Probability in the Engineering and Informational Sciences, **35**, 2021, 331–354.
doi:10.1017/S0269964819000408

ORDERING RESULTS ON EXTREMES OF EXPONENTIATED LOCATION-SCALE MODELS

Sangita Das, Suchandan Kayal and Debajyoti Choudhuri
Department of Mathematics, National Institute of Technology Rourkela, Rourkela 769008, India
E-mail: sangitadas118@gmail.com, kayals@nitrkl.ac.in, suchandan.kayal@gmail.com, dc.iit12@gmail.com

In this paper, we consider exponentiated location-scale model and obtain several ordering results between extreme order statistics in various senses. Under majorization type partial order-based conditions, the comparisons are established according to the usual stochastic order, hazard rate order and reversed hazard rate order. Multiple-outlier models are considered. When the number of components are equal, the results are obtained based on the ageing faster order in terms of the hazard rate and likelihood ratio orders. For unequal number of components, we develop comparisons according to the usual stochastic order, hazard rate order, and likelihood ratio order. Numerical examples are considered to illustrate the results.

Keywords: exponentiated location-scale family, majorization, multiple-outlier model, order statistics, stochastic orders

Jel Classification: 60E15, 62G30, 60K10

1. INTRODUCTION

Order statistics have received a great amount of attention from various authors. It plays an important role in statistical theory and methodology. Let $X_1, \ldots, X_n$ be n independent and identically distributed observations taken from a population with cumulative distribution function (CDF) F_X and probability density function (PDF) f_X. The ordered sample values $X_{1:n} \leq \cdots \leq X_{n:n}$ are called the order statistics. The order statistic $X_{n-k+1:n}$ represents the reliability of a k-out-of-n system. Thus, to study a k-out-of-n system, it is sufficient to study $(n-k+1)$th order statistic and vice versa. Note that k-out-of-n system reduces to series and parallel systems for $k=1$ and n, respectively. There are several applications of extreme values in hydrology (floods and droughts), aeronautics (gust loads), oceanography (waves and tides), material strength (weakest link theory), and meteorology (extremes of temperature, pressure, wind velocity, and precipitation). Order statistics occur naturally in life testing. Suppose n similar items are simultaneously placed on a life test. Then, the life of the first to fail is the first-order statistic of the sample of size n from the life distribution, second to fail is the second-order statistic, and so on. For more details, please refer to [1,13]. At the beginning, most of the studies focused mainly on the case when order statistics are from independent and identically distributed random variables. The studies

 0269-9648/19 $25.00

of order statistics from heterogeneous samples started in early 70s motivated by the issues of robustness. It is worth pointing that heterogeneous samples arise in various practical situations. For example, a complex engineering system is often constructed based on several different types of electrical components. Thus, naturally, heterogeneous samples arise while investigating the reliability of such system. Here, the samples represent the failure times of electrical components, which can be collected from the experiments. Further, in insurance, a total claim for the portfolio of an insurer may be consisted of different sub-claims. These are of different distributions. Because of its usefulness, the topic of heterogenous samples attracts researchers from different areas. As a result a wide variety of work has been completed in single-outlier models and multiple-outlier models on order statistics constructed from heterogeneous samples.

There are many distributions for which comparisons can not be done based on the standard measures such as mean and standard deviation. Thus, more informative methods are required. Again, the distributions can be expressed by many functional forms. These are survival function, hazard rate function, and reverse hazard rate function. Comparisons based on these functional forms of the underlying distributions often establish partial orders, which are called as stochastic orders. Stochastic orders have been used in several areas of probability and statistics such as reliability theory, queueing theory, survival analysis, and operations research. We refer to [38] for elaborate discussion on this topic. Stochastic orderings and related inequalities are used in reliability theory for various aims. We present few of these below.

- In reliability theory, it is always desirable to deduce probabilistic properties of a system from available information regarding its structure. This is a difficult job to the practitioners. Thus, to get approximations of the original system, easier systems are required. This results in comparisons between stochastic performance processes. Useful bounds for the characteristics of the original system are obtained by establishing a stochastic order between these processes.
- Sometimes, it is also required to examine which modifications of a system result in an improvement. A system is improved if one uses shorter repair times. Again, the repair times are stochastic variables. Thus, stochastic ordering is needed to define meaning of shorter.

In this direction, we refer to [7,11,18,19,25,35]. There is an extensive literature for the use of stochastic orderings in reliability to compare locations of the lifetime, residual lifetime, or inactivity time of the systems. See, for instance, Khaledi and Shaked [24]. We also see some other types of stochastic orderings for the measurement of variability and spread. The use of these measures has become classical in insurance literature. In this direction, we refer the readers to [14,15] and the references therein. Sometimes, we find its application in statistical inference while studying the robustness of estimators of reliability parameters of a statistical model when independent random observations are taken from heterogeneous distributions (see [37]). For more applications of the stochastic orderings, we refer to [10,30,32].

Motivated by these, various authors have considered problems of stochastic comparisons of the lifetimes of parallel and series systems when their components are comprising of independent and heterogeneous distributions. Few recent references in this direction are [5,12,36,44]. However, there have been recent interest on studying stochastic comparisons of extreme order statistics for various general statistical models. Khaledi et al. [22] studied conditions under which the series and parallel systems consisting of components with lifetimes from scale family of distributions are ordered in terms of the hazard rate and reversed hazard rate orderings, respectively. Kochar and Torrado [27] revisited the problem and obtained a stronger result than the result presented in [22] for the largest order statistics. They established likelihood ratio ordering of the largest order statistics for general scale model. Bashkar et al. [9] discussed stochastic comparisons of extreme order statistics from independent heterogeneous exponentiated scale models. Barmalzan et al. [8] provided stochastic comparison between the smallest claim amounts in the sense of the usual stochastic and hazard rate orders. These are done using the concept of vector majorization and related orders. Torrado [40] obtained various ordering results for the comparisons of two extreme order statistics from scale models when one set of scale parameters majorizes the other. Hazra et al. [20] obtained various stochastic comparisons of the maximum order statistics from the location-scale family of distributions. Recently, Hazra et al. [21] considered stochastic comparisons of the minimum order statistics from the location-scale family of distributions. Under certain conditions, they showed that the minimum order statistic of one set of random variables dominates that of other set of random variables with respect to various stochastic orders. We also refer to [3,16,17,39] for more results in this direction.

Consider two sets of independent random samples drawn from heterogeneous exponentiated location-scale family of distributions. Comparisons between the maximum and the minimum order statistics arising from these sets have not been considered so far in the literature. This motivates our study in this direction. The established results generalize and strengthen some known results in the literature. Further, it is well-known that the lifetimes of the series and parallel systems are described by the minimum and the maximum order statistics. So, our investigation finds various relations among parameters for which one parallel/series system dominates the other with respect to various stochastic orders. A random variable X is said to follow the exponentiated location-scale model if its cumulative distribution function (CDF) is given by

$$F_X(x) \equiv F_X(x;\lambda,\theta,\alpha) = \left[F\left(\frac{x-\lambda}{\theta}\right)\right]^{\alpha} = F^{\alpha}\left(\frac{x-\lambda}{\theta}\right), \quad x > \lambda, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\alpha > 0$, $\theta > 0$ and F is the baseline distribution function. Here, λ, θ, and α are respectively known as the location, scale, and shape parameters. We write $X \sim \mathcal{ELS}(\lambda,\theta,\alpha)$ if X has the distribution function given by (1.1). The probability density function (PDF) of the exponentiated location-scale model with CDF (1.1) is denoted by f_X. Further, the hazard rate and reversed hazard rate functions of this model are given by $r_X = f_X/\bar{F}_X$ and $\widetilde{r}_X = f_X/F_X$, respectively, where $\bar{F}_X = 1 - F_X$. The PDF, hazard rate, and reversed hazard rate of a baseline distribution with CDF F are denoted by $f, r^* = f/\bar{F}$ and $\widetilde{r}^* = f/F$, respectively. When $\alpha = 1$, (1.1) reduces to the location-scale family of distributions. Further, when $(\alpha,\lambda) = (1,0)$ and $(\alpha,\theta) = (1,1)$, then (1.1) becomes scale and location models, respectively.

The plan of this paper is described as follows. In the next section, we provide definitions and preliminary results. Section 3 is devoted to obtain comparisons between the extreme order statistics in terms of the usual stochastic order when sets of parameters are related with the majorization-based orders. In Section 4, we obtain results in terms of the hazard rate and reversed hazard rate orders. Multiple-outlier models are considered in Section 5, where we establish various stochastic orders. In Section 6, some special cases of our main results are added. Section 7 provides applications of the established results. Finally, in Section 8, we include some concluding remarks.

2. PRELIMINARY RESULTS AND NOTATIONS

In this section, we review some definitions and well-known notions of majorization concepts and stochastic orders. We only focus on the nonnegative random variables. Throughout the article, the terms "increasing" and "decreasing" are used in nonstrict sense. First, we present definitions of various stochastic orders.

2.1. Stochastic orders

Assume that X and Y are two nonnegative random variables with PDFs f_X and f_Y, CDFs F_X and F_Y, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, hazard rates $r_X = f_X/\bar{F}_X$ and $r_Y = f_Y/\bar{F}_Y$, reversed hazard rates $\tilde{r}_X = f_X/F_X$ and $\tilde{r}_Y = f_Y/F_Y$, respectively.

Definition 2.1: *A random variable X is said to be smaller than Y in the*

- *usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x.*
- *hazard rate order (denoted by $X \leq_{hr} Y$) if $r_X(x) \geq r_Y(x)$ for all x.*
- *reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ for all x.*
- *likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in x.*
- *aging faster order in terms of hazard rate order (denoted by $X \leq_{R-hr} Y$) if $r_X(x)/r_Y(x)$ is increasing in x for which the ratio is well defined, for all x.*

For greater details on different stochastic orderings, we refer the reader to [38]. The notion of majorization plays an important role in studying stochastic inequalities among various order statistics. Below, we present a few majorization and related orders which will be useful for the subsequent sections. We refer to [31] for detailed discussion on this topic.

2.2. Majorization and some related orders

Let $\mathbb{A} \subset \mathbb{R}^n$. Here $\mathbb{R}^n$ is an n-dimensional Euclidean space. Further, let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two points in $\mathbb{A}$. Assume that $x_{1:n} \leq \ldots \leq x_{n:n}$ and $y_{1:n} \leq \ldots \leq y_{n:n}$ denote the order coordinates of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

Definition 2.2: *A vector $\boldsymbol{x}$ is said to be*

- *majorized by another vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq^m \boldsymbol{y}$, if for each $k = 1, \ldots, n-1$, we have $\sum_{i=1}^k x_{i:n} \geq \sum_{i=1}^k y_{i:n}$ and $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$.*
- *weakly submajorized by another vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq_w \boldsymbol{y}$, if for each $k = 1, \ldots, n$, we have $\sum_{i=k}^n x_{i:n} \leq \sum_{i=k}^n y_{i:n}$.*
- *weakly supermajorized by another vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq^w \boldsymbol{y}$, if for each $k = 1, \ldots, n$, we have $\sum_{i=1}^k x_{i:n} \geq \sum_{i=1}^k y_{i:n}$.*
- *p-larger than the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \succeq^p \boldsymbol{y}$ if $\prod_{i=1}^k x_{i:n} \leq \prod_{i=1}^k y_{i:n}$ for $k = 1, \ldots, n$.*
- *reciprocally majorized by another vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq^{rm} \boldsymbol{y}$, if $\sum_{i=1}^k x_{i:n}^{-1} \leq \sum_{i=1}^k y_{i:n}^{-1}$ for all $k = 1, \ldots, n$.*

The implication chain is well-known: $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{p}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{rm}{\succeq} \boldsymbol{y}$. Next, we present definition of the Schur-convex and Schur-concave functions.

DEFINITION 2.3: *A function* $\xi : \mathbb{R}^n \to \mathbb{R}$ *is said to be Schur-convex (Schur-concave) on* $\mathbb{R}^n$ *if*

$$\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \Rightarrow \xi(\boldsymbol{x}) \geq (\leq) \xi(\boldsymbol{y}) \quad \textit{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Throughout the paper, we will use the following notations:

NOTATION 2.1: *(i)* $\mathcal{D}_+ = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$, *(ii)* $\mathcal{E}_+ = \{(x_1, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \ldots \leq x_n\}$.

Denote $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n), \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$. Next, we provide lemmas which are required to establish the main results. The proofs follow after some simple algebraic derivations, hence omitted. Note that the first part of the following lemma is already proved by Torrado [39].

LEMMA 2.1: *Let* $w_1 : (0, \infty) \times (0, 1) \to (0, \infty)$ *be defined as* $w_1(\alpha, t) = \alpha t^\alpha/(1 - t^\alpha)$. *Then, (i) for each* $t \in (0, 1)$, $w_1(\alpha, t)$ *is decreasing with respect to* α, *(ii) for each* $\alpha \in (0, \infty)$, $w_1(\alpha, t)$ *is increasing with respect to* t.

LEMMA 2.2: *Let* $w_2 : (0, \infty) \times (0, 1) \to (0, \infty)$ *be defined as* $w_2(\alpha, t) = \alpha t^\alpha \ln t/(1 - t^\alpha)$. *Then, (i) for each* $t \in (0, 1)$, $w_2(\alpha, t)$ *is increasing with respect to* α, *(ii) for each* $\alpha \in (0, \infty)$, $w_2(\alpha, t)$ *is decreasing with respect to* t.

3. THE USUAL STOCHASTIC ORDER

Let two sets of n independent random variables be taken from heterogeneous exponentiated location-scale family of distributions. This section deals with the comparisons for extreme order statistics in terms of the usual stochastic ordering. For simplicity of the presentation of the subsequent results, we first state the following assumption.

ASSUMPTION 3.1: *Let* $\{X_1, \ldots, X_n\}$ *and* $\{Y_1, \ldots, Y_n\}$ *be two sets of* n *independent random variables such that* $X_i \sim \mathcal{ELS}(\lambda_i, \theta_i, \alpha_i)$ *and* $Y_i \sim \mathcal{ELS}(\mu_i, \delta_i, \beta_i)$, *where* $\lambda_i, \theta_i, \alpha_i, \mu_i, \delta_i, \beta_i > 0$, $i = 1, \ldots, n$. *For convenience, we denote* $\boldsymbol{X} \sim \mathcal{ELS}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\alpha})$ *and* $\boldsymbol{Y} \sim \mathcal{ELS}(\boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{\beta})$, *where* $\boldsymbol{X} = (X_1, \ldots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$.

Under the above assumption, the distribution function of $X_{n:n}$ and the survival function of $X_{1:n}$ are respectively given by

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} F^{\alpha_i}\left(\frac{x - \lambda_i}{\theta_i}\right) \quad \text{and} \quad \bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n} \left[1 - F^{\alpha_i}\left(\frac{x - \lambda_i}{\theta_i}\right)\right], \tag{3.1}$$

where $x > \max\{\lambda_i,\ i = 1, \ldots, n\}$. We assume that λ_i's are positive. Below, we provide comparison results between the largest order statistics $X_{n:n}$ and $Y_{n:n}$. The proofs of the results are presented in Appendix A. The first part shows that the largest order statistic of one set of independent random variables dominate that of the other set when the set of shape parameters of the first set weakly submajorized that of the second set. Second part shows that the dominance relation gets reversed when the set of shape parameters of the first set is weakly supermajorized by that of the second set. Third part establishes dominance result similar to the first part when the set of reciprocal of shape parameters of

https://doi.org/10.1017/S0269964819000408 Published online by Cambridge University Press

the first set is weakly supermajorized by that of the second set. We assume same sets of location and scale parameters.

THEOREM 3.1: *Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of n independent random variables as in Assumption 3.1 with $\lambda_i = \mu_i$ and $\theta_i = \delta_i$, $i = 1, \ldots, n$.*

(i) *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+(\mathcal{D}_+)$. Then, $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

(ii) *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}_+(\mathcal{E}_+)$ and $\boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+(\mathcal{D}_+)$. Then, $\boldsymbol{\alpha} \succeq^w \boldsymbol{\beta} \Rightarrow Y_{n:n} \geq_{st} X_{n:n}$.*

(iii) *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+(\mathcal{D}_+)$. Then, $\boldsymbol{\alpha}^{-1} \succeq^w \boldsymbol{\beta}^{-1} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

To illustrate first two parts of the above theorem, we present the following example.

Example 3.1: (i) Let (X_1, X_2, X_3) be a vector of heterogeneous $\mathcal{ELS}(\lambda_i, \theta_i, \alpha_i)$ with parameters vectors $\boldsymbol{\lambda} = (1, 1.5, 1.7)$, $\boldsymbol{\theta} = (2, 2.1, 2.3)$ and $\boldsymbol{\alpha} = (2, 4, 6)$. Let (Y_1, Y_2, Y_3) be another vector of heterogeneous $\mathcal{ELS}(\lambda_i, \theta_i, \beta_i)$ with $\boldsymbol{\beta} = (1, 3, 5)$. Consider a baseline distribution with distribution function $F(x) = 1 - e^{-x}$, $x > 0$. Evidently, $(\alpha_1, \alpha_2, \alpha_3) \succeq_w (\beta_1, \beta_2, \beta_3)$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+$. Thus, as an application of Theorem 3.1(i), we have $X_{3:3} \geq_{st} Y_{3:3}$.

(ii) Let us take two sets of random vectors (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) as in Example 3.1 such that $\boldsymbol{\alpha} = (2, 2.5, 3)$, $\boldsymbol{\lambda} = \boldsymbol{\mu} = (0.3, 0.2, 0.1)$, $\boldsymbol{\beta} = (3, 3.5, 4)$, $\boldsymbol{\theta} = \boldsymbol{\delta} = (1.3, 1.2, 1.1)$. In this case, it is easy to see that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{E}_+$ and $\boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{D}_+$. Further, $(\alpha_1, \alpha_2, \alpha_3) \succeq^w (\beta_1, \beta_2, \beta_3)$. Thus, from Theorem 3.1$(ii)$, we have $Y_{3:3} \geq_{st} X_{3:3}$.

In the following theorem, we show that $X_{n:n}$ is larger than $Y_{n:n}$ in the sense of the usual stochastic ordering when a vector of reciprocal of scale parameters is p-larger than that of another vector of the reciprocal of scale parameters with some additional conditions. Similar results also hold under reciprocally majorized based conditions among the reciprocal of scale parameters. We assume that two sets of random variables have the same sets of location and shape parameters.

THEOREM 3.2: *Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of n independent random variables as in Assumption 3.1 with $\lambda_i = \mu_i$, $\alpha_i = \beta_i$, $i = 1, \ldots, n$ and $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta} \in \mathcal{D}_+(\mathcal{E}_+)$.*

(i) *Let $u\tilde{r}^*(u)$ be decreasing in u. Then, $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

(ii) *Let $u^2\tilde{r}^*(u)$ be decreasing in u. Then, $\boldsymbol{\theta}^{-1} \succeq^{rm} \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

Remark 3.1: Note that if all the location and shape parameters are known and set to zero and one, respectively, then the assumption made in Theorem 3.2(i) can be relaxed as shown in Theorem 3.2 of Khaledi et al. [22].

Remark 3.2: Let $\boldsymbol{\alpha} = \boldsymbol{\beta} = (1, \ldots, 1)$ and $\boldsymbol{\lambda} = \boldsymbol{\mu}$. Then, Theorems 3.2$(i)$ and 3.2(ii) respectively reduce to Theorems 3(i) and 3(ii) of Hazra et al. [20].

An application of Theorem 3.2(i) is as follows.

Example 3.2: Consider two three-component parallel Systems A and B. Assume (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are the component lifetimes of Systems A and B, respectively. Let the random lifetimes of the components follow exponentiated location-scale model with baseline distribution function $F(x) = 1 - \exp\{1 - x^{\alpha}\}$, $x \geq 1$, $\alpha > 0$. This is known as the lower truncated Weibull distribution. In this case, when $\alpha = 2$, clearly, $x\tilde{r}^*(x)$

is decreasing. Set $(\alpha_1, \alpha_2, \alpha_3) = (1, 5, 7.1)$, $(\lambda_1, \lambda_2, \lambda_3) = (2, 5.2, 8.1)$, $(\theta_1, \theta_2, \theta_3) = (3, 5, 6)$, and $(\delta_1, \delta_2, \delta_3) = (2, 4, 9)$. All the conditions of Theorem 3.2(i) are satisfied. Thus, as an application of Theorem 3.2(i), we conclude that System A has better performance than System B in the sense of the usual stochastic ordering.

Further, it may be of interest to examine the existence of the usual stochastic order between $X_{n:n}$ and $Y_{n:n}$ for two completely different sample sets with different sets of parameters. In this regard, we have the following results.

THEOREM 3.3: *Let* $\{X_1, \ldots, X_n\}$ *and* $\{Y_1, \ldots, Y_n\}$ *be two sets of* n *independent random variables as in Assumption 3.1 Further, let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{E}_+(\mathcal{D}_+)$. *Then,*

(i) $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1}, \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}, \boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$, *provided* $u\tilde{r}^*(u)$ *is decreasing in* u.
(ii) $\boldsymbol{\theta}^{-1} \succeq^{rm} \boldsymbol{\delta}^{-1}, \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}, \boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$, *provided* $u\tilde{r}^*(u)$ *and* $u^2\tilde{r}^*(u)$ *are decreasing in* u.

Remark 3.3: Let $(\alpha_1, \ldots, \alpha_n) = (1, \ldots, 1)$ and $(\beta_1, \ldots, \beta_n) = (1, \ldots, 1)$. Then, Theorems 3.3(i) and 3.3(ii) respectively reduce to Theorems 5(i) and 5(ii) in [20].

Next, we provide comparison results with respect to the usual stochastic ordering between the smallest order statistics $X_{1:n}$ and $Y_{1:n}$. We show that the smallest order statistic of one set dominates that of other set when the set of shape parameters of the first set weakly supermajorized that of the second set. Similar result is also shown when two sets of shape parameters are related to p-larger order.

THEOREM 3.4: *Let* $\{X_1, \ldots, X_n\}$ *and* $\{Y_1, \ldots, Y_n\}$ *be two sets of* n *independent random variables as in Assumption 3.1 such that* $\lambda_i = \mu_i$ *and* $\theta_i = \delta_i$, $i = 1, \ldots, n$.

(i) *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+(\mathcal{D}_+)$. *Then,* $\boldsymbol{\alpha} \succeq^w \boldsymbol{\beta} \Rightarrow Y_{1:n} \geq_{st} X_{1:n}$.
(ii) *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+$. *Then,* $\boldsymbol{\alpha} \succeq^p \boldsymbol{\beta} \Rightarrow Y_{1:n} \geq_{st} X_{1:n}$.

The following result deals with the comparison of the lifetimes of series systems in terms of the usual stochastic order when two sets of reciprocal of scale parameters are related to p-larger order and sub-weakly majorization order.

THEOREM 3.5: *Consider two sets of random variables* $\{X_1, \ldots, X_n\}$ *and* $\{Y_1, \ldots, Y_n\}$ *as described in Assumption 3.1. Assume* $\lambda_i = \mu_i$ *and* $\alpha_i = \beta_i$, $i = 1, \ldots, n$. *Further, let* $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta} \in \mathcal{D}_+(\mathcal{E}_+)$. *Then,*

(i) $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1} \Rightarrow X_{1:n} \geq_{st} Y_{1:n}$, *provided* $u\tilde{r}^*(u)$ *is decreasing in* u.
(ii) $\boldsymbol{\theta}^{-1} \succeq_w \boldsymbol{\delta}^{-1} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$, *provided* $u\tilde{r}^*(u)$ *is increasing in* u.

Remark 3.4: Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$. Further, assume that $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_n^+$. Then, it is known that $\boldsymbol{u} \succeq_w \boldsymbol{v} \Leftarrow \boldsymbol{u} \succeq^m \boldsymbol{v} \Rightarrow \boldsymbol{u} \succeq^w \boldsymbol{v} \Rightarrow \boldsymbol{u} \succeq^p \boldsymbol{v} \Rightarrow \boldsymbol{u} \succeq^{rm} \boldsymbol{v}$. Thus, under the assumptions made as in Theorems 3.4(ii) and 3.5, the desired usual stochastic order holds under the majorization-based conditions on the same sets of parameters.

Next, we establish the usual stochastic order between $X_{1:n}$ and $Y_{1:n}$. This can be proved after combining Theorems 3.4(i) and 3.5(ii).

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THEOREM 3.6: *Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of n independent random variables as in Assumption 3.1. Further, let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\lambda} \in \mathcal{D}_+(\mathcal{E}_+)$ and $\lambda_i = \mu_i,\ i = 1, \ldots, n$. Then, $\boldsymbol{\theta}^{-1} \succeq_w \boldsymbol{\delta}^{-1}, \boldsymbol{\alpha} \succeq^w \boldsymbol{\beta} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$, provided $u\tilde{r}^*(u)$ is increasing in u.*

To illustrate the usefulness of Theorem 3.6, we consider the following example.

Example 3.3: Consider two Systems I and II each having three components connected in series. Let the component lifetimes of System I be denoted by (X_1, X_2, X_3) and that for System II by (Y_1, Y_2, Y_3). Further, assume that the random lifetimes of the components have heterogeneous exponentiated location-scale models with baseline distribution $F(x) = (x/a)^l,\ 0 < x < a$ and $l > 0$. Set $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.71, 0.91)$, $(\beta_1, \beta_2, \beta_3) = (0.51, 0.81, 1)$, $(\theta_1, \theta_2, \theta_3) = (0.2, 0.31, 0.4)$, and $(\delta_1, \delta_2, \delta_3) = (0.3, 0.5, 0.6)$. Clearly, the assumptions are satisfied. Thus, from Theorem 3.6, we can conclude that System II has better performance than System I in the sense of the usual stochastic ordering.

Most of the results presented above can be used to get bounds of survival functions of a parallel and a series systems. For example, Theorem 3.4 can be used to compute an upper bound for the survival function of a series system consisting of independent heterogeneous exponentiated location-scale components, in terms of the corresponding functions of a series system comprising homogeneous exponentiated location-scale components. It is known that $(\alpha_1, \ldots, \alpha_n) \succeq^m (\bar{\alpha}, \ldots, \bar{\alpha})$, where $\bar{\alpha} = n^{-1} \sum_{i=1}^n \alpha_i$. Further, let $\lambda_i = \mu_i = \lambda$ and $\theta_i = \delta_i = \theta$. Then, as an application of Theorem 3.4(i), for the baseline distribution function $F(x) = e^{-1/x},\ x > 0$, one can derive the following upper bound of $X_{1:n}$ as

$$\bar{F}_{X_{1:n}}(x) \leq \left(1 - e^{-\theta\bar{\alpha}/(x-\lambda)}\right)^n, \quad x > \lambda.$$

Remark 3.5: It is noted that the usual stochastic order implies cumulative residual entropy order (see [42]). Thus the results developed in this section enable us to compare two parallel and series systems with heterogeneous exponentiated location-scale components in the sense of the cumulative residual entropy order.

Remark 3.6: Consider two electronic systems with random lifetimes X and Y. In this section, we have obtained various results based on the usual stochastic order, say $X \leq_{st} Y$. Now, suppose that at time $t > 0$, both systems are observed to be working. Then, one might conjecture that their residual lives would also be stochastically ordered. However, this is not true in general. Please refer to [31,33] in this direction. Hence, a stronger concept than the usual stochastic order is needed. Further, we assume that the systems have failed before the inspection time $t > 0$. Now, if $X \leq_{st} Y$, it is not necessarily true that the inactivity time of X is stochastically larger than that of Y. To resolve this issue, the concept of reversed hazard rate order is required. In the subsequent section, we consider comparison based on these stronger ordering concepts.

4. HAZARD RATE AND REVERSED HAZARD RATE ORDERS

Here, we obtain comparison results for the extreme order statistics in terms of the hazard rate and reversed hazard rate orderings. The proofs of the theorems of this section are presented in Appendix *B*. Note that for a set of n independent random variables $\{X_1, \ldots, X_n\}$

with $X_i \sim \mathcal{ELS}(\lambda_i, \theta_i, \alpha_i)$, the hazard rate and reversed hazard rate functions of $X_{1:n}$ and $X_{n:n}$ are given by

$$r_{1:n}(x) = \sum_{i=1}^{n} \frac{\alpha_i}{\theta_i} \tilde{r}^* \left(\frac{x - \lambda_i}{\theta_i} \right) \left[\frac{F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})}{1 - F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})} \right] \quad \text{and} \quad \tilde{r}_{n:n}(x) = \sum_{i=1}^{n} \frac{\alpha_i}{\theta_i} \tilde{r}^* \left(\frac{x - \lambda_i}{\theta_i} \right), \tag{4.1}$$

respectively. We denote $s_{1:n}(x)$ and $\tilde{s}_{n:n}(x)$ for the hazard rate and reversed hazard rate functions of the smallest and the largest order statistics of the set of independent random observations $\{Y_1, \ldots, Y_n\}$. First, we consider comparisons between the maximum order statistics. The following theorem shows that the largest order statistic of a set of independent random observations is smaller or larger than that of another set in terms of the reversed hazard rate ordering depending upon different sufficient conditions.

THEOREM 4.1: *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be two random vectors as mentioned in Assumption 3.1. Also, let $u\tilde{r}^*(u)$ be decreasing in u. Then, $\{\boldsymbol{\alpha} \leq (\geq)\boldsymbol{\beta}, \boldsymbol{\delta} \geq (\leq)\boldsymbol{\theta}, \boldsymbol{\mu} \geq (\leq)\boldsymbol{\lambda}\} \Rightarrow X_{n:n} \leq_{rh} (\geq_{rh}) Y_{n:n}$.*

Remark 4.1: Let $F(x) = 1 - e^{-x},\ x > 0$ be the baseline distribution. Then, Theorem 4.1 generalizes Corollary 4.1 of Kundu et al. [29].

The first part of the following result states that the largest order statistic of one set of independent observations dominates that of the other set when the vector of location parameters of the second set is weakly submajorized by that of the first set. We assume that two sets have the same sets of scale and shape parameters. Second part of the theorem shows similar result for the same sets of location and shape parameters.

THEOREM 4.2: *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors as in Assumption 3.1.*

(i) *Let $\theta_i = \delta_i,\ \alpha_i = \beta_i,\ i = 1, \ldots, n$ and $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta} \in \mathcal{E}_+$ or $\mathcal{D}_+$. If $u^2\tilde{r}^{*\prime}(u)$ and $\tilde{r}^*(u)$ are respectively increasing and decreasing in u, then $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$.*
(ii) *Let $\lambda_i = \mu_i, \alpha_i = \beta_i,\ i = 1, \ldots, n$ and $\boldsymbol{\alpha} \in \mathcal{D}_+$ (or $\mathcal{E}_+$), $\boldsymbol{\lambda}, \boldsymbol{\delta}, \boldsymbol{\theta} \in \mathcal{E}_+$ (or $\mathcal{D}_+$). If $u\tilde{r}^*(u)$ is convex in $u > 0$, then $\boldsymbol{\theta}^{-1} \succeq^m \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$.*

Remark 4.2: For $\boldsymbol{\alpha} = (1, \ldots, 1)$, Theorem 4.2(ii) reduces to Theorem 8(i) of Hazra et al. [20].

Next result generalizes Theorem 8(iii) of Hazra et al. [20]. We assume that two sets of independent random variables have the same sets of location and shape parameters, but different sets of scale parameters.

THEOREM 4.3: *Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of n independent random variables such that $X_i \sim \mathcal{ELS}(\mu_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}(\mu_i, \delta_i, \alpha_i)$ with $\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{\theta} \in \mathcal{D}_+$ (or $\mathcal{E}_+$).*

(i) *Then, $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$ if $u\tilde{r}^*(u)$ and $u[\tilde{r}^*(u) + u\tilde{r}^{*\prime}(u)]$ are decreasing and increasing in u, respectively.*
(ii) *Then, $\boldsymbol{\theta}^{-1} \succeq^{rm} \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$ provided $u\tilde{r}^*(u)$ is decreasing and $u^2\tilde{r}^*(u)$ is convex in u.*

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Remark 4.3: On using $\boldsymbol{\alpha} = (1,\ldots,1)$, Theorem 4.3(i) reduces to Theorem 8(iii) of Hazra et al. [20].

Note that Theorem 4.3(ii) can be thought of an extension of Theorem 8(iv) of Hazra et al. [20]. They established similar result to Theorem 4.3(ii) for the location-scale family of distributions with the same set of location parameters. Evidently, Theorem 4.3(ii) reduces to Theorem 8(iv) of Hazra et al. [20] for $\boldsymbol{\alpha} = (1,\ldots,1)$. Below, we obtain reversed hazard rate ordering between the largest order statistics when the vector of reciprocal of scale parameters of a set weakly supermajorized that of the other set. The proof is similar to Theorem 8(ii) of Hazra et al. [20]. Thus, it is omitted.

THEOREM 4.4: *Let $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_n\}$ be two sets of n independent random variables as in Assumption 3.1 with $\lambda_i = \mu_i, \alpha_i = \beta_i,\ i = 1,\ldots,n$. Further, let $\boldsymbol{\alpha} \in \mathcal{D}_+$ (or $\mathcal{E}_+$), $\boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{\theta} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, $\boldsymbol{\theta}^{-1} \succeq^{w} \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$ provided $u\tilde{r}^*(u)$ is convex and decreasing in u.*

Remark 4.4: For $\boldsymbol{\alpha} = (1,\ldots,1)$, Theorem 4.4 reduces to Theorem 8(ii) of Hazra et al. [20].

In the first part of the following theorem, we consider that both sets of random variables have the same set of scale and location parameters. The second part is a consequence of Theorems 4.2(i) and 4.5(i). Thus, the proof is omitted. We assume the same set of scale parameters for both sets of the random variables.

THEOREM 4.5: *Take two sets of independent random variables $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_n\}$ as in Assumption 3.1.*

(i) *Let $\lambda_i = \mu_i$ and $\theta_i = \delta_i,\ i = 1,\ldots,n$. Further, assume that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\mu} \in \mathcal{E}_+$ (or $\mathcal{D}_+$). Then, $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$ provided, $u\tilde{r}^*(u)$ is decreasing in u.*

(ii) *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\mu} \in \mathcal{E}_+$ or $\mathcal{D}_+$. Then, $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$ and $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}$ imply $X_{n:n} \geq_{rh} Y_{n:n}$ if $u\tilde{r}^*(u)$ and $\tilde{r}^*(u)$ are decreasing, and $u^2\tilde{r}^{*\prime}(u)$ is increasing in u.*

The following example illustrates Theorem 4.5(i).

Example 4.1: Let $F(x) = (1-\exp\{-x\})^{\beta},\ x > 0,\ \beta > 0$. In this case, $u\tilde{r}^*(u)$ is decreasing in $u > 0$ for any value of β. Let (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) be the random lifetimes of three components of System A and System B, respectively. Assume that the components are connected parallel. Further, let the component lifetimes follow exponentiated location-scale model with exponentiated exponential baseline distribution function. The values of the parameters are taken as $(\lambda_1, \lambda_2, \lambda_3) = (\mu_1, \mu_2, \mu_3) = (9.1, 9.2, 9.3)$, $(\theta_1, \theta_2, \theta_3) = (\delta_1, \delta_2, \delta_3) = (8.1, 8.3, 8.7)$, $(\alpha_1, \alpha_2, \alpha_3) = (5.1, 6.1, 7.1)$, and $(\beta_1, \beta_2, \beta_3) = (2.1, 3.1, 4.1)$. Thus, from Theorem 4.5(i), one can easily conclude that System A is better than System B in the sense of the reversed hazard rate ordering.

Analogous to Theorem 3.3, here, we obtain reversed hazard rate ordering between $X_{n:n}$ and $Y_{n:n}$ for two completely different sample sets with different sets of parameters. The proof of the first part follows after combining the results in Theorems 4.2(i), 4.3(ii), and then with Theorem 4.5(i). The second part follows from Theorems 4.2(i), 4.3(i), and then with Theorem 4.5(i).

THEOREM 4.6: *Let us take two sets of independent random variables $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_n\}$ as in Assumption 3.1. Further, let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}, \boldsymbol{\delta} \in \mathcal{D}_+(\mathcal{E}_+)$. Then,*

(i) $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}, \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}, \boldsymbol{\theta}^{-1} \succeq^{rm} \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$, *provided* $\tilde{r}^*(u)$ *and* $u\tilde{r}^*(u)$ *are decreasing,* $u^2\tilde{r}^{*\prime}(u)$ *is increasing and* $u^2\tilde{r}^*(u)$ *is convex in* u.

(ii) $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}, \boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}, \boldsymbol{\theta}^{-1} \succeq^{p} \boldsymbol{\delta}^{-1} \Rightarrow X_{n:n} \geq_{rh} Y_{n:n}$, *provided* $\tilde{r}^*(u)$ *and* $u\tilde{r}^*(u)$ *are decreasing,* $u^2\tilde{r}^{*\prime}(u)$ *and* $u[\tilde{r}^*(u) + u\tilde{r}^{*\prime}(u)]$ *are increasing in* u.

It can be shown that for any nonnegative random variables the statement "$u^2\tilde{r}^{*\prime}(u)$ is increasing" and "$u\tilde{r}^*(u)$ is convex" are equivalent. Using this result, in the following remark, we show that Theorem 4.6 extends last two parts of [20, Thm. 10].

Remark 4.5: Let $\boldsymbol{\alpha} = (1, \ldots, 1) = \boldsymbol{\beta}$. Then, Theorems 4.6(i) and 4.6(ii) reduce to Theorems 10(iv) and 10(iii), respectively in [20].

The following example shows an application of Theorem 4.6(i).

Example 4.2: Consider two parallel systems having three components each. Let the lifetimes of the components of the first system be denoted by (X_1, X_2, X_3) and that of the second system be denoted by (Y_1, Y_2, Y_3). Assume that the lifetimes follow exponentiated location-scale models with baseline distribution function $F(x) = 1 - \exp\{1 - x^{\alpha}\}$, $x \geq 1$, $\alpha > 0$. Let $(\lambda_1, \lambda_2, \lambda_3) = (5, 7, 9.1)$, $(\mu_1, \mu_2, \mu_3) = (4, 4.5, 6)$, $(\alpha_1, \alpha_2, \alpha_3) = (5, 5.6, 7)$, $(\beta_1, \beta_2, \beta_3) = (3, 3.9, 4)$, $(\theta_1, \theta_2, \theta_3) = (6.11, 6.51, 6.71)$, and $(\delta_1, \delta_2, \delta_3) = (3.15, 3.45, 3.65)$. Thus, clearly $(\lambda_1, \lambda_2, \lambda_3) \succeq_w (\mu_1, \mu_2, \mu_3)$, $(\alpha_1, \alpha_2, \alpha_3) \succeq_w (\beta_1, \beta_2, \beta_3)$, and $(\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1}) \succeq^{rm} (\delta_1^{-1}, \delta_2^{-1}, \delta_3^{-1})$. Further, for $\alpha = 2$, assumptions on $\tilde{r}^*(u)$, $u\tilde{r}^*(u)$, $u^2\tilde{r}^{*\prime}(u)$ and $u^2\tilde{r}^*(u)$ are satisfied. Thus, from Theorem 4.6(i), we say that first system performs better than the second system in the sense of the reversed hazard rate ordering.

Now, we consider comparison results for the minimum order statistics of two sets of n independent observations. Take $\boldsymbol{\lambda} = \boldsymbol{\mu} = \lambda$ and $\boldsymbol{\theta} = \boldsymbol{\delta} = \theta$. It is shown that the minimum order statistic of one set of random variables is smaller than that of the other set if the set of shape parameters of the second set is weakly supermajorized by that of the first set.

Theorem 4.7: *Let* $\boldsymbol{X} \sim \mathcal{ELS}(\lambda, \theta, \boldsymbol{\alpha})$ *and* $\boldsymbol{Y} \sim \mathcal{ELS}(\lambda, \theta, \boldsymbol{\beta})$ *with* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}_+$. *Then,* $\boldsymbol{\alpha} \succeq^w \boldsymbol{\beta} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$.

To illustrate Theorem 4.7, we provide the following example.

Example 4.3: Consider (X_1, X_2, X_3) as a vector of heterogeneous $\mathcal{ELS}(1.2, 2, \boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (3, 2.5, 2)$. Further, let (Y_1, Y_2, Y_3) be another set of independent heterogeneous $\mathcal{ELS}(1.2, 2, \boldsymbol{\beta})$ random variables with $\boldsymbol{\beta} = (4, 3.5, 3)$. Assume the baseline distribution as $F(x) = 1 - e^{-x}$, $x > 0$. It is easy to show that $(3, 2.5, 2) \succeq^w (4, 3.5, 3)$. Then, as an application of Theorem 4.7, we get $X_{1:3} \leq_{hr} Y_{1:3}$, which can be verified using graphical plots. We do not present the plots for brevity.

Theorem 4.7 can be used to compute a lower bound of a series system comprising of independent heterogeneous exponentiated location-scale components, in terms of the corresponding functions of a series system consisting of independent homogeneous exponentiated location-scale components. Consider the baseline distribution function as $F(x) = 1 - e^{-x}$, $x > 0$. Because $(\alpha_1, \ldots, \alpha_n) \succeq^m (\bar{\alpha}, \ldots, \bar{\alpha})$, where $\bar{\alpha} = n^{-1}\sum_{i=1}^{n} \alpha_i$, the

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following lower bound for the hazard rate function of $X_{1:n}$ based on Theorem 4.7 can be derived. It is given by

$$r_{X_{1:n}}(x) \geq \frac{n\bar{\alpha}}{\theta} e^{-(x-\lambda)/\theta} \frac{\left(1-e^{-(x-\lambda)/\theta}\right)^{\bar{\alpha}-1}}{1-\left(1-e^{-(x-\lambda)/\theta}\right)^{\bar{\alpha}}}, \quad x > \lambda. \tag{4.2}$$

5. MULTIPLE-OUTLIER MODELS

In real-life situations, we face systems where the components are from multiple-outlier models. A multiple-outlier model is a collection of independent random variables (lifetimes) $X_1, \ldots, X_n$ such that $X_i \overset{st}{=} X,\ i = 1, \ldots, p$ and $X_i \overset{st}{=} X^*,\ i = p+1, \ldots, n$, where $1 \leq p < n$. Here, the notation $X_i \overset{st}{=} X$ means that the distributions of X_i and X are same. These models have been widely used in various fields of statistics, especially in studying robustness of different estimators of parameters of some distributions. Readers may refer to [2] for various theoretical properties of a multiple-outlier model with its applications in robustness issues. Further, for some stochastic comparison results on this model, we refer the readers to [6,41,43]. In the following theorem, we show that under some conditions, the majorization order between the shape parameter vectors of exponentiated location-scale distributed components of parallel system implies R-hr ordering. Note that the proofs of the results of this section are provided in Appendix *C*.

THEOREM 5.1: *Assume two vectors of n independent nonnegative random variables* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *such that* $X_i \sim \mathcal{ELS}(\lambda, \theta, \alpha),\ i = 1, \ldots, p,\ X_i \sim \mathcal{ELS}(\lambda^*, \theta^*, \alpha^*),\ i = p+1, \ldots, p+q\ (= n)$ *and* $Y_i \sim \mathcal{ELS}(\lambda, \theta, \beta),\ i = 1, \ldots, p,\ Y_i \sim \mathcal{ELS}(\lambda^*, \theta^*, \beta^*),\ i = p+1, \ldots, p+q\ (= n)$. *Let* $(\alpha, \alpha^*), (\beta, \beta^*) \in \mathcal{E}_+$ *(or* $\mathcal{D}_+$*). Then,*

$$(\underbrace{\alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \ldots, \alpha^*}_{q}) \succeq^m (\underbrace{\beta, \ldots, \beta}_{p}, \underbrace{\beta^*, \ldots, \beta^*}_{q}) \Rightarrow X_{n:n} \leq_{R-hr} Y_{n:n}$$

if $\lambda \geq (\leq)\lambda^*,\ \theta \geq (\leq)\theta^*,\ u\tilde{r}^*(u)$ *is decreasing and* $u\tilde{r}^{*\prime}(u)/\tilde{r}^*(u)$ *is increasing in u.*

We obtain below sufficient conditions for the likelihood ratio order between the largest order statistics.

THEOREM 5.2: *Consider two vectors of independent random variables as in Theorem 5.1 with* $(\alpha, \alpha^*), (\beta, \beta^*) \in \mathcal{D}_+$. *Then,*

$$(\underbrace{\alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \ldots, \alpha^*}_{q}) \succeq^m (\underbrace{\beta, \ldots, \beta}_{p}, \underbrace{\beta^*, \ldots, \beta^*}_{q}) \Rightarrow X_{n:n} \leq_{lr} Y_{n:n}$$

if $\lambda = \lambda^*,\ \theta \leq \theta^*$ *and* $u\tilde{r}^*(u)$ *is decreasing and* $u\tilde{r}^{*\prime}(u)/\tilde{r}^*(u)$ *is increasing in u.*

We will now establish a result for the comparison of the smallest order statistics constructed from multiple-outlier exponentiated location-scale models. In particular, we obtain likelihood ratio order between $X_{1:n}$ and $Y_{1:n}$ if the shape parameters are ordered according to the majorization order.

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THEOREM 5.3: *Let us assume two vectors of n independent random observations $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_i \sim \mathcal{ELS}(\lambda, \theta, \alpha)$ and $Y_i \sim \mathcal{ELS}(\lambda, \theta, \beta)$ for $i = 1, \ldots, p$ and $X_i \sim \mathcal{ELS}(\lambda, \theta, \alpha^*)$ and $Y_i \sim \mathcal{ELS}(\lambda, \theta, \beta^*)$ for $i = p+1, \ldots, p+q(=n)$, where $(\alpha, \alpha^*), (\beta, \beta^*) \in \mathcal{D}_+$. Then,*

$$(\underbrace{\alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \ldots, \alpha^*}_{q}) \succeq^m (\underbrace{\beta, \ldots, \beta}_{p}, \underbrace{\beta^*, \ldots, \beta^*}_{q}) \Rightarrow X_{1:n} \leq_{lr} Y_{1:n}.$$

To illustrate Theorem 5.3, we consider the following example.

Example 5.1: Let $F(x) = 1 - e^{-x},\ \ x > 0$ be the baseline distribution of X. Further, assume two vectors of three independent random variables (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) such that $X_1 \sim \mathcal{ELS}(\lambda, \theta, \alpha)$ and $Y_1 \sim \mathcal{ELS}(\lambda, \theta, \beta)$; $X_2, X_3 \sim \mathcal{ELS}(\lambda, \theta, \alpha^*)$ and $Y_2, Y_3 \sim \mathcal{ELS}(\lambda, \theta, \beta^*)$. Here, we assume that $p = 1, q = 2, \alpha = 5, \alpha^* = 1, \beta = 3, \beta^* = 2, \lambda = 5$, and $\theta = 6$. Further, it can be shown that $(5, 1, 1) \succeq^m (3, 2, 2)$. Hence, as an application of Theorem 5.3, we obtain $X_{1:3} \leq_{lr} Y_{1:3}$.

Next, we consider multiple-outlier models, where $X_i \overset{st}{=} X,\ \ i = 1, \ldots, p$ and $X_i \overset{st}{=} Y,\ \ i = p+1, \ldots, p+q(=n)$ for a set of independent random variables $X_1, \ldots, X_n$. Let $X_{n:n}(p, q)$ and $X_{n^*:n^*}(p^*, q^*)$ be the largest order statistics from the sets $(X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q=n})$ and $(X_1, \ldots, X_{p^*}, X_{p^*+1}, \ldots, X_{p*+q*=n^*})$, respectively such that $n \neq n^*$. Denote the distribution functions of X and Y as F_1 and F_2, respectively. Then, from Arnold et al. [1], the distribution function of $X_{n:n}(p, q)$ is given by

$$F_{p,q}(x) = [F_1(x)]^p [F_2(x)]^q, \quad x > 0. \tag{5.1}$$

Below, we obtain comparison results for the largest and the smallest order statistics in terms of various stochastic orders. The first result is on the usual stochastic order.

THEOREM 5.4: *Let $X_1, \ldots, X_p$ be a random sample of size p considered from a random variable X with CDF $F^{\alpha_1}((x - \lambda_1)/\theta_1)$ and $X_{p+1}, \ldots, X_n$ be another independent random sample of size q drawn from a random variable Y with CDF $F^{\alpha_2}((x - \lambda_2)/\theta_2)$, where $n = p + q$ and $1 \leq p \leq q$. Further, let $1 \leq p^* \leq q^*$, $p^* \leq p \leq q \leq q^*$, $0 < \alpha_2 \leq \alpha_1$, $0 < \theta_2 \leq \theta_1$, $0 < \lambda_2 \leq \lambda_1$, and $n^* = p^* + q^*$. Then, $(p^*, q^*) \succeq^w (p, q) \Rightarrow X_{n^*:n^*}(p^*, q^*) \leq_{st} X_{n:n}(p, q)$.*

Next, we consider an example to illustrate the result stated in Theorem 5.4.

Example 5.2: Take $(p^*, q^*) = (2, 7)$ and $(p, q) = (5, 6)$. Further, set $(\alpha_1, \alpha_2) = (2, 1.5)$, $(\theta_1, \theta_2) = (4, 3)$, and $(\lambda_1, \lambda_2) = (4, 3.5)$. Here, $p^* \leq p \leq q \leq q^*$. It can be seen that $(p^*, q^*) \succeq^w (p, q)$. Then, for the baseline distribution with distribution function $F(x) = 1 - e^{-x}, x > 0$, we obtain $X_{9:9}(2, 7) \leq_{st} X_{11:11}(5, 6)$. This can be verified using graphical plots, which have been omitted for the sake of conciseness.

THEOREM 5.5: *Let $X_1, \ldots, X_p$ be a random sample of size p considered from a random variable X with CDF $F^{\alpha_1}((x - \lambda_1)/\theta_1)$ and $X_{p+1}, \ldots, X_n$ be another independent random sample of size q drawn from a random variable Y with CDF $F^{\alpha_2}((x - \lambda_2)/\theta_2)$, where $n = p + q$ and $1 \leq p \leq q$. Further, let $1 \leq p^* \leq q^*$, $p^* \leq p \leq q \leq q^*$, $0 < \alpha_2 \leq \alpha_1$, $0 < \theta_2 \leq \theta_1$, $0 < \lambda_2 \leq \lambda_1$, and $n^* = p^* + q^*$. Then, $(p^*, q^*) \succeq^w (p, q) \Rightarrow X_{n^*:n^*}(p^*, q^*) \leq_{rh} X_{n:n}(p, q)$ if $u\tilde{r}^*(u)$ is decreasing in u.*

Next, we establish comparison results between the smallest order statistics for multiple-outlier exponentiated location-scale models.

Theorem 5.6: *Assume $X_1, \ldots, X_p$ be the random sample of size p taken from a random variable X with CDF $F^{\alpha_1}((x-\lambda)/\theta)$ and $X_{p+1}, \ldots, X_n$ be another independent random sample of size q taken from a random variable Y with CDF $F^{\alpha_2}((x-\lambda)/\theta)$ such that $0 < \alpha_1 \le \alpha_2$, $n = p+q$, and $1 \le p \le q$. Further, assume $n^* = p^* + q^*$ and $1 \le p^* \le q^*$, $p^* \le p \le q \le q^*$. Then, $(p^*, q^*) \succeq^w (p, q) \Rightarrow X_{1:n^*}(p^*, q^*) \ge_{hr} X_{1:n}(p, q)$.*

Example 5.3: Set $(p^*, q^*) = (2, 7)$ and $(p, q) = (5, 6)$. Further, set $(\alpha_1, \alpha_2) = (1, 2)$, $\lambda = 1$ and $\theta = 2$. In this case, $p^* \le p \le q \le q^*$ and $(p^*, q^*) \succeq^w (p, q)$. Then, for the baseline distribution as in Example 5.2, we obtain $X_{1:9}(2, 7) \ge_{hr} X_{1:11}(5, 6)$.

6. SOME SPECIAL CASES

In this section, we discuss some special cases of the results established in the previous sections.

6.1. Weibull distribution

Assume that the baseline distribution in (1.1) is the Weibull distribution with CDF $F(x) = 1 - \exp\{-\sqrt{x}\}$, $x > 0$. Then, the corresponding cumulative distribution function of the exponentiated location-scale model is $F_X(x) = (1 - \exp\{-\sqrt{(x-\lambda)/\theta}\})^\alpha$, $x > \lambda$. We denote it by $\mathcal{ELS}^w(\lambda, \theta, \alpha)$.

- Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ be two sets of independent random variables such that $X_i \sim \mathcal{ELS}^w(\lambda_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^w(\mu_i, \delta_i, \beta_i)$, $i = 1, \ldots, n$. Let $\boldsymbol{\alpha} = (1.5, 3.5, 6.5)$, $\boldsymbol{\beta} = (0.5, 2.5, 5.5)$, $\boldsymbol{\lambda} = (2.4, 4.4, 6.4)$, $\boldsymbol{\mu} = (0.4, 3.4, 5.4)$, $\boldsymbol{\theta} = (5, 9.1, 10)$, and $\boldsymbol{\delta} = (2, 3.1, 5)$. Clearly, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}, \boldsymbol{\delta} \in \mathcal{E}_+$, $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}, \boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}$, and $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1}$. Also $u\tilde{r}^*(u)$ is decreasing in u. Thus, as an application of Theorem 3.3(i), we have $X_{3:3} \ge_{st} Y_{3:3}$.
- Consider two sets of independent random variables $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ such that $X_i \sim \mathcal{ELS}^w(\mu_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^w(\mu_i, \delta_i, \alpha_i)$, $i = 1, \ldots, n$. Let $\boldsymbol{\mu} = (0.5, 0.91, 1)$, $\boldsymbol{\theta} = (1.2, 1.5, 1.8)$, $\boldsymbol{\delta} = (1, 1.01, 1.05)$, and $\boldsymbol{\alpha} = (1.3, 1.6, 2.9)$. Clearly, $\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{\theta} \in \mathcal{E}_+$ and $\boldsymbol{\theta}^{-1} \succeq^w \boldsymbol{\delta}^{-1}$. Further, $u\tilde{r}^*(u)$ is convex and decreasing in u. Then, from Theorem 4.4, we have $X_{3:3} \ge_{rh} Y_{3:3}$.

6.2. Log-normal distribution

Let the baseline distribution in the exponentiated location-scale model be the log-normal distribution. The CDF of the log-normal distribution is $F(x) = \int_0^x (1/t\sqrt{2\pi}) \exp\{-(\ln t)^2/2\} dt$, $x > 0$. We denote the exponentiated location-scale model with log-normal as the baseline distribution by $\mathcal{ELS}^{ln}(\lambda, \theta, \alpha)$.

- Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ be two sets of independent random observations with $X_i \sim \mathcal{ELS}^{ln}(\lambda_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^{ln}(\lambda_i, \theta_i, \beta_i)$ for $i = 1, \ldots, n$. Take $\boldsymbol{\lambda} = (2, 2.5, 2.75), \boldsymbol{\theta} = (4, 4.9, 5), \boldsymbol{\alpha} = (2.3, 2.5, 6.5), \boldsymbol{\beta} = (3, 3.5, 4.5)$. Clearly, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{E}_+$ and $\boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathcal{E}_+$. Further, $u\tilde{r}^*(u)$ is decreasing with respect to u. It is also not difficult to see that $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$ holds. Thus, as an application of Theorem 4.5(i), we obtain $X_{3:3} \ge_{rh} Y_{3:3}$.

- We assume that $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ are sets of independent random observations with $X_i \sim \mathcal{ELS}^{ln}(\lambda_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^{ln}(\lambda_i, \delta_i, \alpha_i)$ for $i = 1, \ldots, n$. Let $\boldsymbol{\alpha} = (4.5, 5.1, 6.1)$, $\boldsymbol{\mu} = (5.5, 6.1, 7.2)$, $\boldsymbol{\theta} = (3.5, 6.5, 7.5)$, and $\boldsymbol{\delta} = (2.5, 5.5, 6.5)$. Note that $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$, $\boldsymbol{\theta}$, $\boldsymbol{\delta} \in \mathcal{E}_+$. Further, $u\tilde{r}^*(u)$ is decreasing and $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}$. As a result, from Theorem 3.5(i), we conclude that $X_{1:3} \geq_{st} Y_{1:3}$.

6.3. Mixture distribution

In this subsection, we consider a mixture distribution as the baseline distribution. In particular, we take the mixture of two Weibull distributions. Its distribution function is $F(x) = \frac{1}{3}(1 - e^{-x^{0.1}}) + \frac{2}{3}(1 - e^{-x^{0.8}})$, $x > 0$. The exponentiated location-scale model with this distribution as the baseline distribution is denoted by $\mathcal{ELS}^{mx}(\lambda, \theta, \alpha)$.

- Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ be two sets of independent random variables such that $X_i \sim \mathcal{ELS}^{mx}(\lambda_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^{mx}(\lambda_i, \delta_i, \alpha_i)$, $i = 1, \ldots, n$. Consider $\boldsymbol{\lambda} = (5.5, 6.5, 7.9)$, $\boldsymbol{\delta} = (4.5, 5.5, 6)$, $\boldsymbol{\alpha} = (9.1, 9.5, 9.9)$, and $\boldsymbol{\theta} = (8.2, 8.4, 8.9)$. It is easy to see that these vectors belong to $\mathcal{E}_+$ and $\boldsymbol{\theta}^{-1} \succeq^p \boldsymbol{\delta}^{-1}$. Further, $u\tilde{r}^*(u)$ is decreasing in u. Thus, an application of Theorem 3.2(i) confirms that $X_{3:3} \geq_{st} Y_{3:3}$.
- Let $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2, Y_3\}$ be the sets of independent random observations with $X_i \sim \mathcal{ELS}^{mx}(\mu_i, \theta_i, \alpha_i)$ and $Y_i \sim \mathcal{ELS}^{mx}(\mu_i, \theta_i, \beta_i)$, $i = 1, \ldots, n$. Consider $\boldsymbol{\alpha} = (7.9, 4.6, 3.5)$, $\boldsymbol{\beta} = (9.9, 5.6, 4.5)$, $\mu_i = 0.25$, and $\theta_i = 0.75$. Clearly, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}_+$. Further, $\boldsymbol{\alpha} \succeq^w \boldsymbol{\beta}$. Thus, as an application of Theorem 4.7, we get $X_{1:3} \leq_{rh} Y_{1:3}$.

7. APPLICATIONS IN AUCTION THEORY

There are many real-life applications of the ordering results. In this section, we discuss applications of few of our established results in auction theory. Auction theory has been an interest topic to various scientists because of its usefulness for sale of variety of items or purchasing services. For more details in auction theory, we refer to [26]. In real world, among all types of auctions, the sealed-bid private-value auction is of theoretical interest. Also, this type of auction has been used extensively. In this case, bidders hand in their bids to the auctioneer simultaneously and can neither observe their rivals' bids nor revise their own bids. The bidders having the highest bid wins. The bidders with the lowest bid wins in the reverse auction. Consequently, the bidder pays his own bid in the sealed-bid first-price auction (FPA). Few of our established results could be useful for some new light in the auction theory.

Let the bids follow exponentiated location-scale model. Then, under some conditions, Theorems 3.1(i), 3.1(iii), 3.2(i), and 3.2(ii) respectively conclude that the final price in the FPA with more heterogeneous shape parameters (in the weakly submajorized order), reciprocal of the shape parameters (in the weakly supermajorized order), reciprocal of the scale parameters (in the p-larger and reciprocally majorized orders) is stochastically larger. Theorems 4.2(i), 4.2(ii), and 4.5(i) respectively state that the final price in the FPA with more heterogeneous location parameters (in the subweakly majorization order), reciprocal of scale parameters (in the majorization order), and shape parameters (in the subweakly majorization order) is larger in the sense of the reversed hazard rate ordering. Further, let the bids be from multiple-outlier models. Then, Theorems 5.1 and 5.2 conclude that the final price in the FPA with more heterogeneous shape parameters in the majorization order is smaller in the sense of R-hr and likelihood ratio orderings, respectively. Similar observations for the first price in the FPA can be pointed out from other theorems.

8. CONCLUDING REMARKS

In this paper, we have considered two sets of n independent random lifetimes following heterogeneous exponentiated location-scale models. Various results comparing the extreme order statistics have been established. The comparisons are studied in terms of the usual stochastic, hazard rate, and reversed hazard rate orders. Further, we have considered multiple-outlier models. In this case, we have established some results comparing the extreme order statistics constructed from two sets of n independent random lifetimes. The comparisons are shown based on the usual stochastic order, ageing faster order in terms of the hazard rate, and likelihood ratio orders. The sufficient conditions involve majorization, weak majorization, reciprocal majorization, and p-larger orders. The results obtained in this paper extend some of the existing results. From the established results, we get some useful insights into the determination of a better reliability structure or system under different criteria. Also, this study is useful for placing different components in a structure. Further, these results are also useful in obtaining bounds of various characteristics of a reliability system in terms of that of other system.

There are situations in practice, where the system components are dependent. Hence, we get a set of statistically dependent observations instead of independent observations. One can consider this dependence with the help of the concept of copula (see [34]). Thus, one may study similar problem assuming dependence structure in the components.

Acknowledgments

The authors sincerely wish to thank the Editor, Associate Editor, and the two anonymous referees for the suggestions, which have considerably improved the content and the presentation of the paper. One of the authors, Sangita Das, thanks the financial support provided by the MHRD, Government of India. Suchandan Kayal gratefully acknowledges the partial financial support for this research work under a grant MTR/2018/000350, SERB, India.

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APPENDIX A

Proof of Theorem 3.1(i): The partial derivative of (3.1) with respect to $\alpha_i,\ i=1,\ldots,n$ is obtained as

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \alpha_i} = \ln\left[F\left(\frac{x-\lambda_i}{\theta_i}\right)\right] F_{X_{n:n}}(x), \tag{A.1}$$

which is at most zero. This implies that $F_{X_{n:n}}(x)$ is decreasing with respect to α_i for $i=1,\ldots,n$. Moreover, under the assumption made, for $1\le i\le j\le n$, we have $\lambda_i \le (\ge)\lambda_j$ and $\theta_i\le(\ge)\theta_j$. Hence, $(x-\lambda_i)/\theta_i \ge (\le)(x-\lambda_j)/\theta_j$. As a result, we obtain

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \alpha_i} - \frac{\partial F_{X_{n:n}}(x)}{\partial \alpha_j} \ge (\le) 0, \tag{A.2}$$

which ensures that $\partial F_{X_{n:n}}(x)/\partial\alpha_i$ is decreasing (increasing) in $i=1,\ldots,n$. From Lemma 3.3 (Lemma 3.1) of Kundu et al. [29], it can be shown that $F_{X_{n:n}}(x)$ is Schur-concave with respect to $\alpha_i \in \mathcal{E}_+(\mathcal{D}_+),\ i=1,\ldots,n$. Now, the rest of the proof follows from [31, Thm. A.8]. This completes the proof. ■

Proof of Theorem 3.1(ii): Based on the assumptions made, it is shown in the proof of Theorem 3.1(i) that the partial derivative of $F_{X_{n:n}}(x)$ with respect to α_i is decreasing (increasing) in $i=1,\ldots,n$. Thus, from Lemma 3.1 (Lemma 3.3) of Kundu et al. [29], $F_{X_{n:n}}(x)$ is Schur-convex with respect to $\alpha_i\in\mathcal{D}_+(\mathcal{E}_+),\ i=1,\ldots,n$. Using [31, Thm A.8], the proof readily follows. ■

Proof of Theorem 3.1(iii): The distribution function of $X_{n:n}$ can be expressed as the function of a_i, where $a_i = 1/\alpha_i,\ \ i=1,\ldots,n$. We denote it by $\phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta})$, where $\boldsymbol{a}=(a_1,\ldots,a_n)$. Differentiating $\phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta})$ with respect to a_i partially, we obtain

$$\frac{\partial \phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta})}{\partial a_i} = -\frac{1}{a_i^2}\ln\left[F\left(\frac{x-\lambda_i}{\theta_i}\right)\right]\phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta}), \tag{A.3}$$

which is nonnegative. Thus, $\phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta})$ is increasing with respect to $a_i,\ i=1,\ldots,n$. Let $1\le i\le j\le n$. Then, under the assumptions on the parameters, it can be shown that $\partial\phi_1/\partial a_i$ is increasing (decreasing) in $i=1,\ldots,n$. Hence, the function $\phi_1(\boldsymbol{a};x,\boldsymbol{\lambda},\boldsymbol{\theta})$ is Schur-concave in $a_i\in\mathcal{D}_+(\mathcal{E}_+)$. Now, the proof is completed from [31, Thm A.8]. ■

Proof of Theorem 3.2(i): We write the distribution function of $X_{n:n}$ as $\phi_2(e^{\boldsymbol{v}};x,\boldsymbol{\lambda},\boldsymbol{\alpha})$, where $v_i=-\ln\theta_i,\ i=1,\ldots,n$. The partial derivative of $\phi_2(e^{\boldsymbol{v}};x,\boldsymbol{\lambda},\boldsymbol{\alpha})$ with respect to v_i is given by

$$\frac{\partial \phi_2(e^{\boldsymbol{v}};x,\boldsymbol{\lambda},\boldsymbol{\alpha})}{\partial v_i} = \alpha_i e^{v_i}(x-\lambda_i)\tilde{r}^*((x-\lambda_i)e^{v_i})\phi_2(e^{\boldsymbol{v}};x,\boldsymbol{\lambda},\boldsymbol{\alpha}), \tag{A.4}$$

where $\tilde{r}^*(u)=f(u)/F(u)$. Clearly, the expression in the right-hand side of (A.4) is nonnegative, which implies that $\phi_2(e^{\boldsymbol{v}};x,\boldsymbol{\lambda},\boldsymbol{\alpha})$ is increasing in $v_i,\ i=1,\ldots,n$. Under the assumed conditions, for $1 \leq i \leq j \leq n$, it can be shown that

$$\frac{\partial \phi_2(e^{\boldsymbol{v}}; x, \boldsymbol{\lambda}, \boldsymbol{\alpha})}{\partial v_i} - \frac{\partial \phi_2(e^{\boldsymbol{v}}; x, \boldsymbol{\lambda}, \boldsymbol{\alpha})}{\partial v_j} \geq (\leq) 0. \tag{A.5}$$

This implies that $\partial \phi_2(e^{\boldsymbol{v}}; x, \boldsymbol{\lambda}, \boldsymbol{\alpha})/\partial v_i$ is decreasing (increasing) in $i = 1, \ldots, n$. Thus, $\phi_2(e^{\boldsymbol{v}}; x, \boldsymbol{\lambda}, \boldsymbol{\alpha})$ is Schur-concave in $v_i \in \mathcal{E}_+(\mathcal{D}_+)$. Now, the rest of the proof is completed from [23, Lem. 2.1]. ■

Proof of Theorem 3.2(ii): The proof follows using similar arguments as in the proof of Theorem 3.2(i), and thus it is omitted. ■

Proof of Theorem 3.4(i): Differentiating (3.1) with respect to α_i partially, we get

$$\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} = -\ln\left[F\left(\frac{x - \lambda_i}{\theta_i}\right)\right] \frac{F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})}{1 - F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})} \bar{F}_{X_{1:n}}(x), \tag{A.6}$$

which is nonnegative. Thus, $\bar{F}_{X_{1:n}}(x)$ is increasing in α_i, $i = 1, \ldots, n$. Let $1 \leq i \leq j \leq n$. Then, under the assumption made, $(x - \lambda_i)/\theta_i \geq (\leq)(x - \lambda_j)/\theta_j$. Thus, from [4, Lem. 2], we have

$$\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_j} \geq (\leq) 0, \tag{A.7}$$

which implies that $\partial \bar{F}_{X_{1:n}}(x)/\partial \alpha_i$ is decreasing (increasing) in $i = 1, \ldots, n$. So, from Lemma 3.3 (Lemma 3.1) of Kundu et al. [29], $\bar{F}_{X_{1:n}}(x)$ is Schur-concave with respect to $\alpha_i \in \mathcal{E}_+(\mathcal{D}_+)$. Hence, the desired result readily follows from [31, Thm A.8]. ■

Proof of Theorem 3.4(ii): The distribution function of $X_{1:n}$ is

$$F_{X_{1:n}}(x) = 1 - \prod_{i=1}^{n}\left[1 - e^{-e^{a_i} H\left(\frac{x-\lambda_i}{\theta_i}\right)}\right] = \phi_4(e^{\boldsymbol{a}}; x, \boldsymbol{\lambda}, \boldsymbol{\theta}), \quad \text{(say)}, \tag{A.8}$$

where $H((x - \lambda_i)/\theta_i) = -\ln[F((x - \lambda_i)/\theta_i)]$ and $a_i = \ln \alpha_i$. Further, similar to Theorem 3.2(i), it can be shown that the function $\phi_4(e^{\boldsymbol{a}}; x, \boldsymbol{\lambda}, \boldsymbol{\theta})$ is Schur-convex in $a_i \in \mathcal{E}_+$, $i = 1, \ldots, n$. The proof now follows from [23, Lem. 2.1]. ■

Proof of Theorem 3.5(i): The distribution function of $X_{1:n}$ is given by

$$F_{X_{1:n}}(x) = 1 - \prod_{i=1}^{n}\left[1 - F^{\alpha_i}\left((x - \lambda_i)\, e^{v_i}\right)\right] = \phi_5(e^{\boldsymbol{v}}; x, \boldsymbol{\lambda}, \boldsymbol{\alpha}), \quad \text{say}, \tag{A.9}$$

where $v_i = -\ln \theta_i$, $i = 1, \ldots, n$. The rest of the proof follows similar to that of Theorem 3.4(ii). Thus, it is omitted. We omit the proof of Part *(ii)* for brevity. ■

APPENDIX B

Proof of Theorem 4.1: Denote

$$l(x) = \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)} = \frac{\prod_{i=1}^{n} F^{\alpha_i}\left(\frac{x-\lambda_i}{\theta_i}\right)}{\prod_{i=1}^{n} F^{\beta_i}\left(\frac{x-\mu_i}{\delta_i}\right)}. \tag{B.1}$$

To prove the required result, our goal is to show that $l(x)$ is decreasing (increasing) in x. The derivative of $l(x)$ with respect to x is obtained as

$$l'(x) = \frac{F_{X_{n:n}}(x)}{F_{Y_{n:n}}(x)}\left[\sum_{i=1}^{n}\frac{\alpha_i}{\theta_i}\tilde{r}^*\left(\frac{x-\lambda_i}{\theta_i}\right) - \sum_{i=1}^{n}\frac{\beta_i}{\delta_i}\tilde{r}^*\left(\frac{x-\mu_i}{\delta_i}\right)\right]. \tag{B.2}$$

Now, $l'(x) \le (\ge) 0$ if

$$\sum_{i=1}^{n}\frac{\alpha_i}{\theta_i}\tilde{r}^*\left(\frac{x-\lambda_i}{\theta_i}\right) \le (\ge) \sum_{i=1}^{n}\frac{\beta_i}{\delta_i}\tilde{r}^*\left(\frac{x-\mu_i}{\delta_i}\right). \tag{B.3}$$

Further, under the assumption made, we have $(x-\mu_i)/\delta_i \le (\ge)(x-\lambda_i)/\theta_i$, and hence

$$\left(\frac{x-\lambda_i}{\theta_i}\right)\tilde{r}^*\left(\frac{x-\lambda_{\mathbf{i}}}{\theta_{\mathbf{i}}}\right) \le (\ge) \left(\frac{x-\mu_i}{\delta_i}\right)\tilde{r}^*\left(\frac{x-\mu_i}{\delta_i}\right)$$

since $u\tilde{r}^*(u)$ is decreasing in u. Again, for $i = 1,\ldots,n$, $\alpha_i/(x-\lambda_i) \le (\ge)(\beta_i/(x-\mu_i))$. Combining these, inequalities given in (B.3) follow. Thus, the proof is completed. ■

Proof of Theorem 4.2(i): We prove this theorem when the parameter vectors belong to $\mathcal{E}_+$. The proof is similar for the case when they belong to $\mathcal{D}_+$. Under the assumption made, it can be shown that the reversed hazard rate function of $X_{n:n}$ given by (4.1) is increasing in each $\lambda_i,\ i = 1,\ldots,n$. Further, for $1 \le i \le j \le n$,

$$\frac{\partial \tilde{r}_{n:n}(x)}{\partial \lambda_i} - \frac{\partial \tilde{r}_{n:n}(x)}{\partial \lambda_j} = \sum_{i=1}^{n}\frac{\alpha_i}{\theta_i^2}\tilde{r}^{*\prime}\left(\frac{x-\lambda_i}{\theta_i}\right) - \sum_{j=1}^{n}\frac{\alpha_j}{\theta_j^2}\tilde{r}^{*\prime}\left(\frac{x-\lambda_j}{\theta_j}\right) \ge 0 \tag{B.4}$$

if

$$\frac{\alpha_i}{(x-\lambda_i)^2}\left(\frac{x-\lambda_i}{\theta_i}\right)^2\tilde{r}^{*\prime}\left(\frac{x-\lambda_i}{\theta_i}\right) \ge \frac{\alpha_j}{(x-\lambda_j)^2}\left(\frac{x-\lambda_j}{\theta_j}\right)^2\tilde{r}^{*\prime}\left(\frac{x-\lambda_j}{\theta_j}\right). \tag{B.5}$$

Note that when $1 \le i \le j \le n$, we have $\alpha_i \le \alpha_j, \lambda_i \le \lambda_j$ and $\theta_i \le \theta_j$. Thus, $(x-\lambda_j)/\theta_j \le (x-\lambda_i)/\theta_i$ and $1/(x-\lambda_j)^2 \ge 1/(x-\lambda_i)^2$. Moreover, $u^2\tilde{r}^{*\prime}(u)$ is increasing in $u > 0$. Hence, after some simplification, the inequality in (B.5) holds. This implies that $\tilde{r}_{n:n}(x)$ in (4.1) is Schur-convex with respect to $\boldsymbol{\lambda} \in \mathcal{E}_+$ (from Kundu et al. [29, Lem. 3.3]). Thus, the desired result follows from [31, Thm. A.8]. ■

Proof of Theorem 4.2(ii): The reversed hazard rate of $X_{n:n}$ can be written as

$$\tilde{r}_{n:n}(x) = \phi_6(\boldsymbol{p}; x, \boldsymbol{\mu}, \boldsymbol{\alpha}), \quad \text{say}, \tag{B.6}$$

where $p_i = 1/\theta_i,\ i = 1,\ldots,n$. Let $1 \le i \le j \le n$. Then, under the assumption, $\alpha_i \ge \alpha_j, \theta_i \le \theta_j$ and $\mu_i \le \mu_j$. This imply that $p_j(x-\mu_j) \le p_i(x-\mu_i)$. Further, $u\tilde{r}^*(u)$ is convex, that is, $(d/du)(u\tilde{r}^*(u))$ is increasing in u. Using this, it can be shown that for $1 \le i \le j \le n$, $\partial\phi_6/\partial p_j - \partial\phi_6/\partial p_i \le 0$, that is, $\partial\phi_6/\partial p_i$ is decreasing in $i = 1,\ldots,n$. Thus, from [29, Lem. 3.1], the rest of the proof follows. The proof is similar when $\boldsymbol{\alpha} \in \mathcal{E}_+$ and $\boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{\theta} \in \mathcal{D}_+$. Thus, it is omitted. ■

Proof of Theorem 4.3(i): To prove the result, we denote the reversed hazard rate of $X_{n:n}$ given by (4.1) as $\phi_7(e^{\boldsymbol{p}}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})$, where $p_i = -\ln\theta_i$, $i = 1, \ldots, n$. Since $u\tilde{r}^*(u)$ is decreasing in $u > 0$, $\phi_7(e^{\boldsymbol{p}}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})$ is decreasing in each p_i, $i = 1, \ldots, n$. Now, for $1 \leq i \leq j \leq n$,

$$\frac{\partial\phi_7(e^{\boldsymbol{p}}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})}{\partial p_i} - \frac{\partial\phi_7(e^{\boldsymbol{p}}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})}{\partial p_j} \leq (\geq) 0 \tag{B.7}$$

if

$$\frac{\alpha_i}{x-\mu_i}[-u(\tilde{r}^*(u) + u\tilde{r}^{*\prime}(u))]_{u=(e^{p_i}(x-\mu_i))} \geq (\leq) \frac{\alpha_j}{x-\mu_j}[-u(\tilde{r}^*(x) + x\tilde{r}^{*\prime}(x))]_{x=(e^{p_j}(x-\mu_j))}. \tag{B.8}$$

Further, we have $\alpha_i \geq (\leq)\alpha_j$, $\theta_i \geq (\leq)\theta_j$ and $\mu_i \geq (\leq)\mu_j$ imply $(x-\mu_j)e^{p_j} \geq (\leq)(x-\mu_i)e^{p_i}$. Moreover, it is assumed that $u[\tilde{r}^*(u) + u\tilde{r}^{*'}(u)]$ is increasing in u. Hence, the inequality in (B.8) holds. Now, by Lemma 3.3 (Lemma 3.1) of Kundu et al. [29], it can be shown that $\phi_7(e^{\boldsymbol{p}}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})$ is Schur-convex with respect to $\boldsymbol{p} \in \mathcal{E}_+(\mathcal{D}_+)$. Now, the rest of the proof follows from [23, Lem. 2.1], thus it is omitted. ■

Proof of Theorem 4.3(ii): Here, we denote

$$\tilde{r}_{n:n}(x) = \sum_{i=1}^{n} \frac{\alpha_i}{\theta_i}\tilde{r}^*\left(\frac{x-\mu_i}{\theta_i}\right) = \phi_8(\boldsymbol{\theta}^{-1}; x, \boldsymbol{\mu}, \boldsymbol{\alpha}), \quad \text{say.} \tag{B.9}$$

Under the assumption made, it can be shown that $\phi_8(\boldsymbol{\theta}^{-1}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})$ is increasing in each θ_i, $i = 1, \ldots, n$. Furthermore, analogous to the above proof, it can be shown that $\phi_8(\boldsymbol{\theta}^{-1}; x, \boldsymbol{\mu}, \boldsymbol{\alpha})$ is Schur-convex in $\boldsymbol{\theta} \in \mathcal{D}_+$(or $\mathcal{E}_+$). Thus, the rest of the proof follows from [20, Lem. 2.1]. ■

Proof of Theorem 4.5(i): Note that the reversed hazard rate of $X_{n:n}$, denoted by $\tilde{r}_{n:n}(x)$ is increasing in each α_i, $i = 1, \ldots, n$. Further, under the assumption made, it can be shown that $\partial\tilde{r}_{n:n}(x)/\partial\alpha_i$ is increasing (decreasing) in $i = 1, \ldots, n$. Thus, from Lemma 3.3 (Lemma 3.1) of Kundu et al. [29], $\tilde{r}_{n:n}(x)$ is Schur-convex with respect to $\boldsymbol{\alpha} \in \mathcal{E}_+$(or $\mathcal{D}_+$). Now, the required result follows from [31, Thm. A.8]. ■

Proof of Theorem 4.7: The hazard rate of $X_{1:n}$ can be written as

$$r_{1:n}(x) = \sum_{i=1}^{n} \frac{1}{\theta}\phi(\alpha_i), \tag{B.10}$$

where

$$\phi(\alpha_i) = \alpha_i \frac{F^{\alpha_i-1}(\frac{x-\lambda}{\theta})f(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_i}(\frac{x-\lambda}{\theta})}. \tag{B.11}$$

Differentiating (B.11) with respect to α_i we get

$$\phi'(\alpha_i) = \frac{F^{\alpha_i-1}(\frac{x-\lambda}{\theta})f(\frac{x-\lambda}{\theta})[1 - F^{\alpha_i}(\frac{x-\lambda}{\theta}) + \alpha_i \ln F(\frac{x-\lambda}{\theta})]}{(1 - F^{\alpha_i}(\frac{x-\lambda}{\theta}))^2}, \tag{B.12}$$

which is nonpositive by [28, Lem. 2.2]. Thus, $\phi(\alpha_i)$ is decreasing in α_i. Further, differentiating (B.12) with respect to α_i, we obtain

$$\phi''(\alpha_i) = \frac{F^{\alpha_i-1}(\frac{x-\lambda}{\theta})f(\frac{x-\lambda}{\theta})\ln F(\frac{x-\lambda}{\theta})}{(1 - F^{\alpha_i}(\frac{x-\lambda}{\theta}))^3}\sigma(\alpha_i), \tag{B.13}$$

https://doi.org/10.1017/S0269964819000408 Published online by Cambridge University Press

where $\sigma(\alpha_i) = 2 - 2F^{\alpha_i}((x-\lambda)/\theta) + \ln F^{\alpha_i}((x-\lambda)/\theta) + \alpha_i F^{\alpha_i}((x-\lambda)/\theta)\ln F((x-\lambda_i)/\theta_i)$. Moreover, differentiating $\sigma(\alpha_i)$ with respect to α_i we get

$$\sigma'(\alpha_i) = \ln F\left(\frac{x-\lambda_i}{\theta_i}\right)\eta(\alpha_i), \tag{B.14}$$

where $\eta(\alpha_i) = 1 - F^{\alpha_i}((x-\lambda)/\theta) + \alpha_i F^{\alpha_i}((x-\lambda)/\theta)\ln F((x-\lambda)/\theta)$, which on differentiating again with respect to α_i, gives

$$\eta'(\alpha_i) = \alpha_i F^{\alpha_i}\left(\frac{x-\lambda}{\theta}\right)\left[\ln F\left(\frac{x-\lambda}{\theta}\right)\right]^2 > 0. \tag{B.15}$$

Further, $\eta(\alpha_i)$ is increasing in α_i with $\eta(\alpha_i) = 0$ at $\alpha_i = 0$. Thus, $\eta(\alpha_i) > 0$ for all $\alpha_i > 0$. So, from Equation (B.14), $\sigma(\alpha_i)$ is decreasing in α_i with $\sigma(\alpha_i) = 0$ at $\alpha_i = 0$. Hence $\sigma(\alpha_i) < 0$, for all $\alpha_i > 0$. Finally, from (B.13), $\phi''(\alpha_i) \geq 0$, gives that $\phi(\alpha_i)$ is convex in α_i, $i = 1, \ldots, n$. Now, from [29, Thm. 3.1(b)(ii)], $r_{1:n}(x)$ is Schur-convex in $\boldsymbol{\alpha} \in \mathcal{D}_+$. Further, it is shown that $r_{1:n}(x)$ is decreasing in each α_i. Hence, the proof follows from [31, Thm. A.8]. ■

APPENDIX C

Proof of Theorem 5.1: To prove the required result, we need to show that

$$k_1(x) = \frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)} = \frac{\sum_{i=1}^n \frac{\alpha_i}{\theta_i}\tilde{r}^*(\frac{x-\lambda_i}{\theta_i})}{\sum_{i=1}^n \frac{\beta_i}{\theta_i}\tilde{r}^*(\frac{x-\lambda_i}{\theta_i})} \tag{C.1}$$

is increasing in $x > 0$, where $\lambda_1 = \cdots = \lambda_p = \lambda$, $\theta_1 = \cdots = \theta_p = \theta$, $\lambda_{p+1} = \cdots = \lambda_n = \lambda^*$, $\theta_{p+1} = \cdots = \theta_n = \theta^*$, $\alpha_1 = \cdots = \alpha_p = \alpha$, $\alpha_{p+1} = \cdots = \alpha_n = \alpha^*$, $\beta_1 = \cdots = \beta_p = \beta$, $\beta_{p+1} = \cdots = \beta_n = \beta^*$. On differentiating (C.1) with respect to x, we get

$$k_1'(x) \stackrel{sign}{=} \frac{\sum_{i=1}^n \frac{\alpha_i}{\theta_i^2}\tilde{r}^{*\prime}(\frac{x-\lambda_i}{\theta_i})}{\sum_{i=1}^n \frac{\alpha_i}{\theta_i}\tilde{r}^*(\frac{x-\lambda_i}{\theta_i})} - \frac{\sum_{i=1}^n \frac{\beta_i}{\theta_i^2}\tilde{r}^{*\prime}(\frac{x-\lambda_i}{\theta_i})}{\sum_{i=1}^n \frac{\beta_i}{\theta_i}\tilde{r}^*(\frac{x-\lambda_i}{\theta_i})}. \tag{C.2}$$

Denote $\xi_1(\boldsymbol{\alpha}) = \sum_{i=1}^n(\alpha_i/\theta_i^2)\tilde{r}^{*\prime}((x-\lambda_i)/\theta_i)/\sum_{i=1}^n(\alpha_i/\theta_i)\tilde{r}^*((x-\lambda_i)/\theta_i)$. Differentiating this function with respect to α_i, $i = 1, \ldots, n$ we get

$$\frac{\partial\xi_1(\boldsymbol{\alpha})}{\partial\alpha_i} \stackrel{sign}{=} \frac{1}{\theta_i^2}\tilde{r}^{*\prime}\left(\frac{x-\lambda_i}{\theta_i}\right)\sum_{i=1}^n \frac{\alpha_i}{\theta_i}\tilde{r}^*\left(\frac{x-\lambda_i}{\theta_i}\right) - \frac{1}{\theta_i}\tilde{r}^*\left(\frac{x-\lambda_i}{\theta_i}\right)\sum_{i=1}^n \frac{\alpha_i}{\theta_i^2}\tilde{r}^{*\prime}\left(\frac{x-\lambda_i}{\theta_i}\right).$$

Now, consider the following cases:

Case-(i) For $1 \leq i \leq j \leq p$, let $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$. Then, $\partial\xi_1/\partial\alpha_i - \partial\xi_1/\partial\alpha_j = 0$.

Case-(ii) For $p+1 \leq i \leq j \leq n$, let $\alpha_i = \alpha_j = \alpha^*$ and $\beta_i = \beta_j = \beta^*$. Then, $\partial\xi_1/\partial\alpha_i - \partial\xi_1/\partial\alpha_j = 0$.

Case-(iii) Let $1 \leq i \leq p$ and $p+1 \leq j \leq n$. Further, let $\alpha_i = \alpha, \alpha_j = \alpha^*, \beta_i = \beta$ and $\beta_j = \beta^*$. In this case,

$$\frac{\partial\xi_1(\boldsymbol{\alpha})}{\partial\alpha_i} - \frac{\partial\xi_1(\boldsymbol{\alpha})}{\partial\alpha_j} \leq (\geq)0,$$

if $(1/\theta)\tilde{r}^{*\prime}((x-\lambda)/\theta)/\tilde{r}^*((x-\lambda)/\theta) \leq (\geq)(1/\theta^*)\tilde{r}^{*\prime}((x-\lambda^*)/\theta^*)/\tilde{r}^*((x-\lambda^*)/\theta^*)$, which is ensured from the assumptions made. Thus, from Lemma 3.3 (Lemma 3.1) of Kundu et al. [29], the desired result follows. ■

Proof of Theorem 5.2: We prove the result when $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}_+$. The reversed hazard rate of $X_{n:n}$ can be written as

$$\tilde{r}_{n:n}(x) = \sum_{i=1}^{n} u_i g(\alpha_i), \tag{C.3}$$

where $g(\alpha_i) = \alpha_i$ and $u_i = (1/(x-\mu))((x-\mu)/\theta_i)\tilde{r}^*((x-\mu)/\theta_i)$. Now, under the assumption made, it can be shown that u_i is decreasing whenever $\boldsymbol{\theta} \in \mathcal{E}_+$. Further, since $\boldsymbol{\alpha} \in \mathcal{D}_+$, $g(\alpha_i)$ is decreasing. Also, $g(\alpha_i)$ is convex. Thus, from Kundu et al. [29], it can be concluded that $\tilde{r}_{n:n}(x)$ is Schur-convex in $\boldsymbol{\alpha} \in \mathcal{D}_+$. This implies $X_{n:n} \geq_{rh} Y_{n:n}$. Now, using [38, Thm. 1.C.4], the proof follows if $\tilde{r}_{n:n}(x)/\tilde{s}_{n:n}(x)$ increases in $x > 0$. This readily follows from Theorem 5.1, which completes the proof. ■

Proof of Theorem 5.3: In order to prove the theorem, we have to show that

$$h(x) = \frac{f_{X_{1:n}}(x)}{f_{Y_{1:n}}(x)} \overset{sign}{=} \frac{\sum_{i=1}^{n} \frac{\alpha_i}{\theta} \frac{F^{\alpha_i}(\frac{x-\lambda}{\theta})}{1-F^{\alpha_i}(\frac{x-\lambda}{\theta})}}{\sum_{i=1}^{n} \frac{\beta_i}{\theta} \frac{F^{\beta_i}(\frac{x-\lambda}{\theta})}{1-F^{\beta_i}(\frac{x-\lambda}{\theta})}} \tag{C.4}$$

is decreasing in $x > 0$, where $\alpha_1 = \cdots = \alpha_p = \alpha$, $\alpha_{p+1} = \cdots = \alpha_n = \alpha^*$, $\beta_1 = \cdots = \beta_p = \beta$, $\beta_{p+1} = \cdots = \beta_n = \beta^*$. Now, differentiating $k(x)$ given by (C.4) with respect to x we get

$$\begin{aligned} h'(x) \overset{sign}{=} & \left[\sum_{i=1}^{n} \frac{1}{\theta^2} \frac{\alpha_i^2 F^{\alpha_i}(\frac{x-\lambda}{\theta})}{[1-F^{\alpha_i}(\frac{x-\lambda}{\theta})]^2}\right] \left[\sum_{i=1}^{n} \frac{\beta_i}{\theta} \frac{F^{\beta_i}(\frac{x-\lambda}{\theta})}{1-F^{\beta_i}(\frac{x-\lambda}{\theta})}\right] \\ & - \left[\sum_{i=1}^{n} \frac{1}{\theta^2} \frac{\beta_i^2 F^{\beta_i}(\frac{x-\lambda}{\theta})}{[1-F^{\beta_i}(\frac{x-\lambda}{\theta})]^2}\right] \left[\sum_{i=1}^{n} \frac{\alpha_i}{\theta} \frac{F^{\alpha_i}(\frac{x-\lambda}{\theta})}{1-F^{\alpha_i}(\frac{x-\lambda}{\theta})}\right]. \end{aligned} \tag{C.5}$$

Denote

$$\xi_2(\boldsymbol{\alpha}; x, \boldsymbol{\theta}, \boldsymbol{\lambda}) = \frac{\sum_{i=1}^{n} \frac{\alpha_i^2 F^{\alpha_i}(\frac{x-\lambda}{\theta})}{(1-F^{\alpha_i}(\frac{x-\lambda}{\theta}))^2}}{\sum_{i=1}^{n} \frac{\alpha_i F^{\alpha_i}(\frac{x-\lambda}{\theta})}{1-F^{\alpha_i}(\frac{x-\lambda}{\theta})}} = \frac{\sum_{i=1}^{n} \eta_1(\alpha_i; x)\eta_2(\alpha_i; x)}{\sum_{i=1}^{n} \eta_1(\alpha_i; x)}, \tag{C.6}$$

where $\eta_1(\alpha_i; x) = \alpha_i F^{\alpha_i}((x-\lambda)/\theta)/(1-F^{\alpha_i}((x-\lambda)/\theta))$ and $\eta_2(\alpha_i; x) = \alpha_i/(1-F^{\alpha_i}((x-\lambda)/\theta))$, $i = 1, \ldots, n$. Now, differentiating (C.6) with respect to α_i, $i = 1, \ldots, n$, we obtain

$$\begin{aligned} \frac{\partial \xi_2(\boldsymbol{\alpha})}{\partial \alpha_i} \overset{sign}{=} & [\eta_1'(\alpha_i; x)\eta_2(\alpha_i; x) + \eta_2'(\alpha_i; x)\eta_1(\alpha_i; x)] \sum_{i=1}^{n} \eta_1(\alpha_i; x) \\ & - \eta_1'(\alpha_i; x) \sum_{i=1}^{n} \eta_1(\alpha_i; x)\eta_2(\alpha_i; x). \end{aligned} \tag{C.7}$$

Now, consider the following cases:

Case-(i) For $1 \leq i \leq j \leq p$, let $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$. Then, $\partial \xi_2/\partial \alpha_i - \partial \xi_2/\partial \alpha_j = 0$.

Case-(ii) For $p+1 \leq i \leq j \leq n$, let $\alpha_i = \alpha_j = \alpha^*$ and $\beta_i = \beta_j = \beta^*$. Then, $\partial \xi_2/\partial \alpha_i - \partial \xi_2/\partial \alpha_j = 0$.

Case-(iii) Let $1 \leq i \leq p$ and $p+1 \leq j \leq n$. Further, let $\alpha_i = \alpha, \alpha_j = \alpha^*, \beta_i = \beta$ and $\beta_j = \beta^*$. In this case,

$$\frac{\partial \xi_2(\boldsymbol{\alpha})}{\partial \alpha_i} - \frac{\partial \xi_2(\boldsymbol{\alpha})}{\partial \alpha_j} \overset{sign}{=} \left[(p\eta_1(\alpha; x) + (n-p)\eta_1(\alpha^*; x)) \left(\eta_1(\alpha, x) \frac{\partial \eta_2(\alpha, x)}{\partial \alpha} - \eta_1(\alpha^*, x) \frac{\partial \eta_2(\alpha^*, x)}{\partial \alpha^*} \right) \right] + \left[\eta_1(\alpha; x)\eta_1(\alpha^*; x)(\eta_2(\alpha; x) - \eta_2(\alpha^*, x)) \left(\frac{(n-p)\eta_3(\alpha; x)}{\alpha} + \frac{p\eta_3(\alpha^*; x)}{\alpha^*} \right) \right], \tag{C.8}$$

where $\eta_3(x) = 1 + \alpha \ln F(x)/(1 - F^{\alpha}(x))$. Now, using [28, Lem. 2.5], it can be shown that $\eta_1(\alpha, x)(\partial\eta_2(\alpha, x)/\partial\alpha) < \eta_1(\alpha^*, x)(\partial\eta_2(\alpha^*, x)/\partial\alpha^*)$. Thus, the first bracketed term of (C.8) is negative. Further, from [28, Lem. 2.4], $\partial\eta_1/\partial\alpha = \eta_1\eta_3/\alpha$, where η_1 is decreasing in $\alpha > 0$. Moreover, η_1 is positive valued function, implies that η_3 is negative. Again, from [28, Lem. 2.3], $\eta_2(\alpha)$ is increasing in α and then $\eta_2(\alpha) \geq \eta_2(\alpha^*)$ since $\alpha \geq \alpha^*$. Thus, the second bracketed term is also negative. Hence from (C.8), $\partial\xi_2(\boldsymbol{\alpha})/\partial\alpha_i - \partial\xi_2(\boldsymbol{\alpha})/\partial\alpha_j \leq 0$. This completes the proof. ■

Proof of Theorem 5.4: To prove the result, it is sufficient to show that

$$\alpha_1(p - p^*) \ln \left[F\left(\frac{x - \lambda_1}{\theta_1} \right) \right] \leq \alpha_2(q^* - q) \ln \left[F\left(\frac{x - \lambda_2}{\theta_2} \right) \right]. \tag{C.9}$$

Now, $(p^*, q^*) \succeq^w (p, q) \Rightarrow p - p^* \geq q^* - q \Rightarrow \alpha_1(p - p^*) \ln F((x - \lambda_2)/\theta_2) \leq \alpha_2(q^* - q) \ln F((x - \lambda_2)/\theta_2)$ since $\alpha_1 \geq \alpha_2 > 0$. Further, under the assumption made, $(x - \lambda_1)/\theta_1 \leq (x - \lambda_2)/\theta_2 \Rightarrow \ln F((x - \lambda_1)/\theta_1) \leq \ln F((x - \lambda_2)/\theta_2) \Rightarrow \alpha_1(p - p^*) \ln F((x - \lambda_1)/\theta_1) \leq \alpha_1(p - p^*) \ln F((x - \lambda_2)/\theta_2)$. Thus, clearly, (C.9) holds. This completes the result. ■

Proof of Theorem 5.5: The reversed hazard rate of $X_{n:n}(p, q)$ is obtained as

$$\tilde{r}_{p,q}(x) = p\frac{\alpha_1}{\theta_1}\tilde{r}^*\left(\frac{x - \lambda_1}{\theta_1} \right) + q\frac{\alpha_2}{\theta_2}\tilde{r}^*\left(\frac{x - \lambda_2}{\theta_2} \right). \tag{C.10}$$

Analogously, the reversed hazard rate of $X_{n^*:n^*}(p^*, q^*)$ is

$$\tilde{r}_{p^*,q^*}(x) = p^*\frac{\alpha_1}{\theta_1}\tilde{r}^*\left(\frac{x - \lambda_1}{\theta_1} \right) + q^*\frac{\alpha_2}{\theta_2}\tilde{r}^*\left(\frac{x - \lambda_2}{\theta_2} \right). \tag{C.11}$$

From (C.10) and (C.11), the theorem is proved if $(\alpha_1/\theta_1)(p - p^*)\tilde{r}^*((x - \lambda_1)/\theta_1) \geq (\alpha_2/\theta_2)(q^* - q)\tilde{r}^*((x - \lambda_2)/\theta_2)$, equivalently $(\alpha_1/(x - \lambda_1))(p - p^*)((x - \lambda_1)/\theta_1)\tilde{r}^*((x - \lambda_1)/\theta_1) \geq (\alpha_2/(x - \lambda_2))(q^* - q)((x - \lambda_2)/\theta_2)\tilde{r}^*((x - \lambda_2)/\theta_2)$, which can be achieved after some calculations based on the assumptions taken. Hence, the proof is completed. ■

Proof of Theorem 5.6: The hazard rate of $X_{1:n}(p, q)$ is

$$r_{p,q}(x) = p\frac{\alpha_1}{\theta}r^*\left(\frac{x - \lambda}{\theta} \right) \frac{F^{\alpha_1}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_1}(\frac{x-\lambda}{\theta})} + q\frac{\alpha_2}{\theta}r^*\left(\frac{x - \lambda}{\theta} \right) \frac{F^{\alpha_2}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_2}(\frac{x-\lambda}{\theta})}. \tag{C.12}$$

Similarly, the hazard rate of $X_{1:n^*}(p^*, q^*)$ is

$$r_{p^*,q^*}(x) = p^*\frac{\alpha_1}{\theta}r^*\left(\frac{x - \lambda}{\theta} \right) \frac{F^{\alpha_1}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_1}(\frac{x-\lambda}{\theta})} + q^*\frac{\alpha_2}{\theta}r^*\left(\frac{x - \lambda}{\theta} \right) \frac{F^{\alpha_2}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_2}(\frac{x-\lambda}{\theta})}. \tag{C.13}$$

Thus, to prove the theorem, we need to show the following inequality:

$$\alpha_1(p - p^*)\frac{F^{\alpha_1}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_1}(\frac{x-\lambda}{\theta})} \geq \alpha_2(q^* - q)\frac{F^{\alpha_2}(\frac{x-\lambda}{\theta})}{1 - F^{\alpha_2}(\frac{x-\lambda}{\theta})}.$$

This can be shown using [28, Lem. 2.2] since $(p^*, q^*) \succeq^w (p, q)$ and $\alpha_1 \leq \alpha_2$. ■