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Cauchy problem and periodic homogenization for nonlocal Hamilton–Jacobi equations with coercive gradient terms

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This paper deals with the periodic homogenization of nonlocal parabolic Hamilton–Jacobi equations with superlinear growth in the gradient terms. We show that the problem presents different features depending on the order of the nonlocal operator, giving rise to three different cell problems and effective operators. To prove the locally uniform convergence to the unique solution of the Cauchy problem for the effective equation we need a new comparison principle among viscosity semi-solutions of integrodifferential equations that can be of independent interest.

Keywords: Homogenization; Hamilton–Jacobi equations; integro-differential equations; fractional Laplacians; comparison principle; viscosity solutions

2010 *Mathematics subject classification:* 35R09; 35B27; 35F25; 35D40

1. Introduction

This paper deals with periodic homogenization for nonlocal parabolic Hamilton–Jacobi equations of the form

$$u_t^\epsilon - a\left(x, \frac{x}{\epsilon}\right)\mathcal{I}(u^\epsilon, x) + H\left(x, \frac{x}{\epsilon}, Du^\epsilon\right) = 0 \quad \text{in } Q_T, \tag{1.1}$$

where $Q_T := \mathbb{R}^N \times (0, T]$, for $T > 0$ fixed. We complement this equation with the initial condition

$$u^\epsilon(x, 0) = u_0(x) \quad x \in \mathbb{R}^N, \tag{1.2}$$

where u_0 is a bounded and uniformly continuous function in $\mathbb{R}^N$. The elliptic part of the operator in (1.1) is the term $a(x,y)\mathcal{I}(u,x)$, where $a:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ is a uniformly continuous function and $\mathcal{I}(u,x)$ is a nonlocal operator defined as

$$\mathcal{I}(u,x) := \int_{\mathbb{R}^N}[u(x+z)-u(x)-\mathbf{1}_B(z)\langle Du(x),z\rangle]K^\sigma(z)\mathrm{d}z, \tag{1.3}$$

for suitable functions $u:\mathbb{R}^N\to\mathbb{R}$, with $K^\sigma:\mathbb{R}^N\to\mathbb{R}$ nonnegative and measurable and $\mathbf{1}_B$ the indicator function of the unit ball B centred at 0. The main assumption on this nonlocal operator is the following *ellipticity condition*

$$\begin{gathered}0<a_0\leqslant a(x,y)\leqslant a_0^{-1}\quad \text{for all } x,y\in\mathbb{R}^N,\quad \text{and there exists } \sigma\in(0,2) \text{ such that}\\ \bar{k}(z):=K^\sigma(z)|z|^{N+\sigma} \text{ is bounded in } \mathbb{R}^N, \text{ continuous at the origin, and } \bar{k}(0)>0.\end{gathered} \tag{E}$$

This assumption makes $\mathcal{I}(u,x)$ in (1.3) well-defined for bounded and sufficiently smooth functions u. The parameter σ shall be regarded as the *order* of the operator.

An example of particular interest is the case of the *fractional Laplacian of order* $\sigma\in(0,2)$ defined as

$$(-\Delta)^{\sigma/2}u(x) := -C_{N,\sigma}\int_{\mathbb{R}^N}[u(x+z)-u(x)-\mathbf{1}_B(z)\langle Du(x),z\rangle]|z|^{-(N+\sigma)}\mathrm{d}z, \tag{1.4}$$

where $C_{N,\sigma}>0$ is a suitable normalizing constant, see [**20**].

We will assume $\bar{k}(0)=C_{N,\sigma}$, see assumption (2.3), so the interaction kernel K^σ in (1.3) under assumption (**E**) coincides with the kernel of the fractional Laplacian $(-\Delta)^{\sigma/2}$ multiplied by the function $\bar{k}(z)/\bar{k}(0)$ which is bounded, continuous in 0 and takes value 1 in 0. Therefore, K^σ can be considered a perturbation of the kernel of the fractional Laplacian $(-\Delta)^{\sigma/2}$, and therefore the integro-differential operator $\mathcal{I}$ is a perturbation of $(-\Delta)^{\sigma/2}$.

Concerning the Hamiltonian, we concentrate here on the case where H is superlinear in the gradient variable, see assumption (**H1**). A model problem is

$$u_t^\epsilon - a\left(x,\frac{x}{\epsilon}\right)\mathcal{I}(u^\epsilon,x)+b\left(x,\frac{x}{\epsilon}\right)|Du^\epsilon|^m = f\left(x,\frac{x}{\epsilon}\right)$$

with $m>1$ and $b(x,y)\geqslant b_0>0$, but we do not need any convexity of H with respect to Du. This is a suitable framework because we can exploit available well-posedness and regularity results, especially by Barles, Koike, Ley, and Topp [**11**], to study the behaviour of the family of viscosity solutions $\{u^\epsilon\}_\epsilon$ to (1.1)–(1.2) as $\epsilon\to 0$. The estimates in [**11**] extend to integro-differential equations some interesting results of [**15**] for viscous Hamilton–Jacobi equations.

Our main purpose is to obtain homogenization results for problems of the form (1.1) under periodicity conditions on the 'fast variable' x/ϵ, in the spirit of the celebrated paper of Lions, Papanicolaou, and Varadhan [**24**] and subsequently addressed for first and second-order degenerate elliptic and parabolic equations

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in [1–3, 21, 22], among many others. The goal is finding an effective Hamiltonian $\bar{H} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that u^ϵ converges to a solution of

$$u_t + \bar{H}(x, Du, \mathcal{I}(u, x)) = 0 \quad \text{in } Q_T, \tag{1.5}$$

possibly the unique one satisfying the initial condition

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}^N. \tag{1.6}$$

The basic strategy to identify $\bar{H}$ begins with a formal expansion in powers of ϵ of the form

$$u^\epsilon(x, t) = \bar{u}(x, t) + \epsilon^{1\vee\sigma}\psi(x/\epsilon), \tag{1.7}$$

where $a \vee b = \max\{a, b\}$ and ψ is called *the corrector*. Note that the exponent of ϵ is chosen depending on the order σ of the integral operator $\mathcal{I}$. Plugging the ansatz (1.7) in equation (1.1), some nontrivial calculations in § 4 lead to a *cell problem*, which is an additive eigenvalue problem on the torus $\mathbb{T}^N$ whose solution should be the corrector ψ and the eigenvalue $\bar{H} = \bar{H}(x, p, l)$, where x, p and l are parameters. The presence of the nonlocal term $\mathcal{I}$ produces three different cell problems depending on σ:

- for $\sigma < 1$ the cell problem is the purely first-order PDE

$$-a(x, y)l + H(x, y, p + D\psi(y)) = \bar{H} \quad y \in \mathbb{T}^N.$$

- for $\sigma > 1$ the cell problem is the linear purely nonlocal equation

$$-a(x, y)l + a(x, y)(-\Delta)^{\sigma/2}\psi(y) + H(x, y, p) = \bar{H} \quad y \in \mathbb{T}^N.$$

- for $\sigma = 1$ it has both first-order and nonlocal terms, and an extra drift term $\langle b, D\psi(y)\rangle$

$$\begin{aligned} &- a(x, y)l + a(x, y)[(-\Delta)^{1/2}\psi(y) + \langle b, D\psi(y)\rangle] \\ &\quad + H(x, y, p + D\psi(y)) = \bar{H} \quad \text{in } \mathbb{T}^N, \end{aligned} \tag{1.8}$$

  with $b \neq 0$ if the kernel K^1 is not symmetric (b is explicitly defined in (4.6)).

The solvability of these problems and sufficient regularity of ψ are not difficult in the first two cases, whereas for $\sigma = 1$ they require some fine estimates that we obtain by adapting the methods of [11], [12] and [29], and by strengthening the regularity assumption on H from the general condition (**H2**) to (2.6). We deduce from the cell problems also information about the effective Hamiltonian $\bar{H}$, especially about its modulus of continuity, since $\bar{H}$ is explicit only for $\sigma > 1$.

Adapting in an appropriate way the perturbed test function method introduced by Evans [21, 22], we show that the weak semilimits of the family of solutions $\{u^\epsilon\}_\epsilon$ are a sub- and a supersolution of the effective equation (1.5) and initial condition (1.6). Next we need a comparison principle between a sub- and a supersolution of this Cauchy problem to obtain the locally uniform convergence of the full sequence $\{u^\epsilon\}_\epsilon$. In the nonlocal setting, however, the known theory does not

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cover nonlinearities where the state variable x and the integral operator $\mathcal{I}$ interact. Only the case $\sigma > 1$, where the effective equation is

$$u_t - \frac{\mathcal{I}(u,x)}{\int_{\mathbb{T}^N} 1/a(x,y)\mathrm{d}y} + \int_{\mathbb{T}^N} \frac{H(x,y,Du)}{a(x,y)}\mathrm{d}y = 0$$

can be treated by the methods of Barles and Imbert [**10**]. For the other two cases we prove a new comparison result for (1.5)–(1.6) under the structure condition on the operator that for some $n > 0$

$$|\bar{H}(x_1,p_1,l_1) - \bar{H}(x_2,p_2,l_2)| \leqslant \omega\Big(|l_1 - l_2| + |x_1 - x_2|(1 + |l| + |p|^m)^n + |p_1 - p_2|(1 + |l| + |p|^m)^n\Big), \quad (1.9)$$

and for semicontinuous sub- and supersolutions attaining the initial data continuously uniformly on $\mathbb{R}^N$, i.e.

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^N} |u(x,t) - u_0(x)| = 0, \quad (1.10)$$

and such that at least one of them is Hölder continuous. The proof relies on a new argument for comparison when one knows that the semisolutions are ordered in a small strip $\mathbb{R}^N \times [0, d_0]$ and one of them is Hölder, proposition 3.1. Then one reduces to this case by regularizing in time, and exploiting the regularity results of [**11**] and the initial condition (1.10), see theorem 3.2. We believe this comparison theorem and the method of proof have independent interest and will find other applications.

Finally, we show that $\bar{H}$ satisfies (1.9) with $n = m - 1$ and the weak semilimits verify the assumptions of the comparison theorem, and therefore we get the homogenization result for all σ, as well as a characterization of the limit as the unique solution of (1.5) with the property (1.10).

There are a few other papers on the homogenization of integrodifferential equations in the framework of viscosity solutions. Arisawa [**4**, **5**] addressed stationary equations of the form $u^\epsilon - a(x/\epsilon)\mathcal{I}(u^\epsilon, x) = g(x/\epsilon)$ in a bounded open set Ω, with u^ϵ prescribed in Ω^c. In this problem there is no interaction between $\mathcal{I}$ and gradient terms in H, and the effective equation does not depend on x, so it satisfies the comparison principle by standard theory. In the unpublished paper [**6**] she considered the same equation with the addition of a non-oscillating Hamiltonian $H = \max_{\alpha\in A}\langle f(x,\alpha), Du^\epsilon\rangle$, with A compact, and mere almost periodicity of a and g. In [**26**] Schwab also considered a Dirichlet problem and nonlocal equations without first order terms, which in his case are elliptic and have the Bellman–Isaacs form with oscillating kernels $K^\sigma(x/\epsilon, z)$. In [**26**] the effective equation has nontrivial interaction between the state variable and the nonlocality, but it enjoys translation invariance properties which allow to get a comparison principle by inf/sup convolutive regularizations. Schwab also extended some of these results to stochastic homogenization [**27**]. The very recent preprint [**18**] addresses periodic homogenization of Hamilton–Jacobi–Bellman equations with nonlocal terms and at most linear growth in the gradient. We mention that nonlocal homogenization problems have been addressed also in other contexts, such as divergence-form equations, using

Γ-convergence [23], and semigroup theory [25]. Finally, we point out that a phenomenon related to the appearance of the extra term in (1.8) when the kernel is not symmetric was observed in [16].

This paper is organized as follows. In § 2 we present the main assumption and preliminary results. In § 3 we provide the new comparison principle that is needed in the case $\sigma \leqslant 1$. In § 4 we present the different cell problems associated to the value of $\sigma \in (0, 2)$. Sections 5, 6 and 7 deal, respectively, with the case $\sigma = 1$, $\sigma < 1$ and $\sigma > 1$. Finally, in the appendix we provide two a priori estimates for solution to coercive Hamilton–Jacobi equation with fractional Laplacian of order $1/2$.

2. Preliminaries

2.1. Basic assumptions and examples

First of all we assume that $a : \mathbb{R}^{2N} \to \mathbb{R}$ is uniformly continuous and $H \in C(\mathbb{R}^{3N})$ satisfies

$$\begin{aligned} &|H(\cdot,\cdot,0)|_\infty, |a|_\infty < +\infty, \\ &a(x,\cdot),\ H(x,\cdot,p) \text{ are } \mathbb{Z}^N\text{-periodic, for all } x, p \in \mathbb{R}^N. \end{aligned} \tag{\textbf{H0}}$$

The assumption on the nonlocal operator are given in (**E**). We define $\bar{\omega}$ to be the modulus of continuity of $\bar{k}$ at 0, that is

$$\bar{\omega}(t) = \sup_{|z| \leqslant t} \{|\bar{k}(z) - \bar{k}(0)|\}, \quad t > 0. \tag{2.1}$$

Moreover, in the case $\sigma = 1$, we impose the following extra condition on K^1, when it is not symmetric:

$$\int_0^1 \bar{\omega}(r) r^{-1} \mathrm{d}r < +\infty. \tag{2.2}$$

Regarding (**E**), the second assumption is related to what we call 'the order' of the nonlocal operator, i.e. the number $\sigma \in (0, 2)$. On the other hand, the first assumption is important to get the existence and uniqueness to (1.1). For simplicity, we assume that

$$\bar{k}(0) = C_{N,\sigma} > 0, \tag{2.3}$$

where $C_{N,\sigma} > 0$ is the well-known normalizing constant arising in the definition of fractional Laplacian $(-\Delta)^{\sigma/2}$ (see [20]). This is going to be used in § 4.

We assume that the Hamiltonian is superlinear in the gradient variable in the following sense:

$$\begin{aligned} \exists b_0, C_0 > 0,\ m > 1 :\ &\mu H(x,y,\mu^{-1}p) - H(x,y,p) \\ &\geqslant (1-\mu)\Big(b_0|p|^m - C_0\Big), \quad \forall \mu \in (0,1), \end{aligned} \tag{\textbf{H1}}$$

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for all $x, y, p \in \mathbb{R}^N$. Moreover, we assume there exists a modulus of continuity ω such that

$$|H(x,y,p) - H(x',y',p')| \leqslant \omega(|x-x'| + |y-y'|)(1+R^m) + \omega(|p-p'|)(1+R^{m-1}), \tag{\textbf{H2}}$$

for all $R > 0$, all $x, x', y, y' \in \mathbb{R}^N$ and $p, p' \in \mathbb{R}^N$ with $|p|, |p'| \leqslant R$. Since it is not restrictive to assume $\omega(r) \leqslant C_1 r$ for all $r \geqslant 1$, (**H2**) and (**H0**) imply the existence of $C > 0$ such that

$$|H(x,y,p)| \leqslant C(1+|p|^m), \quad \text{for } x, y, p \in \mathbb{R}^N. \tag{2.4}$$

We observe that assumptions (**H0**), (**H1**) and (**H2**) imply the following coercivity condition: for some $C > 1$ and $K \geqslant 0$

$$C^{-1}(1+|p|^m) - K \leqslant H(x,y,p), \quad \text{for } x, y, p \in \mathbb{R}^N. \tag{2.5}$$

A proof of this fact is detailed at the end of the appendix. A model example is

$$H(x,y,p) = b(x,y)|p|^m - f(x,y),$$

with $m > 1$ and f, b bounded and uniformly continuous, with $b \geqslant b_0 > 0$.

Finally, in the case $\sigma = 1$, we require the following extra Lipschitz condition over the data: recalling $m > 1$ arising in (**H1**), we assume the existence of $L > 0$ such that, for all $R > 0$, all $X = (x,y), X' = (x',y') \in \mathbb{R}^{2N}$ and $p, p' \in \mathbb{R}^N$ with $|p|, |p'| \leqslant R$ we have

$$\begin{cases} |H(X,p) - H(X',p')| \leqslant L(1+R^m)|X-X'| + L(1+R^{m-1})|p-p'|, \\ |a(X) - a(X')| \leqslant L|X-X'|. \end{cases} \tag{2.6}$$

We recall briefly the definition of viscosity solutions for nonlocal parabolic equations such as (1.1). For more details we refer to [**10**].

2.2. Notion of solution

We describe the notion of solution for slightly more general Cauchy problems of the form

$$\begin{cases} u_t + F(x, Du, \mathcal{I}(u,x)) = 0 & \text{in } Q_T \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N. \end{cases} \tag{2.7}$$

Here, $F \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$ is *degenerate elliptic in the nonlocal variable*, that is

$$F(x,p,l_1) \leqslant F(x,p,l_2) \quad \text{for all } x, p \in \mathbb{R}^N, \ l_1, l_2 \in \mathbb{R}, \ \text{such that } l_1 \geqslant l_2.$$

We introduce some notation. Let $\delta \in (0,1)$, and we denote with B_δ the ball centred at 0 of radius δ, with B the ball of radius 1, and with B_δ^c, B^c the complements

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of such sets. Finally $B_\delta(x)$ will indicate the ball centred at x of radius δ. For $\phi \in C^2(\mathbb{R}^N \times (0,T))$ and $x \in \mathbb{R}^N$, $t \in (0,T)$, we define the localized operator

$$\mathcal{I}[B_\delta](\phi, x) = \int_{B_\delta} [\phi(x+z,t) - \phi(x,t) - \langle D\phi(x,t), z\rangle] K^\sigma(z)\mathrm{d}z. \tag{2.8}$$

Moreover, for any $u \in L^\infty(\mathbb{R}^N \times (0,T))$, $p \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$, $t \in (0,T)$, we define

$$\mathcal{I}[B_\delta^c](u,p,x) = \int_{B_\delta^c} [u(x+z,t) - u(x,t) - \mathbf{1}_B(z)\langle p, z\rangle] K^\sigma(z)\mathrm{d}z. \tag{2.9}$$

Note that if K^σ is symmetric, that is $K^\sigma(z) = K^\sigma(-z)$, due to its integrability properties we get that the previous operator is independent of $p \in \mathbb{R}^N$, that is

$$\mathcal{I}[B_\delta^c](u,p,x) = \mathcal{I}[B_\delta^c](u,x) = \int_{B_\delta^c} [u(x+z,t) - u(x,t)] K^\sigma(z)\mathrm{d}z. \tag{2.10}$$

Definition 2.1 Viscosity solutions.

- A bounded upper semicontinuous function $u : \mathbb{R}^N \times (0,T] \to \mathbb{R}$ is a viscosity subsolution of (1.1) if for any $(x,t) \in \mathbb{R}^N \times (0,T]$ and any test-function $\phi \in C^2(\mathbb{R}^N \times (0,T])$, such that (x,t) is a maximum point of $u - \phi$ in $B_\delta(x) \times (t-\delta, t+\delta)$, for a small $\delta > 0$, there holds

$$\phi_t(x,t) + F\Big(x, D\phi(x,t), I[B_\delta](\phi,x) + I[B_\delta^c](u, D\phi(x,t), x)\Big) \leqslant 0.$$

- A bounded lower semicontinuous function $u : \mathbb{R}^N \times (0,T] \to \mathbb{R}$ is a viscosity supersolution of (1.1) if for any $(x,t) \in \mathbb{R}^N \times (0,T]$ and any test-function $\phi \in C^2(\mathbb{R}^N \times (0,T])$, such that (x,t) is a minimum point of $u - \phi$ in $B_\delta(x) \times (t-\delta, t+\delta)$, for a small $\delta > 0$, there holds

$$\phi_t(x,t) + F\Big(x, D\phi(x,t), I[B_\delta](\phi,x) + I[B_\delta^c](u, D\phi(x,t), x)\Big) \geqslant 0.$$

- A bounded continuous function $u : \mathbb{R}^N \times (0,T] \to \mathbb{R}$ is a viscosity solution of (1.1) if it is both a subsolution and a supersolution.

2.3. Existence and comparison principle for (1.1)–(1.2)

In this section, we present well known results about existence and uniqueness of solutions to the Cauchy problem (1.1)–(1.2). We point out that we give also a precise estimate on the behaviour of the solutions to the parabolic problem as $t \to 0$, that is estimate (2.12), based on the uniform continuity assumption on the initial data, which will be useful in comparing the weak upper and lower semilimits of u^ϵ as $\epsilon \to 0$.

Proposition 2.2. *Assume* (**E**), (**H0**), (**H1**), (**H2**) *hold and* $u_0 \in BUC(\mathbb{R}^N)$. *Then there exists a unique bounded continuous viscosity solution to the Cauchy*

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problem (1.1)–(1.2). *Moreover,*

$$|u^\epsilon|_{L^\infty(Q_T)} \leqslant |u_0|_\infty + |H(\cdot,\cdot,0)|_\infty T \tag{2.11}$$

and there exists a modulus of continuity $\bar{\omega}$ (*depending on the modulus of* u_0) *such that*

$$\sup_{x\in\mathbb{R}^N} |u^\epsilon(x,t) - u_0(x)| \leqslant \bar{\omega}(t) \quad \textit{for all } t>0,\ \epsilon>0. \tag{2.12}$$

Proof. A comparison principle for bounded viscosity sub and supersolutions which are well-ordered at time $t=0$ is proposition 3.1 in [**11**]. It does not apply directly to (1.1) unless the coefficient a multiplying the nonlocal operator $\mathcal{I}$ is constant. However, in view of assumption (**E**), equation (1.1) can be equivalently formulated as

$$a^{-1}(x,x/\epsilon)u_t - \mathcal{I}(u,x) + a^{-1}(x,x/\epsilon)H(x,x/\epsilon,Du) = 0,$$

so that the nonlocal operator does not interact with the state variables $x, x/\epsilon$. Then, using the continuity of a, we can get the comparison result by a straightforward adaption of the proof in [**11**].

Concerning existence, by (**E**) and (**H0**), if $u_0 \in C^2(\mathbb{R}^N)$ with $|u_0|_{C^2(\mathbb{R}^N)} < \infty$, then we see that the function $U(x,t) = u_0(x) \pm C_0 t$ with C_0 large enough in terms of $|u_0|_{C^2(\mathbb{R}^N)}$ is a supersolution (resp. a subsolution) for the problem solved by u^ϵ. More precisely, C_0 can be chosen of the form

$$C_0 = C_1|D^2u_0|_\infty + C_2|Du_0|^m_\infty,$$

with C_1, C_2 depending only on the constants in the assumptions, thanks to the linearity of $\mathcal{I}$ and the growth (2.4) of H. Therefore, Perron's method leads to the existence of a viscosity solution to this problem. By stability arguments, it is possible to conclude the existence for initial data merely continuous by approximation. Moreover, by comparison principle the unique solution u^ϵ to problem (1.1)–(1.2) is uniformly bounded in Q_T for all $\epsilon > 0$, that is (2.11) holds.

We prove now (2.12). If $|u_0|_{C^2(\mathbb{R}^N)} < \infty$ then (2.12) holds with $\bar{\omega}(t) = C_0 t$. In the general case, we consider a standard mollifier $\rho \in C^\infty(\mathbb{R}^N)$ with support in the unit ball and $\int_B \rho(x)\mathrm{d}x = 1$, and its rescaled version $\rho_h(x) = h^{-N}\rho(x/h)$, $h>0$. Then we define $u_0^h := u_0 * \rho_h$, which is a C^∞ function with $|Du_0^h|_\infty \leqslant Ch^{-1}$ and $|D^2u_0^h|_\infty \leqslant Ch^{-2}$. Notice that for all $x\in\mathbb{R}^N$ we have

$$|u_0^h(x) - u_0(x)| \leqslant h^{-N}\int_{B_h} |u_0(y)-u_0(x)|\rho((x-y)/h)\mathrm{d}y \leqslant \omega_0(h),$$

where ω_0 is the modulus of continuity of u_0. Therefore a function with the form

$$U^h(x,t) = u_0^h(x) + \omega_0(h) + C(h)t,$$

is a supersolution for the problem solved by u^ϵ, with a constant $C(h)$ of the form

$$C(h) = CC_1h^{-2} + CC_2h^{-m} \leqslant C_3h^{-\alpha}, \quad \alpha = 2\vee m, \quad h\leqslant 1$$

where $m > 1$ is the constant appearing in (**H1**), (**H2**). Since a subsolution can be constructed in the same way, we have that

$$\sup_{x\in\mathbb{R}^N} |u^\epsilon(x,t) - u_0(x)| \leqslant \inf_{h>0}\{2\omega_0(h) + C(h)t\} \leqslant 2\omega_0(t^{1/(2\alpha)}) + C_3 t^{1/2} =: \bar{\omega}(t),$$

which proves (2.12) and in particular leads to (1.2). □

3. Comparison principle and uniqueness result for a class of nonlocal Hamilton–Jacobi operators

In this section, we provide a comparison principle among semicontinuous viscosity sub and supersolutions and a uniqueness result for problems of the form (2.7). We need it for the effective problems addressed in §§ 5 and 6 of this paper, which do not fall within the theory of [**11**], different from the ϵ-problem (1.1).

We consider the following continuity assumption: there exists $n > 0$ such that such that for all $x_i, p_i \in \mathbb{R}^N$, $l_i \in \mathbb{R}$, $i = 1, 2$,

$$|F(x_1, p_1, l_1) - F(x_2, p_2, l_2)| \leqslant \omega\Big(|l_1 - l_2| + |x_1 - x_2|(1 + |l| + |p|^m)^n + |p_1 - p_2|(1 + |l| + |p|^m)^n\Big), \quad (3.1)$$

where $m > 1$, ω be a modulus of continuity, and $|p| = \max\{|p_1|, |p_2|\}, |l| = \max\{|l_1|, |l_2|\}$.

The initial condition $u_0 \in BUC(\mathbb{R}^N)$ satisfies (2.12).

Note that the nonlocal operator depends on the state variable: in this setting, the validity of a comparison principle among semicontinuous sub- and supersolutions is an open problem. We provide in theorem 3.2 a comparison principle by exploiting regularization by sup-convolutions in the time variable and the uniform continuity of the initial datum u_0. We will first need a technical result for the case $\sigma = 1$, which requires sufficient regularity either of the subsolution or of the supersolution, and moreover it requires to control the behaviour of sub- and supersolutions in a small neighbourhood of the initial time.

Proposition 3.1. *Assume $\sigma \leqslant 1$. Let u, v bounded, u u.s.c. in $\bar{Q}_T$, v l.s.c. in $\bar{Q}_T$ be, respectively, a viscosity sub- and supersolution to the PDE in (2.7), with F satisfying (3.1). Moreover we assume*

$$u \leqslant v \quad \text{in } \mathbb{R}^N \times [0, d_0], \quad (3.2)$$

for some $0 < d_0 < T$. Then there exists $\alpha_0 = \alpha_0(n, \sigma, m) < 1$ such that, if u or v is in $C^\alpha(\bar{Q}_T)$ for some $\alpha \in (\alpha_0, 1)$, then $u \leqslant v$ in $\bar{Q}_T$.

Proof. We assume that the C^α property corresponds to u. The case in which v is Hölder follows the same lines. By contradiction, we assume that

$$\sup_{\bar{Q}_T}\{u - v\} =: M > 0.$$

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Replacing u by $u - \nu t$ for some $\nu > 0$ small enough in terms of M and T, a classical argument allows us to assume that u in fact satisfies the viscosity inequality

$$u_t + F(x, Du, \mathcal{I}(u)) \leqslant -\nu \quad \text{in } Q_T.$$

Then, we double variables and approximate M as follows:

$$\begin{aligned} M_{\epsilon,\eta,\beta} &= \sup_{\bar{Q}_T \times \bar{Q}_T} \Phi(x, y, s, t) \\ &:= \sup_{\bar{Q}_T \times \bar{Q}_T} (u(x, s) - v(y, t) - \chi_\beta(y) - \epsilon^{-2}|x - y|^2 - \eta^{-1}(s - t)^2), \end{aligned} \tag{3.3}$$

where the parameters $\epsilon, \eta, \beta > 0$ are small parameters that will go to 0, and the function χ_β is constructed as follows, arguing as in the proof of [**10**, theorem 3]. We consider a function $\chi \in C_b^2(\mathbb{R})$ with $\|\chi\|_{C^2} < \infty$, $\chi = 0$ in B_1, $\chi \geqslant |u|_\infty + |v|_\infty + 1$ in B_2^c. For $\beta > 0$ we denote $\chi_\beta(x) = \chi(\beta x)$.

Observe that $\chi_\beta(x) > |u|_\infty + |v|_\infty + 1$ for all $|x| \geqslant 2/\beta$ which ensures that the supremum defining $M_{\epsilon,\eta,\beta}$ is achieved and therefore the function Φ in (3.3) attains its maximum at a point $(\bar{x}, \bar{y}, \bar{s}, \bar{t})$ for all $\beta, \epsilon, \eta > 0$ small enough.

Moreover, again as in [**10**, theorem 3] we get that

$$|D\chi_\beta|_\infty, |\mathcal{I}(\chi_\beta, \cdot)|_\infty \to 0 \quad \text{uniformly in } \mathbb{R}^N \text{ as } \beta \to 0. \tag{3.4}$$

Hence, for $\beta > 0$ small enough in terms of M we have

$$\sup_{\bar{Q}_T} \{u(x, t) - v(x, t) - \chi_\beta(x)\} =: \tilde{M} \geqslant M/2, \tag{3.5}$$

and this supremum is achieved at some point $(\hat{x}, \hat{t}) \in \bar{Q}_T$ with $|\hat{x}| \leqslant 2/\beta$. Using the inequality

$$\Phi(\bar{x}, \bar{y}, \bar{s}, \bar{t}) \geqslant \Phi(\hat{x}, \hat{x}, \hat{t}, \hat{t}) = \tilde{M} > 0, \tag{3.6}$$

we see that $|\bar{x} - \bar{y}| \leqslant C\epsilon$ and $|\bar{s} - \bar{t}| \leqslant C\eta^{1/2}$. Using this and (3.6) again together with the fact that u is C^α, we conclude that

$$\tilde{M} \leqslant u(\bar{y}, \bar{t}) - v(\bar{y}, \bar{t}) + C(\epsilon^\alpha + \eta^{\alpha/2}),$$

for all η, ϵ, β and a constant $C > 0$ not depending on these parameters. Then, for all ϵ, η small enough depending on $\tilde{M}$, assumption (3.2) implies that $\bar{t} \geqslant d_0$ and therefore, taking η smaller if it is necessary, we conclude that $\bar{s}, \bar{t} \geqslant d_0/2$, independent of β.

Thus, we use the viscosity inequality for u at $(\bar{x}, \bar{s})$ and for v at $(\bar{y}, \bar{t})$, for each $\delta > 0$ we can write

$$\begin{aligned} &2\eta^{-1}(\bar{s} - \bar{t}) + F(\bar{x}, \bar{p}, I_{\delta,1} + I_1^\delta) \leqslant -\nu \\ &2\eta^{-1}(\bar{s} - \bar{t}) + F(\bar{y}, \bar{q}, I_{\delta,2} + I_2^\delta) \geqslant 0, \end{aligned} \tag{3.7}$$

where $\bar{p} = 2\epsilon^{-2}(\bar{x} - \bar{y}), \bar{q} = \bar{p} - D\chi_\beta(\bar{y})$. For the nonlocal evaluations denotes as $I_{\delta,i}, I_i^\delta$, $i = 1, 2$, we require some notation to split the analysis depending if $\sigma < 1$

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or $\sigma = 1$. Denote $\mathbb{I}_\sigma = 1$ if $\sigma = 1$, $\mathbb{I}_\sigma = 0$ if $\sigma < 1$, $\phi(x,y) := \epsilon^{-2}|x-y|^2 + \chi_\beta(y)$, and with this the integral terms

$$I_{\delta,1} = \int_{B_\delta} [\phi(\bar{x}+z,\bar{y}) - \phi(\bar{x},\bar{y}) - \mathbb{I}_\sigma \langle \bar{p}, z\rangle] K^\sigma(z) \mathrm{d}z,$$

$$I_{\delta,2} = -\int_{B_\delta} [\phi(\bar{x},\bar{y}+z) - \phi(\bar{x},\bar{y}) - \mathbb{I}_\sigma \langle \bar{q}, z\rangle] K^\sigma(z) \mathrm{d}z,$$

$$I_1^\delta = \int_{B_\delta^c} [u(\bar{x}+z,\bar{s}) - u(\bar{x},\bar{s}) - \mathbb{I}_\sigma \mathbf{1}_B \langle \bar{p}, z\rangle] K^\sigma(z) \mathrm{d}z,$$

$$I_2^\delta = \int_{B_\delta^c} [v(\bar{y}+z,\bar{t}) - v(\bar{y},\bar{t}) - \mathbb{I}_\sigma \mathbf{1}_B \langle \bar{q}, z\rangle] K^\sigma(z) \mathrm{d}z,$$

where we have omitted the dependence of these quantities on the rest of the parameters for simplicity. Subtracting the inequalities in (3.7), by the continuity of F and the respective semicontinuity of u, v we take limit as $\eta \to 0$ to arrive at

$$F(\bar{x}, \bar{p}, I_{\delta,1} + I_1^\delta) - F(\bar{y}, \bar{q}, I_{\delta,2} + I_2^\delta) \leqslant -\nu, \tag{3.8}$$

where $\tau \in [0, T]$ is such that $\bar{s}, \bar{t} \to \tau$ as $\eta \to 0$. We keep using the notation $\bar{x}, \bar{y}$ after taking $\eta \to 0$ for simplicity.

Using that $\Phi(\bar{x}, \bar{y}, \tau, \tau) \geqslant \Phi(\hat{x}, \hat{x}, \hat{t}, \hat{t})$, the definition of ϕ and the property of $\mathcal{I}$ in (3.4) we arrive at

$$I_1^\delta \leqslant I_2^\delta + o_\beta(1),$$

where $o_\beta(1) \to 0$ uniformly on the rest of the parameters. Thus, by the elliptic monotonicity of F in the nonlocal variable, (3.8) leads us to

$$F(\bar{x}, \bar{p}, I_{\delta,1} + I_1^\delta) - F(\bar{y}, \bar{q}, I_{\delta,2} + I_1^\delta + o_\beta(1)) \leqslant -\nu. \tag{3.9}$$

It is direct to check using (3.4) that

$$|I_{\delta,i}| \leqslant o_\beta(1) + C\epsilon^{-2} \begin{cases} \delta & \text{if } \sigma = 1 \\ \delta^{1-\sigma} & \text{if } \sigma < 1, \end{cases}$$

for each $i = 1, 2$. On the other hand, using the C^α assumption for u we see that

$$|I_1^\delta| \leqslant C \int_{B_\delta^c} |z|^{\alpha - N - \sigma} \mathrm{d}z + C\mathbb{I}_\sigma |\bar{p}| \int_{B \setminus B_\delta} |z|^{1-N-\sigma} \mathrm{d}z,$$

from which we get

$$|I_1^\delta| \leqslant C\delta^{\alpha - \sigma} + C\mathbb{I}_\sigma |\bar{p}| |\log(\delta)|.$$

Next we deal first with the case $\sigma = 1$. Using (3.6) once more we see that

$$u(\bar{x}, \tau) - u(\bar{y}, \tau) - \epsilon^{-2}|\bar{x} - \bar{y}|^2 \geqslant 0.$$

Then, applying the C^α continuity of u we conclude that

$$C|\bar{x} - \bar{y}|^\alpha \geqslant \epsilon^{-2}|\bar{x} - \bar{y}|^2,$$

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for some constant depending on α. From here, denoting $\theta = 2/(2-\alpha)$ we conclude that

$$|\bar{x} - \bar{y}| \leqslant C\epsilon^{\theta} \quad \text{and} \quad |\bar{p}| \leqslant C\epsilon^{\theta-2}. \tag{3.10}$$

Notice that $\theta \to 2$ as $\alpha \to 1^-$.

In view of the above estimates, we apply the continuity assumption on the Hamiltonian F in (3.9) to conclude

$$\omega\Big(\epsilon^{-2}\delta + o_\beta(1) + (\epsilon^{\theta} + o_\beta(1))[1 + \delta^{\alpha-1} + \epsilon^{-2}\delta + \epsilon^{\theta-2}|\log(\delta)| + \epsilon^{m(\theta-2)}]^n\Big) \leqslant -\nu \tag{3.11}$$

At this point we choose $\delta = \epsilon^{2+\kappa}$, $\kappa > 0$ and $\beta \ll \epsilon$ to have $o_\beta(1) = \epsilon^{\theta}$ to get, recalling that $m > 1$ and $\theta < 2$,

$$\omega\Big(\epsilon^{\kappa} + \epsilon^{\theta} + \epsilon^{\theta + n(2+\kappa)(\alpha-1)} + \epsilon^{\theta + nm(\theta-2)}\Big) \leqslant -\nu \tag{3.12}$$

where we have replaced $\omega(\cdot)$ by $\omega(C\cdot)$ for $C > 0$ large enough. We show now that we can choose $\kappa > 0$ such that there exists a constant $\alpha_0(n, \sigma, m) \in (0, 1)$ such that for $\alpha > \alpha_0(n, \sigma, m)$ all the exponents of ϵ in (3.12) are positive. This will give a contradiction sending $\epsilon \to 0$ since $\nu > 0$ is fixed.

Indeed, choosing $\kappa = 2$ and recalling that $\theta = 2/(2-\alpha)$ we observe that

$$\theta + 4n(\alpha - 1) > 0 \quad \text{if } \alpha \in \left(\frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{2}{n}}, 1\right)$$

and

$$\theta + mn(\theta - 2) > 0 \quad \text{if and only if} \quad \alpha > 1 - \frac{1}{nm}.$$

This implies the claim for $\sigma = 1$ by choosing

$$\alpha_0(n, 1, m) = \max\left(1 - \frac{1}{nm}, \frac{3}{2} - \frac{1}{2}\sqrt{1 + \frac{2}{n}}\right).$$

In the case $\sigma < 1$ we argue in the same way. Now (3.11) is replaced by

$$\omega\Big(\epsilon^{-2}\delta^{1-\sigma} + o_\beta(1) + (\epsilon^{\theta} + o_\beta(1))\left[1 + \delta^{\alpha-\sigma} + \epsilon^{-2}\delta^{1-\sigma} + \epsilon^{m(\theta-2)}\right]^n\Big) \leqslant -\nu \tag{3.13}$$

and (3.12) is replaced by

$$\omega\Big(\epsilon^{\kappa(1-\sigma)-2\sigma} + \epsilon^{\theta} + \epsilon^{\theta + n(2+\kappa)(\alpha-\sigma)} + \epsilon^{\theta + nm(\theta-2)}\Big) \leqslant -\nu$$

Then we choose $\kappa > 2\sigma/(1-\sigma)$ and observe that

$$\theta + n(2+\kappa)(\alpha - \sigma) > 0 \quad \text{if } \alpha \in (\bar{\alpha}, 1),$$

where $\bar{\alpha} := (2 + \sigma - \sqrt{(2-\sigma)^2 + 8/(n(2+\kappa))})/2$. This proves the claim for $\sigma < 1$ by choosing

$$\alpha_0(n, \sigma, m) = \max\left(1 - \frac{1}{nm}, \bar{\alpha}\right). \qquad \square$$

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The key assumption on the regularity of the subsolution in the previous proposition can be obtained through the gradient dominance. We say that F is *superlinear in the gradient* if there exist $m > 1$ and $C, c > 0$ such that

$$F(x,p,l) \geqslant c|p|^m - C(|l|+1), \quad \text{for all } x, p \in \mathbb{R}^N, \ l \in \mathbb{R}. \tag{3.14}$$

Now we are ready to prove a comparison principle for semicontinuous solutions to problem (2.7) among functions u *attaining uniformly continuously the initial data*, namely, satisfying

$$\sup_{x \in \mathbb{R}^N} |u(x,t) - u_0(x)| \leqslant \omega_0(t), \quad t \geqslant 0, \tag{3.15}$$

for some modulus ω_0 (i.e. $\omega_0(t) \to 0$ as $t \to 0$).

THEOREM 3.2. *Assume that $\sigma \leqslant 1$, F satisfies (3.1), it is degenerate elliptic in the nonlocal variable and superlinear in the gradient (3.14), and $u_0 \in BUC(\mathbb{R}^N)$. Let $\underline{u}$ be a bounded l.s.c. supersolution to (2.7) in Q_T and $\bar{u}$ be a bounded u.s.c. subsolution to (2.7) in Q_T attaining uniformly continuously the initial data u_0. Then, $\bar{u} \leqslant \underline{u}$ in $\bar{Q}_T$.*

In particular, there exists at most one viscosity solution to (2.7) among functions satisfying (3.15).

Proof. It is sufficient to prove that $\bar{u} \leqslant \underline{u}$ in $\bar{Q}_T$, since the uniqueness of the continuous viscosity solution is a direct consequence of this. For $\gamma > 0$ and $(x,t) \in \bar{Q}_T$ we consider

$$\bar{u}^\gamma(x,t) = \sup_{s \in [0,T]} \{\bar{u}(x,s) - \gamma^{-1}|s-t|^2\},$$

and present some well-known properties for this regularization. Since $\bar{u}$ is u.s.c., for each $(x,t) \in \bar{Q}_T$, there exists $\tilde{s}$ depending on x, t and γ such that $\bar{u}^\gamma(x,t) = u(x,\tilde{s}) - \gamma^{-1}|t-\tilde{s}|^2$ and from here, noticing that $\bar{u} \leqslant \bar{u}^\gamma$, it is possible to conclude that $|t-\tilde{s}| \leqslant 2|u|_\infty \sqrt{\gamma}$. Using again the u.s.c. of $\bar{u}$, we see that $\bar{u}^\gamma \to \bar{u}$ as $\gamma \to 0$ locally uniformly in $\bar{Q}_T$.

In particular, we see that for all x we can write

$$\bar{u}^\gamma(x,t) - u_0(x) \leqslant \bar{u}(x,\tilde{s}) - u_0(x) \leqslant \omega_0(t + 2|u|_\infty \sqrt{\gamma}),$$

where ω_0 comes from (3.15), and therefore, that for all $d > 0$ small enough, there exists γ small in terms of d such that

$$\bar{u}^\gamma(x,t) - u_0(x) \leqslant \omega(d), \quad \text{for all } (x,t) \in \mathbb{R}^N \times [0,d],$$

where ω is a modulus of continuity.

At this point, we consider $d > 0$ fixed and define

$$\bar{w}(x,t) := \bar{u}^\gamma(x,t) - 2(\omega(d) + \omega_0(d)). \tag{3.16}$$

Then it is easy to see that

$$\bar{w} \leqslant \underline{u}, \quad \text{in } \mathbb{R}^N \times [0, d/2].$$

On the other hand, standard arguments concerning sup-convolutions lead us to prove that $\bar{w}$ solves

$$w_t + F(x, Dw, \mathcal{I}(w,x)) \leqslant 0 \quad \text{in } \mathbb{R}^N \times (a_\gamma, T],$$

where $a_\gamma > 0$ is such that $a_\gamma \to 0$ as $\gamma \to 0$.

Then we get that for every fixed $d > 0$, there exists $\gamma(d) > 0$ such that for $\gamma < \gamma(d)$, $a_\gamma < d/4$ and $\bar{w} \leqslant \underline{u}$ in $\mathbb{R}^N \times [0, d/2]$. This implies that $\bar{w}, \underline{u}$ are respectively a sub and a supersolution to (2.7) in $\mathbb{R}^N \times [2a_\gamma, T]$ with $\bar{w} \leqslant \underline{u}$ in $\mathbb{R}^N \times [2a_\gamma, d/2]$. Our aim is to apply proposition 3.1 to conclude that $\bar{w} \leqslant \underline{u}$ in $\mathbb{R}^N \times [2a_\gamma, T]$ for all γ small enough. We need to prove that $\bar{w}$ is in $C^\alpha(\mathbb{R}^N \times [2a_\gamma, T])$ for some α sufficiently close to 1.

By definition the function $t \mapsto \bar{u}^\gamma(x,t)$ (and so also $\bar{w}$) is Lipschitz continuous in $[0, T]$, uniformly in x with Lipschitz constant proportional to γ^{-1}. Then $\bar{w}_t$ is bounded by a constant C proportional to γ^{-1} and $\bar{w}$ is a viscosity subsolution to $F(x, Dw, \mathcal{I}w) - C \leqslant 0$ in $\mathbb{R}^N \times [2a_\gamma, T]$. A standard argument in viscosity solution theory, see [**7**, lemma II.5.17], gives that for all t, the function $\bar{w}(\cdot, t)$ is a viscosity subsolution to $F(x, Dw, \mathcal{I}w) - C \leqslant 0$ in $\mathbb{R}^N$. We sketch briefly the argument for completeness. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ such that x is a strict maximum point of $\bar{w}(\cdot, t) - \phi$. We can assume without loss of generality that $\phi \geqslant 1$. We define $\psi_\epsilon(y, s) = \phi(y)(1 + |t - s|^2/\epsilon)$ for $\epsilon > 0$. Let (x_ϵ, t_ϵ) be a maximum point of $\bar{w} - \psi_\epsilon$ in $\mathbb{R}^n \times [2a_\gamma, T]$. Then $(\bar{w} - \psi_\epsilon)(x_\epsilon, t_\epsilon) \geqslant (\bar{w} - \psi_\epsilon)(x_\epsilon, t)$ implies that $|t - t_\epsilon| \leqslant C\epsilon$, where C is the Lipschitz constant of $\bar{w}$ (with respect to time). Using this we get that $(x_\epsilon, t_\epsilon) \to (x, t)$ as $\epsilon \to 0$, and $D\psi_\epsilon(x_\epsilon, t_\epsilon) \to D\phi(x)$. Then the continuity properties of the operator imply $F(x, D\phi(x), \mathcal{I}(\phi, x)) \leqslant C$.

Therefore, we can use the Hölder estimates in [**11**, theorem 2.2] to obtain C^α estimates for $\bar{w}(\cdot, t)$. In fact, if $\sigma = 1$, for each $\alpha \in (0, 1)$, there exists C depending on α and γ such that

$$|\bar{w}(x,s) - \bar{w}(y,t)| \leqslant C(|s-t| + |x-y|^\alpha), \quad \text{for } x, y \in \mathbb{R}^N,\ s, t \in [2a_\gamma, T].$$

If $\sigma < 1$, then there exists $C > 0$ depending on σ and γ such that

$$|\bar{w}(x,s) - \bar{w}(y,t)| \leqslant C(|s-t| + |x-y|), \quad \text{for } x, y \in \mathbb{R}^N,\ s, t \in [2a_\gamma, T].$$

Therefore, in both cases, we fulfil the requirements of proposition 3.1, which allows us to conclude that $\bar{w} \leqslant \underline{u}$ in $\mathbb{R}^N \times (2a_\gamma, T]$ for all γ small enough. Then

$$\bar{u}^\gamma \leqslant \underline{u} + 2(\omega(d) + \omega_0(d)) \quad \text{in } \mathbb{R}^N \times (2a_\gamma, T],$$

which implies, taking $\gamma \to 0$ that $\bar{u} \leqslant \underline{u} + 2(\omega(d) + \omega_0(d))$ in $\bar{Q}_T$. Since $d > 0$ is arbitrary, we arrive to $\bar{u} \leqslant \underline{u}$ in $\bar{Q}_T$. □

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4. The cell problems for the homogenization

We consider the formal asymptotic expansion (1.7) and we plug it in equation (1.1) to get the effective operator, through the solution of the so called cell problem.

We introduce some notation. We will denote $y = x/\epsilon$, $p = D\bar{u}(x,t)$, $c = -\bar{u}_t(x,t)$ and

$$l = \mathcal{I}(\bar{u}(\cdot,t),x) = \int_{\mathbb{R}^N} [\bar{u}(x+z,t) - \bar{u}(x,t) - \mathbf{1}_B(z)\langle D\bar{u}(x,t),z\rangle] K^\sigma(z)\mathrm{d}z.$$

Moreover we denote $\psi_\epsilon(x) = \psi(x/\epsilon)$ and for $e > 0$ we introduce the notation

$$\delta_e(v,x,z) = v(x+z) - v(x) - \mathbf{1}_{B_e}(z)\langle Dv(x),z\rangle, \tag{4.1}$$

where $\mathbf{1}_{B_e} = \mathbf{1}_{B_e}^{(\sigma)}$ denotes the indicator function of B_e, the open ball centred at the origin with radius e if $\sigma \geqslant 1$, and the zero function if $\sigma < 1$.

Plugging the formal asymptotic expansion (1.7) into equation (1.1), we obtain

$$-a(x,y)l - a(x,y)\epsilon^{1\vee\sigma}\mathcal{I}(\psi_\epsilon,x) + H(x,y,p+\epsilon^{0\vee(\sigma-1)}D\psi(y)) = c. \tag{4.2}$$

Performing the change of variables $\xi = z/\epsilon$ we get that

$$\mathcal{I}(\psi_\epsilon,x) = \epsilon^N \int_{\mathbb{R}^N} \delta_{\epsilon^{-1}}(\psi,x/\epsilon,\xi)K^\sigma(\epsilon\xi)\mathrm{d}\xi.$$

Using assumption (**E**) and (2.3) we obtain

$$\mathcal{I}(\psi_\epsilon,x) = \epsilon^{-\sigma}\Big(-(-\Delta)^{\sigma/2}\psi_\epsilon + J(\psi_\epsilon,x)\Big), \tag{4.3}$$

where

$$J(\psi_\epsilon,x) = \int_{\mathbb{R}^N} \delta_{\epsilon^{-1}}(\psi,x/\epsilon,\xi)\Big(\bar{k}(\epsilon\xi) - \bar{k}(0)\Big)|\xi|^{-(N+\sigma)}\mathrm{d}\xi. \tag{4.4}$$

We prove now the following claim:

$$\|J(\psi_\epsilon,x)\|_\infty = \begin{cases} o_\epsilon(1) & \sigma \in (0,2), \sigma \neq 1 \\ \langle b, D\psi(x/\epsilon)\rangle + o_\epsilon(1), & \sigma = 1, \end{cases} \tag{4.5}$$

where

$$b := \lim_{\rho\to 0}\int_{B\setminus B_\rho} \frac{(\bar{k}(z) - \bar{k}(0))}{|z|^{N+1}} z\mathrm{d}z \in \mathbb{R}^N, \tag{4.6}$$

and where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ only depends on $N, \sigma, C^{\sigma+\alpha}$ estimates of ψ, $\alpha > 0$ and $\bar{\omega}$ in (2.2) when $\sigma = 1$. Note that if $\bar{k}$ (and then K^1) is symmetric, then $b = 0$. This means that the nonlocal term develops an extra drift term when the kernel defining it is nonsymmetric and satisfies the integrability condition (2.2) with respect to the kernel of the square root of the Laplacian.

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To prove the claim, we introduce some notation. For $A \subseteq \mathbb{R}^N$ measurable we write

$$J[A] = \int_A \delta_{\epsilon^{-1}}(\psi, x/\epsilon, \xi)(\bar{k}(\epsilon\xi) - \bar{k}(0))|\xi|^{-(N+\sigma)}\mathrm{d}\xi.$$

Then, we split J in (4.4) as

$$J = J[B] + J[B_{1/\epsilon} \backslash B] + J[B^c_{1/\epsilon}],$$

and we estimate each term separately.

For $J[B]$ we perform a second-order Taylor expansion for ψ in the integral term and using that $\bar{k}(\epsilon\xi) - \bar{k}(0) \leqslant \bar{\omega}(\epsilon)$ for $\xi \in B$ together with the fact that $\sigma < 2$ we arrive at

$$|J[B]| \leqslant \frac{1}{2}\bar{\omega}(\epsilon)|D^2\psi|_\infty \int_B |\xi|^{-N-\sigma+2}\mathrm{d}\xi \leqslant C|D^2\psi|_\infty \bar{\omega}(\epsilon),$$

for some constant $C = C(N, \sigma) > 0$ not depending on ϵ.

For $J[B^c_{1/\epsilon}]$ we notice that the compensator term $\mathbf{1}_{B_{\epsilon^{-1}}}(z)\langle Du(x), z\rangle$ is no longer present in the integral and therefore we have that

$$|J[B^c_{1/\epsilon}]| \leqslant 4|\psi|_\infty |\bar{k}|_\infty \int_{B^c_{1/\epsilon}} \frac{\mathrm{d}\xi}{|\xi|^{N+\sigma}} \leqslant C|\psi|_\infty \epsilon^\sigma.$$

It remains to estimate $J[B_{1/\epsilon} \backslash B]$, and at this point we separate the cases $\sigma \neq 1$ and $\sigma = 1$.

For the case $\sigma \neq 1$, we split the remaining integral as

$$J[B_{1/\epsilon} \backslash B] = J[B_{1/\epsilon} \backslash B_{\theta_\epsilon}] + J[B_{\theta_\epsilon} \backslash B],$$

where $\theta_\epsilon \to \infty$ and $\epsilon\theta_\epsilon \to 0$ as $\epsilon \to 0$. With this choice, we see that

$$|J[B_{\theta_\epsilon} \backslash B]| \leqslant \begin{cases} \bar{\omega}(\epsilon\theta_\epsilon)(2|\psi|_\infty + |D\psi|_\infty) \int_{B_{\theta_\epsilon} \backslash B} \frac{\mathrm{d}\xi}{|\xi|^{N+\sigma-1}} \Big) & \\ \quad \leqslant C(|\psi|_\infty + |D\psi|_\infty)\bar{\omega}(\epsilon\theta_\epsilon), & \sigma > 1 \\ \bar{\omega}(\epsilon\theta_\epsilon)2|\psi|_\infty \int_{B_{\theta_\epsilon} \backslash B} \frac{\mathrm{d}\xi}{|\xi|^{N+\sigma}} \leqslant C|\psi|_\infty \bar{\omega}(\epsilon\theta_\epsilon) + o_\epsilon(1) & \sigma < 1 \end{cases}$$

for some $C = C(N, \sigma) > 0$ not depending on ϵ.

Similarly, for $J[B_{1/\epsilon} \backslash B_{\theta_\epsilon}]$ we have

$$|J[B_{1/\epsilon} \backslash B_{\theta_\epsilon}]| \leqslant \begin{cases} 2|\bar{k}|_\infty (2|\psi|_\infty + |D\psi|_\infty) \int_{B^c_{\theta_\epsilon}} \frac{\mathrm{d}\xi}{|\xi|^{N+\sigma-1}} & \\ \quad \leqslant C(|\psi|_\infty + |D\psi|_\infty)\theta_\epsilon^{1-\sigma} & \sigma > 1 \\ 4|\bar{k}|_\infty |\psi|_\infty \int_{B^c_{\theta_\epsilon}} \frac{\mathrm{d}\xi}{|\xi|^{N+\sigma}} \leqslant C|\bar{k}|_\infty |\psi|_\infty \theta_\epsilon^{-\sigma} + o_\epsilon(1) & \sigma < 1. \end{cases}$$

Hence, joining the above estimates we conclude (4.5) if $\sigma \neq 1$.

We consider now the case $\sigma = 1$. First of all note that the estimates for $J[B]$ and $J[B^c_{1/\epsilon}]$ follow the same lines above. Moreover observe that, if $\bar{k}$ is symmetric, then

$$\int_{B_{1/\epsilon} \backslash B} \langle D\psi(x/\epsilon), z\rangle \frac{(\bar{k}(\epsilon z) - \bar{k}(0))}{|z|^{N+1}}\mathrm{d}z = 0,$$

therefore we can estimate $J[B_{1/\epsilon} \backslash B]$ exactly as in the case $\sigma < 1$.

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In the nonsymmetric case, we consider the term θ_ϵ present in the previous analysis for $\sigma < 1$ to write

$$\begin{aligned} J[B_{1/\epsilon}\backslash B] &= \int_{B_{1/\epsilon}\backslash B} [\psi(x/\epsilon+\xi) - \psi(x/\epsilon)] \frac{\bar{k}(\epsilon\xi) - \bar{k}(0)}{|\xi|^{N+1}} \mathrm{d}\xi \\ &\quad + \int_{B_{1/\epsilon}\backslash B} \langle D\psi(x/\epsilon), \xi\rangle \frac{\bar{k}(\epsilon\xi) - \bar{k}(0)}{|\xi|^{N+1}} \mathrm{d}\xi \\ &\leqslant C|\psi|_\infty (\bar{\omega}(\epsilon\theta_\epsilon) + |\bar{k}|_\infty \theta_\epsilon^{-1}) + \left\langle D\psi(x/\epsilon), \int_{B\backslash B_\epsilon} \frac{\bar{k}(z) - \bar{k}(0)}{|z|^{N+1}} z \mathrm{d}z \right\rangle, \end{aligned}$$

where in the last integral we have performed the change of variables $z = \epsilon\xi$. We observe that, by definition (2.1) and assumption (2.2),

$$\left| \frac{\bar{k}(z) - \bar{k}(0)}{|z|^{N+1}} z \right| \leqslant \frac{\bar{\omega}(|z|)}{|z|^N} \in L^1(B).$$

Hence, the dominated convergence theorem allows us to conclude (4.5). This finishes the proof of the claim.

Therefore, using (4.5) in (4.2), we conclude with different cell problems, according to the value of σ.

Case $\sigma < 1$: in this case (4.2) reads

$$-a(x,y)l + a(x,y)\epsilon^{1-\sigma}((-\Delta)^{\sigma/2}\psi(y) + o_\epsilon(1)) + H(x,y,p+D\psi(y)) = c.$$

Therefore, the cell problem is the following: for every $(x,p,l) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ there exists a unique $c = c(x,p,l)$ such that there exists a periodic viscosity solution to

$$-a(x,y)l + H(x,y,p+D\psi(y)) = c \quad y \in \mathbb{T}^N. \tag{4.7}$$

Case $\sigma > 1$: in this case (4.2) reads

$$-a(x,y)l + a(x,y)((-\Delta)^{\sigma/2}\psi(y) + o_\epsilon(1)) + H(x,y,p+\epsilon^{\sigma-1}D\psi(y)) = c.$$

Therefore, the cell problem is the following: for every $(x,p,l) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ there exists a unique $c = c(x,p,l)$ such that there exists a periodic viscosity solution to

$$-a(x,y)l + a(x,y)(-\Delta)^{\sigma/2}\psi(y) + H(x,y,p) = c \quad y \in \mathbb{T}^N. \tag{4.8}$$

Case $\sigma = 1$: in this case (4.2) reads

$$\begin{aligned} &-a(x,y)l + a(x,y)((-\Delta)^{\sigma/2}\psi(y) + \langle b, D\psi(y)\rangle + o_\epsilon(1)) \\ &\quad + H(x,y,p+D\psi(y)) = c. \end{aligned}$$

Therefore, the cell problem is the following: for every $(x,p,l) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ there exists a unique $c = c(x,p,l)$ such that there exists a periodic viscosity

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solution to

$$- a(x,y)l + a(x,y)(-\Delta)^{1/2}\psi(y) + a(x,y)\langle b, D\psi(y)\rangle \\ + H(x,y,p+D\psi(y)) = c \tag{4.9}$$

for $y \in \mathbb{T}^N$, where $b \in \mathbb{R}^N$ is defined in (4.6) (and it is identically 0 if K^1 is symmetric).

REMARK 4.1. Looking at the computations related to $J(\phi_\epsilon, x)$ made above in the case $\sigma = 1$, we see that if we consider nonlocal operators written in the *second order finite differences* form

$$\int_{\mathbb{R}^N} [u(x+z) + u(x-z) - 2u(x)]K^1(z)\mathrm{d}z$$

assumption (2.2) can be dropped.

5. Homogenization for the case $\sigma = 1$

We start studying the cell problem introduced above.

PROPOSITION 5.1 Cell problem. *Assume* (**E**) *with* $\sigma = 1$, (**H0**), (**H1**) *and* (2.6). *If* K^1 *is not symmetric, we additionally assume that condition* (2.2) *holds.*

Then, for each x, p, l, *there exists a unique constant* $c = \bar{H}(x,p,l)$ *such that the cell problem* (4.9) *has a classical solution* $\psi \in C^{1,\alpha}$ *for some* $\alpha \in (0,1)$, *and such solution is unique up to an additive constant.*

Moreover, the following estimate holds

$$|(-\Delta)^{1/2}\psi|_{L^\infty(\mathbb{T}^N)} \leqslant C(1 + |l| + |p|^m)^m, \tag{5.1}$$

where $C > 0$ *does not depend on* x, l *nor* p.

Proof. We concentrate on the case $b \neq 0$. Given $x, p \in \mathbb{R}^N$ and $l \in \mathbb{R}$, and for each $\delta \in (0,1)$ we consider the solution $\psi = \psi^\delta(y)$ for the approximating problem

$$\delta\psi - a(x,y)l + a(x,y)[(-\Delta)^{1/2}\psi - \langle b, D\psi\rangle] + H(x,y,p+D\psi) = 0, \quad y \in \mathbb{T}^N. \tag{5.2}$$

The proper term $\delta\psi$ implies the existence and uniqueness of a solution ψ^δ to this problem, and the following estimate holds

$$|\psi^\delta|_\infty \leqslant \delta^{-1}\Big(|a|_\infty|l| + |H(\cdot,\cdot,p)|_\infty\Big),$$

and in view of (2.6) and (2.5) we have the existence of a constant $C_1 > 0$ such that

$$|\psi^\delta|_\infty \leqslant C_1\delta^{-1}(1 + |l| + |p|^m). \tag{5.3}$$

Then, in view of (2.5) and the fact that $m > 1$, it is direct to see that ψ^δ satisfies, in the viscosity sense, the inequality

$$(-\Delta)^{1/2}\psi^\delta + c|D\psi^\delta|^m \leqslant C(1 + |l| + |p|^m) \quad \text{in } \mathbb{T}^N,$$

from which, by applying theorem 2.2 in [**11**], we get that ψ^δ is Hölder continuous for each exponent $\gamma \in (0,1)$. More precisely, a careful analysis of the proof shows

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that there exists a constant $C_\gamma > 0$ such that

$$|\psi^\delta(y) - \psi^\delta(y')| \leqslant C_\gamma(1 + \operatorname{osc}(\psi^\delta)^{1/m} + (|p|^m + |l|)^{1/m})|y - y'|^\gamma \quad y, y' \in \mathbb{T}^N. \tag{5.4}$$

A sketch of the proof of this estimate is provided in the appendix, lemma 7.4.

From this we deduce the existence of a constant $C > 0$ such that

$$\operatorname{osc}_{\mathbb{T}^N}(\psi^\delta) \leqslant C(1 + |p| + |l|^{1/m}). \tag{5.5}$$

At this point we claim that under the assumptions of the proposition together with (5.3) and (5.5) we get the Lipschitz bound

$$|\psi^\delta(y) - \psi^\delta(y')| \leqslant C(1 + |l| + |p|^m)|y - y'|, \tag{5.6}$$

for some $C > 0$ not depending on δ, x, p or l. This claim is a consequence of theorem 3.1 in [**12**], but we provide a proof in the appendix (lemma 7.3) for completeness.

The application of the above boundedness/regularity results in the periodic setting leads us to the solvability of the cell problem (4.9) by stability results of viscosity solutions by taking $\delta \to 0$. The ergodic constant is characterized as the uniform limit $\lambda = -\lim_{\delta\to 0} \delta\psi^\delta$. The uniqueness of the ergodic constant is achieved as in [**22**] by comparison principle. The uniqueness up to constants of the corrector is a consequence of the strong maximum principle [**11**, proposition 4.1], by a classical argument of Arisawa and Lions, see, e.g. [**2**].

We devote the rest of the proof to get the $C^{1,\alpha}$ regularity. This is a consequence of a 'linearization' argument which is possible by the Lipschitz estimates given by (5.6). In fact, for a fixed $e \in \mathbb{R}^N$ with $|e| > 0$ we define the function

$$v_e(y) = (\psi(y + e) - \psi(y))/|e|.$$

Notice that by (5.6) this function v_e is bounded, with

$$|v_e|_\infty \leqslant C(1 + l + |p|^m). \tag{5.7}$$

In what follows we derive an equation solved by v_e. Using (5.6) together with (2.6) we get the existence of $C > 0$ such that

$$\begin{aligned}|a^{-1}&(x, y + e)H(x, y + e, p + D\psi(y + e)) - a^{-1}(x, y)H(x, y, p + D\psi(y))| \\ &\leqslant C(1 + l + |p|^m)^m|e| + C(1 + l + |p|^m)^{m-1}|D\psi_e(y)|,\end{aligned}$$

where $a^{-1}(x, y) = 1/a(x, y)$ and $\psi_e(y) = \psi(y + e) - \psi(y)$.

Using this estimate, the linearity of the fractional Laplacian, the assumptions on the data, and the uniform bounds on v_e, we conclude that v_e satisfies, in the viscosity sense

$$\begin{aligned}(-\Delta)^{1/2}v_e - A(p, l)|Dv_e| &\leqslant C(p, l), \\ (-\Delta)^{1/2}v_e + A(p, l)|Dv_e| &\geqslant -C(p, l).\end{aligned}$$

for some $A(p, l), C(p, l) > 0$ depending on the parameters p, l and the data, but not on e. From here, we use theorem 6.1 in [**29**] (stated for parabolic problems, but

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easily adapted to the stationary case), or the appendix in [19], to conclude the existence of $\alpha > 0$ (small, depending on the data and $A(p,l), C(p,l)$ but not on e) such that $v_e \in C^{\alpha}$. This concludes the $C^{1,\alpha}$ regularity for the solution of ψ.

Finally, we notice that the Lipschitz bound (5.6) is inherited by ψ via uniform convergence. We use this into the pointwise inequality

$$|(-\Delta)^{1/2}\psi(y)| \leqslant \lambda + C(1 + |l| + |D\psi(y)| + |H_x(y, p + D\psi)|),$$

which leads to (5.1) using (2.5) and (5.3). This concludes the proof. □

Now we present some properties of the effective Hamiltonian. The proof is a straightforward adaptation to the corresponding effective properties given in [21].

Lemma 5.2. *Let $\bar{H}$ be the effective Hamiltonian associated to (4.9). Then*

(i) *There exists $C > 0$ just depending on the data such that*

$$|\bar{H}(x_1, p_1, l_1) - \bar{H}(x_2, p_2, l_2)| \leqslant C\Big(|l_1 - l_2| + |x_1 - x_2|(1 + |l| + |p|^m)^m + |p_1 - p_2|(1 + |l| + |p|^m)^{m-1}\Big),$$

where $|p| = \max\{|p_1|, |p_2|\}$, $|l| = \max\{|l_1|, |l_2|\}$.

(ii) *There exists $b_0, C > 0$ such that for all $x, p \in \mathbb{R}^N, l \in \mathbb{R}$*

$$\bar{H}(x, p, l) \geqslant b_0|p|^m - |a|_\infty|l| - C.$$

(iii) *For all $x, p \in \mathbb{R}^N$, the function $l \mapsto \bar{H}(x, p, l)$ is decreasing.*

Proof. (*i*) Let $x_1, x_2, p_1, p_2 \in \mathbb{R}^N$ and $l_1, l_2 \in \mathbb{R}$ and for $\delta > 0$ and $i = 1, 2$ consider the approximating problems

$$\delta\psi_i - a_i(y)l_i + a_i(y)(-\Delta)^{1/2}\psi_i + H_i(y, p_i + D\psi_i) = 0 \quad \text{in } \mathbb{T}^N,$$

where, with a slight abuse of notation we have written $a_i(y) = a(x_i, y)$ and $H_i(y, p_i + D\psi_i) = H(x_i, y, p_i + D\psi_i)$. Then, we use the equation solved by ψ_2 and assumptions (2.6) and (**H1**) to write

$$\begin{aligned} &\delta\psi_2 - a_1(-\Delta)^{1/2}\psi_2 + H_1(y, p_1 + D\psi_2) \\ &\quad \leqslant C|l_1 - l_2| + C|x_1 - x_2|\Big(1 + |l| + |(-\Delta)^{1/2}\psi|_\infty + L_H(|p|^m + |D\psi_2|^m_\infty)\Big) \\ &\qquad + L_H|p_1 - p_2|(1 + |p|^{m-1} + |D\psi_2|^{m-1}_\infty), \end{aligned}$$

and from this, using the Lipschitz bound (5.6) and the fractional estimate (5.1) we arrive at

$$\begin{aligned} &\delta\psi_2 - a_1(-\Delta)^{1/2}\psi_2 + H_1(y, p_1 + D\psi_2) \\ &\quad \leqslant C\Big(|l_1 - l_2| + |x_1 - x_2|(1 + |l| + |p|^m)^m + L_H|p_1 - p_2|(1 + |l| + |p|^m)^{m-1}\Big), \end{aligned} \tag{5.8}$$

for some $C > 0$ just depending on the data.

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From here, by comparison it is possible to get that

$$\delta(\psi_2^\delta - \psi_1^\delta) \leqslant C(|l_1 - l_2| + |x_1 - x_2|(1 + |l| + |p|^m)^m + |p_1 - p_2|(1 + |l| + |p|^m)^{m-1}),$$

and a similar lower bound can be obtained. Letting $\delta \to 0^+$ and recalling the definition of $\bar{H}$ we conclude the result.

(*ii*) We consider $\delta > 0$ and the approximating problem (5.2). Then, we consider $y_0 \in \mathbb{T}^N$ a maximum point to ψ^δ and using a constant function as a test function to ψ^δ at y_0 we can write

$$\delta\psi^\delta(y_0) - a(x, y_0)l + H(x, y_0, p) \leqslant 0,$$

and using the boundedness of a and coercivity of H we get that

$$-C - |a|_\infty |l| + b_0|p|^m \leqslant -\delta\psi^\delta(y_0),$$

for some C, b_0 depending on H. Thus, recalling that $\delta\psi^\delta(y_0) \to -\bar{H}(x, p, l)$ as $\delta \to 0^+$, we conclude the result taking the limit in the right-side of the last inequality.

(*iii*) We fix x, p, consider $l_1 < l_2$ and assume by contradiction that

$$\bar{H}(x, p, l_1) < \bar{H}(x, p, l_2). \tag{5.9}$$

For $i = 1, 2$, let ψ_i solution to the cell problem

$$-a(x, y)l_i + a(x, y)(-\Delta)^{1/2}\psi_i + H(x, y, p + D\psi_i) = \bar{H}(x, p, l_i), \quad y \in \mathbb{T}^N.$$

We can assume without loss of generality that $\psi_2 < \psi_1$.

Next we claim that ψ_2 satisfies the inequality

$$-a(x, y)l_1 + a(x, y)(-\Delta)^{1/2}\psi_2 + H(x, y, p + D\psi_2) > \bar{H}(x, p, l_1) \tag{5.10}$$

in the viscosity sense. For this, we take $y_0 \in \mathbb{T}^N$ and consider ϕ bounded and smooth such that y_0 is a minimum point for $\psi_2 - \phi$ in $\mathbb{T}^N$. Then, using the equation solved by ψ_2 we get

$$-a(x, y_0)l_2 + a(x, y_0)(-\Delta)^{1/2}\phi(y_0) + H(x, y_0, p + D\phi(y_0)) \geqslant \bar{H}(x, p, l_2).$$

Then, using (5.9), that $l_2 > l_1$ and the nonnegativeness of a we arrive at

$$-a(x, y_0)l_1 + a(x, y_0)(-\Delta)^{1/2}\phi(y_0) + H(x, y_0, p + D\phi(y_0)) > \bar{H}(x, p, l_1),$$

from which the claim follows. The strict inequality in (5.10) allows us to compare to get $\psi_2 \geqslant \psi_1$, which contradicts the assumed reverse inequality. This concludes the proof. □

At this point we present the main result of this section

THEOREM 5.3 Homogenization. *Under the assumptions of proposition 5.1 and for $u_0 \in BUC(\mathbb{R}^N)$, the family of solutions u^ϵ of (1.1)–(1.2) converges locally uniformly to a viscosity solution u of the associated effective problem (1.5) with $\bar{H}$ given in*

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proposition 5.1. Moreover u is the unique solution of (1.5) attaining uniformly continuously the initial data u_0.

Proof. Recalling proposition 2.2, we see that the family of functions $\{u^\epsilon\}_\epsilon$ is uniformly bounded in $\bar{Q}_T$. Then, by half-relaxed limits as in [**13**] we see that the functions $\bar{u} = \limsup^*_\epsilon u^\epsilon$ and $\underline{u} = \liminf^*_\epsilon u^\epsilon$ are respective viscosity sub and supersolution to the effective problem.

To see this we argue over $\bar{u}$, a similar treatment can be done for $\underline{u}$. Let $(x_0, t_0) \in Q_T$ and ϕ be a smooth function such that (x_0, t_0) is a strict global maximum point to $\bar{u} - \phi$. Then, for $x = x_0$, $p = D\phi(x_0)$ and $l = \mathcal{I}(\phi, x_0)$ let ψ be a solution to (4.9). In view of proposition 5.1 we can assume $\psi \in C^{1,\alpha}$.

By the strict maximality of x_0 , the fact that $u^\epsilon \to \bar{u}$ locally uniformly in $\mathbb{R}^N$ and the boundedness of ψ, there exists a sequence $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$, maximum point to $(x, t) \mapsto u^\epsilon(x, t) - (\phi(x, t) + \epsilon\psi(x/\epsilon))$ in the set $B_{R_\epsilon}(x_\epsilon) \times [0, T]$, with $R_\epsilon \to +\infty$ as $\epsilon \to 0$.

Then, we can use $\phi_\epsilon(x, t) = \phi(x, t) + \epsilon\psi(x/\epsilon)$ as test function for u^ϵ at (x_ϵ, t_ϵ) and denoting $y_\epsilon = x_\epsilon/\epsilon$ we can write

$$\begin{aligned} &\phi_t(x_\epsilon, t_\epsilon) - a(x_\epsilon, y_\epsilon)\mathcal{I}[B_{R_\epsilon}](\phi_\epsilon, x_\epsilon) - a(x_\epsilon, y_\epsilon)\mathcal{I}[B^c_{R_\epsilon}](u^\epsilon, x_\epsilon) \\ &\quad + H(x_\epsilon, y_\epsilon, D\phi_\epsilon(x_\epsilon, t_\epsilon)) \leqslant 0, \end{aligned} \tag{5.11}$$

where we have also used the notation introduced before. By the boundedness and smoothness of ϕ and since $R_\epsilon \to \infty$ as $\epsilon \to 0$ we see that

$$\phi_t(x_\epsilon, t_\epsilon) \to \phi_t(x_0, t_0) \quad \mathcal{I}[B_{R_\epsilon}](\phi, x_\epsilon) \to \mathcal{I}(\phi, x_0) \quad \text{as } \epsilon \to 0,$$

meanwhile, by the uniform boundedness and smoothness of ψ we can use (4.5) to conclude that

$$\epsilon\mathcal{I}[B_{R_\epsilon}](\psi(\cdot/\epsilon), x_\epsilon) + (-\Delta)^{1/2}\psi(y_\epsilon) - \langle b, D\psi(y_\epsilon)\rangle = o_\epsilon(1).$$

Plugging this into (5.11) and using the smoothness of ϕ again, and the regularity assumption (2.6) we arrive at

$$\begin{aligned} &\phi_t(x_0, t_0) - a(x_0, y_\epsilon)\mathcal{I}(\phi, x_0) + a(x_0, y_\epsilon)(-\Delta)^{1/2}\psi(y_\epsilon) - a(x_0, y_\epsilon)\langle b, D\psi(y_\epsilon)\rangle \\ &\quad + H(x_0, y_\epsilon, D\phi(x_0, t_0) + D\psi(y_\epsilon)) \leqslant o_\epsilon(1), \end{aligned}$$

and therefore

$$\phi_t(x_0, t_0) + \bar{H}(x_0, D\phi(x_0, t_0), \mathcal{I}(\phi, x_0)) \leqslant o_\epsilon(1),$$

from which we conclude that $\bar{u}$ is a viscosity subsolution of the effective problem using the continuity of $\bar{H}$ and letting $\epsilon \to 0$. Observe that $\bar{H}$ satisfies (3.1) by lemma 5.2.

By definition $\underline{u} \leqslant \bar{u}$, and moreover (2.12) implies that $\underline{u}$ and $\bar{u}$ satisfy (3.15). Therefore, using theorem 3.2 we deduce that $\underline{u} = \bar{u}$ in $\bar{Q}_T$. This concludes the proof. □

6. Homogenization in the case $\sigma < 1$

We recall that when $\sigma < 1$ the compensator term $\mathbf{1}_B(z)\langle Du(x), z\rangle$ in (1.3) is not required, so we consider in this section that

$$\mathcal{I}(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)]K(z)\mathrm{d}z.$$

Then the nonlocal operator has strictly lower order than the gradient term. In the supercritical framework given by assumption (**H1**), this leads to a dominance of the Hamiltonian term that makes the homogenization problem similar to the purely first-order case already addressed in the literature. For this reason, in the current section we mainly remark the new arguments involving the nonlocality. These features would also allow to weaken some assumptions, e.g. to consider kernels K^σ that are integrable and with a direct dependence on x, but we do not pursue these generalizations here.

Proposition 6.1 Cell problem. *Assume* (**E**) *with* $\sigma < 1$, (**H0**), (**H1**), (**H2**). *Then, for all* $x, p \in \mathbb{R}^N, l \in \mathbb{R}$ *there exists a unique constant* $c = \bar{H}(x,p,l)$ *such that problem* (4.7) *has a Lipschitz continuous viscosity solution* ψ.

As in proposition 5.1, the solvability of the cell problem is obtained as the limit as $\delta \to 0$ of $\delta\psi^\delta$ with ψ^δ solving the problem

$$\delta\psi^\delta(y) + H(x,y,p + D\psi^\delta(y)) - a(x,y)l = 0, \quad y \in \mathbb{T}^N.$$

The coercivity of H in the gradient variable leads to the equi-Lipschitz property for the family ψ^δ, see [**11**]. Since (**H0**) gives its equiboundedness, we obtain the needed compactness. From here, the proof follows classical lines.

Lemma 6.2. *Under the assumptions of proposition* 6.1, *the effective Hamiltonian* $\bar{H}$ *associated to problem* (4.7) *satisfies the property*

(i′) *there exists* $C > 0$ *just depending on the data such that*

$$|\bar{H}(x_1,p_1,l_1) - \bar{H}(x_2,p_2,l_2)| \\ \leqslant C|l_1 - l_2| + \omega(|x_1 - x_2|)(1 + |l| + |p|^m) + \omega(|p_1 - p_2|)(1 + |p|^{m-1}),$$

where $|l| = \max\{|l_1|, |l_2|\}$, $|p| = \max\{|p_1|, |p_2|\}$ *and* ω *is a modulus of continuity related to the one in* (**H2**),

as well as the properties (ii) *and* (iii) *of lemma* 5.2.

Proof. We concentrate on (i') to provide explicit bounds. The proof of (ii) and (iii) follow as in lemma 5.2.

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Let $x_1, x_2, p_1, p_2 \in \mathbb{R}^N$ and $l_1, l_2 \in \mathbb{R}$ and for $\delta > 0$ and $i = 1, 2$ consider the approximating problems

$$\delta\psi_i - a(x_i, y)l_i + H(x_i, y, p_i + D\psi_i) = 0 \quad \text{in } \mathbb{T}^N.$$

We use the equation solved by ψ_2, (**H2**), the uniform continuity of a, and the known Lipschitz continuity of ψ_2 [**7**, **8**], to write

$$\begin{aligned} &\delta\psi_2 - a(x_1, y)l_1 + H(x_1, y, p_1 + D\psi_2) \\ &\quad \leqslant C|l_1 - l_2| + \omega(|x_1 - x_2|)(|l| + 1 + (|p| + |D\psi_2|_\infty)^m) \\ &\qquad + \omega(|p_1 - p_2|)\Big(1 + (|p| + |D\psi_2|_\infty)^{m-1}\Big), \end{aligned}$$

where ω is the maximum between the modulus of continuity of a and the modulus appearing in (**H2**). Moreover, condition (**H1**) implies that $|D\psi_2|_\infty \leqslant C|p|^{1/m}$ for some $C > 0$ just depending on the data. From here, we arrive at

$$\begin{aligned} &\delta\psi_2 - a(x_1, y)l_1 + H(x_1, y, p_1 + D\psi_2) \\ &\quad \leqslant C|l_1 - l_2| + \omega(|x_1 - x_2|)(1 + |l| + |p|^{m\vee 1}) + \omega(|p_1 - p_2|)\Big(1 + |p|^{(m-1)\vee 0}\Big), \end{aligned}$$

and therefore, by the comparison principle, we get the existence of $C > 0$ just depending on the data such that

$$\begin{aligned} \delta(\psi_2^\delta - \psi_1^\delta) &\leqslant C|l_1 - l_2| + \omega(|x_1 - x_2|)(1 + |l| + |p|^{m\vee 1}) \\ &\quad + \omega(|p_1 - p_2|)(1 + |p|^{(m-1)\vee 0}). \end{aligned}$$

A similar lower bound can be obtained. Letting $\delta \to 0^+$ and considering the definition of $\bar{H}$ we conclude the result. □

Now we are in position to prove the homogenization result for this case.

THEOREM 6.3 Homogenization. *Under the assumptions of proposition 6.1 and for $u_0 \in BUC(\mathbb{R}^N)$, the family of solutions $\{u^\epsilon\}$ of (1.1)–(1.2) converges locally uniformly to a viscosity solution u of the associated effective problem (1.5) with $\bar{H}$ given in proposition 6.1. Moreover u is the unique solution of (1.5) attaining uniformly continuously the initial data u_0.*

Proof. As in the proof of theorem 5.3, we consider the half-relaxed semilimits $\bar{u}, \underline{u}$. We are able to prove that $\bar{u}, \underline{u}$ are respective viscosity sub and supersolution to the effective problem, the main difference being that ϕ_ϵ cannot be used directly as a test function because ψ is just Lipschitz continuous. Anyway a standard argument by contradiction based on viscosity solution theory (see [**1**, **22**]) can be used to make it rigorous.

The uniqueness of the limit problem comes from theorem 3.2, observing that, by lemma 6.2 and the property $\omega(r) \leqslant Cr$ for all $r \geqslant 1$, the effective operator $\bar{H}$ satisfies (3.1) (possibly with a different modulus ω). □

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7. Homogenization in the case $\sigma > 1$

In this section, we deal with the case $\sigma \in (1, 2)$. Let us mention that the stronger ellipticity nature of this case would allow to weaken some assumptions, e.g. to consider non-coercive Hamiltonians H, but we do not pursue these generalizations here.

The solvability of the cell problem now reads as follows.

PROPOSITION 7.1 Cell problem. *Assume* (**E**) *with* $1 < \sigma < 2$, (**H0**), (**H1**) *and* (**H2**). *Then, for each* x, p, l *there exists a constant* $c = \bar{H}(x, p, l)$ *such that the cell problem* (4.8) *has a classical solution* $C^{1,\alpha}$ *with* $1 + \alpha > \sigma$, *and such solution is unique up to additive constants.*

Moreover, we have the following characterization of the effective Hamiltonian $\bar{H}$:

$$\bar{H}(x, p, l) = -A(x)l + \int_{\mathbb{T}^N} \frac{H(x, y, p)}{a(x, y)} \mathrm{d}y, \quad \textit{for } x, p \in \mathbb{R}^N,\ l \in \mathbb{R}, \tag{7.1}$$

where $A(x) := 1/(\int_{\mathbb{T}^N} 1/(a(x, y))\mathrm{d}y)$.

Proof. Fixed x, p, l, for each $\delta > 0$ we consider the vanishing discount approximation of (4.8)

$$\delta\psi - a(x, y)l + a(x, y)(-\Delta)^{\sigma/2}\psi(y) + H(x, y, p) = 0, \quad y \in \mathbb{T}^N,$$

which can be uniquely solved by a function ψ^δ such that $\delta\psi^\delta$ is bounded. Then, we define the function $\tilde{\psi}^\delta(y) = \psi^\delta(y) - \psi^\delta(0)$ and claim that it is uniformly bounded. The argument is known (see for instance [**9**], sublinear case), but we provide a sketch of the proof for completeness. By contradiction, if $\tilde{\psi}^\delta$ is not bounded, up to subsequences we can consider $|\tilde{\psi}^\delta|_\infty \to \infty$ as $\delta \to 0$ and from here we define $v^\delta = \tilde{\psi}^\delta / |\tilde{\psi}^\delta|_\infty$. By construction, $|v^\delta|_\infty = 1$ for all δ and satisfies, in the viscosity sense, a problem with the form

$$-C(\delta) \leqslant (-\Delta)^{\sigma/2} v^\delta \leqslant C(\delta) \quad \text{in } \mathbb{T}^N,$$

for some constant $C(\delta) \to 0$ as $\delta \to \infty$. Then, by the interior Hölder estimates presented in [**14**] we conclude that the family $\{v^\delta\}$ is equi-Hölder continuous. By stability results in the viscosity theory, and up to subsequences, there exists a function $\bar{v}$ such that $v^\delta \to \bar{v}$ uniformly in the torus, solving the problem $(-\Delta)^{\sigma/2}\bar{v} = 0$ in $\mathbb{T}^n$. Thus, by strong maximum principle, it must be a constant. However, by construction $\bar{v}(0) = 0$ and $|\bar{v}|_\infty = 1$, a contradiction.

Then, using stability results over the family $\{\tilde{\psi}^\delta\}$ we get the existence of a constant c such that (4.8) has a continuous solution (which ends up to be classical by the regularity results in [**28**]). Applying a strong maximum principle in [**17**] we conclude this constant is unique and the solution of the problem is unique up to an additive constant.

Finally, the characterization of the effective Hamiltonian is obtained writing (4.8) as

$$(-\Delta)^{\sigma/2}\psi = a^{-1}(x, y)(\bar{H}(x, p, l) - H(x, y, p)) + l =: f(x, y, p, l).$$

Since the fractional Laplacian is a self adjoint operator and by the strong maximum principle we have that the unique solutions to $(-\Delta)^{\sigma/2}u = 0$ in $\mathbb{T}^N$ are

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constants. By Fredholm alternative the above problem is solvable if and only if

$$\int_{\mathbb{T}^N} f(x, y, p, l) \mathrm{d}y = 0,$$

from which the characterization of $\bar{H}$ follows. □

The above characterization of the effective Hamiltonian allows us to conclude the homogenization result more directly.

THEOREM 7.2 Homogenization. *Under the assumptions of proposition 7.1 and for $u_0 \in BUC(\mathbb{R}^N)$, the family of solutions u^ϵ to problem (1.1)–(1.2) converges locally uniformly to the unique viscosity solution u of the associated effective problem (1.5), with $\bar{H}$ given in proposition 7.1, satisfying $u(x, 0) = u_0(x)$.*

Proof. Also in this case, the proof of the convergence of the family follows the lines provided in theorem 5.3. Now the uniqueness of the effective problem follows at once from the comparison principle in [**10**], noticing that the term A in (7.1) is bounded and uniformly positive and therefore we can divide by it to get rid of the x-dependence of the nonlocality. We omit the details.

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Appendix

We start providing a proof for the Lipschitz bounds leading to (5.6) in the proof of proposition 5.1.

LEMMA 7.3. *Let $\delta \in (0, 1)$, $\tilde{a} \in C(\mathbb{T}^N)$ strictly positive, and $\tilde{H} \in C(\mathbb{T}^N \times \mathbb{R}^N)$ satisfying the assumptions (**H1**) (in the x independent setting). For p, l fixed, let ψ be a continuous solution to the problem*

$$\delta \tilde{a} \psi - l + (-\Delta)^{1/2} \psi + \tilde{H}(y, p + D\psi) = 0 \quad \text{in } \mathbb{T}^N.$$

Then there exists a constant $C > 0$ depending only on the data such that

$$|\psi(x) - \psi(y)| \leqslant C(1 + \mathrm{osc}(\psi) + |l| + |p|^m)|x - y|, \quad \text{for } x, y \in \mathbb{T}^N.$$

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Proof. We follow closely the lines of theorem 3.1 in [12]. We start noticing that by comparison principle, ψ satisfies

$$\delta|\psi|_\infty \leqslant C(1 + |l| + |\tilde{H}(\cdot, p)|_\infty) \leqslant C(1 + |l| + |p|^m),$$

for some $C > 0$ depending on the data.

Now, replacing ψ by $\psi - \inf_{\mathbb{T}^N} \psi + 1$ we can assume $\psi \geqslant 1$ at the expense of deal with the modified problem

$$\delta\tilde{a}\psi - l + (-\Delta)^{1/2}\psi + \tilde{H}(y, p + D\psi) = \tilde{f} \quad \text{in } \mathbb{T}^N,$$

where $\tilde{f}(y) = \theta\tilde{a}(y)$ with $\theta \in \mathbb{R}$ satisfying $|\theta| \leqslant C(1 + |l| + |p|^m)$.

Then, we introduce the change of variables $\psi = e^v$, from which we conclude that v solves the problem

$$\delta\tilde{a} - le^{-v} - J(v, x) + e^{-v}\tilde{H}(y, p + e^v D\psi) = \tilde{f}e^{-v} \quad \text{in } \mathbb{T}^N, \tag{7.2}$$

where J is a nonlinear nonlocal operator with the form

$$J(v, x) = \int_{\mathbb{R}^N} [e^{v(x+z)-v(x)} - 1 - \mathbf{1}_B\langle Dv(x), z\rangle]|z|^{-(N+1)}\mathrm{d}z.$$

Then, for $L > 0$ we consider the function

$$(x, y) \mapsto \Phi(x, y) = v(x) - v(y) - L|x - y|, \quad x, y \in \mathbb{T}^N$$

which attains its maximum at a point $(\bar{x}, \bar{y})$. We prove that for L large enough this maximum is nonpositive from which the result follows.

By contradiction, we assume $\Phi(\bar{x}, \bar{y}) > 0$, from which $\bar{x} \neq \bar{y}$. Then we can use $\bar{x}$ as test point for v (regarded as subsolution to (7.2)) with test function $x \mapsto L|x - \bar{y}|$, and $\bar{y}$ as test point for v (regarded as supersolution to (7.2)) with test function $y \mapsto -L|\bar{x} - y|$. Subtracting the viscosity inequalities and using the maximality of $(\bar{x}, \bar{y})$ together with the definition of J to control the nonlocal terms, we arrive at

$$-\delta|\tilde{a}(\bar{x}) - \tilde{a}(\bar{y})| - l(e^{-v(\bar{x})} - e^{-v(\bar{y})}) + \mathcal{H} \leqslant e^{-v(\bar{x})}\tilde{f}(\bar{x}) - e^{-v(\bar{y})}\tilde{f}(\bar{y}),$$

where

$$\mathcal{H} = e^{-v(\bar{x})}\tilde{H}(\bar{x}, p + Le^{v(\bar{x})}\hat{a}) - e^{-v(\bar{x})}\tilde{H}(\bar{x}, p + Le^{v(\bar{x})}\hat{a}),$$

and $\hat{a} = (x - y)/|x - y|$.

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From now on we denote $\mu = e^{v(y)-v(x)} \in (0,1)$. By the assumptions, the last inequality lead us to

$$- L_{\tilde{a}}|\bar{x}-\bar{y}| - l^+ e^{-v(y)}(1-\mu) + \mathcal{H} \leqslant e^{-v(y)}(|\tilde{f}|(1-\mu) + L_{\tilde{f}}|x-y|), \tag{7.3}$$

From here we focus on $\mathcal{H}$. Notice that

$$\mathcal{H} = e^{-v(y)}\Big(\mu \tilde{H}(x, p+\mu^{-1}L\tilde{p}) - \tilde{H}(y, p+L\tilde{p})\Big), \quad \tilde{p} = e^{v(\bar{y})}\hat{a}.$$

In view of the assumption on $\tilde{H}$ we see that

$$\begin{aligned}\mathcal{H} \geqslant -Ce^{-v(y)}\Big\{ &- L_H(1+|p+L\mu^{-1}\tilde{p}|^m)|x-y| \\ &- L_H(1+|p+L\tilde{p}|^{m-1})|(1-\mu)|p| + (1-\mu)(c|\mu p + L\tilde{p}|^m - C)\Big\}\end{aligned}$$

If we assume that $L \geqslant \max\{1, 4|p|, l^+, |\tilde{f}|_\infty, Cc^{-1}\}$ we can write

$$\mathcal{H} \geqslant (1-\mu)cL^m|\tilde{p}|^m,$$

for some constants $C, c > 0$. Hence, (7.3) reduces to

$$-L_{\tilde{a}}|\bar{x}-\bar{y}| + c(1-\mu)e^{-v(y)}L^m|\tilde{p}|^m \leqslant e^{-v(y)}L_{\tilde{f}}|x-y|.$$

At this point, we notice that the maximality of (x,y) we see that $L|x-y| \leqslant v(x) - v(y)$, which in turn implies that $|x-y| \leqslant L^{-1}\mathrm{osc}(v)$. Then, considering additionally L large enough in terms of $\mathrm{osc}(v)$ ($L \geqslant 2\mathrm{osc}(v)$), by definition of μ we can conclude that

$$1-\mu \geqslant 1 - e^{-L|x-y|} \geqslant e^{-\mathrm{osc}(v)}L|x-y|.$$

Using this and cancelling the common factor $|x-y| > 0$ in the last inequality, and using the definition of $\tilde{p}$ we arrive at

$$-L_{\tilde{a}} + ce^{-osc(v)}e^{(m-1)v(y)}L^{m+1} \leqslant e^{-v(y)}L_{\tilde{f}}.$$

Then, since $m > 1$ we get that additionally assuming that

$$L \geqslant C\max\{c^{-1}e^{osc(v)}, L_a, L_f\}$$

for a large universal constant $C > 1$ we arrive at a contradiction. Finally, recalling the relation $\psi = e^v$ we notice that

$$e^{osc(v)} = \frac{e^{\sup v}}{e^{\inf v}} = \frac{\sup\psi - \inf\psi}{e^{\inf v}} + 1 \leqslant osc(\psi) + 1,$$

from which the dependence on the oscillation is obtained. □

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Next, we provide a sketch of the proof of (5.4), presented as the following

LEMMA 7.4. *Let $c_0, C_0 > 0$, $m > 1$ and u be a bounded, upper semicontinuous viscosity solution to the problem*

$$(-\Delta)^{1/2}u + c_0|Du|^m \leqslant C_0 \quad \text{in } \mathbb{T}^N.$$

Then, for each $\gamma \in (0,1)$, there exists a constant $C_\gamma > 0$ just depending on γ, m, N and c_0 such that

$$|u(x) - u(y)| \leqslant C_\gamma\Big(1 + (\mathrm{osc}(u) + C_0)^{1/m}\Big)|x-y|^\gamma \quad \forall\, x, y \in \mathbb{T}^N.$$

Proof. Fix $\gamma \in (0,1)$ and let $x_0 \in \mathbb{T}^N$. We look for a constant $L > 0$ large enough, not depending on x_0, such that

$$u(x) - u(x_0) \leqslant L|x - x_0|^\gamma \quad \text{for all } x \in \mathbb{T}^N.$$

We proceed by contradiction. Then, for every $L > 0$ there exists $\theta_L > 0$ and $\bar{x} \in \mathbb{T}^N, \bar{x} \neq x_0$ such that

$$u(\bar{x}) - u(x_0) - L|\bar{x} - x_0|^\gamma = \max_{x \in \mathbb{T}^N}\{u(x) - u(x_0) - L|x - x_0|^\gamma\} \geqslant \theta_L.$$

Now, we observe that we can use the function $\phi(x) = L|x - x_0|^\gamma$ as test function for u at $\bar{x}$. Actually we fix $\delta_0 < |\bar{x} - x_0|$ and we consider a smooth function ϕ_0 which coincides with ϕ in $B(\bar{x}, \delta_0)$. So $u - \phi_0$ has a maximum at $\bar{x}$ in $B_{\delta_0}(\bar{x})$ and recalling Definition 2.1, we get that for any $0 < \delta < \min(1, \delta_0)$, there holds

$$-\mathcal{I}[B_\delta](\phi, \bar{x}) - \mathcal{I}[B_\delta^c](u, \bar{x}) + c_0\gamma^m L^m|\bar{x} - x_0|^{m(\gamma-1)} \leqslant C_0, \tag{7.4}$$

where $-\mathcal{I} = (-\Delta)^{1/2}$, and $\mathcal{I}[B_\delta](\phi, \bar{x})$ and $\mathcal{I}[B_\delta^c](u, \bar{x})$ have been defined in (2.8) and (2.10).

Using the fact that $\bar{x}$ is a maximum point to $u - \phi$ we can write

$$\mathcal{I}[B_\delta^c](u, \bar{x}) \leqslant \mathcal{I}[B \backslash B_\delta](\phi, \bar{x}) + \mathcal{I}[B^c](u, \bar{x}).$$

Now it is easy to check that $\mathcal{I}[B^c](u, \bar{x}) \leqslant C\mathrm{osc}(u)$ for some universal constant $C > 0$. Moreover, recalling the definition of $(-\Delta)^{1/2}$ in (1.4), of ϕ and δ, we get that for any $\delta \in (0, \min(1, \delta_0))$,

$$\begin{aligned}\mathcal{I}[B \backslash B_\delta](\phi, \bar{x}) &= LC_{N,1}\int_{\delta<|z|<1}[|\bar{x} + z - x_0|^\gamma - |\bar{x} - x_0|^\gamma]|z|^{-N-1}\mathrm{d}z \\ &\leqslant LC_{N,1}\int_{\delta<|z|<1}|z|^\gamma|z|^{-N-1}\mathrm{d}z \\ &\leqslant CL(\delta^{\gamma-1} - 1) \leqslant CL(|\bar{x} - x_0|^{\gamma-1} - 1)\end{aligned}$$

for some constant $C > 0$, depending on N and γ. On the other hand, if we fix $\delta = |\bar{x} - x_0|/2$, we observe that there exists a constant $C_0 > 0$ depending only on

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γ and N such that for every $z \in B_\delta$ we get

$$\begin{aligned}\phi(\bar{x}+z) - \phi(\bar{x}) - \langle D\phi(\bar{x}), z\rangle &= \frac{1}{2}\int_0^1 (1-t)\langle D^2\phi(\bar{x}+tz)z, z\rangle \mathrm{d}t \\ &\leqslant C_0 L|\bar{x}-x_0|^{\gamma-2}|z|^2.\end{aligned}$$

Therefore, we conclude that

$$\mathcal{I}[B_\delta](\phi, \bar{x}) \leqslant C_{N,1}C_0 L|\bar{x}-x_0|^{\gamma-2}\int_{B_\delta}|z|^2|z|^{-(N+1)}\mathrm{d}z \leqslant CL|\bar{x}-x_0|^{\gamma-1}.$$

for some constant $C>0$ depending only on N, γ.

Joining the above estimates into (7.4) we can write

$$c_0\gamma^m L^m|\bar{x}-x_0|^{m(\gamma-1)}\Big(1 - \frac{C}{c_0}\gamma^{-m}(L|\bar{x}-x_0|^{\gamma-1})^{1-m}\Big) \leqslant C\mathrm{osc}(u) + C_0,$$

and since we can assume $|\bar{x}-x_0| \leqslant \sqrt{N}$ together with the fact that $m>1$, we arrive at

$$c_0\gamma^m L^m|\bar{x}-x_0|^{m(\gamma-1)}\Big(1 - \frac{C}{c_0}\gamma^{-m}(LN^{(\gamma-1)/2})^{1-m}\Big) \leqslant C\mathrm{osc}(u) + C_0.$$

Thus, taking L large enough in terms of c_0, N, γ and m we arrive at

$$c_0\gamma^m L^m|\bar{x}-x_0|^{m(\gamma-1)} \leqslant 2(C\mathrm{osc}(u) + C_0),$$

from which we arrive at a contradiction by taking L sufficiently large in terms of C_0/c_0. □

We finish with the proof of the following

Claim: Conditions (**H0**), (**H1**) *and* (**H2**) *imply* (2.5).

By uniform continuity of H, see assumption (**H2**), from (**H0**) we can get (2.5) in the case of p bounded, by taking $K>0$ large enough. Therefore, we take $K>0$ large enough such that (2.5) holds for all $|p| \leqslant 2$. Thus, from here we concentrate on the case of $|p|>2$.

Consider $q \in \mathbb{R}^N$, $q \neq 0$. Now, for $\mu \in (0,1)$ and $k \in \mathbb{N}$, applying (**H1**) with $p = \mu^{-(k-1)}q$, we have

$$\mu H(x,y,\mu^{-k}q) - H(x,y,\mu^{-(k-1)}q) \geqslant (1-\mu)\Big(b_0\mu^{-m(k-1)}|q|^m - C_0\Big).$$

We multiply the above inequality by μ^{k-1} and sum it up from $k=1$ to n for some $n \in \mathbb{N}$, and we conclude that

$$\begin{aligned}\mu^n H(x,y,\mu^{-n}q) - H(x,y,q) &= \sum_{k=1}^n \Big[\mu^k H(x,y,\mu^{-k}q) - \mu^{k-1}H(x,y,\mu^{-(k-1)}q)\Big] \\ &\geqslant \sum_{k=1}^n \Big[(1-\mu)\Big(b_0\mu^{(1-m)(k-1)}|q|^m - C_0\mu^{k-1}\Big)\Big] \\ &= (1-\mu)b_0|q|^m\frac{\mu^{n(1-m)}-1}{\mu^{1-m}-1} - C_0(1-\mu^n). \qquad (7.5)\end{aligned}$$

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Let us fix $|p| > 2$ and let $n \in \mathbb{N}$ such that $2^n \leqslant |p| \leqslant 2^{n+1}$. Let $\mu = |p|^{-1/n} < 1$. Note that by our choice of n, $\mu \in [1/4, 1/2]$. Then, from (7.5), applied to $q = p/|p|$ and to μ and n as above, so that $p = \mu^{-n}q$, we get that

$$|p|^{-1}H(x,y,p) - H(x,y,q) \geqslant (1-\mu)b_0\frac{|p|^{m-1}-1}{\mu^{1-m}-1} - C_0\left(1-|p|^{-1}\right). \tag{7.6}$$

Observe that $1/2 \leqslant 1-\mu \leqslant 3/4$ and

$$\frac{1}{1/4^{1-m}-1} \leqslant \frac{1}{\mu^{1-m}-1} \leqslant \frac{1}{1/2^{1-m}-1}.$$

So there exist constants $c_m, C_m > 0$ depending only on $m > 1$ such that

$$c_m \leqslant (1-\mu)\frac{1}{\mu^{1-m}-1} \leqslant C_m.$$

Therefore from (7.6) we get

$$H(x,y,p) \geqslant b_0 c_m |p|^m - b_0 C_m |p| + |p| H(x,y,q) - C_0(|p|-1).$$

By (2.4) we have that $H(x,y,q) \geqslant -C$ for all $q \in \mathbb{R}^N$ with $|q| = 1$, therefore, we conclude that

$$H(x,y,p) \geqslant b_0 c_m |p|^m - |p|(b_0 C_m + C + C_0) + C_0 \quad \forall |p| > 2.$$

Therefore, recalling that $m > 1$, we conclude that there exist $\tilde{C} > 0$, and $K > 0$, depending on m, b_0, C_m, c_m, C, C_0 such that

$$H(x,y,p) \geqslant \tilde{C}(|p|^m + 1) - K \quad \forall |p| > 2. \qquad \square$$

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