AN APPLICATION OF THE KONTOROVICH-LEBEDEV TRANSFORM

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1. In this note a simple application of the Kontorovich-Lebedev transform is described. The problem to which the transform is applied is the one considered in a recent paper (1).

An infinitely long line source of uniformly distributed current, whose density is a function of time alone, lies parallel to the edge of a wedge with perfectly conducting walls in a region of conductivity σ, dielectric constant ϵ and permeability μ. Cylindrical coordinates (r, θ, z) with the tip of the wedge coincident with the z axis are used. The angle of the wedge is taken to be α and the line source passes through (r_0, θ_0), $0 \leqslant \theta_0 \leqslant \alpha$. Since the problem is two dimensional, Maxwell's equations (in m.k.s. units) for the region within the wedge are

$$\left.\begin{array}{l} \text{(a)}\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rH_\theta)-\dfrac{1}{r}\dfrac{\partial H_r}{\partial \theta}=\left(\sigma+\epsilon\dfrac{\partial}{\partial t}\right)E_z+M(t)\dfrac{\delta(r-r_0)}{r}\delta(\theta-\theta_0), \\ \text{(b)}\ \dfrac{1}{r}\dfrac{\partial E_z}{\partial \theta}=-\mu\dfrac{\partial H_r}{\partial t}, \quad \text{(c)}\ \dfrac{\partial E_z}{\partial r}=\mu\dfrac{\partial H_\theta}{\partial t}, \end{array}\right\} \quad \ldots\ldots(1)$$

where $r^{-1}M(t)\delta(r-r_0)\delta(\theta-\theta_0)$ represents a current density corresponding to a line source passing through (r_0, θ_0). $M(t)$ is a function of time alone and $\delta(x)$ denotes the Dirac delta function.

The conditions on the boundary are :—

$$\left.\begin{array}{l} E_z=0 \quad \text{on} \quad \theta=0, \alpha, \\ H_r=H_\theta=E_z=0 \quad \text{at} \quad t=0 \quad \text{and as } r\to\infty. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

In the previous paper a Laplace transform in t and a finite Fourier transform in θ were used to reduce the equations (1) to an ordinary differential equation in r. In this paper a Laplace transform in t and a Kontorovich-Lebedev transform in r are used to reduce these equations to an ordinary differential equation in θ.

2. The particular form of the Kontorovich-Lebedev transform chosen is

$$f(r)=\frac{1}{\pi i}\int_{-i\infty}^{i\infty}\lambda I_\lambda(\eta r)\,d\lambda\int_0^\infty f(\xi)K_\lambda(\eta\xi)\frac{d\xi}{\xi}, \quad \eta^2=\mu s(\sigma+\epsilon s), \ldots\ldots\ldots\ldots\ldots(3)$$

where $I_\lambda(z)$ and $K_\lambda(z)$ denote modified Bessel functions of order λ.
A formula of this type has been reported by Lebedev (2).

Hence multiplying equation (1a) by $rK_\lambda(\eta r)e^{-st}$, (1b) by $K_\lambda(\eta r)e^{-st}$, (1c) by

$$r\frac{\partial}{\partial r}K_\lambda(\eta r)e^{-st}$$

and integrating with respect to t and r over the range $(0, \infty)$ we find

$$\left.\begin{aligned} -\mathscr{H}_\theta - \frac{d\mathscr{H}_r}{d\theta} &= (\sigma+\epsilon s)\mathscr{E}_z^{(1)} + \bar{M}(s)K\hat{\lambda}(\eta r_0)\delta(\theta-\theta_0), \\ \frac{d\mathscr{E}_z}{d\theta} &= -\mu s\mathscr{H}_r, \\ -\eta^2\mathscr{E}_z^{(1)} - \lambda^2\mathscr{E}_z &= \mu s\mathscr{H}_\theta, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

where

$$\mathscr{E}_z(\lambda, \theta, s) = \int_0^\infty \bar{E}_z(r, \theta, s)K_\lambda(\eta r)\frac{dr}{r} = \int_0^\infty K_\lambda(\eta r)\frac{dr}{r}\int_0^\infty E_z(r, \theta, t)e^{-st}dt,$$

$$\mathscr{E}_z^{(1)}(\lambda, \theta, s) = \int_0^\infty \bar{E}_z K_\lambda(\eta r)rdr,$$

$$\mathscr{H}_\theta(\lambda, \theta, s) = \int_0^\infty \bar{H}_\theta r\frac{\partial K_\lambda(\eta r)}{\partial r}dr,$$

$$\mathscr{H}_r(\lambda, \theta, s) = \int_0^\infty \bar{H}_r K\ (\eta r)dr,$$

and the bar above a component denotes its Laplace transform with respect to t.

From equations (4) we find that the equation for $\mathscr{E}_z$ is

$$\left(\frac{d^2}{d\theta^2} + \lambda^2\right)\mathscr{E}_z = \mu s\bar{M}(s)K_\lambda(\eta r_0)\delta(\theta-\theta_0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The solution of equation (5) subject to the boundary conditions (2) is readily shown to be

$$\mathscr{E}_z = -\mu s\bar{M}(s)\frac{\sin\lambda(\alpha-\theta_0)\sin\lambda\theta}{\lambda\sin\lambda\alpha}K_\lambda(\eta r_0),\ 0\leqslant\theta<\theta_0. \quad \ldots\ldots\ldots\ldots(6)$$

When $\theta > \theta_0$ the positions of θ and θ_0 are interchanged.

Inverting with respect to the Kontorovich-Lebedev transform (using equation (3)) we have

$$\bar{E}_z(r, \theta, s) = -\frac{\mu s\bar{M}(s)}{\pi i}\int_{-i\infty}^{i\infty}\frac{\sin\lambda(\alpha-\theta_0)\sin\lambda\theta}{\sin\lambda\alpha}K_\lambda(\eta r_0)I_\lambda(\eta r)d\lambda,\ \theta<\theta_0. \quad \ldots(7)$$

We may easily derive an expression for $\bar{E}_z$ in the form of an infinite series. For this purpose we indent at the origin by a small semi-circle in the right half plane (with radius tending to zero) and complete the path of integration $(r<r_0)$ to a closed one by a half circle radius $(N+\frac{1}{2})\frac{\pi}{\alpha}$ in the right half plane, apply the residue theorem and let N tend to infinity through the positive integers. The only simple poles within the closed contour are at

$$\lambda_n = \frac{n\pi}{\alpha}, \quad n = 1, 2, 3\ldots$$

and the residue of the integrand at $\lambda=\lambda_n$ is found to be

$$-\frac{1}{\alpha}K_{\frac{n\pi}{\alpha}}(\eta r_0)I_{\frac{n\pi}{\alpha}}(\eta r)\sin\frac{n\pi\theta_0}{\alpha}\sin\frac{n\pi\theta}{\alpha},\ r<r_0. \qquad \text{(8)}$$

The contribution along the indentation around the origin for a radius of indentation tending to zero is zero. Hence we have

$$\bar{E}_z=-\frac{2\mu s\bar{M}(s)}{\alpha}\sum_{n=1}^{\infty}K_{\frac{n\pi}{\alpha}}(\eta r_0)I_{\frac{n\pi}{\alpha}}(\eta r)\sin\frac{n\pi\theta_0}{\alpha}\sin\frac{n\pi\theta}{\alpha},\ r<r_0. \qquad \text{(9)}$$

When $r>r_0$ the positions of r and r_0 are interchanged.

Inverting equation (9) with respect to the Laplace transform gives

$$E_z(r,\theta,t)=\frac{i\mu}{\pi\alpha}\sum_{n=1}^{\infty}\sin\frac{n\pi\theta_0}{\alpha}\sin\frac{n\pi\theta}{\alpha}\int_{\gamma-i\infty}^{\gamma+i\infty}s\bar{M}(s)K_{\frac{n\pi}{\alpha}}(\eta r_0)I_{\frac{n\pi}{\alpha}}(\eta r)e^{st}ds. \qquad \text{(10)}$$

Equation (10) is identical with the solution obtained in (1, equation (6)).

REFERENCES

(1) A. C. Butcher and J. S. Lowndes, The diffraction of transient electromagnetic waves by a wedge, *Proc. Edin. Math. Soc.*, **11** (1958), 95-103.

(2) N. N. Lebedev, *Doklady Akad. Nauk. S.S.S.R.* (N.S.), **58** (1947), 1007-1010.

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