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On a family of torsional creep problems in Finsler metrics

Maria Fărcăşeanu, Mihai Mihăilescu, and Denisa Stancu-Dumitru

Abstract. The asymptotic behavior of solutions to a family of Dirichlet boundary value problems, involving differential operators in divergence form, on a domain equipped with a Finsler metric is investigated. Solutions are shown to converge uniformly to the distance function to the boundary of the domain, which takes into account the Finsler norm involved in the equation. This implies that a well-known result in the analysis of problems modeling torsional creep continues to hold in this more general setting.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain, $H : \mathbb{R}^N \to [0, \infty)$ be a Finsler norm (see Section 2.1 for details), and $\alpha : \overline{\Omega} \times \mathbb{R} \to (0, \infty)$ be a continuous function for which there exist two positive constants, λ and Λ, such that

$$0 < \lambda \leq \alpha(x,t) \leq \Lambda < +\infty, \quad \forall\, x \in \overline{\Omega},\ \forall\, t \in \mathbb{R}. \tag{1.1}$$

For each real number $p \in (1, \infty)$, we consider the following problem

$$\begin{cases} -\mathrm{div}(\alpha(x,u) H(\nabla u)^{p-2} \mathcal{H}(\nabla u)) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $f : \overline{\Omega} \to (0, \infty)$ is a given continuous function and $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\mathcal{H}_i(\xi) := \frac{\partial}{\partial \xi_i}\left(\frac{1}{2} H(\xi)^2\right), \quad \forall\, \xi \in \mathbb{R}^N,\ \forall\, i \in \{1, \ldots, N\}. \tag{1.3}$$

In the particular case, when $\alpha(x,t)$ and $f(x)$ are positive constant functions and $H(\cdot) = |\cdot|$ is the Euclidean norm on $\mathbb{R}^N$ equation (1.2) reduces to the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = c, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

with c as a positive constant. Equation (1.4) serves as a model for the so-called *torsional creep problem,* which in the case when $N = 2$, has been proposed to describe the behavior under torsion of a prismatic bar with cross section $\Omega \subset \mathbb{R}^2$ for an extended

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period of time at high temperature, which is assumed to be constant (see, e.g., Bhattacharya et al. [7, p. 59] for details regarding the torsional creep problem based on some models proposed by Kachanov [22, 23]). The analysis of the limiting case when $p \to \infty$ is of particular interest in applications because it can model the *perfect plastic torsion*. More precisely, denoting by u_p the unique solution of problem (1.4) it can be proved that the family of solutions converges uniformly over $\overline{\Omega}$, as $p \to \infty$, to the distance function to the boundary of Ω, i.e., $\delta(x) := \inf_{y \in \partial\Omega} |x - y|$, for each $x \in \Omega$ (see Bhattacharya et al. [7] or Kawohl [24]). Similar results were obtained in the recent years when there were published a series of studies concerning the investigation of different generalizations of the classical torsional creep problem to the case of different types of inhomogeneous differential equations, e.g., Pérez-Llanos and Rossi [28], Bocea and Mihăilescu [9], Fărcăşeanu and Mihăilescu [17], and Mihăilescu and Pérez-Llanos [26].

Taking into account the above remarks, we can regard the general case of problem (1.2) as an extension of the classical model to the situation when we deal with an anisotropic material or with the case when the Euclidean distance in Ω is distorted due to the presence of the Finsler norm (see Belloni and Kawohl [5] or Belloni et al. [6] for similar interpretations of the use of Finsler norms). Moreover, the motivation of the presence of a nonconstant function $\alpha(x, u)$, depending on u, in the divergence operator involved in problem (1.2) is to take into account the reaction of this equation to its own state (see, e.g., Chipot [11, p. 160]). In particular, the temperature which was assumed to be constant in Kachanov's model may vary in this new situation. Our goals in this general setting will be, first, to show the existence of a solution of equation (1.2) for each $p > 1$, and, next, to prove the uniform convergence of the family of solutions, as $p \to \infty$, to a distance function to the boundary of Ω, which takes into account the Finsler norm involved in the equation, i.e., $\delta_H(x) := \inf_{y \in \partial\Omega} H^0(x - y)$, for each $x \in \Omega$ (note H^0 stands for the dual norm of H; the definition of H^0 is provided in Section 2.1). In particular, our results complement the works of Ishibashi and Koike [19], Belloni and Kawohl [5], Di Castro et al. [16], and Bianchini and Ciraolo [8].

The notion of solution for equation (1.2) will be understood in the *weak* sense. More precisely, we work under the following definition.

***Definition* 1.1** *We say that u_p is a weak solution of problem (1.2) if $u_p \in W_0^{1,p}(\Omega)$ and it satisfies the following relation*

$$(1.5) \qquad \int_\Omega \alpha(x, u_p) H(\nabla u_p)^{p-2} \langle \mathcal{H}(\nabla u_p), \nabla \varphi \rangle \, dx = \int_\Omega f \varphi \, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega).$$

The main results of this paper are given by the following theorems.

***Theorem* 1.1** *Assume that condition (1.1) is fulfilled. Then, for each $p \in (1, \infty)$, problem (1.2) has a weak solution $u_p \in W_0^{1,p}(\Omega)$ such that $u_p(x) \geq 0$ for a.e. $x \in \Omega$.*

***Theorem* 1.2** *Assume that condition (1.1) is fulfilled. Let $\{p_n\}_n \subset (N, \infty)$ be a sequence of real numbers satisfying $\lim_{n\to\infty} p_n = \infty$. For each $n > 1$ denote by $u_{p_n} \in W_0^{1,p_n}(\Omega)$ the weak, nonnegative solution of problem (1.2) with $p = p_n$ given by* Theorem 1.1. *Then, the sequence $\{u_{p_n}\}_n$ converges uniformly in Ω to the distance function to the boundary of domain Ω given by $\delta_H(x) := \inf_{y \in \partial\Omega} H^0(x - y)$, for each $x \in \Omega$.*

The rest of the paper is organized as follows: In Section 2, we present some auxiliary results regarding the Finsler norms and some known facts on eigenvalue problems involving Finsler norms. Sections 3 and 4 are devoted to the proofs of the main results: Theorems 1.1 and 1.2, respectively.

2 Auxiliary results

The goal of this section is to present, on the one hand, the definition and the main properties of the Finsler norms and, on the other hand, to recall some known results regarding eigenvalue problems involving Finsler norms.

2.1 Finsler norms: definition, properties, and examples

Let $H : \mathbb{R}^N \to [0, \infty)$ be a convex function of class $C^2(\mathbb{R}^N \setminus \{0\})$, even and homogeneous of degree 1, i.e.,

$$H(t\xi) = |t| H(\xi), \quad \forall\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N,$$

such that H^2 is strongly convex (in the sense that $D^2[H^2](\xi)$ is positive definite for $\xi \in \mathbb{R}^N \setminus \{0\}$). We will refer to H as being a *Finsler norm*.

Set

$$K := \{x \in \mathbb{R}^N :\ H(x) \le 1\}$$

and

$$H^\circ(x) := \sup_{\xi \in K} \langle x, \xi \rangle .$$

We will refer to H° as being the support function of K. It is easy to check that $H^\circ : \mathbb{R}^N \to [0, \infty)$ is a convex homogeneous function, and, actually, a Finsler norm, too. We will call H and H° polar to each other in the sense

$$H^\circ(x) := \sup_{\xi \ne 0} \frac{\langle x, \xi \rangle}{H(\xi)}$$

and

$$H(x) := \sup_{\xi \ne 0} \frac{\langle x, \xi \rangle}{H^\circ(\xi)} .$$

The above relations yield

$$|\langle x, \xi \rangle| \le H(x) H^\circ(\xi), \quad \forall\ x,\ \xi \in \mathbb{R}^N . \tag{2.1}$$

Examples of Finsler norms.

1) The Euclidean norm: $H(x) = |x| = \left(\sum_{i=1}^N |x_i|^2\right)^{1/2}$;
2) $H(x) = \langle Ax, x \rangle$, where A is a symmetric, positive definite $N \times N$ matrix;
3) The p-norm $H(x) = \left(\sum_{i=1}^N |x_i|^p\right)^{1/p}$, with $p \in (1, \infty)$;
4) $H(x) = \sqrt{\sqrt{x_1^4 + \cdots + x_N^4} + x_1^2 + \cdots + x_N^2}$.

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We recall some important properties regarding functions H and H° that will be useful in our subsequent analysis

$$H(\nabla_\xi H^\circ(\xi)) = 1 \quad \text{and} \quad H^\circ(\nabla_x H(x)) = 1\,, \tag{2.2}$$

$$\langle \nabla_x H(x), x\rangle = H(x) \quad \textit{and} \quad \langle \nabla_\xi H^\circ(\xi), \xi\rangle = H^\circ(\xi), \quad \forall\ x \ \text{ and } \xi \in \mathbb{R}^N\,. \tag{2.3}$$

We refer to [3] for the proofs of the above relations and to [12, p. 352] for some similar relations obtained in the case when H is a more general Finsler norm.

Furthermore, let us also recall the so-called *fundamental inequality* regarding Finsler norms, namely for each $x \in \mathbb{R}^N$ we have

$$\langle \xi, \nabla H(x)\rangle \le H(\xi), \quad \forall\ \xi \neq 0\,, \tag{2.4}$$

and equality holds if and only if $x = \alpha\xi$ for some $\alpha \ge 0$ (see [1, Theorem 1.2.2, relation (1.2.3)] for more details).

Since any two norms are equivalent on $\mathbb{R}^N$, we infer that for H defined as above there exist two positive constants, a and b, such that

$$a|x| \le H(x) \le b|x|, \quad \forall\ x \in \mathbb{R}^N \tag{2.5}$$

(see, e.g., [4] or [2]).

Using a Finsler norm, we can define for each real number $p \in (1, \infty)$ a differential operator that generalizes the classical p-Laplacian, namely

$$Q_p u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[H(\nabla u)^{p-2}\mathcal{H}_i(\nabla u)\right],$$

where $\mathcal{H}$ was defined in relation (1.3). Note that Q_p is a particular case of the differential operator involved in equation (1.2), which is obtained in the particular case when $\alpha(x,u) = 1$. Moreover, it is useful to observe that $\mathcal{H}(\xi) = H(\xi)\nabla H(\xi)$.

2.2 Eigenvalue problems involving Finsler norms

It is known (see, e.g., Belloni et al. [4] or Belloni et al. [6]) that for each real number $p \in (1, \infty)$ the minimum of the Rayleigh quotient associated to the eigenvalue problem

$$\begin{cases} -Q_p v = \lambda |v|^{p-2} v & \text{if } x \in \Omega \\ v = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{2.6}$$

i.e.,

$$\lambda_1(p) := \inf_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega H(\nabla v)^p\, dx}{\int_\Omega |v|^p\, dx} > 0\,,$$

stands for the lowest eigenvalue of problem (2.6), whose corresponding eigenfunctions are minimizers of $\lambda_1(p)$ that do not change sign in Ω. Moreover, for $p > 1$, a minimizer is C^1-Hölder continuous.

In particular, for each $p > 1$, we have

$$\int_\Omega H(\nabla v)^p\, dx \ge \lambda_1(p) \int_\Omega |v|^p\, dx, \ \ \forall\ v \in W_0^{1,p}(\Omega)\,. \tag{2.7}$$

Furthermore, define the distance function to the boundary of Ω with respect to the dual of the Finsler norm H, i.e., $\delta_H : \overline{\Omega} \to [0, \infty)$ given by

$$\delta_H(x) := \inf_{y \in \partial\Omega} H^0(x - y), \quad \forall\, x \in \Omega .$$

Recall that δ_H is Lipschitz continuous and satisfies $H(\nabla \delta_H(x)) = 1$ for a.e. $x \in \Omega$ (see, e.g., [6, Section 3] or [25] for a more involved discussion regarding the distance function to the boundary in Finsler metrics). Define also

$$\Lambda_\infty := \frac{\|H(\nabla \delta_H)\|_{L^\infty(\Omega)}}{\|\delta_H\|_{L^\infty(\Omega)}} = \|\delta_H\|^{-1}_{L^\infty(\Omega)} .$$

By [6, Lemma 3.1] we know that

$$\lim_{p\to\infty} (\lambda_1(p))^{1/p} = \Lambda_\infty . \tag{2.8}$$

Note that, in the particular case, when we work with the Euclidean norm this result was obtained by Juutinen et al. [21] and Fukagai et al. [18].

3 Proof of Theorem 1.1

Let $p \in (1, \infty)$ be an arbitrary but fixed real number. We start by establishing some auxiliary results that will be useful in obtaining the conclusion of Theorem 1.1.

***Lemma* 3.1** *For each $v \in L^p(\Omega)$, problem*

$$\begin{cases} -div(\alpha(x, v) H(\nabla u)^{p-2} \mathcal{H}(\nabla u)) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{3.1}$$

has a unique weak solution $u \in W_0^{1,p}(\Omega)$, i.e.,

$$\int_\Omega \alpha(x, v) H(\nabla u)^{p-2} \langle \mathcal{H}(\nabla u), \nabla \varphi \rangle \, dx = \int_\Omega f \varphi \, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \tag{3.2}$$

which satisfies $u \geq 0$ a.e. in Ω.

Proof Step 1: *Existence.* Fix $v \in L^p(\Omega)$. By hypotheses (1.1) we get $\alpha(x, v) \in L^\infty(\Omega)$.

Consider the energy functional associated to problem (3.1), $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p} \int_\Omega \alpha(x, v) H(\nabla u)^p \, dx - \int_\Omega f u \, dx.$$

Standard arguments imply that $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ with the derivative given by

$$\langle J'(u), \varphi \rangle = \int_\Omega \alpha(x, v) H(\nabla u)^{p-2} \langle \mathcal{H}(\nabla u), \nabla \varphi \rangle \, dx - \int_\Omega f \varphi \, dx, \ \ \forall\, u, \varphi \in W_0^{1,p}(\Omega) .$$

Thus, the weak solutions of problem (3.1) are exactly the critical points of J.

By relations (1.1), (2.5), and (2.7), as well as using Hölder's inequality, we deduce that for each $p \in (1, +\infty)$ and $u \in W_0^{1,p}(\Omega)$, we have

$$J(u) \geq \frac{\lambda a^p}{p} \int_\Omega |\nabla u|^p \, dx - \|f\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)}$$
$$\geq \frac{\lambda a^p}{p} \|u\|^p_{W_0^{1,p}(\Omega)} - \|f\|_{L^{p'}(\Omega)} (\lambda_1(p))^{-1/p} b \ \|u\|_{W_0^{1,p}(\Omega)},$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p. The above estimates show that J is coercive. On the other hand, it is standard to check that J is weakly lower semicontinuous. Then, the Direct Method in the Calculus of Variations (see, e.g., [29, Theorem 1.2]) guarantees the existence of a global minimum point of J, say $u \in W_0^{1,p}(\Omega)$. It is also standard to prove that u is a weak solution of problem (3.1).

Step 2: *Uniqueness*. Assume there are two weak solutions of problem (3.1), say $u_1, u_2 \in W_0^{1,p}(\Omega)$, which means that u_1 and u_2 are critical points for functional J. Standard regularity theory of elliptic operators assures that $u_1, u_2 \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0,1)$. Define

$$\Omega^+ := \{x \in \Omega : u_1(x) > u_2(x)\}.$$

The continuity of u_1 and u_2 assures that Ω^+ is an open subset of Ω. Since $\langle J'(u_1), \varphi\rangle = 0$ and $\langle J'(u_2), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega)$, working with the extension of $(u_1 - u_2)^+$ to Ω by zero outside Ω^+ as a test function it follows that

$$\int_{\Omega^+} \alpha(x,\nu) H(\nabla u_1)^{p-2} \langle \mathcal{H}(\nabla u_1), \nabla(u_1 - u_2)\rangle \, dx - \int_{\Omega^+} f(u_1 - u_2) \, dx = 0$$

and

$$\int_{\Omega^+} \alpha(x,\nu) H(\nabla u_2)^{p-2} \langle \mathcal{H}(\nabla u_2), \nabla(u_1 - u_2)\rangle \, dx - \int_{\Omega^+} f(u_1 - u_2) \, dx = 0.$$

Subtracting these two equalities term by term, we obtain

$$\int_{\Omega^+} \alpha(x,\nu) \langle H(\nabla u_1)^{p-2} \mathcal{H}(\nabla u_1) - H(\nabla u_2)^{p-2} \mathcal{H}(\nabla u_2), \nabla u_1 - \nabla u_2\rangle \, dx = 0.$$

By the strict convexity of the mapping $\mathbb{R}^N \ni \xi \to H^p(\xi)$ we have $\nabla u_1(x) = \nabla u_2(x)$ for a.e. $x \in \Omega^+$. Since $u_1 = u_2$ on $\partial\Omega^+$ we find that Ω^+ has measure zero. Similarly, the set $\Omega^- := \{x \in \Omega : u_1(x) < u_2(x)\}$ has measure zero, which yields $u_1 = u_2$.

Step 3: *Nonnegativity*. Finally, note that since $J(u) \geq J(|u|)$, for all $u \in W_0^{1,p}(\Omega)$ and J possesses a unique critical point, we must have $u = |u| \geq 0$ a.e. in Ω. The proof of Lemma 3.1 is complete. ∎

Next, for each $v \in L^p(\Omega)$ let $u = T(v) \in W_0^{1,p}(\Omega) \subset L^p(\Omega)$ be the unique weak solution of problem (3.1) given by Lemma 3.1. Thus, we can actually introduce an application

$$T : L^p(\Omega) \to W_0^{1,p}(\Omega)$$

associating to each $v \in L^p(\Omega)$ the unique weak solution of problem (3.1) denoted by $T(v) \in W_0^{1,p}(\Omega)$.

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***Lemma* 3.2** *There exists an universal constant* $\mathcal{C}_p > 0$, *which does not depend on* v, *such that*

$$\int_\Omega H(\nabla T(v))^p \, dx \le \mathcal{C}_p, \quad \forall\, v \in L^p(\Omega). \tag{3.3}$$

Proof Since $T(v)$ is a weak solution of problem (3.1), taking $\varphi = T(v)$ in (3.2) we find

$$\int_\Omega \alpha(x,v) H(\nabla T(v))^{p-2} \langle \mathcal{H}(\nabla T(v)), \nabla T(v) \rangle \, dx = \int_\Omega f\, T(v) \, dx .$$

Using relations (1.1) and (2.3), Hölder's inequality and (2.7), we deduce

$$\begin{aligned} \lambda \int_\Omega H(\nabla T(v))^p \, dx &\le \|f\|_{L^\infty(\Omega)} \int_\Omega T(v) \, dx \\ &\le \|f\|_{L^\infty(\Omega)} \, |\Omega|^{(p-1)/p} \|T(v)\|_{L^p(\Omega)} \\ &\le \|f\|_{L^\infty(\Omega)} \, |\Omega|^{(p-1)/p} \left(\frac{\int_\Omega H(\nabla T(v))^p \, dx}{\lambda_1(p)} \right)^{1/p} , \end{aligned}$$

where $|\Omega|$ stands for the Lebesgue measure of domain Ω. Thus, we have

$$\left(\int_\Omega H(\nabla T(v))^p \, dx \right)^{(p-1)/p} \le \frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \, |\Omega|^{(p-1)/p} \frac{1}{\lambda_1(p)^{1/p}} .$$

Taking

$$\mathcal{C}_p := \left(\frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \right)^{p/(p-1)} |\Omega| \, \frac{1}{\lambda_1(p)^{1/(p-1)}},$$

we obtain inequality (3.3). The proof of Lemma 3.2 is complete. ∎

***Remark* 3.3** Using Lemma 3.2 and inequality (2.7), it follows that there exists a positive constant, $\mathcal{D}_p$, such that

$$\int_\Omega |T(v)|^p \, dx \le \left(\frac{\|f\|_{L^\infty(\Omega)}}{\lambda} \right)^{p/(p-1)} |\Omega| \, \frac{1}{\lambda_1(p)^{p/(p-1)}} := \mathcal{D}_p, \quad \forall\, v \in L^p(\Omega) .$$

Next, we point out an auxiliary result that will be used in establishing our next lemma.

***Remark* 3.4** Let $q \in (1,\infty)$ and denote by $q' := q/(q-1)$ its Hölder conjugate. Assume that $\eta_n \to \eta$ in $L^q(\Omega)$ and $\psi_n \rightharpoonup \psi$ in $L^{q'}(\Omega)$. Then,

$$\lim_{n\to\infty} \int_\Omega \eta_n (\psi_n - \psi) \, dx = 0 \tag{3.4}$$

and

$$\lim_{n\to\infty} \int_\Omega \eta_n \psi_n \, dx = \int_\Omega \eta \psi \, dx . \tag{3.5}$$

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Proof First, we note that simple estimates combined with Hölder's inequality imply

$$\left|\int_\Omega \eta_n(\psi_n-\psi)\,dx\right| = \left|\int_\Omega (\eta_n-\eta)(\psi_n-\psi)\,dx + \int_\Omega \eta(\psi_n-\psi)\,dx\right|$$
$$\le \|\eta_n-\eta\|_{L^q(\Omega)}\|\psi_n-\psi\|_{L^{q'}(\Omega)} + \left|\int_\Omega \eta(\psi_n-\psi)\,dx\right|.$$

Next, using Riesz's representation theorem of linear and continuous maps on the Lebesgue spaces and the hypothesis, we deduce that relation (3.4) holds true. Furthermore, similar estimates yield

$$\left|\int_\Omega (\eta_n\psi_n-\eta\psi)\,dx\right| = \left|\int_\Omega \eta_n(\psi_n-\psi)\,dx + \int_\Omega (\eta_n-\eta)\psi\,dx\right|$$
$$\le \left|\int_\Omega \eta_n(\psi_n-\psi)\,dx\right| + \|\eta_n-\eta\|_{L^q(\Omega)}\|\psi\|_{L^{q'}(\Omega)}.$$

Combining the above pieces of information, we deduce that relation (3.5) holds true, too. ■

***Lemma* 3.5** *The map $T : L^p(\Omega) \to W_0^{1,p}(\Omega)$ is continuous.*

Proof Let $\{v_n\} \subset L^p(\Omega)$ and $v \in L^p(\Omega)$ be such that $\{v_n\}$ converges to v in $L^p(\Omega)$. It follows that, passing eventually to a subsequence, $v_n(x)$ converges to $v(x)$ for a.e. $x \in \Omega$.

Set $u_n := T(v_n)$ for any positive integer n. By Lemma 3.2 and relation (2.5), we infer

$$\int_\Omega |\nabla u_n|^p\,dx = \int_\Omega |\nabla T(v_n)|^p\,dx \le \frac{1}{a^p}\int_\Omega H(\nabla T(v_n))^p\,dx \le \frac{\mathcal{C}_p}{a^p},\quad \forall\, n,$$

that is, the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. It follows that there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $W_0^{1,p}(\Omega)$ and $\{u_n\}$ converges strongly to u in $L^p(\Omega)$. On the other hand, we have that u_n is a weak solution of problem (3.1) and thus by (3.2), we get

(3.6)
$$\int_\Omega \alpha(x,v_n)H(\nabla u_n)^{p-2}\langle \mathcal{H}(\nabla u_n),\nabla\varphi\rangle\,dx = \int_\Omega f\varphi\,dx,\quad \forall\,\varphi \in W_0^{1,p}(\Omega),\quad \forall\, n.$$

Testing with $\varphi = u_n - u$ in (3.6) and taking into account the above pieces of information, we find

$$\lim_{n\to\infty}\int_\Omega \alpha(x,v_n)H(\nabla u_n)^{p-2}\langle \mathcal{H}(\nabla u_n),\nabla u_n-\nabla u\rangle\,dx = 0. \tag{3.7}$$

Next, since $v_n(x) \to v(x)$ for a.e. $x \in \Omega$ and α is continuous on $\overline{\Omega}\times\mathbb{R}$, we deduce that $\alpha(x,v_n(x)) \to \alpha(x,v(x))$ for a.e. $x \in \Omega$. Using that fact and Lebesgues's dominated convergence theorem, we infer that

$$\alpha(x,v_n)H(\nabla u)^{p-2}\mathcal{H}(\nabla u) \to \alpha(x,v)H(\nabla u)^{p-2}\mathcal{H}(\nabla u)\quad \text{in } (L^{p'}(\Omega))^N,$$

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where $p' := p/(p-1)$. On the other hand, we know that $\nabla u_n \rightharpoonup \nabla u$ in $(L^p(\Omega))^N$. Then, using relation (3.4), from Remark 3.4, we get

$$\lim_{n\to\infty}\int_\Omega \alpha(x,v_n)H(\nabla u)^{p-2}\langle \mathcal{H}(\nabla u), \nabla u_n - \nabla u\rangle\, dx = 0\,. \tag{3.8}$$

Combining (3.7) and (3.8), we find

$$\lim_{n\to\infty}\int_\Omega \alpha(x,v_n)\langle H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n) - H(\nabla u)^{p-2}\mathcal{H}(\nabla u), \nabla u_n - \nabla u\rangle\, dx = 0\,.$$

In particular, $\alpha(x,v_n)\langle H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n) - H(\nabla u)^{p-2}\mathcal{H}(\nabla u), \nabla u_n - \nabla u\rangle \geq 0$ and it converges to 0 in $L^1(\Omega)$. Thus, up to a subsequence, $\alpha(x,v_n)\langle H(\nabla u_n)^{p-2}$ $\mathcal{H}(\nabla u_n) - H(\nabla u)^{p-2}\mathcal{H}(\nabla u), \nabla u_n - \nabla u\rangle \to 0$ a.e. in Ω. Therefore, $\nabla u_n \to \nabla u$ a.e. in Ω.

Using the fact that $\nabla u_n \to \nabla u$ a.e. in Ω it follows that

$$H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n) \to H(\nabla u)^{p-2}\mathcal{H}(\nabla u) \quad \text{a.e in } \Omega\,.$$

On the other hand, using (2.2), we find that

$$H^\circ(H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n)) = H(\nabla u_n)^{p-1}\,,$$

which shows that the sequence $\{H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n)\}$ is bounded in $(L^{p'}(\Omega))^N$. Using the above pieces of information and [30, Theorem 10.36] we deduce that

$$H(\nabla u_n)^{p-2}\mathcal{H}(\nabla u_n) \rightharpoonup H(\nabla u)^{p-2}\mathcal{H}(\nabla u) \quad \text{in } (L^{p'}(\Omega))^N\,.$$

This last weak convergence combined with relation (3.5) from Remark 3.4 allows us to pass to the limit as $n\to\infty$ in relation (3.6) and to obtain

$$\int_\Omega \alpha(x,v)H(\nabla u)^{p-2}\langle \mathcal{H}(\nabla u), \nabla\varphi\rangle\, dx = \int_\Omega f\varphi\, dx, \quad \forall\ \varphi \in W_0^{1,p}(\Omega)\,. \tag{3.9}$$

Thus, u is actually the unique weak solution of equation (3.1). Since the possible limit of the sequence u_n is uniquely determined, the whole sequence u_n converges toward u in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$. In order to end the proof of this lemma, it remains to establish the strong convergence of u_n to u in $W_0^{1,p}(\Omega)$. With that end in view, let us observe first, that the convexity of the mapping $\mathbb{R}^N \ni \xi \to H^p(\xi)$ yields

$$\begin{aligned}\int_\Omega \alpha(x,v_n)H(\nabla u)^p\, dx &\geq \int_\Omega \alpha(x,v_n)H(\nabla u_n)^p\, dx\\ &+ p\int_\Omega \alpha(x,v_n)H(\nabla u_n)^{p-2}\langle \mathcal{H}(\nabla u_n), \nabla u_n - \nabla u\rangle\, dx, \quad \forall\ n\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \alpha(x,v_n)H(\nabla u_n)^p\, dx &\geq \int_\Omega \alpha(x,v_n)H(\nabla u)^p\, dx\\ &+ p\int_\Omega \alpha(x,v_n)H(\nabla u)^{p-2}\langle \mathcal{H}(\nabla u), \nabla u - \nabla u_n\rangle\, dx, \quad \forall\ n\,.\end{aligned}$$

These estimates and relations (3.7) and (3.8) imply

$$\liminf_{n\to\infty}\int_\Omega \alpha(x,v_n)H(\nabla u)^p\, dx \geq \limsup_{n\to\infty}\int_\Omega \alpha(x,v_n)H(\nabla u_n)^p\, dx$$

and

$$\liminf_{n\to\infty} \int_\Omega \alpha(x, v_n) H(\nabla u_n)^p \, dx \geq \limsup_{n\to\infty} \int_\Omega \alpha(x, v_n) H(\nabla u)^p \, dx .$$

The last two relations show that

$$\lim_{n\to\infty} \int_\Omega \alpha(x, v_n) H(\nabla u_n)^p \, dx = \int_\Omega \alpha(x, v) H(\nabla u)^p \, dx . \tag{3.10}$$

Consider now the sequence $\{g_n\}_n$ in $L^1(\Omega)$ defined pointwise in Ω by

$$\begin{aligned} g_n(x) := \; & \frac{\alpha(x, v_n(x)) H(\nabla u_n(x))^p + \alpha(x, v_n(x)) H(\nabla u(x))^p}{2} \\ & - \alpha(x, v_n(x)) H\left(\frac{\nabla u_n(x) - \nabla u(x)}{2}\right)^p \geq 0 . \end{aligned}$$

It is clear that $g_n \to \alpha(x, v) H(\nabla u)^p$ a.e. in Ω. Then, by Fatou's Lemma and (3.10), we have

$$\begin{aligned} \int_\Omega \alpha(x, v) H(\nabla u)^p \, dx & \leq \liminf_{n\to\infty} \int_\Omega g_n \, dx \\ & = \int_\Omega \alpha(x, v) H(\nabla u)^p \, dx \\ & \quad - \limsup_{n\to\infty} \int_\Omega \alpha(x, v_n) H\left(\frac{\nabla u_n - \nabla u}{2}\right)^p \, dx . \end{aligned}$$

We conclude that $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$, which means application T is continuous.

The proof of Lemma 3.5 is complete. ∎

Remark 3.6 Since $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, which is the inclusion operator $i : W_0^{1,p}(\Omega) \to L^p(\Omega)$ is compact, it follows by Lemma 3.5 that the operator $S : L^p(\Omega) \to L^p(\Omega)$ defined by $S = i \circ T$ is compact.

Proof of Theorem 1.1 For each $p \in (1, \infty)$, let $\mathcal{D}_p$ be the positive constant given by Remark 3.3. We have

$$\|S(v)\|_{L^p(\Omega)} \leq \sqrt[p]{\mathcal{D}_p}, \quad \forall\, v \in L^p(\Omega) .$$

Define the set in $L^p(\Omega)$,

$$B_{\sqrt[p]{\mathcal{D}_p}}(0) := \left\{ v \in L^p(\Omega) : \; \|v\|_{L^p(\Omega)} \leq \sqrt[p]{\mathcal{D}_p} \right\} .$$

Clearly, $B_{\sqrt[p]{\mathcal{D}_p}}(0)$ is a convex, closed subset of $L^p(\Omega)$ and $S\left(B_{\sqrt[p]{\mathcal{D}_p}}(0)\right) \subset B_{\sqrt[p]{\mathcal{D}_p}}(0)$. Moreover, by Remark 3.6, we have that $S\left(B_{\sqrt[p]{\mathcal{D}_p}}(0)\right)$ is relatively compact in $B_{\sqrt[p]{\mathcal{D}_p}}(0)$.

Finally, by Lemma 3.5 and Remark 3.6, we have that $S : B_{\sqrt[p]{\mathcal{D}_p}}(0) \to B_{\sqrt[p]{\mathcal{D}_p}}(0)$ is a continuous map. Thus, we can apply the Schauder's fixed-point theorem to obtain that S possesses a fixed point u_p. This gives us a weak solution $u_p \in W_0^{1,p}(\Omega)$ of problem (1.2), which is nonnegative in Ω.

The proof of Theorem 1.1 is finally complete. ∎

Remark 3.7 For each $p > 1$ there exists $u_p \in W_0^{1,p}(\Omega) \subset L^p(\Omega)$ such that $T(u_p) = u_p$, where u_p is obtained by applying Schauder's fixed point theorem, and u_p is the unique minimizer of the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$J(u) = \frac{1}{p} \int_\Omega \alpha(x, u_p) H(\nabla u)^p \, dx - \int_\Omega f u \, dx.$$

4 Proof of Theorem 1.2

Let $\{p_n\}_{n \geq 1} \subset (N, \infty)$ be a sequence of real numbers such that $\lim_{n\to\infty} p_n = +\infty$.

For each $n \geq 1$, we consider u_{p_n} to be the weak solution of problem (1.2) with $p = p_n$, which is obtained by applying Schauder's-fixed point theorem. We have that u_{p_n} is the unique weak solution of problem (3.1) with $p = p_n$ and $v = u_{p_n}$. By Remark 3.7, we have that u_{p_n} is the unique minimizer of the functional $J_n : W_0^{1,p_n}(\Omega) \to \mathbb{R}$, defined by

$$J_n(u) = \frac{1}{p_n} \int_\Omega \alpha(x, u_{p_n}) H(\nabla u)^{p_n} \, dx - \int_\Omega f u \, dx\,.$$

4.1 A Γ-convergence result

We start this subsection of our paper by recalling the definition of the concept of Γ-convergence (introduced in [14, 15]) in metric spaces. The reader is referred to [13] and [10] for a comprehensive introduction to the topic.

***Definition* 4.1** *Let X be a metric space. A sequence $\{F_n\}$ of functionals $F_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to $\Gamma(X)$-converges to $F : X \to \overline{\mathbb{R}}$, and we write $\Gamma(X) - \lim_{n\to\infty} F_n = F$, if the following hold*

(i) *for every $u \in X$, and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have*

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(ii) *for every $u \in X$ there exists a recovery sequence $\{u_n\} \subset X$ such that $u_n \to u$ in X and*

$$F(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

For each $n \geq 1$, we introduce $I_n : L^1(\Omega) \to [0, \infty]$ defined by

$$I_n(u) := \begin{cases} \frac{1}{p_n} \int_\Omega \alpha(x, u_{p_n}) H(\nabla u)^{p_n} \, dx, & \text{if } u \in W_0^{1,p_n}(\Omega), \\ +\infty, & \text{if } u \in L^1(\Omega) \setminus W_0^{1,p_n}(\Omega). \end{cases}$$

The main result of this subsection gives the following Γ-convergence result for the sequence $\{I_n\}$.

***Lemma* 4.1** *Define $I_\infty : L^1(\Omega) \to [0,\infty]$ by*

$$I_\infty(u) := \begin{cases} 0, & \text{if } u \in X_0 \text{ and } H(\nabla u(x)) \le 1 \text{ for a.e. } x \in \Omega, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $X_0 = W^{1,\infty}(\Omega) \cap \left(\cap_{q>1} W_0^{1,q}(\Omega)\right)$. Then $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} I_n = I_\infty$.

Proof Let $v \in L^1(\Omega)$ and $\{v_n\} \subset L^1(\Omega)$ be such that $v_n \to v$ in $L^1(\Omega)$. We may assume without loss of generality that $\{v_n\} \subset W_0^{1,p_n}(\Omega)$ and

$$\liminf_{n\to\infty} I_n(v_n) = \lim_{n\to\infty} I_n(v_n) < \infty. \tag{4.1}$$

Let $x \in \Omega$ be a Lebesgue point for $\nabla v \in (L^1(\Omega))^N$. For any ball $B_r(x) \subset \Omega$ and for any integer $n > 1$ such that $p_n \ge 2$ we have, by Hölder's inequality,

$$\int_{B_r(x)} H(\nabla v_n(y))\, dy \le \|H(\nabla v_n)\|_{L^{p_n}(\Omega)} \|\chi_{B_r(x)}\|_{L^{\frac{p_n}{p_n-1}}(\Omega)}. \tag{4.2}$$

Also, we have

$$\|\chi_{B_r(x)}\|_{L^{\frac{p_n}{p_n-1}}(\Omega)} = |B_r(x)|^{\frac{p_n-1}{p_n}}. \tag{4.3}$$

On the other hand, note that

$$\int_\Omega H(\nabla v_n)^{p_n}\, dx \le \frac{1}{\lambda}\; p_n I_n(v_n)$$

or

$$\|H(\nabla v_n)\|_{L^{p_n}(\Omega)} \le \lambda^{-1/p_n} p_n^{1/p_n} [I_n(v_n)]^{1/p_n}. \tag{4.4}$$

By (4.2), (4.3), and (4.4), we obtain

$$\int_{B_r(x)} H(\nabla v_n(y))\, dy \le \lambda^{-1/p_n} {p_n}^{1/p_n} [I_n(v_n)]^{1/p_n} |B_r(x)|^{\frac{p_n-1}{p_n}}$$

which in view of (4.1) implies

$$\limsup_{n\to\infty} \int_{B_r(x)} H(\nabla v_n(y))\, dy \le |B_r(x)|. \tag{4.5}$$

Let $q \ge 1$ be an arbitrary real number. For each $n > 1$ such that $q < p_n$, using Hölder's inequality, we have

$$\begin{aligned} \int_\Omega H(\nabla v_n)^q\, dx &\le \left(\int_\Omega H(\nabla v_n)^{p_n}\, dx\right)^{\frac{q}{p_n}} |\Omega|^{\frac{p_n-q}{p_n}} \\ &\le \lambda^{-q/p_n} p_n^{q/p_n} [I_n(v_n)]^{q/p_n} |\Omega|^{\frac{p_n-q}{p_n}} \end{aligned}$$

and thus, using (2.5),

$$a\| \, |\nabla v_n| \, \|_{L^q(\Omega)} \le \|H(\nabla v_n)\|_{L^q(\Omega)} \le [\lambda^{-1} p_n I_n(v_n)]^{\frac{1}{p_n}} |\Omega|^{\frac{1}{q} - \frac{1}{p_n}}.$$

We obtain that the sequence $\{\nabla v_n\}$ is bounded in $L^q(\Omega; \mathbb{R}^N)$, for any $q \ge 1$. It follows that the sequence $\{v_n\}$ is bounded in $W_0^{1,q}(\Omega)$, and thus we may extract a subsequence, still denoted by $\{v_n\}$, such that v_n converges weakly to v in $W_0^{1,q}(\Omega)$.

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In particular, we find that $v \in \cap_{q>1} W_0^{1,q}(\Omega)$. On the other hand, a well-known weak lower semicontinuity result implies

$$\int_{B_r(x)} H(\nabla v(y))\, dy \leq \liminf_{n\to\infty} \int_{B_r(x)} H(\nabla v_n(y))\, dy \leq \limsup_{n\to\infty} \int_{B_r(x)} H(\nabla v_n(y))\, dy$$

which, in view of (4.5), yields

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} H(\nabla v(y))\, dy \leq 1, \quad \forall\ r > 0.$$

Since almost every $x \in \Omega$ is a Lebesgue point for ∇v, passing to the limit as $r \to 0^+$ in the above inequality yields $H(\nabla v(x)) \leq 1$ for a.e. $x \in \Omega$. Now, since $v \in \cap_{q>1} W_0^{1,q}(\Omega)$ we deduce by (2.7) that

$$\int_\Omega H(\nabla v)^q\, dx \geq \lambda_1(q) \int_\Omega |v|^q\, dx, \quad \forall\ q > 1.$$

In view of (2.8), the above relation implies that

$$1 \geq \|H(\nabla v)\|_{L^\infty(\Omega)} \geq \Lambda_\infty \|v\|_{L^\infty(\Omega)},$$

and, thus, we deduce $v \in W^{1,\infty}(\Omega)$ and, consequently, $v \in X_0$. It follows that $I_\infty(v) = 0$ and thus we obtain

$$I_\infty(v) \leq \liminf_{n\to\infty} I_n(v_n).$$

It remains to prove the existence of a recovery sequence for the Γ-limit. Let $v \in L^1(\Omega)$. Note, if $I_\infty(v) = +\infty$ there is nothing to prove, because the inequality holds true for any sequence $v_n \to v$ strongly in $L^1(\Omega)$. On the other hand, if $I_\infty(v) < \infty$ we must have $I_\infty(v) = 0$ and, consequently, $v \in X_0 \subset \cap_{q>1} W_0^{1,q}(\Omega)$ and $H(\nabla v(x)) \leq 1$ for a.e. $x \in \Omega$. For each integer $n > 1$ define $v_n := v$ and note that we have

$$\begin{aligned}\limsup_{n\to\infty} I_n(v_n) &= \limsup_{n\to\infty} \frac{1}{p_n} \int_\Omega \alpha(x, u_{p_n}) H(\nabla v(x))^{p_n}\, dx \\ &\leq \limsup_{n\to\infty} \frac{\Lambda}{p_n} |\Omega| = 0 = I_\infty(v).\end{aligned}$$

The proof of Lemma 4.1 is complete. ■

In the end of this subsection, we also recall the following well-known result, which can be found in [20, Corollary 6.1.1].

***Proposition* 4.2** *Let X be a topological space satisfying the first axiom of countability, and assume that the sequence $\{F_n\}$ of functionals $F_n : X \to \overline{\mathbb{R}}$, Γ-converges to $F : X \to \overline{\mathbb{R}}$. Let z_n be a minimizer for F_n. If $z_n \to z$ in X, then z is a minimizer of F, and*

$$F(z) = \liminf_{n\to\infty} F_n(z_n).$$

This result will prove to be extremely useful in obtaining the result from Theorem 1.2 of our manuscript.

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4.2 Asymptotic behavior

We start by establishing a result which in the particular case when $\alpha(x,t)$ and $f(x)$ are positive constant functions and $H(\cdot) = |\cdot|$ is the Euclidean norm on $\mathbb{R}^N$ was proved by Payne and Philippin in [27].

***Proposition* 4.3** *For each $n \in \mathbb{N}$, let $u_{p_n} \in W_0^{1,p_n}(\Omega)$ be the weak solution of the problem (1.2) with $p = p_n$ given by Schauder's fixed-point theorem. Then*

$$\lim_{n\to\infty} \int_\Omega f u_{p_n}\, dx = \int_\Omega f \delta_H\, dx.$$

The proof of this proposition will be based on two auxiliary results, which are established in the next two lemmas.

***Lemma* 4.4** *The sequences $\{\int_\Omega f u_{p_n}\, dx\}$ and $\{\int_\Omega u_{p_n}\, dx\}$ are bounded.*

Proof First, we prove that the sequence $\{\int_\Omega f u_{p_n}\, dx\}$ is bounded. For every $p_n \geq 2$, using Hölder's inequality, we have

$$\int_\Omega f u_{p_n}\, dx \leq \|f\|_{L^\infty(\Omega)} \int_\Omega u_{p_n}\, dx \leq \|f\|_{L^\infty(\Omega)} \left(\int_\Omega |u_{p_n}|^{p_n}\, dx\right)^{1/p_n} |\Omega|^{\frac{p_n-1}{p_n}}.$$

Taking $p = p_n$ and $v = u_{p_n}$ in (2.7), we have

$$\int_\Omega |u_{p_n}|^{p_n}\, dx \leq \frac{\int_\Omega H(\nabla u_{p_n})^{p_n}\, dx}{\lambda_1(p_n)}.$$

Combining the above two inequalities, we deduce

$$\left(\int_\Omega f u_{p_n}\, dx\right)^{p_n} \leq \|f\|^{p_n}_{L^\infty(\Omega)} \frac{\int_\Omega H(\nabla u_{p_n})^{p_n}\, dx}{\lambda_1(p_n)} |\Omega|^{p_n-1}. \tag{4.6}$$

Since u_{p_n} is a weak solution for problem (1.2) with $p = p_n$, relation (1.5) with $\varphi = u_{p_n}$ gives

$$\int_\Omega \alpha(x, u_{p_n}) H(\nabla u_{p_n})^{p_n-2} \langle \mathcal{H}(\nabla u_{p_n}), \nabla u_{p_n}\rangle\, dx = \int_\Omega f u_{p_n}\, dx,$$

and taking into account (2.3) and (1.1), it follows that

$$\int_\Omega H(\nabla u_{p_n})^{p_n}\, dx = \int_\Omega H(\nabla u_{p_n})^{p_n-2} \langle \mathcal{H}(\nabla u_{p_n}), \nabla u_{p_n}\rangle\, dx \leq \frac{1}{\lambda} \int_\Omega f u_{p_n}\, dx. \tag{4.7}$$

By inequalities (4.6) and (4.7), we get

$$\left(\int_\Omega f u_{p_n}\, dx\right)^{p_n-1} \leq \|f\|^{p_n}_{L^\infty(\Omega)} \frac{1}{\lambda_1(p_n)} |\Omega|^{p_n-1} \frac{1}{\lambda}$$

or

$$\int_\Omega f u_{p_n}\, dx \leq \|f\|^{p_n/(p_n-1)}_{L^\infty(\Omega)} \frac{\lambda^{-1/(p_n-1)}}{\left(\sqrt[p_n]{\lambda_1(p_n)}\right)^{p_n/(p_n-1)}}\ |\Omega|.$$

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On the other hand, by (2.8), we know that $\lim_{n\to\infty} \sqrt[p_n]{\lambda_1(p_n)} = \|\delta_H\|^{-1}_{L^\infty(\Omega)}$, which implies that the right-hand side in the last inequality above is bounded and, consequently, $\{\int_\Omega f u_{p_n}\,dx\}$ is bounded.

Finally, since $f : \overline{\Omega} \to (0,\infty)$ is continuous, we deduce that $\{\int_\Omega u_{p_n}\,dx\}$ is bounded, too. The proof of Lemma 4.4 is complete. ∎

***Lemma* 4.5** *There exists $u_\infty \in X_0$ with $u_\infty \geq 0$ in Ω and $\|H(\nabla u_\infty)\|_{L^\infty(\Omega)} \leq 1$ and a subsequence of $\{u_{p_n}\}$ (not relabeled) such that $u_{p_n} \to u_\infty$ uniformly in Ω.*

Proof Fix $q > N$ be an arbitrary real number. Since $\lim_{n\to\infty} p_n = +\infty$, it follows that $q < p_n$ for sufficiently large $n \in \mathbb{N}$.

For each $q < p_n$, using Hölder's inequality, recalling the fact that $\langle J_n'(u_{p_n}), u_{p_n}\rangle = 0$ and taking into account (4.7), we deduce

$$\begin{aligned}\int_\Omega H(\nabla u_{p_n})^q\,dx &\leq \left(\int_\Omega H(\nabla u_{p_n})^{p_n}\,dx\right)^{\frac{q}{p_n}} |\Omega|^{\frac{p_n-q}{p_n}}\\ &\leq \left(\frac{1}{\lambda}\int_\Omega f u_{p_n}\,dx\right)^{\frac{q}{p_n}} |\Omega|^{\frac{p_n-q}{p_n}}\\ &\leq \frac{1}{\lambda^{q/p_n}}\left(\int_\Omega f u_{p_n}\,dx\right)^{\frac{q}{p_n}} |\Omega|^{\frac{p_n-q}{p_n}}.\end{aligned}$$

By Lemma 4.4, there exists a positive constant M such that

$$\int_\Omega f u_{p_n}\,dx \leq M$$

for all $n \in \mathbb{N}$ sufficiently large. Thus, using also (2.5), for such $n \in \mathbb{N}$ we must have

$$a\|\,|\nabla u_{p_n}|\,\|_{L^q(\Omega)} \leq \|H(\nabla u_{p_n})\|_{L^q(\Omega)} \leq \lambda^{-1/p_n} M^{1/p_n} |\Omega|^{1/q-1/p_n}.$$

Thus, $\{\nabla u_{p_n}\}_n$ is uniformly bounded in $L^q(\Omega;\mathbb{R}^N)$. The fact that $q > N$ guarantees that the embedding of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ is compact. Taking into account the reflexivity of the Sobolev space $W_0^{1,q}(\Omega)$, it follows that there exists a subsequence (not relabeled) of $\{u_{p_n}\}$ and a function $u_\infty \in C(\overline{\Omega})$ such that $u_{p_n} \rightharpoonup u_\infty$ weakly in $W_0^{1,q}(\Omega)$ and $u_{p_n} \to u_\infty$ uniformly in Ω. Moreover, the fact that $u_{p_n} \geq 0$ a.e. in Ω for each $p_n > N$ implies that $u_\infty \geq 0$ a.e. in Ω.

Finally, since there exists u_∞ such that $u_\infty = \lim_{n\to\infty} u_{p_n}$ in $L^1(\Omega)$ in view of Proposition 4.2 (with $X = L^1(\Omega)$, $F_n = I_n$, $F_\infty = I_\infty$, $z_n = u_{p_n}$) and Lemma 4.1 (and taking into account that for each positive integer n the minimizer u_{p_n} of J_n minimizes I_n, too), we conclude that u_∞ must be a minimizer for I_∞ and, in particular $\|H(\nabla u_\infty)\|_{L^\infty(\Omega)} \leq 1$ and $u_\infty \in X_0$. This concludes the proof of Lemma 4.5. ∎

Proof of Proposition 4.3 Fix an arbitrary subsequence of $\{u_{p_n}\}$, still denoted by $\{u_{p_n}\}$. Similar arguments as those used in the proof of Lemma 4.5 can be considered to prove that this subsequence contains, in its turn, a subsequence, say $\{u_{p_{n_k}}\}$, which converges uniformly in Ω to a certain limit $u_\infty \in X_0$ with $\|H(\nabla u_\infty)\|_{L^\infty(\Omega}$ ≤ 1. In order to get the conclusion of Proposition 4.3, it is enough to establish that $\lim_{k\to\infty} \int_\Omega f u_{p_{n_k}}\,dx = \int_\Omega f\delta_H\,dx$. In other words, we will show that the limit of all

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possible subsequences of $\{\int_\Omega f u_{p_n}\,dx\}$ is $\int_\Omega f\delta_H\,dx$ and, consequently, the limit of the full sequence should also be $\int_\Omega f\delta_H\,dx$.

In the sequel, for simplicity, we will write u_{p_n} instead of $u_{p_{n_k}}$.

Since $\delta_H \in X_0 \subset \cap_{q>1} W_0^{1,q}(\Omega)$ and $H(\nabla\delta_H(x)) = 1$ for a.e. $x\in\Omega$, and u_{p_n} is a minimizer of J_n in $W_0^{1,p_n}(\Omega)$, we deduce that for each positive integer $n \in \mathbb{N}$ we have

$$\begin{aligned} J_n(u_{p_n}) \le J_n(\delta_H) &= \frac{1}{p_n}\int_\Omega \alpha(x,u_{p_n})H(\nabla\delta_H(x))^{p_n}\,dx - \int_\Omega f\delta_H\,dx \\ &= \frac{1}{p_n}\int_\Omega \alpha(x,u_{p_n})\,dx - \int_\Omega f\delta_H\,dx \\ &\le \frac{\Lambda}{p_n}|\Omega| - \int_\Omega f\delta_H\,dx. \end{aligned}$$

Taking into account that $p_n\to\infty$ as $n\to\infty$ the above estimates imply

$$\int_\Omega f\delta_H\,dx \le \liminf_{n\to\infty}\int_\Omega f u_{p_n}\,dx = \int_\Omega f u_\infty\,dx\,. \tag{4.8}$$

Next, for each $x\in\Omega$ fix $y\in\partial\Omega$ such that $H^0(x-y)=\delta_H(x)$. Define $h:[0,1]\to\mathbb{R}$ by $h(t) := u_\infty(tx+(1-t)y)$. By the mean value theorem, we deduce that there exists $t_x\in(0,1)$ such that

$$h'(t_x) = h(1)-h(0)$$

or

$$u_\infty(x) = u_\infty(x) - u_\infty(y) = h(1)-h(0) = \langle\nabla u_\infty(t_x x + (1-t_x)y),(x-y)\rangle\,,$$

and by (2.1) we deduce

$$u_\infty(x) \le H^0(x-y)\sup_{z\in[x,y]} H(\nabla u_\infty(z)) \le \delta_H(x)\,. \tag{4.9}$$

Multiplying by f and integrating over Ω, we get

$$\lim_{n\to\infty}\int_\Omega f u_{p_n}\,dx = \int_\Omega f u_\infty\,dx \le \int_\Omega f\delta_H\,dx\,.$$

Recalling (4.8) it follows that $\lim_{n\to\infty}\int_\Omega f u_{p_n}\,dx = \int_\Omega f\delta_H\,dx$. Thus, the proof of Proposition 4.3 is complete. ∎

Proof of Theorem 1.2 As in Proposition 4.3, we fix an arbitrary subsequence of the solutions $\{u_{p_n}\}$ (not relabeled). Similar arguments as those used in Lemma 4.5 ensure that $\{u_{p_n}\}$ converges uniformly to a certain limit

$$u_\infty \in X_0 \quad \text{with} \quad \|H(\nabla u_\infty)\|_{L^\infty(\Omega)} \le 1\,.$$

Thus, it just remains to see that $u_\infty = \delta_H$. Notice that, since $\{u_{p_n}\}$ is arbitrary, this means that δ_H is indeed the limit of the full sequence $\{u_{p_n}\}$. Recall that by (4.9) we have $u_\infty(x)\le\delta_H(x)$, for each $x\in\Omega$. Furthermore, since we have $u_{p_n}(x)\ge 0$ for a.e. $x\in\Omega$ and for every integer $n>1$ for which $p_n\ge 2$, we deduce that $u_\infty(x)\ge 0$ for a.e. $x\in\Omega$. Finally, applying Proposition 4.3 and taking into account the fact that $u_{p_n}\to u_\infty$

uniformly in Ω, we find that

$$\int_\Omega f\delta_H\,dx = \lim_{n\to\infty}\int_\Omega f u_{p_n}\,dx = \int_\Omega f u_\infty\,dx\,.$$

Recalling the continuity of f, δ_H, and u_∞, the last equalities yield $u_\infty = \delta_H$.

The proof of Theorem 1.2 is complete. ∎

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The University of Sydney, Sydney, Australia
e-mail: farcaseanu.maria@yahoo.com

University of Craiova, Craiova, Romania
e-mail: mmihailes@yahoo.com

University Politehnica of Bucharest, Bucharest, Romania
e-mail: farcaseanu.maria@yahoo.com denisa.stancu@yahoo.com

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