mathbf{0}$ is meet irreducible in $\mathrm{Con}(\mathbb{S})$. We will now give some first-order properties of $\mathbb{S}_\omega$ which we will make use of.

Let

$$A = \{\alpha \in S_\omega \mid \exists \beta[\alpha \cdot \beta \neq 0]\} \qquad \text{and} \qquad B = \{\beta \in S_\omega \mid \exists \alpha[\alpha \cdot \beta \neq 0]\}.$$

Then in $\mathbb{S}_\omega$, for each $\alpha \in A$ there is a unique $\beta \in B$ such that $\alpha \cdot \beta \neq 0$, and for each $\beta \in B$, there is a unique $\alpha \in A$ such that $\alpha \cdot \beta \neq 0$. This gives us that $|A| = |B|$. We also have

$$A \cap B = \emptyset \qquad \text{and} \qquad S_\omega = A \cup B \cup \{0\}.$$

For $b \in B$, let

$$A^n \cdot b = \{\alpha_1 \cdots \alpha_m \cdot b \mid 0 \le m \le n \text{ and } \alpha_1, \dots, \alpha_m \in A\}.$$

Then $|A^n \cdot b| = n + 2$. Furthermore, for $b, c \in B$,

$$\text{if } (A^n \cdot b) \cap (A^m \cdot c) \neq \{0\} \qquad \text{then} \qquad b \in A^m \cdot c \text{ or } c \in A^n \cdot b.$$

Lastly, if $F(x)$ is a fundamental translation in $\mathbb{S}_\omega$, then $F(A^n \cdot b) \subseteq A^{n+1} \cdot b$. All of these sets and properties are first-order definable and hold in $\mathbb{S}_\omega$, so their analogues hold in $\mathbb{S}$ as well.

We will now begin to examine $\mathbb{S}$. For $b \in B$, define the *orbit of* b to be

$$b^A = \bigcup_{n \in \mathbb{N}} A^n \cdot b.$$

Since $|A^n \cdot b| = n + 2$, the set b^A is countable. Suppose now that there are $b, c \in B$ such that $b^A \cap c^A = \{0\}$. Then by the properties above, $\mathrm{Cg}^{\mathbb{S}}(b, 0)$ relates the orbit of b to 0 and is the identity relation elsewhere. A similar statement is true of $\mathrm{Cg}^{\mathbb{S}}(c, 0)$. It follows that the two congruences meet to $\mathbf{0}$, which contradicts $\mathbf{0}$ being meet irreducible in $\mathrm{Con}(\mathbb{S})$. It follows that for all $b, c \in B$, $b^A \cap c^A \neq \{0\}$.

Pick distinct $b, c \in B$. Then $b^A \cap c^A \neq \{0\}$, so by the properties above, we have that either $b \in c^A$ or $c \in b^A$. Without loss of generality, assume that $b \in c^A$. There is a finite number n and $\alpha_1, \dots, \alpha_n \in A$ such that $\alpha_1 \cdots \alpha_n \cdot c = b$, so since this is true for all b, c, we have that $\bigcup_{b \in B} b^A$ is countable. Since $B = \bigcup_{b \in B} b^A$, it must be that B is countable. The property that $|A| = |B|$ and $S = A \cup B \cup \{0\}$ therefore gives us that $\mathbb{S}$ is also countable. Since nonprincipal ultrapowers of infinite structures are uncountable, this implies that the ultrapower is principal and $\mathbb{S} \cong \mathbb{S}_\omega$. ⊣

Corollary 6.2. *$\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ does not have definable principal subcongruences if $\mathcal{T}$ does not halt.*

Proof. Suppose that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences witnessed by Γ and ψ, and let

$$\zeta = \forall a, b, a', b' \left[(a \neq b) \wedge (a' \neq b') \rightarrow \exists c, d, c', d' \left[\Gamma(c, d, a, b) \wedge \Gamma(c', d', a', b') \right.\right.$$
$$\left.\left. \wedge \exists r, s \left[r \neq s \wedge \psi(r, s, c, d) \wedge \psi(r, s, c', d')\right]\right]\right].$$

For $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$, we have that $\mathbb{B} \models \zeta$ if and only if $\mathbb{B}$ is finitely subdirectly irreducible (that is, FSI's are axiomatized by ζ). This contradicts Theorem 6.1, so $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ cannot have definable principal subcongruences as we assumed. ⊣

§7. Conclusion. Theorem 5.26 and Corollaries 5.28 and 6.2 yield the Theorem 7.1.

Theorem 7.1. *The following are equivalent.*

(1) *$\mathcal{T}$ halts.*
(2) *$\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences.*
(3) *$\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is finitely based.*

This completes the proof that DPSC is an undecidable property, and provides another negative answer to Tarski's well-known finite basis problem.

REFERENCES

[1] Kirby A. Baker, *Finite equational bases for finite algebras in a congruence-distributive equational class*. ***Advances in Mathematics***, vol. 24 (1977), no. 3, pp. 207–243.

[2] Kirby A. Baker and Ju Wang, *Definable principal subcongruences*. ***Algebra Universalis***, vol. 47 (2002), no. 2, pp. 145–151.

[3] E. Fried, G. Grätzer, and R. Quackenbush, *Uniform congruence schemes*. ***Algebra Universalis***, vol. 10 (1980), no. 2, pp. 176–188.

[4] David Hobby and Ralph McKenzie, *The structure of finite algebras*, ***Contemporary Mathematics***, vol. 76, American Mathematical Society, Providence, RI, 1988.

[5] Ralph McKenzie, *Para primal varieties: A study of finite axiomatizability and definable principal congruences in locally finite varieties*. ***Algebra Universalis***, vol. 8 (1978), no. 3, pp. 336–348.

[6] ———, *The residual bound of a finite algebra is not computable*. ***International Journal of Algebra and Computation***, vol. 6 (1996), no. 1, pp. 29–48.

[7] ———, *The residual bounds of finite algebras*. ***International Journal of Algebra and Computation***, vol. 6 (1996), no. 1, pp. 1–28.

[8] ———, *Tarski's finite basis problem is undecidable*. ***International Journal of Algebra and Computation***, vol. 6 (1996), no. 1, 49–104.

[9] Matthew Moore, *The variety generated by $\mathbb{A}(\mathcal{T})$ – two counterexamples*, available at http://arxiv.org/abs/1304.4659.

[10] Ross Willard, *Tarski's finite basis problem via $A(\mathcal{T})$*, ***Transactions of the American Mathematical Society***, vol. 349 (1997), no. 7, pp. 2755–2774.

[11] ———, *A finite basis theorem for residually finite, congruence meet-semidistributive varieties*, this Journal, vol. 65 (2000), no. 1, pp. 187–200.

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