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Computable Følner monotilings and a theorem of Brudno

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Abstract. A theorem of Brudno says that the Kolmogorov–Sinai entropy of an ergodic subshift over $\mathbb{N}$ equals the asymptotic Kolmogorov complexity of almost every word in the subshift. The purpose of this paper is to extend this result to subshifts over computable groups that admit computable regular symmetric Følner monotilings, which we introduce in this work. For every $d \in \mathbb{N}$, the groups $\mathbb{Z}^d$ and $\mathsf{UT}_{d+1}(\mathbb{Z})$ admit computable regular symmetric Følner monotilings for which the required computing algorithms are provided.

Key words: entropy, Kolmogorov complexity, measure-preserving dynamical systems
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1. *Introduction*

It was proved by Brudno in [3] that the Kolmogorov–Sinai entropy of an ergodic $\mathbb{N}$-dynamical system equals almost everywhere the Kolmorogov complexity of its orbits. An important special case is that of subshifts over $\mathbb{N}$. This says that if $\mathbf{X} = (\mathrm{X}, \mu, \mathbb{N})$ is an ergodic subshift over $\mathbb{N}$ with some finite alphabet Λ, then for μ-almost every (a.e.) $\omega \in \mathrm{X}$ we have

$$h(\mathbf{X}) = \limsup_{n\to\infty} \frac{\mathrm{K}(\omega|_{[1,\ldots,n]})}{n},$$

where $h(\mathbf{X})$ is the Kolmogorov–Sinai entropy of $\mathbf{X}$ and $\mathrm{K}(\omega|_{[1,\ldots,n]})$ is the *Kolmogorov complexity* of the word $\omega|_{[1,\ldots,n]}$ of length n. Roughly speaking, $\mathrm{K}(\omega|_{[1,\ldots,n]})$ is the length of the shortest description of $\omega|_{[1,\ldots,n]}$ for an 'optimal decompressor' which takes finite binary words as the input and produces finite words over the alphabet Λ as the output. A similar result can be easily proved for ergodic subshifts over the group $\mathbb{Z}$ of integers, and a partial generalization to the case of ergodic subshifts over $\mathbb{Z}^d$ was given by Simpson in [15]. The question remains if one can generalize this theorem beyond the $\mathbb{Z}$ and $\mathbb{Z}^d$ cases.

The purpose of this paper is to extend Brudno's theorem to the case of ergodic subshifts over computable groups that admit computable regular symmetric Følner monotilings (see §2.6 for the definition). The class of all such groups includes, for every $d \in \mathbb{N}$, the group $\mathbb{Z}^d$ and the group of unipotent upper-triangular matrices $\mathsf{UT}_{d+1}(\mathbb{Z})$ with integer entries of dimension $d + 1$.

The paper is structured as follows. We devote §2.1 to the general preliminaries on amenable groups and entropy theory. Regular Følner monotilings, which are a special type of *Følner monotilings* from the work of Weiss [16], are introduced in §2.2. We provide some basic notions from the theory of computability and Kolmogorov complexity in §2.3, and in §2.4 we define computable spaces, word presheaves and (asymptotic) Kolmogorov complexity of sections of these presheaves. Section 2.5, based on the work [14], contains the definition of a computable group and some basic examples. We proceed by introducing computable Følner monotilings in §2.6 and explaining why the groups $\mathbb{Z}^d$ and $\mathsf{UT}_{d+1}(\mathbb{Z})$ do admit computable regular symmetric Følner monotilings for every $d \geq 1$. The main result of this paper (Theorem 3.1) is proved in §3.

This paper is based on the preprint [13], which has some overlap with the previous preprint [12], where the original results [2] of Brudno about the topological entropy of subshifts and Kolmogorov complexity were extended. It is also worth mentioning that an alternative approach towards generalizing the original results of Brudno has been suggested in a more recent preprint [1]. We will not discuss these results in this work.

2. *Preliminaries*

2.1. *Amenable groups and ergodic theory.*

In this section we will remind the reader of the classical notion of amenability and state some results from ergodic theory of amenable group actions. We stress that all the groups that we consider are discrete and countably infinite. In what follows we shall rely mostly on [10, 17].

Let Γ be a group with the counting measure $|\cdot|$. A sequence of finite sets $(F_n)_{n\geq 1}$ is called:

(1) a *left weak Følner sequence* (respectively, *right weak Følner sequence*) if for every finite set $K \subseteq \Gamma$ one has

$$\frac{|F_n \triangle K F_n|}{|F_n|} \to 0 \quad \left(\text{respectively, } \frac{|F_n \triangle F_n K|}{|F_n|} \to 0\right);$$

(2) a *left strong Følner sequence* (respectively, *right strong Følner sequence*) if for every finite set $K \subseteq \Gamma$ one has

$$\frac{|\partial_K^{\mathrm{L}}(F_n)|}{|F_n|} \to 0 \quad \left(\text{respectively, } \frac{|\partial_K^{\mathrm{R}}(F_n)|}{|F_n|} \to 0\right),$$

where the sets

$$\partial_K^{\mathrm{L}}(F) := K^{-1}F \cap K^{-1}F^{\mathrm{c}}$$

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and

$$\partial^{\mathrm{r}}_K(F) := FK^{-1} \cap F^{\mathrm{c}}K^{-1}$$

are the *left K-boundary of F* and the *right K-boundary of F*, respectively;

(3) a *(C-)tempered sequence* if there is a constant C such that for every j one has

$$\left| \bigcup_{i<j} F_i^{-1} F_j \right| < C|F_j|.$$

One can show that a sequence of sets $(F_n)_{n\geq 1}$ is a weak left Følner sequence if and only if it is a strong left Følner sequence (see [4], §5.4), hence we will simply call it a left Følner sequence. The same holds for right Følner sequences. If we call a sequence of sets a *Følner sequence* without saying if it is 'left' or 'right', we always mean a left Følner sequence. A sequence of sets $(F_n)_{n\geq 1}$ which is simultaneously a left and a right Følner sequence is called a *two-sided Følner sequence*. A group Γ is called *amenable* if it admits a left Følner sequence. It can be shown that every amenable group admits a two-sided Følner sequence. Since Γ is infinite, for every Følner sequence $(F_n)_{n\geq 1}$ we have $|F_n| \to \infty$ as $n \to \infty$. For a finite subset $K \subseteq \Gamma$ and a subset $F \subseteq \Gamma$ the sets

$$\mathrm{int}^{\mathrm{L}}_K(F) := F \setminus \partial^{\mathrm{L}}_K(F)$$

and

$$\mathrm{int}^{\mathrm{R}}_K(F) := F \setminus \partial^{\mathrm{R}}_K(F)$$

are called the *left K-interior of F* and the *right K-interior of F*, respectively. It is clear that if a sequence of finite sets $(F_n)_{n\geq 1}$ is a left (respectively, right) Følner sequence, then for every finite $K \subseteq \Gamma$ one has

$$|\mathrm{int}^{\mathrm{L}}_K(F_n)|/|F_n| \to 1 \ \ (\text{respectively, } |\mathrm{int}^{\mathrm{R}}_K(F_n)|/|F_n| \to 1)$$

as $n \to \infty$.

One of the reasons why Følner sequences are of interest in this work is that they are 'good' for averaging group actions. In what follows all group actions are left actions. We denote the averages by $\mathbb{E}_{g\in F} := 1/|F| \sum_{g\in F}$. The following important theorem was proved by Lindenstrauss in [10].

THEOREM 2.1. *Let* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ *be a measure-preserving dynamical system, where the group* Γ *is amenable and* $(F_n)_{n\geq 1}$ *is a tempered left Følner sequence. Then for every* $f \in \mathrm{L}^1(\mathrm{X})$ *there is a* Γ*-invariant* $\overline{f} \in \mathrm{L}^1(\mathrm{X})$ *such that*

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f(g \cdot \omega) = \overline{f}(\omega)$$

for μ*-a.e.* $\omega \in \mathrm{X}$. *If the system* $\mathbf{X}$ *is ergodic, then*

$$\lim_{n\to\infty} \mathbb{E}_{g\in F_n} f(g \cdot \omega) = \int f \, d\mu$$

for μ*-a.e.* $\omega \in \mathrm{X}$.

We will need a weighted variant of this result. A function c on Γ is called a *good weight* for pointwise convergence of ergodic averages along a tempered left Følner sequence $(F_n)_{n\geq 1}$ in Γ if for every measure-preserving system $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ and every $f \in \mathrm{L}^\infty(\mathrm{X})$

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the averages

$$\mathbb{E}_{g\in F_n} c(g) f(g\cdot\omega)$$

converge as $n\to\infty$ for μ-a.e. $\omega\in \mathrm{X}$.

We will use a special case of [17, Theorem 1.3].

THEOREM 2.2. *Let Γ be a group with a tempered Følner sequence $(F_n)_{n\geq 1}$. Then for every ergodic measure-preserving system $\mathbf{X}=(\mathrm{X},\mu,\Gamma)$ and every $f\in \mathrm{L}^\infty(\mathrm{X})$ there exists a full-measure subset $\widetilde{X}\subseteq X$ such that for every $x\in\widetilde{X}$ the map $g\mapsto f(g\cdot x)$ is a good weight for the pointwise ergodic theorem along $(F_n)_{n\geq 1}$.*

We will now briefly remind the reader of the notion of the Kolmogorov–Sinai entropy for amenable group actions. Let $\alpha=\{A_1,\ldots,A_n\}$ be a finite measurable partition of a probability space X endowed with a probability measure μ. The function $\omega\mapsto\alpha(\omega)$, mapping a point $\omega\in\mathrm{X}$ to the atom of the partition α containing ω, is defined almost everywhere. The *information function* of α is defined as

$$I_\alpha(\omega):=-\sum_{i=1}^n \mathbf{1}_{A_i}(\omega)\log\mu(A_i)=-\log(\mu(\alpha(\omega))).$$

Then $I_\alpha\in\mathrm{L}^\infty(\mathrm{X})$. The *Shannon entropy* of a partition α is defined by

$$h_\mu(\alpha):=-\sum_{i=1}^n \mu(A_i)\log(\mu(A_i))=\int I_\alpha\,d\mu.$$

The entropy of a partition is always a non-negative real number. If α,β are two finite measurable partitions of X, then

$$\alpha\vee\beta:=\{A\cap B : A\in\alpha,\ B\in\beta\}$$

is a finite measurable partition of X as well. Given a measure-preserving dynamical system $\mathbf{X}=(\mathrm{X},\mu,\Gamma)$, where the discrete amenable group Γ acts on X, we can also define the (dynamical) entropy of a partition. First, for every element $g\in\Gamma$ and every finite measurable partition α we define a finite measurable partition $g^{-1}\alpha$ by

$$g^{-1}\alpha=\{g^{-1}A : A\in\alpha\}.$$

Next, for every finite subset $F\subseteq\Gamma$ and every partition α we define the *F-refinement* of α by

$$\alpha^F:=\bigvee_{g\in F} g^{-1}\alpha.$$

Let $(F_n)_{n\geq 1}$ be a Følner sequence in Γ and α be a finite measurable partition of X. Then the limit

$$h_\mu(\alpha,\Gamma):=\lim_{n\to\infty}\frac{h_\mu(\alpha^{F_n})}{|F_n|}$$

exists, and it is a non-negative real number independent of the choice of a Følner sequence due to the lemma of Ornstein and Weiss (see [6, 9]). The limit $h_\mu(\alpha,\Gamma)$ is called the *dynamical entropy of α*. We define the *Kolmogorov–Sinai entropy* of a measure-preserving

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system $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ by

$$h(\mathbf{X}) := \sup\{h_\mu(\alpha, \Gamma) : \alpha \text{ a finite measurable partition of X}\}.$$

We will need the Shannon–McMillan–Breiman theorem for amenable group actions. For the proof see [10].

THEOREM 2.3. *Let $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be an ergodic measure-preserving system and α be a finite partition of* X. *Assume that $(F_n)_{n\geq 1}$ is a tempered Følner sequence in Γ such that $|F_n|/\log n \to \infty$ as $n \to \infty$. Then there is a constant $h'_\mu(\alpha, \Gamma)$ such that*

$$\frac{I_{\alpha^{F_n}}(\omega)}{|F_n|} \to h'_\mu(\alpha, \Gamma) \tag{1}$$

as $n \to \infty$ for μ-a.e. $\omega \in \mathrm{X}$ and in $\mathrm{L}^1(\mathrm{X})$.

Integrating both sides of equation (1) with respect to μ, we deduce that

$$\frac{h_\mu(\alpha^{F_n})}{|F_n|} \to h_\mu(\alpha, \Gamma) = h'_\mu(\alpha, \Gamma)$$

as $n \to \infty$. The Shannon–McMillan–Breiman theorem has the following important corollary that will be used in the proof of Theorem 3.3 [4].

COROLLARY 2.1. *Let $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be an ergodic measure-preserving system, and $(F_n)_{n\geq 1}$ be a tempered Følner sequence in Γ such that $|F_n|/\log n \to \infty$ as $n \to \infty$. Let α be a finite partition. Then, given $\varepsilon > 0$ and $\delta > 0$, there exists n_0 such that the following assertions hold.*

(a) *For all $n \geq n_0$,*

$$2^{-|F_n|(h_\mu(\alpha,\Gamma)+\varepsilon)} \leq \mu(A) \leq 2^{-|F_n|(h_\mu(\alpha,\Gamma)-\varepsilon)}$$

for all atoms $A \in \alpha^{F_n}$ with the exception of a set of atoms whose total measure is less than δ.

(b) *For all $n \geq n_0$,*

$$2^{-|F_n|(h_\mu(\alpha,\Gamma)+\varepsilon)} \leq \mu(\alpha^{F_n}(\omega)) \leq 2^{-|F_n|(h_\mu(\alpha,\Gamma)-\varepsilon)}$$

for all but at most δ fraction of elements $\omega \in \mathrm{X}$.

Proof. By Theorem 2.3, $I_{\alpha^{F_n}}(\omega)/|F_n| \to h_\mu(\alpha, \Gamma)$ for a.e. ω and hence also in measure. Thus, given $\varepsilon, \delta > 0$ as above, there is n_0 such that for all $n \geq n_0$ we have

$$\mu\left\{\omega \in \mathrm{X} : \left|\frac{I_{\alpha^{F_n}}(\omega)}{|F_n|} - h_\mu(\alpha, \Gamma)\right| \geq \varepsilon\right\} < \delta.$$

It is now clear that both assertions follow. □

2.2. *(Regular) Følner monotilings.* The purpose of this section is to discuss the notion of a *Følner monotiling*, which was introduced by Weiss in [16]. However, in this article we have to introduce both 'left' and 'right' monotilings, while the original notion introduced by Weiss is a 'left' monotiling. The (new) notion of a *regular Følner monotiling*, central to the results of this paper, will also be suggested below.

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A *left monotiling* $[F, \mathcal{Z}]$ in a discrete group Γ is a pair of a finite set $F \subseteq \Gamma$, which we call *a tile*, and a set $\mathcal{Z} \subseteq \Gamma$, which we call a set of *centers*, such that $\{Fz : z \in \mathcal{Z}\}$ is a covering of Γ by disjoint translates of F. Similarly, Given a *right monotiling* $[\mathcal{Z}, F]$, we require that $\{zF : z \in \mathcal{Z}\}$ is a covering of Γ by disjoint translates of F. A *left Følner monotiling* (respectively, *right Følner monotiling*) is a sequence of monotilings $([F_n, \mathcal{Z}_n])_{n\geq 1}$ (respectively, $([\mathcal{Z}_n, F_n])_{n\geq 1}$) such that $(F_n)_{n\geq 1}$ is a left (respectively, right) Følner sequence in Γ. A left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is called *symmetric* if for every $k \geq 1$ the set of centers $\mathcal{Z}_k$ is symmetric, that is, $\mathcal{Z}_k^{-1} = \mathcal{Z}_k$. It is clear that if $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling, then $([\mathcal{Z}_n, F_n^{-1}])_{n\geq 1}$ is a right Følner monotiling.

We begin with a basic example.

Example 2.4. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$ and the Følner sequence $(F_n)_{n\geq 1}$ in $\mathbb{Z}^d$ given by

$$F_n := [0, 1, 2, \dots, n-1]^d.$$

Furthermore, for every n let

$$\mathcal{Z}_n := n\mathbb{Z}^d.$$

It is easy to see that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling of $\mathbb{Z}^d$, and that $(F_n)_{n\geq 1}$ is a tempered two-sided Følner sequence.

A less trivial example is given by Følner monotilings of the discrete Heisenberg group $\mathsf{UT}_3(\mathbb{Z})$. We will return to Følner monotilings of $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$ later.

Example 2.5. Consider the group $\mathsf{UT}_3(\mathbb{Z})$, the discrete Heisenberg group H_3. By definition,

$$\mathsf{UT}_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

To simplify the notation, we will denote a matrix

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{UT}_3(\mathbb{Z})$$

by the corresponding triple (a, b, c) of its entries. Then the products and inverses in $\mathsf{UT}_3(\mathbb{Z})$ can be computed by the formulas

$$(a, b, c)(x, y, z) = (a + x, b + y, c + z + ya),$$
$$(a, b, c)^{-1} = (-a, -b, ba - c).$$

For every $n \geq 1$, consider the subgroup

$$\mathcal{Z}_n := \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : a, b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}.$$

This is a finite-index subgroup, and it is easy to see that for every n the finite set

$$F_n := \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

is a fundamental domain for $\mathcal{Z}_n$. One can show (see [11]) that $(F_n)_{n\geq 1}$ is a left Følner sequence, and a similar argument shows that it is a right Følner sequence as well.

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$([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a symmetric Følner monotiling. In order to check the temperedness of $(F_n)_{n\geq 1}$, note that for every $n > 1$,

$$\bigcup_{i<n} F_i^{-1} F_n \subseteq F_n^{-1} F_n,$$

where

$$F_n^{-1} \subseteq \{(a, b, c) : -n < a, b \leq 0, -n^2 < c < n^2\}.$$

It is easy to see that for every $n > 1$,

$$F_n^{-1} F_n \subseteq \{(a, b, c) : -n < a, b < n, -3n^2 < c < 3n^2\}.$$

Since $|F_n| = n^4$ for every n, the sequence $(F_n)_{n\geq 1}$ is tempered.

For the purposes of this work we need to introduce special Følner monotilings where one can 'average' along the intersections $F_n \cap \mathcal{Z}_k$ for every fixed k and $n \to \infty$. This, together with some other requirements, leads to the following definition. We call a left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ *regular* if the following assumptions hold:

(a) the sequence $(F_n)_{n\geq 1}$ is a tempered two-sided Følner sequence;
(b) for every k the function $\mathbf{1}_{\mathcal{Z}_k} \in \mathrm{L}^\infty(\Gamma)$ is a good weight for pointwise convergence of ergodic averages along the sequence $(F_n)_{n\geq 1}$;
(c) $|F_n|/\log n \to \infty$ as $n \to \infty$;
(d) $\mathrm{e} \in F_n$ for every n.

Of course, our motivating example for the notion of a regular Følner monotiling is Example 2.4. Below we explain why the corresponding indicator functions $\mathbf{1}_{\mathcal{Z}_k}$ are good weights for every k. Checking the remaining conditions for the regularity of the Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is straightforward.

Example 2.6. Let Γ be an amenable group with a fixed tempered Følner sequence $(F_n)_{n\geq 1}$, and $H \leq \Gamma$ be a finite-index subgroup. Let $F \subseteq \Gamma$ be the fundamental domain for left cosets of H. Then $[F, H]$ is a left monotiling of Γ. Furthermore, the indicator function $\mathbf{1}_H$ is a good weight. To see this, consider the ergodic system $\mathbf{X} := (\Gamma/H, |\cdot|., \Gamma)$, where Γ acts on the left on the finite set Γ/H with the normalized counting measure $|\cdot|.$ by

$$g(fH) := gfH, \quad f \in F, g \in \Gamma.$$

Let $f := \mathbf{1}_{\mathrm{e}H} \in \mathrm{L}^\infty(\Gamma/H)$ and $x := \mathrm{e}H \in \Gamma/H$. Then $\mathbf{1}_H(g) = f(g \cdot x)$ for all $g \in \Gamma$ and the statement follows from Theorem 2.2. However, one can also prove this directly without referring to Theorem 2.2.

In what follows we will need the following simple proposition.

PROPOSITION 2.1. *Let $([F_n, \mathcal{Z}_n])_{n\geq 1}$ be a left Følner monotiling of Γ such that $\mathrm{e} \in F_n$ for every n. Then for every fixed $k \geq 1$,*

$$\frac{|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap \mathcal{Z}_k|}{|F_n|} \to \frac{1}{|F_k|} \tag{2}$$

and

$$\frac{|F_n \cap \mathcal{Z}_k|}{|F_n|} \to \frac{1}{|F_k|} \tag{3}$$

as $n \to \infty$. *If, additionally,* $(F_n)_{n\geq 1}$ *is a two-sided Følner sequence, then for every fixed* k,

$$\frac{|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k|}{|F_n|} \to \frac{1}{|F_k|} \tag{4}$$

as $n \to \infty$.

Proof. Observe first that, under the initial assumptions of the theorem, for every set $A \subseteq \Gamma$, $k \geq 1$ and $g \in \Gamma$ we have

$$g \in \mathrm{int}^{\mathrm{L}}_{F_k}(A) \Leftrightarrow F_k g \subseteq A$$

and

$$g \in \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(A) \Leftrightarrow g F_k^{-1} \subseteq A.$$

Let $k \geq 1$ be fixed. For every $n \geq 1$, consider the finite set $A_{n,k} := \{g \in \mathcal{Z}_k : F_k g \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \neq \varnothing\}$. Then the translates $\{F_k z : z \in A_{n,k}\}$ form a disjoint cover of the set $\mathrm{int}^{\mathrm{L}}_{F_k}(F_n)$. It is easy to see that

$$\Gamma = \mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \sqcup \partial^{\mathrm{L}}_{F_k}(F_n) \sqcup \mathrm{int}^{\mathrm{L}}_{F_k}(F_n^{\mathrm{c}}).$$

Since $A_{n,k} \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n^{\mathrm{c}}) = \varnothing$, we can decompose the set of centers $A_{n,k}$ as follows:

$$A_{n,k} = (A_{n,k} \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n)) \sqcup (A_{n,k} \cap \partial^{\mathrm{L}}_{F_k}(F_n)).$$

Since $(F_n)_{n\geq 1}$ is a Følner sequence,

$$\frac{|F_k(A_{n,k} \cap \partial^{\mathrm{L}}_{F_k}(F_n))|}{|F_n|} = \frac{|F_k| \cdot |A_{n,k} \cap \partial^{\mathrm{L}}_{F_k}(F_n)|}{|F_n|} \to 0$$

and $|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n)|/|F_n| \to 1$ as $n \to \infty$. Then from the inequalities

$$\begin{aligned}\frac{|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n)|}{|F_n|} &\leq \frac{|F_k(A_{n,k} \cap \partial^{\mathrm{L}}_{F_k}(F_n))|}{|F_n|} + \frac{|F_k(A_{n,k} \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n))|}{|F_n|} \\ &\leq \frac{|F_k(A_{n,k} \cap \partial^{\mathrm{L}}_{F_k}(F_n))|}{|F_n|} + 1\end{aligned}$$

we deduce that

$$\frac{|F_k| \cdot |A_{n,k} \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n)|}{|F_n|} \to 1 \tag{5}$$

as $n \to \infty$. It remains to note that $A_{n,k} \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n) = \mathcal{Z}_k \cap \mathrm{int}^{\mathrm{L}}_{F_k}(F_n)$ and the first statement follows. The second statement follows trivially from the first. To obtain the last statement, observe that $|\mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n)|/|F_n| \to 1$ as $n \to \infty$ since $(F_n)_{n\geq 1}$ is a right Følner

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sequence, thus

$$\lim_{n\to\infty} \frac{|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap F_n \cap \mathcal{Z}_k|}{|F_n|} = \lim_{n\to\infty} \frac{|\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k|}{|F_n|} = \frac{1}{|F_k|}. \qquad \square$$

This proposition has an important corollary, which we now state.

THEOREM 2.7. *Let* $([F_n, \mathcal{Z}_n])_{n\geq 1}$ *be a regular Følner monotiling. Then for every measure-preserving system* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$*, every* $f \in \mathrm{L}^\infty(\mathrm{X})$ *and every* $k \geq 1$ *the limits*

$$\begin{aligned} |F_k| \lim_{n\to\infty} \mathbb{E}_{g\in F_n} \mathbf{1}_{\mathcal{Z}_k} f(g\cdot\omega) &= \lim_{n\to\infty} \mathbb{E}_{g\in F_n\cap\mathcal{Z}_k} f(g\cdot\omega) \\ &= \lim_{n\to\infty} \mathbb{E}_{g\in \mathrm{int}^{\mathrm{L}}_{F_k}(F_n)\cap\mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n)\cap\mathcal{Z}_k} f(g\cdot\omega) \end{aligned}$$

exist and coincide for μ*-a.e.* $\omega \in \mathrm{X}$.

Proof. The existence of the limit on the left-hand side follows from the definition of a good weight and the definition of a regular Følner monotiling, and equality of the limits follows from the previous proposition. $\square$

Later, in §2.6, we will add a *computability* requirement to the notion of a regular Følner monotiling. The central result of this paper says that Brudno's theorem holds for groups admitting a computable regular symmetric Følner monotiling.

2.3. *Computability and Kolmogorov complexity.* In this section we will discuss the standard notions of computability and Kolmogorov complexity that will be used in this work. We refer to [7, Ch. 7] for details, more definitions and proofs.

For a natural number k, a k-ary *partial function* is any function of the form $f : D \to \mathbb{N} \cup \{0\}$, where D, the *domain of definition*, is a subset of $(\mathbb{N} \cup \{0\})^k$ for some natural k. A k-ary partial function is called *computable* if there exists an algorithm which takes a k-tuple of non-negative integers $(a_1, a_2, \ldots, a_k)$, prints $f(a_1, a_2, \ldots, a_k)$ and terminates if $(a_1, a_2, \ldots, a_k)$ is in the domain of f, while yielding no output otherwise (in particular, it might fail to terminate). A function is called *total* if it is defined everywhere.

The term *algorithm* above stands, informally speaking, for a computer program. One way to formalize it is through introducing the class of *recursive functions*, and the resulting notion coincides with the class of functions computable on *Turing machines*. We do not focus on these question in this work, and we will think about computability in an 'informal' way.

A set $A \subseteq \mathbb{N}$ is called *recursive* (or *computable*) if the indicator function $\mathbf{1}_A$ of A is computable. It is easy to see that finite and cofinite subsets of $\mathbb{N}$ are computable. Furthermore, for computable sets $A, B \subseteq \mathbb{N}$ their union and intersection are also computable. If a total function $f : \mathbb{N} \to \mathbb{N}$ is computable and $A \subseteq \mathbb{N}$ is a computable set, then $f^{-1}(A)$, the full preimage of A, is computable. The image of a computable set via a total computable bijection is computable, and the inverse of such a bijection is a computable function.

A sequence of subsets $(F_n)_{n\geq 1}$ of $\mathbb{N}$ is called *computable* if the total function $\mathbf{1}_{F_\cdot} : (n, x) \mapsto \mathbf{1}_{F_n}(x)$ is computable. It is easy to see that a total function $f : \mathbb{N} \to \mathbb{N}$ is

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computable if and only if the sequence of singletons $(f(n))_{n\geq 1}$ is computable in the sense above.

It is very often important to have a numeration of elements of a set by natural numbers. A set $A \subseteq \mathbb{N}$ is called *enumerable* if there exists a total computable surjective function $f : \mathbb{N} \to A$. If the set A is infinite, we can also require f to be injective. This leads to an equivalent definition because an algorithm computing the function f can be modified so that no repetitions occur in its output. Finite and cofinite sets are enumerable. It can be shown [7, Proposition 7.44] that a set A is computable if and only if both A and $\mathbb{N} \setminus A$ are enumerable. Furthermore, for a set $A \subsetneq \mathbb{N}$ the following are equivalent:

(i) A is enumerable;

(ii) A is the domain of definition of a partial recursive function.

Finally, we can introduce the Kolmogorov complexity for finite words. Let A be a computable partial function defined on a domain D of finite binary words with values in the set of all finite words over a finite alphabet Λ. Of course, we have defined computable functions on subsets of $(\mathbb{N} \cup \{0\})^k$ with values in $\mathbb{N} \cup \{0\}$ above, but this can be easily extended to (co)domains of finite words over finite alphabets. We can think of A as a 'decompressor' that takes compressed binary descriptions (or 'programs') in its domain, and decompresses them to finite words over alphabet Λ. Then we define the *Kolmogorov complexity* of a finite word ω with respect to A as follows:

$$\mathrm{K}_A^0(\omega) := \inf\{l(p) : A(p) = w\},$$

where $l(p)$ denotes the length of the description. If some word ω_0 does not admit a compressed version, then we let $\mathrm{K}_A^0(\omega_0) = \infty$. The *average Kolmogorov complexity* with respect to A is defined by

$$\overline{\mathrm{K}}_A^0(\omega) := \frac{\mathrm{K}_A^0(\omega)}{l(\omega)},$$

where $l(\omega)$ is the length of the word ω. Intuitively speaking, this quantity tells us how effective the compressor A is when describing the word ω.

Of course, some decompressors are intuitively better than others. This is formalized by saying that A_1 is *not worse* than A_2 if there is a constant c such that for all words ω,

$$\mathrm{K}_{A_1}^0(\omega) \leq \mathrm{K}_{A_2}^0(\omega) + c. \tag{6}$$

A theorem of Kolmogorov says that there exist a decompressor A^* that is optimal, that is, for every decompressor A there is a constant c such that for all words ω we have

$$\mathrm{K}_{A^*}^0(\omega) \leq \mathrm{K}_A^0(\omega) + c.$$

An optimal decompressor is not unique, so from now on we let A^* be a fixed optimal decompressor.

The notion of Kolmogorov complexity can be extended to words defined on finite subsets of $\mathbb{N}$, and this will be essential in the following sections. More precisely, let $X \subseteq \mathbb{N}$ be a finite subset, $\iota_X : X \to \{1, 2, \ldots, \mathrm{card}\, X\}$ an increasing bijection, Λ a finite alphabet, A a decompressor and $\omega \in \Lambda^Y$ a word defined on some set $Y \supseteq X$. Then we let

$$\mathrm{K}_A(\omega, X) := \mathrm{K}_A^0(\omega \circ \iota_X^{-1}). \tag{7}$$

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and

$$\overline{\mathrm{K}}_A(\omega, X) := \frac{\mathrm{K}_A^0(\omega \circ \imath_X^{-1})}{\operatorname{card} X}. \tag{8}$$

We call $\mathrm{K}_A(\omega, X)$ the *Kolmogorov complexity* of ω over X with respect to A, and $\overline{\mathrm{K}}_A(\omega, X)$ is called the *mean Kolmogorov complexity* of ω over X with respect to A. If a decompressor A_1 is not worse than a decompressor A_2 with some constant c, then for all X, ω as above,

$$\mathrm{K}_{A_1}(\omega, X) \leq \mathrm{K}_{A_2}(\omega, X) + c.$$

If $X \subseteq \mathbb{N}$ is an infinite subset and $(F_n)_{n\geq 1}$ is a sequence of finite subsets of X such that $\operatorname{card} F_n \to \infty$, then the asymptotic Kolmogorov complexity of $\omega \in \Lambda^X$ with respect to $(F_n)_{n\geq 1}$ and a decompressor A is defined by

$$\widehat{\mathrm{K}}_A(\omega) := \limsup_{n\to\infty} \overline{\mathrm{K}}_A(\omega|_{F_n}, F_n).$$

The dependence on the sequence $(F_n)_{n\geq 1}$ is omitted in the notation. It is easy to see that for every decompressor A and $\omega \in \Lambda^X$,

$$\widehat{\mathrm{K}}_{A^*}(\omega) \leq \widehat{\mathrm{K}}_A(\omega). \tag{9}$$

From now on, we will (mostly) use the optimal decompressor A^* and write $\mathrm{K}(\omega, X)$, $\overline{\mathrm{K}}(\omega, X)$ and $\widehat{\mathrm{K}}(\omega)$ omitting any explicit reference to A^*.

When estimating the Kolmogorov complexity of words we will often have to encode non-negative integers using binary words. We will now fix some notation that will be used later. When n is a non-negative integer, we write $\underline{\mathrm{n}}$ for the *binary encoding* of n and $\overline{\mathrm{n}}$ for the *doubling encoding* of n, that id, if $b_l b_{l-1} \cdots b_0$ is the binary expansion of n, then $\underline{\mathrm{n}}$ is the binary word $\mathrm{b}_l \mathrm{b}_{l-1} \cdots \mathrm{b}_0$ of length $l+1$ and $\overline{\mathrm{n}}$ is the binary word $\mathrm{b}_l \mathrm{b}_l \mathrm{b}_{l-1} \mathrm{b}_{l-1} \cdots \mathrm{b}_0 \mathrm{b}_0$ of length $2l+2$. We denote the length of the binary word w by $l(\mathrm{w})$, and is clear that $l(\underline{\mathrm{n}}) \leq \lfloor \log n \rfloor + 1$ and $l(\overline{\mathrm{n}}) \leq 2\lfloor \log n \rfloor + 2$.

2.4. *Computable spaces, word presheaves and complexity.* The goal of this section is to introduce the notions of *computable space*, *computable function* between computable spaces and *word presheaf* over computable spaces. The complexity of sections of word presheaves and asymptotic complexity of sections of word presheaves are introduced in this section as well.

An *indexing* of a set X is an injective mapping $\imath : X \to \mathbb{N}$ such that $\imath(X)$ is a computable subset. Given an element $x \in X$, we call $\imath(x)$ the *index* of x. If $i \in \imath(X)$, we denote by x_i the element of X having index i. A *computable space* is a pair $(X, \imath)$ of a set X and an indexing $\imath$. Preimages of computable subsets of $\mathbb{N}$ under $\imath$ are called *computable subsets* of $(X, \imath)$. Each computable subset $Y \subseteq X$ can be seen as a computable space $(Y, \imath|_Y)$, where $\imath|_Y$ is the restriction of the indexing function. Of course, the set $\mathbb{N}$ with identity as an indexing function is a computable space, and the computable subsets of $(\mathbb{N}, \mathrm{id})$ are precisely the computable sets of $\mathbb{N}$ in the sense of §2.3.

Let $(X_1, \imath_1), (X_2, \imath_2), \ldots, (X_k, \imath_k), (Y, \imath)$ be computable spaces. A (total) function $f : X_1 \times X_2 \times \cdots \times X_k \to Y$ is called *computable* if the function $\widetilde{f} : \imath_1(X_1) \times \imath_2(X_2) \times$

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$\cdots \times \iota_k(X_k) \to \iota(Y)$ determined by the condition

$$\widetilde{f}(\iota_1(x_1), \iota_2(x_2), \ldots, \iota_k(x_k)) = \iota(f(x_1, x_2, \ldots, x_k))$$

for all $(x_1, x_2, \ldots, x_k) \in X_1 \times X_2 \times \cdots \times X_k$ is computable. This definition extends the standard definition of computability from §2.3 when the computable spaces under consideration are $(\mathbb{N}, \mathrm{id})$. A computable function $f : (X, \iota_1) \to (Y, \iota_2)$ is called a *morphism* between computable spaces. This yields the definition of the *category of computable spaces*. Let $(X, \iota_1), (X, \iota_2)$ be computable spaces. The indexing functions ι_1 and ι_2 of X are called *equivalent* if $\mathrm{id} : (X, \iota_1) \to (X, \iota_2)$ is an isomorphism. It is clear that the classes of computable functions and computable sets do not change if we pass to equivalent indexing functions.

Given a computable space (X, ι), we call a sequence of subsets $(F_n)_{n\geq 1}$ of X *computable* if the function $\mathbf{1}_{F_\cdot} : \mathbb{N} \times X \to \{0, 1\}, (n, x) \mapsto \mathbf{1}_{F_n}(x)$ is computable. We will also need a special notion of computability for sequences of *finite* subsets of (X, ι). A sequence of finite subsets $(F_n)_{n\geq 1}$ of X is called *canonically computable* if there is an algorithm that, given n, prints the set $\iota(F_n)$ and halts. One way to make this more precise is by introducing the canonical index of a finite set. Given a finite set $A = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{N}$, we call the number $\mathrm{I}(A) := \sum_{i=1}^{k} 2^{x_i}$ the *canonical index* of A. Hence a sequence of finite subsets $(F_n)_{n\geq 1}$ of X is canonically computable if and only if the total function $n \mapsto \mathrm{I}(\iota(F_n))$ is computable. Of course, a canonically computable sequence of finite sets is computable, but the converse is not true due to the fact that there is no effective way of determining how large a finite set with a given computable indicator function is. It is easy to see that the class of canonically computable sequences of finite sets does not change if we pass to an equivalent indexing. The proof of the following proposition is straightforward.

PROPOSITION 2.2. *Let (X, ι) be a computable space.*

(a) *If $(F_n)_{n\geq 1}, (G_n)_{n\geq 1}$ are computable (respectively, canonically computable) sequences of sets, then the sequences of sets $(F_n \cup G_n)_{n\geq 1}$, $(F_n \cap G_n)_{n\geq 1}$ and $(F_n \setminus G_n)_{n\geq 1}$ are computable (respectively, canonically computable).*

(b) *If $(F_n)_{n\geq 1}$ is a canonically computable sequence of sets and $(G_n)_{n\geq 1}$ is a computable sequence of sets, then the sequence of sets $(F_n \cap G_n)_{n\geq 1}$ is canonically computable.*

Let (X, ι) be a computable space and Λ be a finite alphabet. A *word presheaf* $\mathcal{F}_\Lambda$ on X consists of

(1) a set $\mathcal{F}_\Lambda(U)$ of Λ-valued functions defined on the set U for every computable subset $U \subseteq X$;

(2) a restriction mapping $\rho_{U,V} : \mathcal{F}_\Lambda(U) \to \mathcal{F}_\Lambda(V)$ for each pair U, V of computable subsets such that $V \subseteq U$, which takes functions in $\mathcal{F}_\Lambda(U)$ and restricts them to the subset V.

It is easy to see that the standard 'presheaf axioms' are satisfied: $\rho_{U,U}$ is the identity on $\mathcal{F}_\Lambda(U)$ for every computable $U \subseteq X$, and for every triple $V \subseteq U \subseteq W$ we have that $\rho_{W,V} = \rho_{U,V} \circ \rho_{W,U}$. Elements of $\mathcal{F}_\Lambda(U)$ are called *sections* over U, or *words* over U. We will often write $s|_V$ for $\rho_{U,V}s$, where $s \in \mathcal{F}_\Lambda(U)$ is a section.

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We have introduced Kolmogorov complexity of words supported on subsets of $\mathbb{N}$ in the previous section, now we want to extend this by introducing complexity of sections. Let (X, ι) be a computable space and let $\mathcal{F}_\Lambda$ be a word presheaf over (X, ι). Let $U \subseteq X$ be a finite set and $\omega \in \mathcal{F}_\Lambda(U)$. Then we define the *Kolmogorov complexity* of $\omega \in \mathcal{F}_\Lambda(U)$ by

$$\mathrm{K}(\omega, U) := \mathrm{K}(\omega \circ \iota^{-1}, \iota(U)), \tag{10}$$

and the *mean Kolmogorov complexity* of $\omega \in \mathcal{F}_\Lambda(U)$ by

$$\overline{\mathrm{K}}(\omega, U) := \overline{\mathrm{K}}(\omega \circ \iota^{-1}, \iota(U)). \tag{11}$$

The quantities on the right-hand side here are defined in equations (7) and (8), respectively (which are special cases of the more general definition when the computable space X is $(\mathbb{N}, \mathrm{id})$).

Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of X such that card $F_n \to \infty$. Then we define the *asymptotic Kolmogorov complexity* of a section $\omega \in \mathcal{F}_\Lambda(X)$ along the sequence $(F_n)_{n\geq 1}$ by

$$\widehat{\mathrm{K}}(\omega) := \limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n}, F_n).$$

Dependence on the sequence $(F_n)_{n\geq 1}$ is omitted in the notation for $\widehat{\mathrm{K}}$, but it will be always clear from the context which sequence we take.

We close this section with an interesting result on invariance of asymptotic Kolmogorov complexity. It says that asymptotic Kolmogorov complexity of a section $\omega \in \mathcal{F}_\Lambda(X)$ does not change if we pass to an equivalent indexing.

THEOREM 2.8. (Invariance of asymptotic complexity) *Let ι_1, ι_2 be equivalent indexing functions of a set X. Let $(F_n)_{n\geq 1}$ be a sequence of finite subsets of X such that:*

(a) *$(F_n)_{n\geq 1}$ is a canonically computable sequence of sets in (X, ι_1);*

(b) card $F_n/\log n \to \infty$ *as* $n \to \infty$.

Let $\omega \in \mathcal{F}_\Lambda(X)$. Then

$$\limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n} \circ \iota_1^{-1}, \iota_1(F_n)) = \limsup_{n\to\infty} \overline{\mathrm{K}}(\omega|_{F_n} \circ \iota_2^{-1}, \iota_2(F_n)),$$

that is, asymptotic Kolmogorov complexity of ω does not change when we pass to an equivalent indexing.

Proof. Since the indexing functions ι_1, ι_2 are equivalent, there is a computable bijection $\phi : \iota_2(X) \to \iota_1(X)$ such that $\phi(\iota_2(x)) = \iota_1(x)$ for all $x \in X$. Furthermore, the sequence $(F_n)_{n\geq 1}$ is canonically computable in (X, ι_2).

Let n be fixed. By definition,

$$\overline{\mathrm{K}}(\omega|_{F_n} \circ \iota_1^{-1}, \iota_1(F_n)) = \frac{\mathrm{K}^0_{A^*}((\omega|_{F_n} \circ \iota_1^{-1}) \circ \iota_{\iota_1(F_n)}^{-1})}{\mathrm{card}\, F_n},$$

where $\omega|_{F_n} \circ \iota_1^{-1}$ is seen as a word on $\iota_1(F_n) \subseteq \mathbb{N}$ and $\widetilde{\omega}_1 := (\omega|_{F_n} \circ \iota_1^{-1}) \circ \iota_{\iota_1(F_n)}^{-1}$ is a word on $\{1, 2, \ldots, \mathrm{card}\, F_n\} \subseteq \mathbb{N}$. Let p_1 be an optimal description of $(\omega|_{F_n} \circ \iota_1^{-1}) \circ \iota_{\iota_1(F_n)}^{-1}$. Similarly, $\widetilde{\omega}_2 := (\omega|_{F_n} \circ \iota_2^{-1}) \circ \iota_{\iota_2(F_n)}^{-1}$ is a word on $\{1, 2, \ldots, \mathrm{card}\, F_n\}$. It is clear

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that $\widetilde{\omega}_1$ is a permutation of $\widetilde{\omega}_2$, hence we can describe $\widetilde{\omega}_2$ by giving the description of $\widetilde{\omega}_1$ and saying how to permute it to obtain $\widetilde{\omega}_2$. We make this intuition formal below.

We define a new decompressor A'. The domain of definition of A' consists of the ‘proper’ programs of the form

$$\bar{\mathrm{l}}01\mathrm{p}, \tag{12}$$

where $\bar{\mathrm{l}}$ is a doubling encoding of an integer l and p is an input for A^*. It will be clear in a moment what we mean by ‘proper’.

The decompressor A' works as follows. Compute the subsets $\iota_1(F_l)$ and $\iota_2(F_l)$ of $\mathbb{N}$. We let $\overline{\phi}$ be the element of $\mathrm{Sym}_{\mathrm{card}\ F_l}$ such that the diagram

$$\begin{array}{ccc} \iota_1(F_l) & \xleftarrow{\quad\phi\quad} & \iota_2(F_l) \\ \downarrow{\scriptstyle \iota_{\iota_1(F_l)}} & & \downarrow{\scriptstyle \iota_{\iota_2(F_l)}} \\ \{1, 2, \ldots, \mathrm{card}\ F_l\} & \xleftarrow{\quad\overline{\phi}\quad} & \{1, 2, \ldots, \mathrm{card}\ F_l\} \end{array}$$

commutes. We compute the word $\omega' := A^*(\mathrm{p})$. If $\mathrm{card}\ F_l \neq l(\omega')$, then the input is not ‘proper’ and the algorithm terminates without producing output. Otherwise, the word $\omega' \circ \overline{\phi}$ is printed. It follows that there is a constant c such that the following holds: for all $l \in \mathbb{N}$ and for all words ω' of length $\mathrm{card}\ F_l$ we have

$$\mathrm{K}^0_{A^*}(\omega' \circ \overline{\phi}) \leq \mathrm{K}^0_{A^*}(\omega') + 2 \log l + c,$$

where $\overline{\phi}$ is the permutation of $\{1, 2, \ldots, \mathrm{card}\ F_l\}$ defined above.

Finally, consider the program $\mathrm{p}' := \bar{\mathrm{n}}01\mathrm{p}_1$. Then $A'(\mathrm{p}') = \widetilde{\omega}_2$. We deduce that $\mathrm{K}^0_{A^*}(\widetilde{\omega}_2) \leq \mathrm{K}^0_{A^*}(\widetilde{\omega}_1) + 2 \log n + c$. The statement of the theorem follows trivially. □

To simplify the notation in the following sections, we adopt the following convention. We say explicitly what indexing function we use when introducing a computable space, but later, when the indexing is fixed, we often omit the indexing function from the notation and think about computable spaces as computable subsets of $\mathbb{N}$. Words defined on subsets of a computable space become words defined on subsets of $\mathbb{N}$. This will help to simplify the notation without introducing much ambiguity.

2.5. *Computable groups.* In this section we provide the definitions of a computable group and a few related notions, connecting results from algebra with computability. This section is based on [14].

Let Γ be a group with respect to the multiplication operation $*$. An indexing ι of Γ is called *admissible* if the function $* : (\Gamma, \iota) \times (\Gamma, \iota) \to (\Gamma, \iota)$ is a computable function in the sense of §2.4. A *computable group* is a pair (Γ, ι) of a group Γ and an admissible indexing ι.

Of course, the groups $\mathbb{Z}^d$ and $\mathsf{UT}_d(\mathbb{Z})$ possess ‘natural’ admissible indexings. More precisely, for the group $\mathbb{Z}$ we fix the indexing

$$\iota : n \mapsto 2|n| + \mathbf{1}_{n \geq 0},$$

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which is admissible. Next, it is clear that for every $d > 1$ the group $\mathbb{Z}^d$ possesses an admissible indexing function such that all coordinate projections onto $\mathbb{Z}$, endowed with the indexing function $\imath$ above, are computable. Similarly, for every $d \geq 2$ the group $\mathsf{UT}_d(\mathbb{Z})$ possesses an admissible indexing function such that for every pair of indexes $1 \leq i, j \leq d$ the evaluation function sending a matrix $g \in \mathsf{UT}_d(\mathbb{Z})$ to its (i, j)th entry is a computable function to $\mathbb{Z}$. We leave the details to the reader. It does not matter which admissible indexing function of $\mathbb{Z}^d$ or $\mathsf{UT}_d(\mathbb{Z})$ we use as long as it satisfies the conditions above, so from now on we assume that this choice is fixed.

The following lemma from [**14**] shows that in a computable group taking the inverse is also a computable operation.

LEMMA 2.1. *Let* $(\Gamma, \imath)$ *be a computable group. Then the function* inv : $(\Gamma, \imath) \to (\Gamma, \imath)$, $g \mapsto g^{-1}$ *is computable.*

$(\Gamma, \imath)$ is a computable space, and we can talk about computable subsets of $(\Gamma, \imath)$. A subgroup of Γ which is a computable subset will be called a *computable subgroup*. A homomorphism between computable groups that is computable as a map between computable spaces will be called a *computable homomorphism*. The proof of the following proposition is straightforward.

PROPOSITION 2.3. *Let* $(\Gamma, \imath)$ *be a computable group. Then the following assertions hold.*

(1) *Given a computable set* $A \subseteq \Gamma$ *and a group element* $g \in \Gamma$, *the sets* A^{-1}, gA *and* Ag *are computable.*
(2) *Given a computable (respectively, canonically computable) sequence* $(F_n)_{n\geq 1}$ *of subsets of* Γ *and a group element* $g \in \Gamma$, *the sequences* $(gF_n)_{n\geq 1}$, $(F_n g)_{n\geq 1}$ *are computable (respectively, canonically computable).*

It is interesting to see that a computable version of the 'first isomorphism theorem' also holds.

THEOREM 2.9. *Let* $(G, \imath)$ *be a computable group and let* $(H, \imath|_H)$ *be a computable normal subgroup, where* $\imath|_H$ *is the restriction of the indexing function* $\imath$ *to* H. *Then there is a compatible indexing function* $\imath'$ *on the factor group* G/H *such that the quotient map* $\pi : (G, \imath) \to (G/H, \imath')$ *is a computable homomorphism.*

For the proof we refer the reader to [**14**, Theorem 1].

2.6. *Computable Følner sequences and computable monotilings.* The notions of an amenable group and a Følner sequence are well known, but, since we are working with computable groups, we need to develop their 'computable' versions.

Let $(\Gamma, \imath)$ be a computable group. A left Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ of Γ is called *computable* if the following assertions hold:

(a) $(F_n)_{n\geq 1}$ is a canonically computable sequence of finite subsets of Γ;
(b) $(\mathcal{Z}_n)_{n\geq 1}$ is a computable sequence of subsets of Γ.

First of all, let us show that the regular symmetric monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ of $\mathbb{Z}^d$ from Example 2.4 is computable.

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Example 2.10. Consider the group $\mathbb{Z}^d$ for some $d \geq 1$. We remind the reader that it is endowed with an admissible indexing such that all the coordinate projections $\mathbb{Z}^d \to \mathbb{Z}$ are computable. Then the Følner sequence $F_n = [0, 1, 2, \ldots, n-1]^d$ is canonically computable. Furthermore, the corresponding sets of centers equal $n\mathbb{Z}^d$ for every n, hence $([\mathcal{Z}_n, F_n])_{n\geq 1}$ is a computable regular symmetric Følner monotiling.

Next, we return to Example 2.5.

Example 2.11. Consider the group $\mathsf{UT}_3(\mathbb{Z})$ and the monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$ from Example 2.5 given by

$$\mathcal{Z}_n = \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : a, b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}$$

and

$$F_n = \{(a, b, c) \in \mathsf{UT}_3(\mathbb{Z}) : 0 \leq a, b < n, 0 \leq c < n^2\}$$

for every $n \geq 1$. We define the projections $\pi_1, \pi_2, \pi_3 : \mathsf{UT}_3(\mathbb{Z}) \to \mathbb{Z}$ as follows. For every $g = (a, b, c) \in \mathsf{UT}_3(\mathbb{Z})$ we let

$$\begin{aligned} \pi_1(g) &:= a, \\ \pi_2(g) &:= b, \\ \pi_3(g) &:= c. \end{aligned}$$

The functions π_1, π_2, π_3 are computable. By definition, for every $(n, g) \in \mathbb{N} \times \mathsf{UT}_3(\mathbb{Z})$,

$$\mathbf{1}_{\mathcal{Z}_\cdot}(n, g) = 1 \Leftrightarrow (\pi_1(g) \in n\mathbb{Z}) \wedge (\pi_2(g) \in n\mathbb{Z}) \wedge (\pi_3(g) \in n^2\mathbb{Z}),$$

hence the sequence of sets $(\mathcal{Z}_n)_{n\geq 1}$ is computable. It is also trivial to show that the sequence $(F_n)_{n\geq 1}$ is canonically computable.

It follows that $([F_n, \mathcal{Z}_n])_{n\geq 1}$ is a computable regular symmetric Følner monotiling.

In general, checking the temperedness of a given canonically computable Følner sequence is not trivial. Lindenstrauss in [**10**] proved that every Følner sequence has a tempered Følner subsequence. Furthermore, the construction of a tempered Følner subsequence from a given Følner sequence is 'algorithmic'. We provide his proof below, and we will use this result later in this section when discussing Følner monotilings of $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$.

PROPOSITION 2.4. *Let $(F_n)_{n\geq 1}$ be a canonically computable Følner sequence in a computable group $(\Gamma, \imath)$. Then there is a computable function $i \mapsto n_i$ such that the subsequence $(F_{n_i})_{i\geq 1}$ is a canonically computable tempered Følner subsequence.*

Proof. We define n_i inductively as follows. Let $n_1 := 1$. If $n_1, \ldots, n_i$ have been determined, we set $\widetilde{F}_i := \bigcup_{j\leq i} F_{n_j}$. Take for n_{i+1} the first integer greater than $i+1$ such that

$$|F_{n_{i+1}} \triangle \widetilde{F}_i^{-1} F_{n_{i+1}}| \leq \frac{1}{|\widetilde{F}_i|}.$$

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The function $i \mapsto n_i$ is total computable. It follows that

$$\left| \bigcup_{j \leq i} F_{n_j}^{-1} F_{n_{i+1}} \right| \leq 2|F_{n_{i+1}}|,$$

hence the sequence $(F_{n_i})_{i \geq 1}$ is 2-tempered. Since the Følner sequence $(F_n)_{n \geq 1}$ is canonically computable and the function $i \mapsto n_i$ is computable, the Følner sequence $(F_{n_i})_{i \geq 1}$ is canonically computable and tempered. □

In case of the discrete Heisenberg group $\mathsf{UT}_3(\mathbb{Z})$ we were able to give simple formulas for the sequences $(F_n)_{n \geq 1}$ and $(\mathcal{Z}_n)_{n \geq 1}$; in particular, checking the computability was trivial. This is no longer the case when $d > 3$, and we will need the following lemma to check the computability of the sequence $(\mathcal{Z}_n)_{n \geq 1}$.

PROPOSITION 2.5. *Let* $(\Gamma, \imath)$ *be a computable group. Let* $([F_n, \mathcal{Z}_n])_{n \geq 1}$ *be a left Følner monotiling of* Γ *such that* $(F_n)_{n \geq 1}$ *is a canonically computable sequence of finite sets and* $\mathrm{e} \in F_n$ *for all* $n \geq 1$*. Then the following assertions are equivalent.*

(i) *There is a total computable function* $\phi : \mathbb{N}^2 \to \Gamma$ *such that*

$$\mathcal{Z}_n = \{\phi(n, 1), \phi(n, 2), \ldots\}$$

for every $n \geq 1$.

(ii) *The sequence of sets* $(\mathcal{Z}_n)_{n \geq 1}$ *is computable.*

Proof. One implication is clear. For the converse, note that to prove computability of the function $\mathbf{1}_{\mathcal{Z}.}$ we have to devise an algorithm that, given $n \in \mathbb{N}$ and $g \in \Gamma$, decides whether $g \in \mathcal{Z}_n$ or not. Let $\phi : \mathbb{N}^2 \to \Gamma$ be the function from assertion (i). Then the following algorithm answers the question. Set $i := 1$ and compute the set $F_n = \{\mathrm{e}, h_{1,n}, \ldots, h_{k,n}\}$. This is possible since $(F_n)_{n \geq 1}$ is a canonically computable sequence of finite sets; in particular, $k = k(n)$ above is a computable function of n.

The main loop begins by computing $\mathrm{e}\phi(n, i), h_{1,n}\phi(n, i), \ldots, h_{k,n}\phi(n, i)$. If $g = \mathrm{e}\phi(n, i)$, then the answer is 'yes' and we stop the program. If $g = h_{j,n}\phi(n, i)$ for some $1 \leq j \leq k$, then the answer is 'no' and we stop the program. If neither is true, then we set $i := i + 1$ and go to the beginning of the main loop.

Since $\Gamma = F_n \mathcal{Z}_n$ for every n, the algorithm terminates for every input. □

In this last example we will explain, referring to the work [7] for details, why the groups $\mathsf{UT}_d(\mathbb{Z})$ for $d > 3$ have computable regular symmetric Følner monotilings as well.

Example 2.12. Let d be fixed. Let u_{ij} be the matrix whose entry with the indexes (i, j) is 1, and where all the other entries are zero. Let $T_{ij} := I + u_{ij}$. Let p be a prime number. For every m consider the subgroup $\mathcal{Z}_m$ generated by $T_{ij}^{p^{m(j-i)}}$ for all indexes (i, j), $i < j$. Then $\mathcal{Z}_m$ is an enumerable subset. There exists a total computable function $\phi : \mathbb{N}^2 \to \mathsf{UT}_d(\mathbb{Z})$ such that

$$\mathcal{Z}_m = \{\phi(m, 1), \phi(m, 2), \phi(m, 3), \ldots\}$$

for all $m \geq 1$.

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$\mathcal{Z}_m$ is a finite-index subgroup of $\mathrm{UT}_d(\mathbb{Z})$ for every m. The fundamental domain ρ_m for $\mathcal{Z}_m$ can be written as

$$\rho_m := \{T_{d-1,d}^{k_{d-1,d}} \cdots T_{1,d}^{k_{1,d}} : \\ l_{d-1,d}(m) \leq k_{d-1,d} \leq L_{d-1,d}(m), \ldots, l_{1,d}(m) \leq k_{1,d} \leq L_{1,d}(m)\},$$

where

$$l_{i,j}(m) = -\left\lfloor \frac{p^{m(j-i)}}{2} \right\rfloor, \quad L_{i,j}(m) = \left\lfloor \frac{p^{m(j-i)}+1}{2} \right\rfloor.$$

It is clear that the sequence of sets $m \mapsto \rho_m$ is canonically computable. Furthermore, it is shown in [7] that $(\rho_m)_{m\geq 1}$ is a two-sided Følner sequence. The computability of the Følner monotiling $([\rho_m, \mathcal{Z}_m])_{m\geq 1}$ follows from Proposition 2.5.

The fact that the Følner monotiling $([\rho_m, \mathcal{Z}_m])_{m\geq 1}$ is symmetric is clear since $\mathcal{Z}_m$ is a subgroup for every m. The fact that for each m the function $\mathbf{1}_{\mathcal{Z}_m}$ is a good weight along a tempered subsequence of $(\rho_m)_{m\geq 1}$ follows from Example 2.6. It is clear that we can ensure the growth conditions by picking a subsequence $(n_i)_{i\geq 1}$ computably such that $([\rho_{n_i}, \mathcal{Z}_{n_i}])_{i\geq 1}$ is a computable regular symmetric Følner monotiling.

3. *A theorem of Brudno*

We are now ready to prove the main theorem of this paper. First, we will explain some definitions.

By a *subshift* (X, Γ) we mean a closed Γ-invariant subset X of Λ^Γ, where Λ is the finite *alphabet* of X. The left action of the group Γ on X is given by

$$(g \cdot \omega)(x) := \omega(xg) \quad \text{for all } x,\ g \in \Gamma,\ \omega \in \mathrm{X}.$$

The words consisting of letters from the alphabet Λ will be often called Λ-*words*. Of course, we can assume without loss of generality that $\Lambda = \{1, 2, \ldots, z\}$ for some z. We denote by $\mathrm{M}_\Gamma^1(\mathrm{X})$ the space of invariant probability measures on X. When a measure $\mu \in \mathrm{M}_\Gamma^1(\mathrm{X})$ is fixed, we will often denote by $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ the associated measure-preserving system.

If the group Γ is endowed with an admissible indexing function ι, then (Γ, ι) is a computable space and we can talk about word presheaves over this space. In particular, if (X, Γ) is a subshift, we associate a word presheaf $\mathcal{F}_\Lambda$ to the subshift (X, Γ) by setting

$$\mathcal{F}_\Lambda(F) := \{\omega|_F : \omega \in \mathrm{X}\}$$

for every computable set $F \subseteq \Gamma$. Every word $\omega \in \mathrm{X}$, viewed as an element of $\mathcal{F}_\Lambda(\Gamma)$, is a section, hence one can define its asymptotic Kolmogorov complexity with respect to some sequence of finite sets $(F_n)_{n\geq 1}$ such that card $F_n \to \infty$ as $n \to \infty$.

We now state the main result of this paper.

THEOREM 3.1. *Let (Γ, ι) be a computable group with a fixed computable regular symmetric Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$. Let (X, Γ) be a subshift on Γ, $\mu \in \mathrm{M}_\Gamma^1(\mathrm{X})$ be an ergodic measure and $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be the associated measure-preserving system.*

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Then

$$\widehat{\mathrm{K}}(\omega) = h(\mathbf{X})$$

for μ*-a.e.* $\omega \in \mathrm{X}$*, where the asymptotic complexity is computed with respect to the sequence* $(F_n)_{n\geq 1}$.

The proof is split into two parts, establishing respective inequalities in Theorems 3.3 and 3.4. From now on, we more or less follow the strategy of Brudno's original paper [3].

Given a subshift $\mathrm{X} \subseteq \Lambda^\Gamma$ with an invariant measure μ on the alphabet $\Lambda = \{1, \ldots, z\}$, we define the partition

$$\alpha_\Lambda := \{A_1, \ldots, A_z\}, \quad A_i := \{\omega \in \mathrm{X} : \omega(\mathrm{e}) = i\} \quad \text{for all } i = 1, \ldots, z.$$

Then α_Λ is clearly a generating partition. We will use the following well-known proposition.

PROPOSITION 3.1. *Let* $\mathrm{X} \subseteq \Lambda^\Gamma$ *be a subshift,* $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ *be an invariant probability measure,* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ *be the associated measure-preserving system and* α_Λ *be the partition defined above. Then*

$$h_\mu(\alpha_\Lambda, \Gamma) = h(\mathbf{X}).$$

Before we proceed to the proofs, we make the following observation. The alphabet $\Lambda = \{1, 2, \ldots, z\}$ is finite, so we fix an encoding of Λ by binary words of length $\lfloor \log z \rfloor + 1$. That is, we have an injective map

$$c_\Lambda : \Lambda \to \{0, 1\}^{\lfloor \log z \rfloor + 1}.$$

If c_Λ is not surjective, then we agree to assign the remaining elements of $\{0, 1\}^{\lfloor \log z \rfloor + 1}$ to the first letter of Λ. Then, for every $N \geq 1$, each binary word of length $N(\lfloor \log z \rfloor + 1)$ is unambiguously interpreted as a Λ-word of length N.

3.1. *Part A.* The first step is proving that the Kolmogorov complexity of a word over Γ is shift-invariant. In the proof below it will become apparent why we need the computability structure on the group and why we require the Følner sequence to be computable. We follow the convention suggested at the end of §2.4, that is, we view Γ as a computable subset of $\mathbb{N}$ such that the multiplication is computable.

THEOREM 3.2. (Shift invariance) *Let* (Γ, ι) *be a computable amenable group with a fixed canonically computable right Følner sequence* $(F_n)_{n\geq 1}$ *such that* $|F_n|/\log n \to \infty$ *as* $n \to \infty$*. Let* (X, Γ) *be a subshift and* $\omega \in \mathrm{X}$ *be a word on* Γ*. Then for every* $g \in \Gamma$,

$$\widehat{\mathrm{K}}(\omega) = \widehat{\mathrm{K}}(g \cdot \omega),$$

where the asymptotic complexity is computed with respect to the sequence $(F_n)_{n\geq 1}$.

Proof. We will prove the following claim: for arbitrary $g \in \Gamma$,

$$\widehat{\mathrm{K}}(g \cdot \omega) = \limsup_{n\to\infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \iota^{-1}_{F_n})}{|F_n|} \leq \widehat{\mathrm{K}}(\omega).$$

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It is trivial to see that the statement of the theorem follows from this claim. Speaking informally, our idea behind the proof of the claim is that the sets F_n and $F_n g^{-1}$ are almost identical for large enough n. The word $(g \cdot \omega)|_{F_n \cap F_n g^{-1}}$ can be encoded using the knowledge of the word $\omega|_{F_n}$ and the *computable* action by g that 'permutes' a part of the word $\omega|_{F_n}$. To encode the word $(g \cdot \omega)|_{F_n}$ we also need to treat the part outside the intersection. We use the fact that our Følner sequence is computable, that is, there is an algorithm that, given n, will print the set F_n. But then we also know the remainder $F_n \setminus F_n g^{-1}$, which is endowed with the ambient numbering of $\Gamma \subseteq \mathbb{N}$. Hence we can simply list additionally the $|F_n \setminus F_n g^{-1}|$ corrections we need to make, which takes little space compared to $|F_n|$. Taken together, this implies that the complexity of $(g \cdot \omega)|_{F_n}$ can be asymptotically bounded by the complexity of $\omega|_{F_n}$. Below we make this intuition formal.

Recall that A^* is a fixed asymptotically optimal decompressor in the definition of Kolmogorov complexity K. We now introduce a new decompressor $A^\dagger$ on the domain of 'proper' programs of the form

$$\overline{\mathrm{s}}01\mathrm{w}01\overline{\mathrm{n}}01\overline{\mathrm{m}}01\mathrm{p}, \tag{13}$$

where $\overline{\mathrm{s}}$ is a doubling encoding of a non-negative integer s and w is a binary encoding of a Λ-word υ of length s, hence $l(\mathrm{w}) = s(\lfloor \log z \rfloor + 1)$. Next, $\overline{\mathrm{n}}$ and $\overline{\mathrm{m}}$ are doubling encodings of some natural numbers n, m. The remainder p is required to be a valid input for A^*. It will be clear in a moment what we mean by 'proper' above. The programs of this form (equation (13)) are unambiguously interpreted, that is, one can uniquely determine the integers s, m, n and the binary words w and p.

The decompressor $A^\dagger$ is defined as follows. Let $g := g_m$ be the element of the computable group (Γ, ι) with index m, and let $F := F_n$ be the nth element of the canonically computable Følner sequence $(F_n)_{n \geq 1}$. We compute the set $D := F \setminus F g^{-1}$, which is seen as a subset of $\mathbb{N}$ with induced ordering. Further, we compute the word $\widetilde{\omega}_1 := A^*(\mathrm{p})$. The increasing bijection $\iota_F : F \to \{1, 2, \ldots, |F|\}$ maps the subsets $F \cap F g^{-1}$ and $F g \cap F$ of F to subsets Y_1, Y_2 of $\{1, 2, \ldots, |F|\}$. The right multiplication R_g on Γ is computable and restricts to a bijection from $F \cap F g^{-1}$ to $F g \cap F$, so let $\widetilde{R_g}$ be the bijection making the diagram

$$\begin{array}{ccc} F \cap F g^{-1} & \xrightarrow{\iota_F} & Y_1 \\ {\scriptstyle R_g}\downarrow & & \downarrow{\scriptstyle \widetilde{R_g}} \\ F g \cap F & \xrightarrow{\iota_F} & Y_2 \end{array}$$

commute. The output of $A^\dagger$ is produced as follows. If $\widetilde{R_g}(x) > l(\widetilde{\omega}_1)$ for some $x \in Y_1$ or $|F| - \operatorname{card} Y_1 \neq s$, then the input is not 'proper' and the algorithm terminates without producing output. Otherwise, for every $x \in Y_1 \subseteq \{1, 2, \ldots, |F|\}$ we set

$$\widetilde{\omega}_2(x) := \widetilde{\omega}_1(\widetilde{R_g}(x)).$$

It remains to describe $\widetilde{\omega}_2$ on the remainder $Y_0 := \{1, 2, \ldots, |F|\} \setminus Y_1$. We let

$$\widetilde{\omega}_2|_{Y_0} := \upsilon \circ \iota_{Y_0},$$

where $\iota_{Y_0} : Y_0 \to \{1, 2, \ldots, \text{card } Y_0\}$ is an increasing bijection. The word $\widetilde{\omega}_2$ is printed as the output.

Let $n \geq 1$ be arbitrary, and let $(g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}$ be the word on $\{1, 2, \ldots, |F_n|\}$ that we want to encode, where $g \in \Gamma$ has index m. Let p_n be an optimal description for $\omega|_{F_n} \circ \iota_{F_n}^{-1}$ with respect to A^*. Let υ be the word $(g \cdot \omega)|_{F_n \setminus F_n g^{-1}} \circ \iota_{F_n \setminus F_n g^{-1}}^{-1}$. To encode the word $(g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}$ using $A^\dagger$, consider the program

$$\widetilde{\mathrm{p}}_n := \bar{\mathrm{s}}01\mathrm{w}01\bar{\mathrm{n}}01\bar{\mathrm{m}}01\mathrm{p}_n,$$

where w is the binary encoding of the Λ-word υ and $s = |F_n \setminus F_n g^{-1}|$. It is trivial to see that $A^\dagger(\widetilde{\mathrm{p}}_n) = (g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}$. The length of the program $\widetilde{\mathrm{p}}_n$ can be estimated by

$$l(\widetilde{\mathrm{p}}_n) \leq |F_n \backslash F_n g^{-1}| \cdot (\log z + 1) + 2 \log|F_n \backslash F_n g^{-1}| + 8 + 2 \log n + 2 \log m + l(\mathrm{p}_n).$$

By the definition of complexity of sections

$$\widehat{\mathrm{K}}(g \cdot \omega) = \limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1})}{|F_n|}.$$

Using that the optimal decompressor A^* is not worse than $A^\dagger$, we conclude that for every $n \geq 1$,

$$\begin{aligned}\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}) &\leq \mathrm{K}^0_{A^\dagger}((g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}) + c \\ &\leq |F_n \setminus g^{-1} F_n| \cdot (\log z + 1) + 2 \log|F_n \setminus F_n g^{-1}| \\ &\quad + 2 \log n + l(\mathrm{p}_n) + c'\end{aligned}$$

for some constants c, c' independent of n and ω. Taking the limits yields

$$\limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}((g \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1})}{|F_n|} \leq \limsup_{n \to \infty} \frac{\mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \iota_{F_n}^{-1})}{|F_n|}.$$

This completes the proof of the claim, and therefore the proof of the theorem. □

Of course, in the proof above we have not used that X is closed. From now on we will omit explicit reference to the sequence $(F_n)_{n \geq 1}$ when talking about $\widehat{\mathrm{K}}$. The proof of the following proposition is essentially similar to the original one in [3].

PROPOSITION 3.2. *Let* (Γ, ι) *be a computable amenable group with a fixed canonically computable right Følner sequence* $(F_n)_{n \geq 1}$ *such that* $|F_n|/\log n \to \infty$ *as* $n \to \infty$. *Let* (X, Γ) *be a subshift. For every* $t \in \mathbb{R}_{\geq 0}$ *the sets*

$$\begin{aligned} E_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) = t\}, \\ L_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) < t\}, \\ G_t &:= \{\omega \in \mathrm{X} : \widehat{\mathrm{K}}(\omega) > t\} \end{aligned}$$

are Borel measurable and shift-invariant.

Proof. Invariance of the sets above follows from the previous proposition. We will now prove that the set L_t is measurable; the measurability of the other sets is proved in a similar

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manner. Observe that

$$L_t := \{\omega : \widehat{\mathrm{K}}(\omega) < t\} = \bigcup_{N\geq 1} \bigcap_{n>N} \{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \iota^{-1}_{F_n}) < t|F_n|\},$$

where, for every $n \geq 1$, the set $\{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_n} \circ \iota^{-1}_{F_n}) < t|F_n|\}$ is measurable as a finite union of cylinder sets. The measurability of L_t follows. □

We are now ready to prove the first inequality. The proof below is a slight adaption of the original one from [3].

THEOREM 3.3. *Let* (Γ, ι) *be a computable group with a canonically computable tempered two-sided Følner sequence* $(F_n)_{n\geq 1}$ *such that* $|F_n|/\log n \to \infty$. *Let* (X, Γ) *be a subshift over* Γ, $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ *be an ergodic invariant probability measure and* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ *be the associated measure-preserving system. Then* $\widehat{\mathrm{K}}(\omega) \geq h(\mathbf{X})$ *for* μ*-a.e.* ω.

Proof. Suppose this is false, and let

$$R := \{\omega : \widehat{\mathrm{K}}(\omega) < h(\mathbf{X})\} \subseteq \mathrm{X}$$

be the measurable set of words whose complexity is strictly smaller than the entropy $h(\mathbf{X})$. By assumption, $\mu(R) > 0$. The measure μ is ergodic and the set R is invariant, hence $\mu(R) = 1$. For every $i \geq 1$ let

$$R_i := \left\{\omega : \widehat{\mathrm{K}}(\omega) < h(\mathbf{X}) - \frac{1}{i}\right\}.$$

Then $R = \bigcup_{i\geq 1} R_i$ and the sets R_i are measurable and invariant for all i. It follows that there exists an index i_0 such that $\mu(R_{i_0}) = 1$. For every $l \geq 1$ define the set

$$Q_l := \left\{\omega : \mathrm{K}^0_{A^*}(\omega|_{F_i} \circ \iota^{-1}_{F_i}) < \left(h(\mathbf{X}) - \frac{1}{i_0}\right)|F_i| \text{ for all } i \geq l\right\}.$$

Then Q_l is a measurable set for every $l \geq 1$ and $R_{i_0} = \bigcup_{l\geq 1} Q_l$. Let $1 > \delta > 0$ be fixed. The sequence of sets $(Q_l)_{l\geq 1}$ is monotone increasing, hence there is l_0 such that for all $l \geq l_0$ we have $\mu(Q_l) > 1 - \delta$.

Let $\varepsilon < \min(1/i_0, 1 - \delta)$ be positive. Let $n_0 := n_0(\varepsilon) \geq l_0$ such that for all $n \geq n_0$ we have the decomposition $\mathrm{X} = A_n \sqcup B_n$, where $\mu(B_n) < \varepsilon$ and for all $\omega \in A_n$ the inequality

$$2^{-|F_n|(h(\mathbf{X})+\varepsilon)} \leq \mu(\alpha^{F_n}_\Lambda(\omega)) \leq 2^{-|F_n|(h(\mathbf{X})-\varepsilon)} \tag{14}$$

holds. Such n_0 exists due to Corollary 2.1. For every $l \geq n_0$, we partition the sets Q_l in the following way:

$$Q^A_l := Q_l \cap A_l,$$
$$Q^B_l := Q_l \cap B_l.$$

It is clear that for every $l \geq n_0$,

$$\mu(Q^B_l) < \varepsilon,$$
$$\mu(Q^A_l) \geq 1 - \delta - \varepsilon > 0.$$

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By the definition of the set Q_l^A, for all $l \geq n_0$ and all $\omega \in Q_l^A$ we have

$$\mathrm{K}^0_{A^*}(\omega|_{F_l} \circ \iota_{F_l}^{-1}) \leq |F_l|\bigg(h(\mathbf{X}) - \frac{1}{i_0}\bigg).$$

This allows us, for every $l \geq n_0$, to estimate the cardinality of the set of all restrictions of words in Q_l^A to F_l as

$$|\{\omega|_{F_l} : \omega \in Q_l^A\}| \leq 2^{|F_l|(h(\mathbf{X})-1/i_0)+1},$$

which can be seen by counting all binary programs of length at most $|F_l|(h(\mathbf{X}) - 1/i_0)$. Combining this with equation (14), we deduce that

$$\mu(Q_l^A) \leq 2^{|F_l|(h(\mathbf{X})-1/i_0)+1} \cdot 2^{-|F_l|(h(\mathbf{X})-\varepsilon)} \leq 2^{|F_l|(\varepsilon-1/i_0)+1}.$$

This implies that $\mu(Q_l^A) \to 0$ as $l \to \infty$, since $|F_l| \to \infty$ and $\varepsilon - 1/i_0 < 0$. This contradicts the estimate

$$\mu(Q_l^A) \geq 1 - \delta - \varepsilon$$

for all $l \geq n_0$ above. □

3.2. *Part B.* In this part of the proof we shall derive the other inequality. We begin with a preliminary lemma.

LEMMA 3.1. *Let $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ be an ergodic measure-preserving system, where the discrete group Γ admits a regular symmetric Følner monotiling $([F_n, \mathcal{Z}_n])_{n\geq 1}$. Let $(\beta_k)_{k\geq 1}$ be a sequence of finite partitions of* X*, where $\beta_k = \{B_1^k, B_2^k, \ldots, B_{M_k}^k\}$ for all $k \geq 1$. For all $k \geq 1$, $h \in \Gamma$, $m \in \{1, 2, \ldots, M_k\}$ let*

$$\pi_{n,m}^{k,h}(\omega) := \mathbb{E}_{g\in F_n\cap\mathcal{Z}_k} \mathbf{1}_{B_m^k}((gh)\cdot\omega) \tag{15}$$

and

$$\tilde{\pi}_{n,m}^{k,h}(\omega) := \mathbb{E}_{g\in \mathrm{int}^{\mathrm{L}}_{F_k}(F_n)\cap \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n)\cap\mathcal{Z}_k} \mathbf{1}_{B_m^k}((gh)\cdot\omega). \tag{16}$$

Then the following assertions hold.

(a) *For μ-a.e. $\omega \in \mathrm{X}$ the limit*

$$\pi_m^{k,h}(\omega) := \lim_{n\to\infty} \pi_{n,m}^{k,h}(\omega) = \lim_{n\to\infty} \tilde{\pi}_{n,m}^{k,h}(\omega)$$

exists for all $k \geq 1$, $m \in \{1, 2, \ldots, M_k\}$ and $h \in \Gamma$.

(b) *For μ-a.e. $\omega \in \mathrm{X}$ and all $k \geq 1$ there exists $h := h_k(\omega) \in F_k^{-1}$ such that*

$$-\sum_{m=1}^{M_k} \pi_m^{k,h}(\omega) \log \pi_m^{k,h}(\omega) \leq h_\mu(\beta_k).$$

Proof. The first assertion follows from the definition of a regular Følner monotiling, Theorem 2.7 and countability of Γ.

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For the second assertion, observe that for μ-a.e. ω, all $k \geq 1$ and all $m \in \{1, 2, \ldots, M_k\}$,

$$\frac{1}{|F_k|} \sum_{h \in F_k^{-1}} \pi_m^{k,h}(\omega) = \lim_{n\to\infty} \mathbb{E}_{g\in F_n} \mathbf{1}_{B_m^k}(g \cdot \omega),$$

since, for every $k \geq 1$, $[\mathcal{Z}_k, F_k^{-1}]$ is a right monotiling,

$$(\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k)F_k^{-1} \subseteq F_n$$

for all $n \geq 1$ and

$$\frac{|(\mathrm{int}^{\mathrm{L}}_{F_k}(F_n) \cap \mathrm{int}^{\mathrm{R}}_{F_k^{-1}}(F_n) \cap \mathcal{Z}_k)F_k^{-1}|}{|F_n|} \to 1$$

as $n \to \infty$.

Using the ergodicity of $\mathbf{X}$, we deduce that for μ-a.e. ω, all $k \geq 1$ and all $m \in \{1, 2, \ldots, M_k\}$,

$$\frac{1}{|F_k|} \sum_{h \in F_k^{-1}} \pi_m^{k,h}(\omega) = \lim_{n\to\infty} \mathbb{E}_{g\in F_n} \mathbf{1}_{B_m^k}(g \cdot \omega) = \mu(B_m^k),$$

and the second assertion follows by the concavity of the entropy. □

We are now ready to prove the converse inequality. The proof is based on essentially the same idea of 'frequency encoding' from [3], but the technical details differ quite a bit.

THEOREM 3.4. *Let* (Γ, ι) *be a computable group with a fixed computable regular symmetric Følner monotiling* $([F_n, \mathcal{Z}_n])_{n\geq 1}$. *Let* (X, Γ) *be a subshift on* Γ, $\mu \in \mathrm{M}^1_\Gamma(\mathrm{X})$ *be an ergodic measure and* $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ *be the associated measure-preserving system. Then* $\widehat{\mathrm{K}}(\omega) \leq h(\mathbf{X})$ *for* μ*-a.e.* ω.

Proof. We describe a decompressor $A^!$ that will be used to encode the restrictions of the words in X. The decompressor $A^!$ is defined on the domain of 'proper' programs of the form

$$\mathrm{p} := \bar{\mathrm{s}}01\bar{\mathrm{t}}01\bar{\mathrm{f}}_1 01 \cdots \bar{\mathrm{f}}_L 0110\bar{\mathrm{r}}01\mathrm{w}01\underline{\mathrm{N}}. \tag{17}$$

Here $\bar{\mathrm{s}}, \bar{\mathrm{t}}, \bar{\mathrm{r}}$ are doubling encodings of some natural numbers s, t, r. Words $\bar{\mathrm{f}}_1, \ldots, \bar{\mathrm{f}}_L$, where we require that $L = z^{|F_s|}$, are doubling encodings of non-negative integers $f_1, \ldots, f_L$. The word w encodes a Λ-word υ of length r. The word $\underline{\mathrm{N}}$ encodes a natural number N via binary encoding. It will become clear what we mean by 'proper' in a moment. The programs of this form (equation (17)) are unambiguously interpreted. Let

$$\{\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_L\}$$

be the list of all Λ-words of length $|F_s|$ ordered lexicographically. It is clear that, given s, such a list can be computed.

The decompressor $A^!$ works as follows. From s and t compute the finite subsets

$$F_s, F_t, \mathrm{int}^{\mathrm{L}}_{F_s}(F_t) \cap \mathrm{int}^{\mathrm{R}}_{F_s^{-1}}(F_t)$$

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of $\mathbb{N}$. Compute the finite set

$$I_{s,t} := \mathrm{int}^{\mathrm{L}}_{F_s}(F_t) \cap \mathrm{int}^{\mathrm{R}}_{F_s^{-1}}(F_t) \cap \mathcal{Z}_s$$

of centers of the monotiling $[F_s, \mathcal{Z}_s]$. Next, for every $h \in I_{s,t}$ compute the tile $T_h := F_s h \subseteq F_t$ centered at h. We compute the union

$$\Delta_{s,t} := \bigcup_{h \in I_{s,t}} T_h \subseteq F_t$$

of all such tiles.

We will construct a Λ-word σ on the set F_t, then $\widetilde{\sigma} := \sigma \circ \iota_{F_t}^{-1}$ yields a word on $\{1, 2, \ldots, |F_t|\}$ which is printed as the output. The word σ is computed as follows. First, we describe how to compute the restriction $\sigma|_{\Delta_{s,t}}$. For every $h \in I_{s,t}$ the word $\sigma \circ \iota_{T_h}^{-1}$ is a word on $\{1, 2, \ldots, |F_s|\}$, hence it coincides with one of the words

$$\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_L$$

introduced above. We require that the word $\widetilde{\omega}_i$ occurs exactly f_i times for every $i \in \{1, \ldots, L\}$. This amounts to saying that the word $\sigma|_{\Delta_{s,t}}$ has the *collection of frequencies* $f_1, f_2, \ldots, f_L$. Of course, this does not determine $\sigma|_{\Delta_{s,t}}$ uniquely, but only up to a certain permutation. Let $\mathcal{F}_{\Lambda,\mathrm{p}}$ be the set of all Λ-words on $\Delta_{s,t}$ having collection of frequencies $f_1, f_2, \ldots, f_L$. If $\sum_{j=1}^{L} f_j \neq |I_{s,t}|$, then the input is not 'proper' and the algorithm terminates, yielding no output. Otherwise, $\mathcal{F}_{\Lambda,\mathrm{p}}$ is non-empty. The set $\mathcal{F}_{\Lambda,\mathrm{p}}$ is ordered lexicographically (recall that $\Delta_{s,t}$ is a subset of $\mathbb{N}$). It is clear that

$$\mathrm{card}\, \mathcal{F}_{\Lambda,\mathrm{p}} = \frac{|I_{s,t}|!}{f_1!\, f_2! \cdots f_L!}. \tag{18}$$

Thus to encode $\sigma|_{\Delta_{s,t}}$ it suffices to give the index $N_{\mathcal{F}_{\Lambda,\mathrm{p}}}(\sigma|_{\Delta_{s,t}})$ of $\sigma|_{\Delta_{s,t}}$ in the set $\mathcal{F}_{\Lambda,\mathrm{p}}$. We require that $N_{\mathcal{F}_{\Lambda,\mathrm{p}}}(\sigma|_{\Delta_{s,t}}) = N$, and this together with the collection of frequencies $f_1, f_2, \ldots, f_L$ determines the word $\sigma|_{\Delta_{s,t}}$ uniquely. If $N > \mathrm{card}\, \mathcal{F}_{\Lambda,\mathrm{p}}$, then the input is not 'proper' and the algorithm terminates without producing output.

We now compute the restriction $\sigma_{F_t \setminus \Delta_{s,t}}$. Since $F_t \setminus \Delta_{s,t}$ is a finite subset of $\mathbb{N}$, we can simply list the values of σ in the order they appear on $F_t \setminus \Delta_{s,t}$. If $r \neq \mathrm{card}(F_t \setminus \Delta_{s,t})$, then the input is not 'proper' and the algorithm terminates without producing output. Otherwise, we require that

$$\sigma|_{F_t \setminus \Delta_{s,t}} \circ \iota_{F_t \setminus \Delta_{s,t}}^{-1} = \upsilon,$$

which determines the word σ completely.

For all $k \geq 1$, let

$$\{\widetilde{\omega}_1^k, \widetilde{\omega}_2^k, \ldots, \widetilde{\omega}_{M_k}^k\}$$

be the list of all Λ-words of length $|F_k|$ ordered lexicographically. Here $M_k = (\mathrm{card}\, \Lambda)^{|F_k|}$ for all k. For all $k \geq 1$ and $i \in \{1, \ldots, M_k\}$ define the cylinder sets

$$B_i^k := \{\omega \in \mathrm{X} : \omega|_{F_k} \circ \iota_{F_k}^{-1} = \widetilde{\omega}_i^k\},$$

and let $\beta_k := \{B_1^k, B_2^k, \ldots, B_{M_k}^k\}$ be the corresponding partition of X into cylinder sets for every k. We apply Lemma 3.1 to the system $\mathbf{X} = (\mathrm{X}, \mu, \Gamma)$ and the sequence of partitions

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$(\beta_k)_{k\geq 1}$. This yields a full-measure subset $\mathrm{X}_0 \subseteq \mathrm{X}$ such that for all $\omega \in \mathrm{X}_0$ and all $k \geq 1$ there is an element $h' := h_k(\omega) \in F_k^{-1}$ such that

$$-\sum_{m=1}^{M_k} \pi_m^{k,h'}(\omega) \log \pi_m^{k,h'}(\omega) \leq h_\mu(\beta_k). \tag{19}$$

Let $\omega \in \mathrm{X}_0$, $k \geq 1$ be arbitrarily fixed and $h' := h_k(\omega) \in F_k^{-1}$ be the group element given by Lemma 3.1. Because of the shift-invariance of Kolmogorov complexity we have $\widehat{\mathrm{K}}(h' \cdot \omega) = \widehat{\mathrm{K}}(\omega)$. We will show that

$$\widehat{\mathrm{K}}(\omega) = \widehat{\mathrm{K}}(h' \cdot \omega) \leq \frac{h_\mu(\beta_k)}{|F_k|}.$$

Then, since $h_\mu(\beta_k) = h_\mu(\alpha_\Lambda^{F_k})$ for all $k \geq 1$, taking the limit as $k \to \infty$ completes the proof of the theorem.

For the moment let n be arbitrary fixed. Observe that for all $i \in \{1, \ldots, M_k\}$,

$$\widetilde{\pi}_{n,i}^{k,h'}(\omega) = \frac{1}{|I_{k,n}|} \sum_{h \in I_{k,n}} \mathbf{1}_{B_i^k}(h \cdot (h' \cdot \omega)),$$

that is, $|I_{k,n}|\widetilde{\pi}_{n,i}^{k,h'}(\omega)$ equals the number of times the translates of the word $\widetilde{\omega}_i^k$ along the set $I_{k,n}$ appear in the word $(h' \cdot \omega)|_{\Delta_{k,n}}$. It follows by the definition of the algorithm $A^!$ that the following program describes the word $(h' \cdot \omega)|_{F_n} \circ \iota_{F_n}^{-1}$:

$$\mathrm{p}_n := \bar{\mathrm{k}}01\bar{\mathrm{n}}01\bar{\mathrm{f}}_1 01 \cdots \bar{\mathrm{f}}_{M_k}0110\bar{\mathrm{r}}01\mathrm{w}01\underline{\mathrm{N}}.$$

Here $\bar{\mathrm{f}}_i$ is the doubling encoding of $|I_{k,n}|\widetilde{\pi}_{n,i}^{k,h'}(\omega)$ for all $i \in \{1, \ldots, M_k\}$. The binary word w encodes the word $\upsilon = (h' \cdot \omega)|_{F_n \setminus \Delta_{k,n}} \circ \iota_{F_n \setminus \Delta_{k,n}}^{-1}$ of length r and $\underline{\mathrm{N}}$ encodes the index of $(h' \cdot \omega)|_{\Delta_{k,n}}$ in the set $\mathcal{F}_{\Lambda,\mathrm{p}_n}$.

We will now estimate the length $l(\mathrm{p}_n)$ of the program p_n above. We begin by estimating the length of the word $\bar{\mathrm{f}}_1 01 \cdots \bar{\mathrm{f}}_{M_k}$. Observe that

$$f_j \leq |I_{k,n}| \leq \frac{|F_n|}{|F_k|} \quad \text{for every } j = 1, \ldots, M_k,$$

hence $l(\bar{\mathrm{f}}_1 01 \cdots \bar{\mathrm{f}}_{M_k}) = o(|F_n|)$. Next, we estimate the length of the word w. Since $(F_n)_{n\geq 1}$ is a Følner sequence, we conclude that $l(\mathrm{w}) = o(|F_n|)$. It is clear that $l(\bar{\mathrm{n}}) \leq 2\lfloor \log n \rfloor + 2 = o(|F_n|)$, since $|F_n|/\log n \to \infty$. Finally, we estimate $l(\underline{\mathrm{N}})$. Of course, $l(\underline{\mathrm{N}}) \leq \log |I_{k,n}|!/f_1!\, f_2! \cdots f_{M_k}! + 1$. We use Stirling's approximation to deduce that

$$\log \frac{|I_{k,n}|!}{f_1!\, f_2! \cdots f_{M_k}!} \leq -\sum_{j=1}^{M_k} f_j \log \frac{f_j}{|I_{k,n}|} + o(|F_n|).$$

Hence we can estimate the length of p_n as

$$l(\mathrm{p}_n) \leq o(|F_n|) - \sum_{j=1}^{M_k} f_j \log \frac{f_j}{|I_{k,n}|}.$$

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Since $f_i = |I_{k,n}|\widetilde{\pi}^{k,h'}_{n,i}(\omega)$ for every $i = 1, \ldots, M_k$, we deduce that

$$l(\mathrm{p}_n) \leq o(|F_n|) - |I_{k,n}| \sum_{j=1}^{M_k} \widetilde{\pi}^{k,h'}_{n,j}(\omega) \log \widetilde{\pi}^{k,h'}_{n,j}(\omega).$$

Dividing both sides by $|F_n|$ and taking the limit as $n \to \infty$, we use Lemma 3.1 and Proposition 2.1 to conclude that

$$\limsup_{n\to\infty} \frac{\mathrm{K}^0_{A^!}((h' \cdot \omega)|_{F_n} \circ \iota^{-1}_{F_n})}{|F_n|} \leq \frac{h_\mu(\beta_k)}{|F_k|}.$$

By the optimality of A^* we deduce that

$$\widehat{\mathrm{K}}(h' \cdot \omega) = \limsup_{n\to\infty} \frac{\mathrm{K}^0_{A^*}((h' \cdot \omega)|_{F_n} \circ \iota^{-1}_{F_n})}{|F_n|} \leq \frac{h_\mu(\beta_k)}{|F_k|}$$

and the proof is complete. □

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