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ARTICLE

Multisets, heaps, bags, families: What is a multiset?‡

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Abstract

Is the current formulation of multiset theory, which is based on sets and multiplicities of their elements, adequate? We exhibit both mathematical and metamathematical reasons which should cause one to rethink the definition. Some problems with multiset theory in its accepted formulation concern even the basic operations of union, intersection, and complement; others, more deeply rooted, concern Cartesian products, relations, or morphisms. We compare current definitions and conclude that the problems of multiset theory need to be resolved at the fundamental level of sets and mappings (or equivalent constructs) with multiplicities introduced only as a secondary concept. As a consequence, we propose to define multisets as families. A mapping establishes the connection to the familiar theory of multisets. Without losing anything, our proposal is simple and provides for an elegant mathematical theory.

Keywords: Multiset; generalised multiset; categories

1. An Unforeseen Problem

A multiset is a collection of objects, called elements, in which the elements are allowed to repeat finitely often (see, e.g., Blizard 1991). Typically, a multiset is specified by its *underlying set* S and, for each element $s \in S$, its *multiplicity* $\mu(s)$, a nonnegative integer. In a paper which we wrote a few years ago, we found it convenient to use multisets as a tool. In the given context, we considered the usual definition of multisets via multiplicities as being too clumsy. Instead, we opted for a seemingly equivalent definition, up to finiteness, that of a family. As usual, a *family* over a set S consists of a set I, called *index set* and a mapping $\iota : I \to S$. The multiplicity by which an element $s \in S$ appears in the multiset is the cardinality of $\iota^{-1}(s)$. A reviewer of that paper found this definition rather strange and, as this choice of words was not important in the given context we fixed the problem with a kind of first-aid bandage. However, while doing so at one spot, we discovered that the bandage began to seep at another spot: fixing one problem just revealed many more elsewhere like the profuse heads of the hydra.

This experience was the motivation to scrutinize the notion of *multiset*. After having done so, we are even more convinced now that the concept of families is far better suited to express the intended notion of multiset than the usual definition involving elements with multiplicities. In this context, it will be important to note that the notions of *objects* and *elements* mentioned above

†Deceased.
‡A preliminary version of this paper, without proofs and details, was presented at the International Conference on Recent Advances in Pure and Applied Mathematics, ICRAPAM 2016, held on 19–23 May, 2016, in Bodrum, Turkey (Jürgensen 2016).

and considered as equivalent in Blizard (1991) must be distinguished. We argue that doing so is inevitable for an intuitively sound theory.[1]

Intuitively, the notion of multiset is meant to generalize that of sets. Consequently, there exists a unique *empty* multiset and every set has a unique interpretation as a multiset. It seems natural to require that this connection be afforded by *forgetting* multiplicities and that it be factorial. We show that this cannot be achieved without violating some part of the intuition.

The original definition of multisets postulates that the multiplicities are nonnegative integers. Once a sound theory involving only these as multiplicities has been established, a formal generalization to other multiplicity domains is easy following the approaches used to define fuzzy sets over various domains like $\mathbb{R}$, the set of real numbers or $\mathbb{L}$, an arbitrary lattice (Peeva and Kyosev 2004). Especially for min–max fuzzy sets and multisets, the connection is formally close; however, as will be explained below, intuitively these notions are quite different depending on the concept of distinguishability between elements.

Our initial problem concerned the complement of a multiset; some aspects of a mathematical theory of multisets are discussed in Jürgensen (2017). Suppose one has one orange in a set. How many oranges are in its complement? The answer depends on what we mean. In a paper on *If no two oranges are "distinguishable,"* then, clearly, none is in the complement – a rather unexpected answer. But is this correct if we have a box with 25 oranges in it? By their very arrangement in the box, they are distinguishable. If one of them is in the set, then there are 24 not in the set. One definitely knows what is not in the complement of a set, but one cannot know what is in its complement unless one refers to a "universal set," or rather a "reference set" to which the given set is compared. The problem is compounded by the fact that there is a notion of indistinguishability – or *sameness* as discussed by Marcus (2001) – which needs to be defined in a meaningful way. However, what is in the complement of a set is rarely a critical issue as the choice of the reference set is, usually, naturally implied by the context. In contrast, in multiset theory, the complement of a multiset with one orange in it may contain any number of oranges. The reference set becomes more arbitrary, and the analogy to negation in logic is lost.

Our original problem was even more complicated than that as it involved potentially unbounded finite multiplicities. Stripped to the essentials, the problem was as follows. Let S be a set. Consider countably many multisets M_i over S with $i = 1, 2, \ldots$ such that, for each i and each $s \in S$, the multiplicity of s in M_i is unbounded. What is the complement of M_i? What is the complement of $\bigcup_{i=0}^{\infty} M_i$? In standard multiset theory, one assumes that the multiplicity of an element is finite, that is, a cardinal number less than $\aleph_0$. To pinpoint the problem, one can focus on a single multiset M_i – that one has to deal with an infinite number of them might just make the argument more convincing – such that the multiplicity of some $s \in S$ in M_i is unpredictable. Staying within standard multiset theory suggests that the complement be taken with respect to some fixed multiset M. The multiplicity of s in M would have to be at least that of s in M_i. Otherwise, what would one do? One could introduce negative multiplicities[2] or some other kind of unconvincing arithmetic acrobatics. One could also leave the realm of multisets to include infinite multiplicities. In the latter case, one has to deal with the subtraction of infinite cardinal numbers. Neither solution is intuitively acceptable.

The complementation problem has plagued the theory of multisets (Blizard 1986, 1989; Bogatiryova 2011; Hickman 1980; Ibrahim 2010; Jena et al. 2001; Singh et al. 2002, 2008, 2011, 2016; Syropoulos 2001; Wildberger 2003). Singh et al. (2011) conclude: *None of the existing approaches succeeds if the goal is to augment the theory of multisets endowed with a boolean algebraic structure without assuming some contrived situations.*

However, complementation is not the only and not the most serious trouble in the theory of multisets which begs for a conceptually convincing solution, albeit being most conspicuous. Other heads of the Hydra appear when one considers multiset theory in the context of the foundations of set theory. The very basics of multiset theory then requires a re-examination of its mathematical and conceptual relevance. In essence, we need to clarify the relation between *object* and *element* as mentioned. In this paper, we propose a clear distinction, possibly relativized to different worlds. Mathematically, this can be achieved quite elegantly using families rather than multisets.

This paper occasionally relies on discussing notions or assumptions which are not completely defined, possibly contentious, and even outside the reach of mathematics. To distinguish such cases from straightforward mathematics, we use the term of "convention" rather than those of "definition" or "assumption." Of course, the implicit meaning is that conventions serve as definitions or assumptions for the respective rest of this paper.

There is no really difficult mathematical result in this paper which would deserve being called a theorem or the like. Hence, we use the term of "observation." The only exception is Proposition 22 in which we propose a solution to a structure problem which has plagued the theory of multisets. This result is important enough because it provides a partial answer to an elusive question.

The flow of thoughts in this paper is roughly as follows. We introduce multisets as usual, review some basic results, and discuss ways in which the category of multisets could be defined. We then turn to exhibiting problems with the notion of "multiset" as apparent from the disjoint union of sets and even basic operations on multisets. This leads us to fundamental questions in set theory, first by looking at a rather simple everyday example and then by briefly explaining the philosophical background of those mathematicians who formulated modern set theory about 150 years ago. Our arguments may be controversial. However, they seem to lead to a formulation of set theory which Cantor, Dedekind, and many others had in mind – and which was not doubted until multisets came around.[3] The point of contention is, whether there can be two truly distinguishable objects in the world. We agree with Leibniz's postulate that this is impossible, but also suggest a pragmatic approach introducing world views, by which distinguishability is relativized. Mathematically, this suggests using families instead of multisets. We revisit the problems of multisets with families instead and find that, as in set theory, the concept of a reference family is missing. With this added, all the problems of multiset theory seem to go away. But we now deal with families, not multisets. However, a very simple mapping carries one from families to multisets.

There is a huge amount of literature on multisets and their relatives. Among the papers concerning multisets and their relatives which we know of, we cite only those which are directly relevant to the problem at hand.

We are indebted to the work by Blizard (1986, 1989, 1991, 1993), Singh (1994), Singh and Isah (2013), Singh and Singh (2008), Singh et al. (2002, 2008, 2011, 2013), and Ibrahim (2010), and especially the unpublished paper by Singh et al. (2016) who did much of the groundwork. We hope that the fundamental, but as yet unpublished paper (Singh et al. 2016) gets published soon because it contains important ideas not found in such a comprehensive coverage anywhere else.

2. Notation and Some Basic Notions

By $\mathbb{N}$ we denote the set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{N}_0$, the difference $m \dot{-} n$ is defined as $m - n$ for $m \geq n$ and as 0 for $m < n$.

We use standard set-theoretic notation. In particular, the power set of a set S is denoted by $\mathsf{P}S$. When S is a subset of a set X, the complement of S with respect to X is the set $\mathsf{C}_X(S) = X \setminus S$; in certain cases, when the superset is not specified, but implied, we may write $\mathsf{C}(S)$ instead. The disjoint union of two sets S and T is denoted by $S \mathbin{\dot\cup} T$. For any objects x and y, $\langle x, y\rangle$ denotes the ordered pair composed of x and y. For a set S, $|S|$ is the cardinality of S. For a set S and an equivalence relation ϱ on it, X/ϱ is the set of equivalence classes modulo ϱ and, for $s \in S$, s/ϱ is the equivalence class of s. Consider two sets S and T. Then $S \times T = \{\langle s, t\rangle \mid s \in S, t \in T\}$ is the direct (or Cartesian) product of S and T. A *relation* f from S to T is a subset of $S \times T$. For a relation f and an element $s \in S$, $f(s) = \{t \mid t \in T, \langle s, t\rangle \in f\}$. The inverse of a relation f is the set

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$f^{-1} = \{\langle t, s\rangle \mid t \in T, s \in S, \langle s, t\rangle \in f\}$. Thus, for $t \in T$, $f^{-1}(t) = \{s \mid s \in S, \langle s, t\rangle \in f\}$. A relation f is a mapping, if $f(s)$ is a singleton set for all $s \in S$. It is an injective mapping, if also f^{-1} is a mapping.

Consider a mapping $f : S \to T$. For a set U and a mapping $g : T \to U$, the composite mapping $g \circ f$ is the mapping $h : S \to U$ defined by $h(s) = g(f(s))$ for all $s \in S$.

Here and in the sequel (Jürgensen 2017), whenever it is convenient and not compromising accuracy, we shall not distinguish notationally between singleton sets and their elements. Thus, if f as above is not just a relation, but even a mapping, $f(s) = \{t\}$ would be written as $f(s) = t$. In some cases, it would be preferable to write all mappings from the right, that is, sf instead of $f(s)$. We chose the latter for consistency and readability.

In addition to the usual assumptions for set theory, we postulate that there is a universe.

Definition 1 (Grothendieck and Verdier 1972). *A universe is a non-empty set U with the following properties:*

1. *if $x \in U$ and $y \in x$ then $y \in U$;*
2. *if $x, y \in U$ then also $\{x, y\} \in U$;*
3. *if $x \in U$ then also $\mathsf{P}x \in U$;*
4. *if $\{x_\alpha\}_{\alpha \in I}$ is a family such that $I \in U$ and $x_\alpha \in U$ for all $\alpha \in I$ then $\bigcup_{\alpha \in I} x_\alpha \in U$.*

The elements of U are called U-sets; the subsets of U are called U-classes. Throughout this article, we assume a fixed universe U with $\mathbb{N}_0 \in U$ and with all sets under consideration being elements of this universe; hence, by "set" we mean "U-set." Further notation and notions will be introduced as needed.

3. Multisets and Their Properties

Before we continue, we need a definition of a multiset in the usual sense and a review of operations on and properties of multisets. Nearly, everything in this section is folklore; we only provide specific references to help the reader with the not-so-common items.

Definition 2 (Multiset). *Let X be a set. A* multiset *over X is a mapping $\mu : X \to \mathbb{N}_0$.*

Typically, a multiset μ is described by the graph of μ, $\{\langle x, \mu(x)\rangle \mid x \in X\}$, where $\mu(x)$ is called the *multiplicity*[4] of x. Let us give this set a name, say M_μ. Intuitively, $\mu(x)$ says, how often $x \in X$ appears in M_μ. Occasionally, we write (X, μ) instead of just M_μ when the set X needs to be specified.

A subset S of X can be viewed as a multiset with the characteristic function

$$\chi_S : X \to \{0, 1\} : x \mapsto \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases}$$

defining the multiplicities. On the other hand, the *carrier* of a multiset M_μ over X is the set $\{x \mid x \in X, \mu(x) > 0\}$.

To distinguish between the usual relations or operations on sets and those on multisets, we use "ms" as an indicator. Further such distinctions will be introduced as needed.

The idea of x appearing in the multiset M_μ with multiplicity $\mu(x)$ is that there are $\mu(x)$ occurrences of x in M_μ. Sometimes, we need to speak about specific occurrences of x. For lack of a better term, we call them *(identical) copies* of x. This is slightly misleading as the copies of an object x would be distinguishable while, in a multiset, the identical object x occurs $\mu(x)$ times. We discuss the subtleties of the notion of *distinguishability* and of related notions in Section 10 below.

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Definition 3 (Multiset relations). *Let M_μ be a multiset over X.*

1. *Let $x \in X$. Then $x \in^{\mathsf{ms}} M_\mu$ if and only if $\mu(x) > 0$.*
2. *Let M_ν be a multiset over X. Then $M_\nu \subseteq^{\mathsf{ms}} M_\mu$ if and only if $\nu(x) \leq \mu(x)$ for all $x \in X$.*

If $M_\nu \subseteq^{\mathsf{ms}} M_\mu$, then M_ν is said to be a *submultiset* of M_μ. Some subtleties regarding the definition of $\subseteq^{\mathsf{ms}}$, not relevant to the issues addressed in the present paper, are discussed in Singh and Singh (2008).

Definition 4 (Multiset properties). *Let M_μ be a multiset over X.*

1. *M_μ is said to be* empty, *$M_\mu = \emptyset^{\mathsf{ms}}$, if and only if $\mu(x) = 0$ for all $x \in X$.*

As mentioned, multiplicities of 0 are sometimes confusing. Let (X, μ) and (X', μ') be multisets such that $X \subseteq X'$, and $\mu'(x) = \mu(x)$ for $x \in X$ and $\mu'(x) = 0$ otherwise. These two multisets are different objects.[5] Moreover, not even $(X, \mu) \subseteq^{\mathsf{ms}} (X', \mu')$ holds. On the other hand, the two multisets seem to be the same. The elements of multiplicity 0 are causing this problem. With some considerable technical effort, this issue can be resolved in a formally clean way by considering two multisets as equivalent if and only if they are equal when the elements with multiplicity 0 are ignored. For this paper, we choose to evade this problem by assuming that the set X is large enough.

Definition 5 (Basic multiset operations). *Let M_μ and M_ν be multisets over X.*

1. *$M_\mu \cap^{\mathsf{ms}} M_\nu$ is the multiset M_κ with $\kappa(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.*
2. *$M_\mu \cup^{\mathsf{ms}} M_\nu$ is the multiset M_κ with $\kappa(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$.*
3. *$M_\mu \uplus M_\nu$ is the multiset M_κ with $\kappa(x) = \mu(x) + \nu(x)$*

The operation $\cap^{\mathsf{ms}}$ is called *multiset intersection*. The operations $\cup^{\mathsf{ms}}$ and $\uplus$ are called *multiset union*.

Observation 6 (Properties of the basic multiset relations and operations; Knuth 1997, vol. 2, pp. 694–695). Let $x \in X$ and let M_μ and M_ν be multisets over X. The following statements hold true.

1. If $x \in^{\mathsf{ms}} M_\mu$ and $M_\mu \subseteq^{\mathsf{ms}} M_\nu$, then $x \in^{\mathsf{ms}} M_\nu$.
2. The relation $\subseteq^{\mathsf{ms}}$ is a partial order on the set of all multisets over X.
3. $M_\mu \subseteq^{\mathsf{ms}} M_\nu$ if and only if $M_\mu \cap^{\mathsf{ms}} M_\nu = M_\mu$.
4. $M_\mu \subseteq^{\mathsf{ms}} M_\nu$ if and only if $M_\mu \cup^{\mathsf{ms}} M_\nu = M_\nu$.
5. The operations $\cap^{\mathsf{ms}}$, $\cup^{\mathsf{ms}}$, and $\uplus$ are commutative, associative, and distributive.
6. The operations $\cap^{\mathsf{ms}}$ and $\cup^{\mathsf{ms}}$ are idempotent.
7. Absorption:
   (a) $M_\mu \cup^{\mathsf{ms}} (M_\mu \cap^{\mathsf{ms}} M_\nu) = M_\mu$;
   (b) $M_\mu \cap^{\mathsf{ms}} (M_\mu \cup^{\mathsf{ms}} M_\nu) = M_\mu$;
   (c) $M_\mu \cap^{\mathsf{ms}} (M_\mu \uplus M_\nu) = M_\mu$;
   (d) $M_\mu \cup^{\mathsf{ms}} (M_\mu \uplus M_\nu) = M_\mu \uplus M_\nu$.
8. $\emptyset^{\mathsf{ms}} \uplus M_\mu = \emptyset^{\mathsf{ms}} \cup^{\mathsf{ms}} M_\mu = M_\mu$.
9. $\emptyset^{\mathsf{ms}} \cap^{\mathsf{ms}} M_\mu = \emptyset^{\mathsf{ms}}$.
10. $M_\mu \uplus M_\nu = (M_\mu \cup^{\mathsf{ms}} M_\nu) \uplus (M_\mu \cap^{\mathsf{ms}} M_\nu)$.

Definition 7 (Derived multiset operations). *Let M_μ and M_ν be multisets over X.*

1. *The multiset difference $M_\mu \setminus^{\mathrm{ms}} M_\nu$ is the multiset M_κ with $\kappa(x) = \mu(x) \dot{-} \nu(x)$ for all $x \in X$.*
2. *The multiset symmetric difference $M_\mu \, \Delta^{\mathrm{ms}} \, M_\nu$ is the multiset M_κ with $\kappa(x) = |\mu(x) - \nu(x)|$ for all $x \in X$.*

Further operations were introduced as needed for specific applications. The operations of $\setminus^{\mathrm{ms}}$ and Δ^{ms} are already close to the general concept of complementation, which motivated our investigation originally.

4. The General Idea of a Multiset and Expectations

Without doubt, the concept of multisets defined by multiplicities is most useful and has many applications, in particular as a conceptually simplifying tool in many areas (see, e.g., Bogatiryova 2011; Red'ko et al. 2015; Singh et al. 2007, 2016).

The mathematical theory of multisets defined by multiplicities alone is clumsy and has many formal or intuitive inconsistencies, which we shall demonstrate in the sequel. Blizard's (1986, 1989) axiomatization cleaned up much of the elementary mess, but failed to exhibit a manageable algebraic structure of the class of multisets, for instance in the form of a category. In our opinion, this is not a failure of Blizard's axiomatics, but an inherent problem of multisets defined by multiplicities, and that it cannot be resolved at that level.

Alternative definitions of multisets exist (see, e.g., Monro 1987) which lead to a cleaner mathematical theory. The cost is that one loses some of the immediate intuitive applicability, which makes multisets in their original definition attractive.

With this in mind, we propose to use the concept of families instead of that of multisets. This does not distract from the applications, where the original definition might fit far better, but also establishes a mathematical framework in which to deal with multisets uniformly and consistently.

5. The Category of Multisets

Do multisets form a category? The answer should be simple, but is not. It depends on the precise definition of a multiset. The key is the definition of morphisms between multisets. Morphisms need to have intuitively justified properties and also meet the formal criteria for a category. Several types of proposals exist. Some modify the definition of multisets, unsatisfactory to multiset theory for this very reason. Others stay close to the definition of multisets through multiplicities, but do not capture the intuitive idea. Others, while intuitively convincing, break down because the composition of such morphisms is not a morphism in the same sense.

One of the universally accepted assumptions about multiset morphisms[6] states

Convention 8 *Let $M_\mu = (X, \mu)$ and $M_\nu = (Y, \nu)$ be multisets and let f be a morphism of M_μ into M_ν. Then, for all $x \in X$, if x' is a copy of x then $f(x')$ is a copy of $f(x)$.*

The word "copy" is used in a metaphorical sense as explained above. Convention 8 implies that,[7] in the worst case, $\nu(f(x)) \geq \mu(x)$. Suppose there is also an element $\bar{x} \in X$ such that $f(\bar{x})$ is a copy of $f(x)$. Then, for each copy $\bar{x}'$ of $\bar{x}$, $f(\bar{x}')$ is also a copy of $f(x)$. Hence, also $\nu(f(x)) \geq \mu(\bar{x})$ in the worst case. In general, for all $x \in X$,

$$\nu(f(x)) \geq \sum_{\substack{\bar{x} \in X \\ f(\bar{x}) = f(x)}} \mu(\bar{x})$$

is a tight, but unrealistic, worst-case bound. The bound, if it holds for every x implies a kind of injectivity of f. However, for morphisms in general, one may encounter more complex cases where such a simple numerical bound does not suffice. We illustrate the situation by an example.

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Example 9 Consider two multisets $M_\mu = (X, \mu)$ and $M_\nu = (Y, \nu)$ as follows:

1. $X = \{a, b, c\}$ with multiplicities 2, 2, and 3, respectively.
2. $Y = \{A, B\}$ with multiplicities 1 and 4.

Let f be the mapping of X into Y such that $a \mapsto A$, $b \mapsto B$, and $c \mapsto B$. There is no problem with a as both its occurrences must map to the single occurrence of A. Suppose the two occurrences of b map onto two occurrences of B, and that the three occurrences of c map onto three occurrences of B. As there are only four occurrences of B, there must be occurrences of b and c which have the same image.

The example shows that defining morphisms of multisets just on the basis of multiplicities is a very difficult task because there is no general connection between the multiplicities in the domain and those in the codomain. One can take this as another strong argument in favor of our claim that to define multisets, the family-based approach is preferable to the one based on multiplicities.

As mentioned, there have been many attempts to define morphisms of multisets. Some, including three of ours in drafts of the present paper, all convincing, turned out to be wrong after some careful examination.

Simple and at the first glance convincing is the approach taken by Singh et al. (2008) and Ibrahim (2010): a multiset morphism (function) is defined to be a mapping between the carriers without any further restrictions on the multiplicities. This definition respects Convention 8 above. There is no problem with the composition of morphisms. However, it is not convincing intuitively as it makes no reference to multiplicities. Indeed, every such morphism maps every multiset into every multiset with the appropriate carriers.

Monro (1987) defines a multiset as pair (Σ, σ) such that Σ is a set and σ is an equivalence relation[8] on Σ. This definition is unorthodox as it does not refer to cardinal numbers. It is more akin to our proposal in which we use families instead of multisets.[9] The multiplicities are, however, a key ingredient in the theory of multisets, and without them, an important property of multisets is missed. In Monro's sense, a morphism of a multiset (Σ, σ) to a multiset (T, τ) is a mapping $f : \Sigma \to T$ such that $(x, x') \in \sigma$ implies $\big(f(x), f(x')\big) \in \tau$. Monro's definition is consistent with Convention 8. Singh et al. (2013) and Singh and Isah (2013) also use Monro's definition.

Observation 10 (Monro's (1987) category of multisets). The multisets in the sense of Monro form a category.

Even though Monro's definition does not rely on multiplicities, we use it as a yardstick against which to compare other definitions, as it seems to capture the intended intuition precisely.

Another type of definition of the category of multisets is proposed by Syropoulos (2001, 2003). That approach is rather cryptic at first glance. In essence, it is equivalent to Monro's approach, but needs a considerable amount of mathematical machinery to establish the definition. To prove its equivalence to Monro's definition requires even more of that. The mathematical machinery invoked and needed in this case seems to be overkill for "the working mathematician."

For the purpose of the present paper, we only need to establish that the multisets form something akin to a category. We do so using Monro's definition, well knowing that this is unsatisfactory. We leave a detailed analysis of the proposed definition of morphisms via Cartesian products and relations to a successor paper (Jürgensen 2017) in which the necessary groundwork is laid.

Monro (1987) also introduces the concept of *multinumber*. These are the multisets in the usual sense. He writes: "The concept of multinumber is related to that of multiset in the same way that the concept of (cardinal) number is related to the ordinary concept of set. I think that the concept of multiset is more fundamental than that of multinumber, but the concept of multinumber may be the more useful of the two. In any event the concepts should be distinguished" (Monro 1987, p. 176). He defines the obvious mapping of his multisets to his multinumbers.

Observation 11 (Mapping of multisets in the sense of Monro to usual multisets). The following mapping $\Phi^{\mathsf{M}\to\mathsf{ms}}$ establishes the natural connection between multisets in the sense of Monro and multisets in the usual sense.

Let (Σ, σ) be a multiset in the sense of Monro. Then $\Phi^{\mathsf{M}\to\mathsf{ms}}(\Sigma, \sigma) = (\Sigma/\sigma, \mu)$ where $\mu(x/\sigma) = |x/\sigma|$ for all $x \in \Sigma$.

The mapping $\Phi^{\mathsf{M}\to\mathsf{ms}}$ is just a mapping, not a functor, as we have no acceptable definition of multiset morphisms. For the purpose of this paper, this is not essential. It just shows that the hydra grows ever more heads.

6. Disjoint Union of Sets

The disjoint union of two sets seems to result in a multiset. Given sets R and S, subsets of X, to form the disjoint union $R \uplus S$ of R and S one takes two disjoint sets

$$\tilde{R} = \{x_R \mid x \in R\} \text{ and } \tilde{S} = \{x_S \mid x \in S\},$$

such that there are bijective mappings

$$\iota_R : \tilde{R} \to R : x_R \mapsto x \text{ and } \iota_S : \tilde{S} \to S : x_S \mapsto x.$$

Then $R \uplus S$ is defined as $\tilde{R} \cup \tilde{S}$. Thus, the disjoint union is not unique, but is defined only up to the mappings ι_R and ι_S. Technically, the disjoint union can be considered as the union of disjoint sets of *names of objects*. Intuitively, however, one thinks of $R \uplus S$ as a union of R and S in which those elements, which occur in both R and S, appear twice. In this sense, $R \uplus S$ is a multiset given by $\kappa : X \to \mathbb{N}_0$ with

$$\kappa(x) = \begin{cases} 0, & \text{if } x \notin R \cup S, \\ 1, & \text{if } x \in (R \cup S) \setminus (R \cap S), \\ 2, & \text{if } x \in R \cap S. \end{cases}$$

Mathematically, the disjoint union $R \uplus S$ and the corresponding multiset M_κ are not the same objects. Considering the complement of $R \uplus S$ reveals one of the key problems.

For the complement of $R \uplus S$, the set with respect to which the complement is taken could be $X \uplus X$. To resolve the ambiguities involved, one would proceed as follows. Let X_R and X_S be disjoint sets such that $\tilde{R} \subseteq X_R$, $\tilde{S} \subseteq X_S$, and $|X_R| = |X_S| = |X|$. The typical elements of X_R and X_S are called x_R and x_S, respectively, where $x \in X$. Extend ι_R and ι_S to bijections

$$\iota_R : X_R \to X : x_R \mapsto x \text{ and } \iota_S : X_S \to X : x_S \mapsto x.$$

One can then define $X \uplus X = X_R \cup X_S$ and thus the complement of $R \uplus S$ is $(X \uplus X) \setminus (R \uplus S) = (X_R \cup X_S) \setminus (\tilde{R} \cup \tilde{S})$.

The multiset counterpart of this construction looks as follows. Consider the multiset M_λ over X with $\lambda(x) = 2$ for all $x \in X$. With M_κ being the multiset corresponding to $R \uplus S$, the multiset $M_\lambda \setminus^{\mathsf{ms}} M_\kappa$ corresponds to the complement of $R \uplus S$.

Other definitions of the complement of $R \uplus S$ are equally convincing. In some contexts, it could even make sense to define the complement of $R \uplus S$ to be $X \setminus (R \cup S)$.

We learn from the example of disjoint unions that intuition about even some of the simplest constructions in set theory can lead to unexpected complications when applied to multisets.

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Table 1. Partial student lists as multisets: “mlt.” means “multiplicity”; the symbols $\cap^{\text{true}}$ and $\cup^{\text{true}}$ mean the usual intersection and union, respectively, when the actual students, not just their first names are considered

Form 1		Form 2		operation				
Name	mlt.	Name	mlt.	$\cap^{ms}$	$\cup^{ms}$	$\uplus$	$\cap^{\text{true}}$	$\cup^{\text{true}}$
Peter	3	Peter	1	1	3	4		
s1, s2, s3		s4					0	4
Susan	2	Susan	2	2	2	4		
s5, s6		s5, s7					1	3
Jonas	2	Jonas	1	1	2	3		
s8, s9		s8					1	2

7. How to Plan a School Trip – More Problems

Mrs. T teaches two forms[10] at her school. She refers to the students by first names. In Form 1, there are three students named Peter, two named Susan, and two named Jonas; in Form 2, there is one Peter, one Jonas, and there are two students named Susan. She is planning a joint school trip to a theatre for the two forms and needs to determine how many tickets to reserve. There may be other students in the forms whom we ignore for the sake of the argument. The list of students and some further information are shown in Table 1.

If Mrs. T were not such a good teacher, this would be the only information available to her. Multiset theory would suggest that she buy 11 tickets to be safe – she might hope to sell spare ones at the entrance? Or should she buy seven tickets as suggested by $\cup^{ms}$?

Fortunately, Mrs. T not only knows her students well (she also knows mathematics): she can distinguish them as individuals, and she also has up-to-date lists with names and student numbers. The student numbers are shown in the table as sn, and so are the sizes of the resulting “true” intersections and unions. With their help, the teacher’s problem is solved: she purchases nine tickets.

Obviously, the true number for the intersection is between 0 and that of $\cap^{ms}$, and that for the union is between that of $\cup^{ms}$ and $\uplus$.

This, however, is not that important. Mrs. T’s dilemma is rooted in the very foundations of set theory: Is each of her forms a set or a multiset? In the former case, there are students in her forms who are distinguishable despite the fact that their first names are the same; hence, intersection and union are those of sets; this results in correct numbers. Otherwise, the students with the same first names are completely indistinguishable, and the multiset computations seem useless.[11]

Mrs. T’s dilemma is far from being new. What is a set? Does she deal with sets or with some other kind of mathematical entity? Her dilemma can be found in several areas of mathematics, most prominently combinatorics and probability theory (explained further below), where the problem is dealt with pragmatically by relying on no-rigorous language. The key issue seems to be the concept of distinguishability of objects. We discuss this concept in detail and put it into some historical context below. In the next section, we consider the problems of the oranges miraculously changing their identities and of the disappearing balls.

We repeat an earlier quotation: *A multiset is a collection of objects (called elements) in which elements are allowed to repeat* (see, e.g., Blizard 1991). The distinction between *objects* and *elements* may become important. Are objects the same as elements as Blizard implies?

8. How to Distribute Oranges

Can a set contain repeated elements? A naive answer would be *yes* – even in mathematical contexts. Doubts arise when mathematical theory spoils the intuition.

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We use an example adapted from Blizard (1986) to show how paradoxes can arise in multiset theory.

Take a box of 25 oranges to be distributed "fairly" among 5 children. The oranges form a set, do not they? What we mean by "fairly" is agreed upon by the children. They adopt the following strategy:

> Arrange the oranges in random order along a single line. Arrange the children in random order in a line. The first child in the line takes the first orange in the line and moves to the end of the children's line to wait for his/her next pick. The next child takes the next orange, and so on.

Whether this strategy is truly fair is irrelevant. Each child i receives a subset S_i containing 5 oranges of the original set S of 25 oranges. Being just sets of oranges, these subsets of five oranges should be equal by the definition of multisets, but obviously they are disjoint. While no distinction of the elements in the original set of oranges is made, we distinguish them one by one when they are distributed. A miracle!

The example of distributing oranges may look too unmathematical to the reader's taste. Replace oranges, apples, and whichever fruit you like by identically shaped balls of various colors in a probabilistic urn experiment. The problem persists, and another one is added.[12]

As for the urn experiments, one kind of them requires that one takes balls out of the urn and then returns them back to the urn.[13] Remember that these balls are indistinguishable originally. Once taken out of the urn, are they suddenly distinguishable from the others? Once returned, did the balls lose their individualities again? That is, how would returning the balls replenish the urn, if the set of balls were considered as a multiset. We hope the following example helps to clarify what the problem is: suppose the urn contains 10 blue balls. When considered in multiset theory, these balls are indistinguishable (see Blizard 1986). Now remove three of them. There are now two multisets of seven and three balls each, all indistinguishable. Now put the three balls back into the urn. How many blue balls are in the urn? Obviously, $\max(7, 3) = 7$, not 10[14]. If all the balls were distinguishable already initially, then set theory would deal with this issue correctly. If they were not, then they would have become distinguishable and indistinguishable again during the experiment, a rather unconvincing idea. We discuss this issue further below in the context of the historical concept of *identity*.

The urn paradigm and similar ones are predominantly used in combinatorics and probability theory. These theories have developed their own terminologies, not resorting to multisets. Whether they could gain from multiset theory is a matter of speculation. We should argue "no," as the present simple tools of sets and mappings seem to suffice for economical and mathematically elegant presentations.

Many puzzling situations arise, mainly concerning the distinguishability or indistinguishability of elements in a set. To resolve these problems, we examine the concepts meant by the terms of *set* and *multiset*, the concept of *distinguishability*, and the terms of *object* and *element*. We argue that, as a basic concept, that of multisets is redundant and that the standard theory of sets and functions suffices, and that multisets can be treated elegantly and adequately as images of families under a very simple mapping.

9. Generalized Multisets

Several generalizations of the concept of *multiset* have been proposed (see Blizard 1986): to allow for negative multiplicities, to permit real-valued multiplicities similar to fuzzy sets, or to allow for arbitrary cardinal numbers as multiplicities. The latter is the most interesting case in our context. Thus, a *generalized multiset* is a set with multiplicities which can be any cardinal numbers in the given universe U. In the sequel, we use the following definition.

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Convention 12 (Generalized multiset). *A multiset in U is a pair* (X, μ) *such that X is a set in U and* μ *is a mapping of X into the set of cardinal numbers of sets in U.*

This definition coincides with the usual one for finite cardinalities. Unless stated otherwise, by *multiset,* we mean a *generalized multiset* in the sequel.

10. History

The history of multisets is described in Blizard's thesis and subsequent publications of his papers (Blizard 1986, 1989, 1991, 1993). In our view, the most important contribution of Blizard's work is to clarify the focus of multiset theory and establishing an axiomatic framework for multisets which allows one to assess the comparison with other related theories. Some important additional historical remarks by Singh are found in Singh (1994). In particular, Singh refers to publications by Brink (1988) and Hailperin (1986) regarding the general principles of multiset theory and to Angelelli (1965) regarding Leibniz's theory (see below). These historical references are interesting and may serve as motivations. They do not lend exceptional credence to any one opinion.

When reviewing the history of multisets, we are quite aware of the fact that there have been many similar ideas over the centuries. We focus on the philosophical ideas which would have likely influenced the thinking of leading mathematicians of the time when the foundations of set theory were laid. Leibniz and Kant dominated the discussion and it is unimaginable that leading mathematicians of the time would not have been familiar with those ideas.

Multisets can be represented in quite a few different ways (e.g., Blizard 1986; Bogatiryova 2011; Eilenberg 1974; Monro 1987; Singh et al. 2016; Tella and Daniel 2011; Wildberger 2003). The introduction of multisets as a mathematical concept worth studying on its own is usually attributed to Knuth (1997, vol. 2, p. 483 and pp. 694–695); in his brief notes, originally of 1969, he mentions, without references, that the idea has been around for some time. He also recalls some of the discussions he had regarding the name for the concept.[15] The origin of the concept of multiset may never be determined, even the origin of names for this concept is unclear (see, e.g., Jena et al. 2001 and Knuth 1997, vol. 2). The concept, indirectly, had a key role in the foundations of set theory; the name for it evolved independently in various branches of science. For our purposes, the detailed history is irrelevant. However, some historical facts could help us with understanding the intended meaning of set-theoretic terminology.

The choice of the formalism to represent multisets is often accidental, sometimes also based on convenience. A notable exception is Eilenberg's representation of multiplicities using formal power series with coefficients in $\mathbb{N}_0$ or, more generally, in some suitable semiring (Eilenberg 1974, Chapter VI). Eilenberg's account of 1974 abstracts from techniques already widely in use at the time in the theories of automata, formal languages, codes, and combinatorics.[16]

The two types of problems concerning the concept of multiset illustrated so far have different roots and, thus, need to be addressed differently. The complementation problem is mathematical in nature and will be addressed at that level. The problem exemplified by students, oranges, or balls in an urn is philosophical: Are there indistinguishable nonidentical objects? Or, what is a set?

The discussion as to what constitutes an individual object[17] has been ongoing for centuries.[18] An important principle is expressed in the so-called *Leibniz' law*, the *principium identitatis indiscernibilium,*[19] as stated in his note starting with *primæ veritates*[20] (Leibniz 1961, pp. 519–520): *Sequitur etiam hinc non dari posse ⟨in natura⟩ duas res singulares solo numero differentes: utique enim oportet rationem reddi posse cur [dicantur] ⟨sint⟩ diversae, quae ex aliqua in ipsis differentia petenda est.*[21] Objects are identical if and only if, all their properties are the same, in symbols:

$$\forall x \forall y \Big(\big(\forall P (P(x) \leftrightarrow P(y)\big) \leftrightarrow x = y \Big),$$

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where P is a unary predicate symbol. Indistinguishability is absolute, not just what we may perceive. Leibniz explains "indistinguishability" further: *Notio completa seu perfecta substantiæ singularis involvit omnia eius prædicata præterita, præsentia ac futura. Utique enim prædicatum futurum esse futurum jam nunc verum est, itaque in rei notione continetur.*[22] Thus, the oranges to be distributed are distinct – fortunately, as the children might feel cheated otherwise – for the simple reason that the oranges can be put in a line or even more simply grabbed. Similarly, 10 blue balls in an urn are distinct; otherwise, taking three out and returning them would leave us with seven blue balls in the urn.

Blizard's (1986) notion of multiset hinges on the definition of distinguishability. If one accepts Leibniz's definition, then Blizard's example of oranges (Blizard 1986, pp. 4–5) is wrong. Even much of his discussion of particle physics becomes dubious under this assumption. We as humans at this time may not be able to distinguish particles by measurements. But with better equipment or better insight, we may. For instance, a single particle may not be at exactly the same spot at exactly the same time in exactly the same condition. Leibniz postulates absolute distinguishability, possibly invoking God. We may relativize these ideas to accommodate different views of the world in a well-defined manner.

Of course, Leibniz's view is not the only one possible, and we shall revisit this point. Indistinguishability, as Leibniz and the philosophical tradition conceived it, was absolute. One can safely assume that subsequent philosophers and mathematicians, like Cantor and Dedekind, while acquainted with also Kant and Hegel grew up in this tradition. On this basis, it seems likely that much of the foundational work on set theory was influenced by these ideas.

Cantor defines a set (Menge) as follows: *Unter einer "Mannigfaltigkeit" oder "Menge" verstehe ich jedoch allgemein jedes Viele, welches sich als Eines denken läßt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann, und ich glaube hiermit etwas zu definieren, was verwandt ist mit dem Platonischen e>idos oder >id'ea, wie auch mit dem, was Platon in seinem Dialoge "Philebos oder das höchste Gut" mikt'on nennt* (Cantor 1883, Section 1, footnote; Zermelo 1962, p. 204).[23] Clearly, Cantor accepts that an object can appear multiple times in a set, but as distinct elements. This is not in conflict with Leibniz's law. Objects, like oranges are distinguishable simply by the way they can be treated. Each "instance" of such an object then can be an element of the set. Hence, to turn this into a multiset, one would require some trivial counting which was an issue of no interest at the time.

Also Dedekind, often cited as implicitly proposing something like multisets, ought to be read within this broader philosophical context (Dedekind 1888, p. 172, Schlußbemerkung). In these final remarks, he considers the cardinality of a set Σ and that of a set of equivalence classes on that set. The cardinality of the equivalence class of $x \in \Sigma$ is the *Häufigkeitszahl* (multiplicity) of x. This looks like Monro's definition. Given that this is a speculative afterthought to Dedekind's paper and that everywhere else in that paper the elements of a set are assumed to be distinct, we doubt that this paragraph merits to be treated as an important modern contribution to multiset theory. In essence, for Dedekind too the 25 oranges would form a set.

Undeniably, there are problems with the idea of distinguishability. Consider the two roots of the polynomial $x^2 + 2x + 1$. Do they form a set? If so what distinguishes them? Or do they form a multiset? In normal mathematical language, we speak about the set of roots of a polynomial and their respective multiplicities. By splitting hairs, one can probably find a distinction between the roots, for instance, a certain algorithm finds one before the other one. Knuth (1997) mentions the roots of a polynomial or the prime factors of a positive integer as examples of true multisets. Many of the standard textbooks will refer to these as just as sets, and most likely mention multiplicities, but not take the next step to call them multisets. The existing terminology suffices as no critical set-theoretic operations are applied to such sets.

The point of this section on the history was to establish that the accepted concept of a set consisting of nonidentical elements in no way contradicts the presence of repetitions. Difficulties arise, when distinguishability seems to become unnatural as in the case of the roots of a polynomial.

Historically, repeated elements were just distinct objects. Hence, the concept of a set sufficed. Contexts, however, may change the view. In one view of the world, we may consider certain objects as distinguishable, in another one as indistinguishable – we cannot be God all the time. A relativization to world views seems to require more subtle tools than the simple transition from sets to multisets.

We propose to replace the concept of *multiset* by that of *family*. The transition from families to multisets is simple, and families are mathematically far better behaved than multisets. Moreover, using families, one can easily introduce different world views and, thus, relativize the notion of indistinguishability.

11. Families and the School Trip Again

Undeniably, as the students in Mrs. T's two forms are distinguishable, the two forms are sets, this being obscured by the fact that the *names* of some students are indistinguishable. Thus, to deal with the teacher's dilemma, one has to distinguish, carefully, between *objects* and their *names*. The concept of *name* can be replaced by something like *classification criterion* or *sort* or, simply, *equivalence class*. For now, we assume that the elements of a set are distinguishable in the strict sense of Leibniz. Later, we indicate how this condition can be relativized to restricted world views so that "God" need not be invoked in every single definition of a set. From here on, we use the terms *set*, *element*, *object*, and *name* with the following understanding.

Convention 13 (Objects). *Objects are parts of a "world" and, by the criteria valid in that world, they are distinguishable.*

Convention 14 (Sets, objects, elements). *Only objects of some fixed "world" can be elements of a set.*

Note that it says "set," not "multiset" in the convention.

Thus, with respect to the world under consideration, the elements of a set are distinguishable. A set cannot contain the same element more than once. It may, however, contain several objects which look the same under another view of the world. This idea is captured in part by the concept of *name* and explained in detail further below.

Convention 15 (Names). *Let X and N be sets. A* naming *is a binary relation ν of X into N. Then $\nu(x)$ is the set of* names *of $x \in X$, and $\nu^{-1}(n)$ is the set of $x \in X$ having the name n.*

These conventions leave open what we mean by *world*. As a default, we assume Leibniz's ideal view. This can be replaced, however, by restricted world views. Thus, the students in Mrs. T's two forms are elements of two sets, and so are their first names. In a set, one cannot have repeated objects as elements. However, a set can be viewed in different ways as determined by the respective *world* under consideration.

Using a naming relation ν can have several effects including the following ones:

1. If ν is an injective partial mapping of X into N, then one can identify the names in $\nu(X)$ with the objects in $\nu^{-1}(\nu(X))$.
2. If ν is a partial mapping of X into N, then every element of the domain of ν has a name, and there could be multiple objects with the same name.

The latter case is the more interesting one in the present context. Each[24] object in X has a name attached to it. This is the scenario of Mrs. T's forms. Therefore, to model Mrs. T's situation in which she works with first names, but also relies on student numbers, we explore the idea of using (mathematical) families.

Abstractly, we have a set Σ, a set X, and a mapping $\sigma : \Sigma \to X$. For Mrs. T, the set Σ might be the set of student numbers she has memorized, X could be the set of all possible first names, and $\sigma(x)$ would be the first name of student $x \in \Sigma$. In this way, a form of Mrs. T would be treated as a family. For example, $\Sigma = \{\mathrm{s1}, \ldots, \mathrm{s9}\}$, $\{\text{Peter, Susan, Jonas}\} \subseteq X$, and

$$\sigma(x) = \begin{cases} \text{Peter}, & \text{for } x \in \{\mathrm{s1}, \ldots, \mathrm{s4}\}, \\ \text{Susan}, & \text{for } x \in \{\mathrm{s5}, \ldots, \mathrm{s7}\}, \\ \text{Jonas}, & \text{for } x \in \{\mathrm{s8}, \mathrm{s9}\}. \end{cases}$$

This mapping captures both of her forms and is deceivingly simple.

Pursuing this idea, we provide a formal definition of some concepts concerning families[25] and show that Mrs. T could still be out of luck.

Definition 16 (Family). *A* family *(over X) is a triple $F = (\Sigma, X, \sigma)$ such that Σ and X are sets and $\sigma : \Sigma \to X$ is a mapping.*

Let $\mathfrak{F}_X$ be the class of families over X for a fixed set X.

Given a family F, the set Σ is usually called the index set, and σ is called the index mapping. Every family defines a multiset over X such that the multiplicity of $x \in X$ is equal to the cardinality of the pre-image $\sigma^{-1}(x)$ of x under σ. This is not quite true as that pre-image could be infinite, while multiplicities of elements of multisets are required to be finite.[26]

In our context, the importance of the roles of the components of a family are somehow inverted: the set Σ consists of the "real" objects of the world, and the function σ gives "names" in X to them. Thus, in the school example, Σ might be the set of students in the school district and X might be the set of their first names. This idea is explained further in our discussion of worlds and world views below.

We now consider elementary properties of and operations on families. We indicate that we work with families rather than sets or multisets by the superscript "f."

Definition 17 *Let $F = (\Sigma, X, \sigma)$ be a family. For $x \in X$, we say that x is an element of F, $x \in^{\mathrm{f}} F$, if and only if $\sigma^{-1}(x) \neq \emptyset$.*

Definition 18 *Let X be a set, and let $F_1 = (\Sigma_1, X, \sigma_1)$ and $F_2 = (\Sigma_2, X, \sigma_2)$ be two families over X. F_1 is a* subfamily *of F_2, $F_1 \subseteq^{\mathrm{f}} F_2$, if and only if $\Sigma_1 \subseteq \Sigma_2$ and σ_1 is the restriction of σ_2 to Σ_1.*

Observation 19 Let X be a set, let $x \in X$ and $F_1, F_2 \in \mathfrak{F}_X$.

1. $\subseteq^{\mathrm{f}}$ is a partial order on $\mathfrak{F}_X$.
2. $x \in^{\mathrm{f}} F_1 \subseteq^{\mathrm{f}} F_2$ implies $x \in^{\mathrm{f}} F_2$.

Let F_1 and F_2 be defined as in Definition 18. We now attempt to define the family theoretic counterparts $\cup^{\mathrm{f}}$ and $\cap^{\mathrm{f}}$ of set union and set intersection, respectively. Let $F = (\Sigma, X, \sigma)$ denote the family meant by $F_1 \cup^{\mathrm{f}} F_2$. It seems natural to assume that $\Sigma = \Sigma_1 \cup \Sigma_2$ and to define σ as follows:

$$\sigma(s) = \begin{cases} \sigma_1(s), & \text{if } s \in \Sigma_1 \setminus \Sigma_2, \\ \sigma_2(s), & \text{if } s \in \Sigma_2 \setminus \Sigma_1, \\ ?, & \text{if } s \in \Sigma_1 \cap \Sigma_2, \end{cases}$$

for $s \in \Sigma$. The undefined case can be removed by assuming that the restrictions of σ_1 and σ_2 to the intersection $\Sigma_1 \cap \Sigma_2$ are the same. A similar problem arises for the family intersection

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$\cap^{\mathrm{f}}$ and related operations. But this is unsatisfactory (another head of the hydra!) as union and intersection now are only partial operations. Moreover, we still do not have a convincing answer to the complementation problem.

In the case of the example of teacher Mrs. T, the index mappings for the two forms coincide and her current problem is solved. To solve the problem in general terms, we are still missing a crucial idea.

12. Reference Families

In everyday language, when we speak about a set S, we do not say what is outside S. For example, when we speak about the rodents in Ontario, it is not clear what would be included in the complement: the rodents outside Ontario? the animals other than rodents? the bridges in Ontario? etc. Context, colloquial conventions and other implicit means usually clarify what the complement is. Implicitly, the set S is often considered as a subset of a "universal," better, a "reference" set R for the current discourse.

Mathematical terminology, while attempting to be precise, is often just as ambiguous. Rarely, when talking about some set, is the reference set defined, simply because there is no need for it. Often the reference set is implicit. Certainly, to define complementation of sets, the reference set must be identified.

For families, this suggests the notion of a *reference family* in which all family operations are to be interpreted. In set theory, the introduction of a reference set turns the set of all subsets of it into a Boolean algebra; similarly, in the theory of families, when a reference family is present, the subfamilies form a Boolean algebra. In contrast, in multiset theory, every attempt to find a decent structure of the set of submultisets of a multiset failed.

Convention 20 (Reference family). *A family is a reference family if every family under consideration is a subfamily of it.*

Let $F = (\Sigma, X, \sigma)$. Every subfamily $F_i = (\Sigma_i, X, \sigma_i)$ of F has the property that σ_i is the restriction of σ to Σ_i with $i = 1, 2$. In particular, this means that σ_1 and σ_2 coincide on $\Sigma_1 \cap \Sigma_2$. Hence we define the analogues of the set-theoretic operations for families with respect to a given reference family.

Definition 21 (Family operations). *Let $F = (\Sigma, X, \sigma)$ be a family and, for $i = 1, 2$, let $F_i = (\Sigma_i, X, \sigma_i)$ be subfamilies of F. The family operations of intersection, union, and complement are defined as follows:*

1. *Union: The family $F_1 \cup^{\mathrm{f}} F_2 = F_3 = (\Sigma_3, X, \sigma_3)$ is given by $\Sigma_3 = \Sigma_1 \cup \Sigma_2$ and $\sigma_3(s) = \sigma(s)$ for $s \in \Sigma_3$.*
2. *Intersection: The family $F_1 \cap^{\mathrm{f}} F_2 = F_4 = (\Sigma_4, X, \sigma_4)$ is given by $\Sigma_4 = \Sigma_1 \cap \Sigma_2$ and $\sigma_4(s) = \sigma(s)$ for $s \in \Sigma_4$.*
3. *Complement: The family $F_5 = (\Sigma_5, X, \sigma_5) = F \setminus^{\mathrm{f}} F_1$ is given by $\Sigma_5 = \Sigma \setminus \Sigma_1$ and $\sigma_5(s) = \sigma(s)$ for $s \in \Sigma_5$.*

Formally, it might be more convenient and lead to a more elegant theory if index mappings were partial. We have not pursued this idea.

Proposition 22 (Boolean algebra of families). *Under the operations of $\cup^{\mathrm{f}}$, $\cap^{\mathrm{f}}$, and $\setminus^{\mathrm{f}}$, the set of subfamilies of a family is a Boolean algebra.*

Proof. The proof is an immediate consequence of the definitions. □

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If we were to replace multisets by families, this would be the end of the discussion, albeit quite unsatisfactory. However, multisets are conceptually different from families. This connection still needs to be explained.

13. Worlds

We left the terms of *world* and *world view* undefined so far. When Adam and Eve saw the animals, plants, and many other things new to them[27] in paradise, Eve started to name them, usually by species.[28] A dodo is called a "dodo" because it looks like a dodo. She even gave names to unique objects.[29]

Abstracting from not just this, but many other examples, we suggest to treat worlds and world views in the following simple way.[30]

Convention 23 (World). *A world is a set.*

Convention 24 (World view). *Let* Σ *be a world. A* (world) view *of* Σ *is a family* (Σ, X, σ) *where* X *is a set and* σ *is a mapping of* Σ *into* X.

These conventions may be insufficient in other contexts. For the purpose of clarifying the notion of *multiset*, they seem to provide exactly the right tools.

Of course, according to the definition, a world view is just a family. But often terminology matters to guide intuition.

We could go a step further by choosing representatives of the equivalence classes. This would reduce the highly differentiated world of the paradise to the world Old MacDonald: he had a cow, a pig, a duck, and quite a few other single animals, but not, fortunately, a crocodile. The song does not say anything about a dodo or Niagara Falls either.

This is the kind of situation Mrs. T finds herself in. Her world is that of all her students (distinguishable objects). The view available to her is that of students mapped to their first names.

14. The Category of Families and Its Relation to that of Multisets

Observation 25 (Category of Families). The class of families forms a category.

Proof. Consider two families $F_1 = (\Sigma_1, X_1, \sigma_1)$ and $F_2 = (\Sigma_2, X_2, \sigma_2)$. A morphism from F_1 to F_2 is a pair (φ, χ) of mappings $\varphi : \Sigma_1 \to \Sigma_2$ and $\chi : X_1 \to X_2$ such that $\sigma_2(\varphi(s)) = \chi(\sigma_1(s))$ for all $s \in \Sigma_1$. The rest is obvious. □

To achieve the transition from families to generalized multisets, one could either use the composition of the rather trivial functor from families to Monro's multisets and from there on the mapping to multisets or one could go directly. These two approaches lead to superficially different, but equivalent, results. To spare us the technicalities, we go the direct way.

Observation 26 (Mapping from families to multisets). The following mapping $\Phi^{\mathsf{f}\to\mathsf{ms}}$ establishes the natural connection between families and multisets in the usual sense

Let (Σ, X, σ) be a family.

Then $\Phi^{\mathsf{f}\to\mathsf{ms}}(\Sigma, X, \sigma) = (X, \mu)$ where $\mu(x) = |\sigma^{-1}(x)|$ for all $x \in X$.

The mapping $\Phi^{\mathsf{f}\to\mathsf{ms}}$ is just a mapping, not a functor, as we have no acceptable definition of multiset morphisms.

15. Putting Things Together

Undeniably, the theory of (generalized) multisets provides useful tools for many applications (see Bogatiryova 2011; Red'ko et al. 2015; Singh et al. 2007, 2016, for example). In many situations,

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however, multiplicities are inadequate for the problem at hand, as demonstrated by the example of Mrs. T's school trip. Moreover, there does not seem to exist an intuitively convincing way by which to define the complement of a multiset. For example, take the multiset S consisting of one occurrence of the symbols a and b each. How many occurrences of these symbols are in the complement of S? None? Seven occurrences of a and two of b? Or, using generalized multisets, $\aleph_0$ occurrences of a and $\aleph_1$ occurrences of b?

We continue with Mrs. T's dilemma in order to illustrate how the connection between families and multisets works. Suppose that the set of distinct names of students at her school is X with multiplicities μ. Let $F = (\Sigma, X, \sigma)$ be the corresponding family. Each form under Mrs. T's care is modeled by a subfamily of F. The complement of each such subfamily with respect to F determines the complement of the multiset under consideration in the whole school's student population.

We propose that families should be used instead of multisets: (1) the problems with the analogues of the set-theoretic operations just go away; (2) one can control the arbitrariness of the complementation operation in great detail; (3) the subfamilies of a family form a Boolean algebra; (4) the transition from families to multisets is afforded by a simple mapping which forgets some of the family structure.

One ought to admit that in some applications using families instead of multisets may make the presentation unnecessarily complex. Choosing the most adequate tool for a given context should never be contentious. Our proposal aims at clarifying what we really mean by "a set with repeated elements."

Why not use Monro's multisets instead of families? After all, the two approaches seem to be equivalent. Our main issue with Monro's concept is that the elements of his multisets, the equivalence classes, are anonymous, while in families they get names. The difference in the concepts is evident from the mappings to the class of multisets.

After an examination of Blizard's (1986) arguments and that of others, we maintain that the definition of set should remain unchanged. There is no set with repeated elements. We introduced the notion of world view to deal with the fact, that distinct objects may seem to be indistinguishable. This should bridge the gap between an ideal world in the sense of Leibniz and that observable by our limited capabilities. Again, families are the key concepts in this argument.

While we agree with Singh et al. (2016) completely that multisets are an important subject for study in science, we suggest that the study of families might be more foundational: (1) it requires fewer basic concepts in mathematics, only sets and mappings; (2) it does not need to deal with the arithmetic of cardinal numbers; (3) the transition to multisets is trivial.

There are some examples where multisets seem to be the most adequate tools. Knuth (1997) mentions the roots of polynomials as an example. They seem to be indistinguishable objects. If x_0 is a multiple root, one might argue that by some algorithm the occurrences are discovered in some order. Does this make them distinguishable? We doubt it. But the mere fact that we could put them into a table separately would make them distinguishable in Leibniz's sense. Clearly, in this case, the concept of world views helps.

In summary, we claim that there is no need for a notion of *multisets* as a new basic one. Those of sets and mappings suffice, once combined into that of a *family*. Of course, this type of argument is nonmathematical in nature. Our paper should be viewed as a contribution to the discussion of which formal setting is most adequate mathematically and philosophically for the intuitive concept of multiset.

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Notes

1 Intuitively, objects are parts of a world. Elements are parts of a set. How these concepts relate to each other is a matter of an ongoing philosophical discussion into which we do not want to enter any more than absolutely required.
2 A theory of multisets with negative multiplicities is explored in Blizard (1990).
3 When we saw the term of multiset for the first time some 45 years ago, we considered it to be a negligible patch for a singular situation. We certainly did not imagine that multisets would become the focus of a mathematical theory.
4 Some papers fail to take into account the multiplicity of 0. Consider sets $X = \{a\}$ and $Y = \{a, b\}$ and multisets (X, μ) and (Y, ν) with $\mu(a) = \nu(a) = 2$ and $\nu(b) = 0$. Intuitively, these multisets are the same, formally they are not. The literature is not clear on how to deal with this situation: to work with equivalent classes of multisets or to work with their representatives; or to handle the case of multiplicity 0 as special. We chose the latter route.
5 Also in the axiomatic theory of Blizard (1989, Axiom II, extensionality), the elements with multiplicity 0 are not ignored.
6 Also called functions or mappings.
7 To avoid a potential misunderstanding, $\nu(f(x))$ is the multiplicity of the element $f(x) \in Y$ in M_ν, not the number of elements of Y which are images of copies of x.
8 The equivalence classes are called *sorts* by Monro.
9 Monro's equivalence classes are anonymous; in our school example, using Monros's definition, even the first names of students would be missing. When families are used instead, each equivalence gets a "name."
10 We do not call them *classes* to avoid a conflict with the mathematical term of *class*.
11 It has been argued that Mrs. T would know how many tickets she needed when she distributes them to the students and that using multiset operations in this case is inadequate. We agree. But the example assumes that Mrs. T has to order, not distribute, the tickets and that she can rely only on the multiset information, hence, must use the multiset operations. The point is that this kind of dilemma inevitably arises when decisions have to rely solely on multiplicity-based multisets and when there is no formal indicator concerning the applicability of multiset operations. With our proposed definition of multisets, the dilemma simply vanishes.
12 The fruit we just distribute. Through the distribution process, each single fruit gains its individuality. This permits us to say that the sets of oranges of the children are mutually disjoint.
13 A reader of an earlier version of this paper asked how the balls could be indistinguishable. This is precisely the point. By taking them out, they seem to become distinguishable. This is a mathematically unexplained miracle or a flaw in the theory of multisets. We claim it is the latter.
14 For union, the operation $\uplus$ cannot be used because the balls are indistinguishable. One has to use $\cup^{\mathrm{ms}}$.
15 The contenders included, for example, *bag* and *heap*, which actually appear in the literature as meaning *multiset* (Hailperin 1986; Jena et al. 2001; Yager 1986), and many others, which never made it.
16 Eilenbergs technique generalizes that of *generating functions*, also known as *Z-transforms* in much of the applied literature. His presentation, however, clarifies many of the underlying principles and removes much of the unnecessary assumptions of the theory of functions.
17 Here and in the sequel Jürgensen (2017), we use terms loosely with just enough rigour to explain and compare the concepts.
18 For the argument made in this paper, historical authority is irrelevant except to illustrate that the discussions about the basic concepts are far less recent than one might assume. See for instance Angelelli (2001) and a comment on this in Angelelli (1965).
19 The principle that indistinguishable objects are identical. A concise analysis of how this principle fits into Leibniz's philosophy and the thinking of the time is presented by Kauppi (1966). See also Kümmel (2016).
20 Literally *first truths*, better *fundamental truths*.
21 The text is transcribed from a manuscript. $\langle A \rangle$ means that A has been added by Leibniz. $[B]\langle A \rangle$ means that Leibniz replaced B by A. Translation: "It follows from here that it is impossible to give, in nature, two separate things which differ only in their numbers: for, therefore, it is required that a reason can be given why they are diverse, which has to be sought from some difference in them."
22 Leibniz (1961, p. 520). Translation: "The complete and perfect notion of a single substance includes all its past, present, and future properties. For, even as a property may only hold in the future it obtains now and is, therefore, contained in the object's notion."
23 Here, Cantor uses the terms *Mannigfaltigkeit* (manifold) and *Menge* (set) as synonyms. He refers to Plato's theory of ideas and specifically to the dialog *Philebos* (Plato 1925). For a tentative explanation of this connection, see Menzel's (1984) paper. For our purposes, it suffices to register this connection without exploring its details. A proper translation of the German original is very difficult; we paraphrase the essence of the statement (and even that is not quite correct): "By *set* I mean any multitude of objects which can be thought of as a single object, that is, the subsumption of certain elements which, by some common law, can be bundled together. I believe I define something akin to Plato's e*̃*idos or >id'ea, and also what Plato calls mikt'on in his dialogue *Philebos*.

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24 Totality of ν is not important; it just simplifies the statements.
25 Surprisingly, there seems not to be any mathematical study of families *per se*.
26 Removing this restriction has been discussed (see, e.g., Blizard 1986) and would make even the present theory of multisets more elegant.
27 How could they not be new to them?
28 Mark Twain, Eve's Diary.
29 From Mark Twain, Adam's Diary: it is the finest thing on the estate, I think. The new creature calls it Niagara Falls, why, I am sure I do not know. Says it looks like Niagara Falls.
30 In other contexts, the detailed mathematical structure of a *world* would have to be considered.

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