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ARTICLE

A cubical Squier's theorem

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Abstract
The homotopical Squier's theorem relates rewriting properties of a presentation of a monoid with homotopical invariants of this monoid. This theorem has since been extended by Guiraud and Malbos, yielding a so-called polygraphic resolution of a monoid starting from a presentation with suitable rewriting properties. In this article, we argue that cubical categories are a more natural setting in which to express and possibly extend Guiraud and Malbos construction. As a proof-of-concept, we give a new proof of Squier's homotopical theorem using cubical categories.

Keywords: Cubical categories; coherence; rewriting

1. Introduction

The aim of this paper is to introduce methods of 2-cubical categories into rewriting theory related to Squier's theorems. A substantial account of rewriting theory using globular methods has been given in a series of papers by Guiraud and Malbos. In particular, a modern exposition of Squier's theorem in terms of globular categories can be found in Guiraud and Malbos (2018).

Convergent rewriting systems originated as tools in the study of the word problem. In particular, a presentation of a monoid by a finite convergent rewriting system gives an algorithm to decide the word problem for this monoid. In Squier (1987) and Squier et al. (1994), the authors proved that there exists a finitely presented monoid whose word problem is decidable but which did not admit a finite convergent presentation. Squier's homotopical theorem has since been extended to higher dimensions in Guiraud and Malbos (2012b), but this result is still ill-understood. In particular, a similar result is expected for structures more general than monoids, but this generalization seems so far out of reach.

Independently, cubical categories in dimension 2 were shown to be equivalent to globular 2-categories in Brown and Mosa (1999), using the notions of connections and thin structures. These results were later extended to all dimensions by Al-Agl, Brown, and Steiner. The cubical geometry seems better suited for rewriting than the globular one, and we expect this advantage to increase in higher dimensions. In this paper, however, we stick to low dimensions and present a new proof of Squier's theorem using cubical methods. The proof loosely follows that of Guiraud and Malbos (2018) but is self contained. As a result, this account can be read independently of the previous work.

Let us first recall Squier's homotopical theorem in the traditional globular setting. Let (G, R) be a convergent (i.e., confluent and terminating: see Definition 3.9) presentation of a monoid M. Squier's construction yields a set of syzygies S corresponding to relations between the relations.

For example, let us consider the following presentation of the braid monoid B_3^+ is given by:

$$\langle a, s, t \,|\, ta = as, sa = a, sas = aa, saa = aat \rangle$$

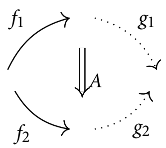
In order to apply rewriting techniques to this presentation, one first needs to choose a name and an orientation for each of the relations, giving for example:

$$\Sigma := \langle a, s, t \,|\, \alpha : ta \to as, \beta : sa \to a, \gamma : sas \to aa, \delta : saa \to aat \rangle$$

Such a presentation Σ is called a monoidal polygraph (see Definition 3.2 for a formal definition). The generating relations induce rewriting steps: relations of the form $ufv : uxv \to uyv$, where $u, v \in \Sigma^*$ and $f : x \to y$ is a generating relation. Those rewriting steps form a graph, whose set of vertices is the set of words on the generators. Taking the free category generated by this graph yields (up to some equations) a strict monoidal category, that we denote Σ^*. Finally, $\Sigma^\top$ denotes the free strict monoidal groupoid generated by Σ.

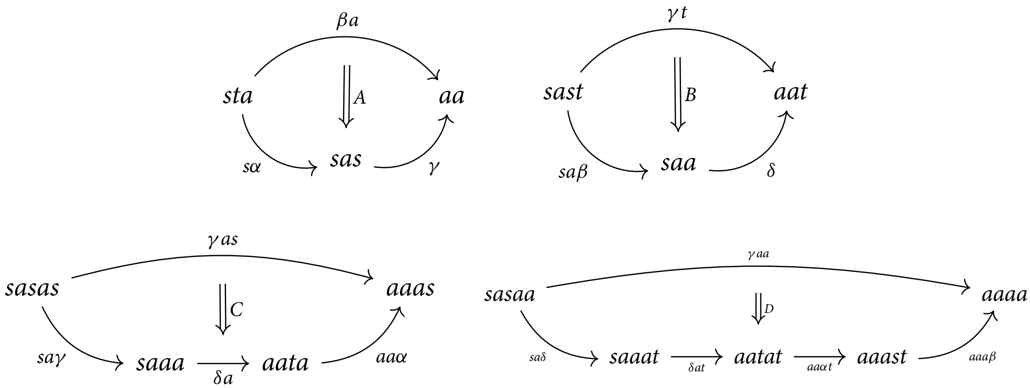
Squier's theorem yields a *coherent* presentation of B_3^+. This means adding generating 2-cells to Σ, such that the monoidal 2-groupoid it generates (still denoted $\Sigma^\top$) satisfies the following property: for any couple of parallel 1-cells f and g in $\Sigma^\top$, there exists a 2-cell $A : f \Rightarrow g$ in $\Sigma^\top$. Squier's homotopical theorem gives a sufficient condition on Σ for this property to hold, based on the notion of critical pair. A pair of distinct rewriting steps $(u_1 f_1 v_1, u_2 f_2 v_2)$ of same source u forms a critical pair if any letter in u is rewritten by f_1 or f_2, and at least one letter is rewritten by both. In the presentation of B_3^+ previously given, there are exactly four critical pairs: $(\beta a, s\alpha)$, $(\gamma t, sa\beta)$, $(\gamma as, sa\gamma)$, and $(\gamma aa, sa\delta)$.

Theorem 1.1. (**Squier et al. 1994**). *Let Σ be a convergent monoidal 2-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exist two 1-cells g_1 and g_2 in Σ^* and a 2-cell A in $\Sigma^\top$ of the following shape:*



Then Σ is coherent.

Let us delay the definition of convergence for now (see Definition 3.9). In our case, Σ is convergent, and Squier's theorem therefore asserts that the following extension of Σ into a 2-polygraph is coherent:



In other words, Squier's theorem reduced the commutativity (up to 2-cells) of all the diagrams that one can form in $\Sigma_1^\top$, to the commutativity of the four aforementioned diagrams.

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Squier's theorem has recently been expanded in higher dimensions (see Guiraud and Malbos 2012b), where coherence is replaced by acyclicity, and critical pairs are replaced by critical n-tuples. However, a lot of calculations from Guiraud and Malbos (2012b) are very complicated. As a result, although Squier's theorem has been generalised to various structures in addition to monoids (see, e.g., Guiraud and Malbos 2012a), a generalisation of the full resolution constructed in Guiraud and Malbos (2012b) seems out of reach.

Cubical categories were introduced in Al-Agl et al. (2002). Although they are equivalent to globular ω-categories, we identify two key advantages of cubical methods that should streamline a lot of the constructions of Guiraud and Malbos (2012b):

- First, the authors rely on the construction of a contracting homotopy, which is nothing else than an ω-natural transformation, that is a natural transformation between functors between ω-categories. Such an object is more easily described in cubical terms (see, e.g., Lucas 2018).
- Any analogue of Squier's theorem should describe the shape of the cells filling the critical n-branchings. The authors solve this problem by mutual induction with the aforementioned contracting homotopy. However, the confluence diagram associated to an n-branching can be readily expressed in cubical terms. As a consequence, we expect that in the cubical setting we should be able to separate the construction of the higher dimensional cells and that of the contracting homotopy, leading to a simpler, more modular proof.

In this paper, however, we limit ourselves to Squier's theorem and leave a cubical analogous of Guiraud and Malbos (2012b) for a future paper. To this end, we define the notion of coherence in the cubical setting and prove a cubical Squier's theorem with this definition.

Theorem 4.2. *Let Σ be a convergent cubical 2-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exists (up to exchange of f_1 and f_2) a 2-cell in $\Sigma_2^\top$ of the following shape*

$$\begin{array}{ccc} \cdot & \xrightarrow{f_1} & \cdot \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$$

Then Σ is coherent.

In Section 2, we introduce cubical categories in low dimensions. In Section 3, we recall some standard notions from word rewriting. Finally, in Section 4, we prove our version of Squier's theorem. Finally, in a last section, we give examples where the cubical geometry would be helpful when extending Squier's theorem in higher dimensions.

2. Cubical 2-categories

The equivalence between globular and cubical ω-groupoids was announced in Brown and Higgins (1977) and proved in Brown and Higgins (1981). The case of ω-categories was covered in (**B02**). Finally, the description of cubical (ω, p)-categories and their equivalence with their globular counterparts was done in Lucas (2018). Here, we focus on cubical 2-categories, (2, 1)-categories, and 2-groupoids.

Definition 2.1. *A cubical* 2*-set consists of:*

- *Sets $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$, whose elements are, respectively, called the* 0-, 1-, *and* 2*-cells.*
- *Functions $\partial^+, \partial^- : \mathcal{C}_1 \to \mathcal{C}_0$.*
- *Functions $\partial_1^+, \partial_1^-, \partial_2^+, \partial_2^- : \mathcal{C}_2 \to \mathcal{C}_1$, called the* faces *of* 2*-cells.*

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Satisfying the following relations for any $\alpha, \beta \in \{+, -\}$:

$$\partial^{\alpha}\partial_2^{\beta} = \partial^{\beta}\partial_1^{\alpha}.$$

Notation 2.2. *We represent a 1-cell f in the following way : $\partial^- f \xrightarrow{f} \partial^+ f$, and a 2-cell A as:*

$$\begin{array}{ccc} & \xrightarrow{\partial_1^- A} & \\ \partial_2^- A \big\downarrow & A & \big\downarrow \partial_2^+ A \\ & \xrightarrow[\partial_1^+ A]{} & \end{array}$$

Cubical 2-categories. Cubical ω-categories were defined in Al-Agl et al. (2002). We here give an elementary description of the 2-dimensional case. A cubical 2-category consists of a cubical 2-set $\mathcal{C}$ equipped with the following structure:

- An operation $\star$ sending any two 1-cells $x \xrightarrow{f} y \xrightarrow{g} z$ to a 1-cell $x \xrightarrow{f \star g} z$.
- An operation ϵ sending any 0-cell x to a 1-cell $x \xrightarrow{\epsilon x} x$, which we usually represent by $x =\!=\!= x$.
- An operation $\star_1$ (respectively, $\star_2$) associating, to any 2-cells $\begin{array}{ccc} & \longrightarrow & \\ \big\downarrow & A & \big\downarrow \\ & \longrightarrow & \end{array}$ and $\begin{array}{ccc} & \longrightarrow & \\ \big\downarrow & B & \big\downarrow \\ & \longrightarrow & \end{array}$ satisfying $\partial_1^+ A = \partial_1^- B$ (respectively, $\partial_2^+ A = \partial_2^- B$), 2-cells

$$\begin{array}{ccc} & \longrightarrow & \\ \big\downarrow & & \big\downarrow \\ & A \star_1 B & \\ \big\downarrow & & \big\downarrow \\ & \longrightarrow & \end{array} \qquad \begin{array}{ccccc} & \longrightarrow & & \longrightarrow & \\ \big\downarrow & & A \star_2 B & & \big\downarrow \\ & \longrightarrow & & \longrightarrow & \end{array}$$

- Operations $\epsilon_1, \epsilon_2 : \mathcal{C}_1 \to \mathcal{C}_2$ sending any 1-cell $\xrightarrow{f}$ to 2-cells $\begin{array}{ccc} & \xrightarrow{f} & \\ \Big\| & \epsilon_1 f & \Big\| \\ & \xrightarrow[f]{} & \end{array}$ and $\begin{array}{ccc} & = & \\ f \big\downarrow & \epsilon_2 f & \big\downarrow f \\ & = & \end{array}$.
- Operations $\Gamma^-, \Gamma^+ : \mathcal{C}_1 \to \mathcal{C}_2$ sending any 1-cell $\xrightarrow{f}$ to 2-cells $\begin{array}{ccc} & \xrightarrow{f} & \\ f \big\downarrow & \Gamma^- f & \Big\| \\ & = & \end{array}$ and $\begin{array}{ccc} & = & \\ \Big\| & \Gamma^+ f & \big\downarrow f \\ & \xrightarrow[f]{} & \end{array}$.

Those operations have to satisfy a number of axioms:

- Both $(\mathcal{C}_0, \mathcal{C}_1, \partial^-, \partial^+, \star, \epsilon)$ and $(\mathcal{C}_1, \mathcal{C}_2, \partial_i^-, \partial_i^+, \star_i, \epsilon_i)$ (for $i = 1, 2$) are categories.
- For any 2-cells A, B, C, D such that $\partial_1^+ A = \partial_1^- B$, $\partial_2^+ A = \partial_2^- C$, $\partial_1^+ C = \partial_1^- D$ and $\partial_2^+ B = \partial_2^- D$, the equality $(A \star_1 B) \star_2 (C \star_1 D) = (A \star_2 C) \star_1 (B \star_2 D)$ holds. In other words, the following composite is uniquely defined:

$$\begin{array}{ccccc} & \longrightarrow & & \longrightarrow & \\ \downarrow & A & \downarrow & C & \downarrow \\ & \longrightarrow & & \longrightarrow & \\ \downarrow & B & \downarrow & D & \downarrow \\ & \longrightarrow & & \longrightarrow & \end{array} \qquad (1)$$

- Any two thin 2-cells A and B sharing the same faces are equal, where a 2-cell is said to be *thin* if it is a composite of cells of the form $\epsilon_i f$ and $\Gamma^\alpha f$ for $i = 1, 2$, $\alpha = \pm$ and $f \in \mathcal{C}_1$.

Remark 2.3. This last property in particular implies that for all $\alpha = \pm$ and $i = 1, 2$, $\Gamma^\alpha \epsilon = \epsilon_i \epsilon$. It also gives expressions for cells $\Gamma^\alpha(f \star g)$ in terms of $\Gamma^\alpha f$ and $\Gamma^\alpha g$. Finally since thin cells are completely characterized by their faces, we will omit them when the context is clear in the rest of this paper.

This definition of cubical categories in terms of thin cells is nonstandard, but Higgins (2005) shows that it is equivalent to the one used in Al-Agl et al. (2002).

Cubical (2,1)-categories. A cubical $(2, 1)$-category is given by a cubical 2-category $\mathcal{C}$ equipped with an operation $T : \mathcal{C}_2 \to \mathcal{C}_2$ sending any 2-cell $\begin{array}{ccc} & \xrightarrow{\partial_1^- A} & \\ \partial_2^- A \downarrow & A & \downarrow \partial_2^+ A \\ & \xrightarrow[\partial_1^+ A]{} & \end{array}$ to a 2-cell of shape $\begin{array}{ccc} & \xrightarrow{\partial_2^- A} & \\ \partial_1^- A \downarrow & TA & \downarrow \partial_1^+ A \\ & \xrightarrow[\partial_2^+ A]{} & \end{array}$

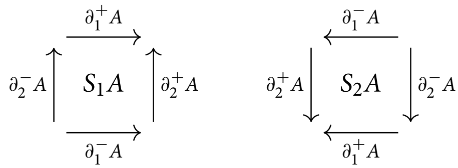
such that $T^2 = \mathrm{id}_{\mathcal{C}_2}$ and:

$$\begin{array}{ccccc} & = & & \longrightarrow & \\ \Vert & & \downarrow & TA & \downarrow \\ & \longrightarrow & & \longrightarrow & \\ \downarrow & A & \downarrow & & \Vert \\ & \longrightarrow & & = & \end{array} = \begin{array}{ccccc} & \longrightarrow & & = & \\ \downarrow & & \Vert & & \Vert \\ & = & & = & \\ \Vert & & \Vert & & \downarrow \\ & = & & \longrightarrow & \end{array} \qquad (2)$$

$$\begin{array}{ccccc} & = & & \longrightarrow & \\ \Vert & & \downarrow & A & \downarrow \\ & \longrightarrow & & \longrightarrow & \\ \downarrow & TA & \downarrow & & \Vert \\ & \longrightarrow & & = & \end{array} = \begin{array}{ccccc} & \longrightarrow & & = & \\ \downarrow & & \Vert & & \Vert \\ & = & & = & \\ \Vert & & \Vert & & \downarrow \\ & = & & \longrightarrow & \end{array} \qquad (3)$$

Remark 2.4. The operation $A \mapsto TA$ corresponds to the operation $A \mapsto A^{-1}$ in a globular setting. The equation $T^2 = \mathrm{id}_{\mathcal{C}_2}$ corresponds to the equality $(A^{-1})^{-1} = A$, and the axioms (2) and (3), respectively, correspond to the relations $A \star_1 A^{-1} = 1$ and $A^{-1} \star_1 A$. See Lucas (2018) for more details.

Cubical 2-groupoid. A cubical 2-groupoid is a cubical 2-category such that $(\mathcal{C}_0, \mathcal{C}_1)$ is a groupoid (we denote by $\xleftarrow{f}$ the inverse of a cell $\xrightarrow{f}$) equipped with operations $S_1, S_2 : \mathcal{C}_2 \to \mathcal{C}_2$, sending any 2-cell $\begin{array}{ccc} & \xrightarrow{\partial_1^- A} & \\ \partial_2^- A \downarrow & A & \downarrow \partial_2^+ A \\ & \xrightarrow[\partial_1^+ A]{} & \end{array}$ to 2-cells of shape:



So that $(\mathcal{C}_1, \mathcal{C}_2, \partial_i^-, \partial_i^+, \star_i, \epsilon_i, S_i)$ is a groupoid for $i = 1, 2$.

Though the proof is not as straightforward as in the globular case, we still have the following expected result (see Lucas 2018):

Proposition 2.5. *A cubical* 2*-groupoid is a cubical* (2, 1)*-category.*

In particular, the operation T of cubical (2, 1)-categories can be defined in any cubical 2-groupoid using the operations S_1 and S_2.

3. Word Rewriting

In this section, we redefine some of the standard concepts of higher dimensional rewriting in our cubical setting (see Guiraud and Malbos 2018 for a more detailed exposition).

Definition 3.1. *A monoidal cubical (n, k)-category is a monoid object in the category of cubical (n, k)-categories (with respect to the cartesian product). In other words, it is the data of a cubical (n, k)-category $\mathcal{C}$ together with an associative and unitary functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.*

Let $\mathcal{C}$ be a monoidal cubical (2, 1)*-category (respectively,* 2*-category) and $f : u \to u'$ and $g : v \to v'$ be cells in $\mathcal{C}_1$. Then, the monoid structure gives a* 1*-cell $fg : uv \to u'v'$ in $\mathcal{C}_1$. We write simply fv (respectively, vf) for the cell $f(\epsilon v) : uv \to u'v$ (respectively, $(\epsilon v)f : vu \to vu'$). There is also a product of* 2*-cells in a similar fashion. Finally, these products are compatible with the identity maps which give, for example, the equation $\epsilon_i(fg) = (\epsilon_i f)(\epsilon_i g)$.*

Polygraphs (Burroni 1993) are presentations for higher dimensional globular categories and were introduced by Street under the name of computads (see Street 1976, 1987). We adapt them here to present monoidal cubical (n, k)-categories using ideas from Batanin (1998) and Garner (2010).

Definition 3.2. *For any set E, we denote by E^* the free monoid on E. A monoidal* 1*-polygraph Σ is given by two sets Σ_0, Σ_1, together with maps $\partial^\alpha : \Sigma_1 \to \Sigma_0^*$ (for $\alpha = \pm$). We denote by Σ^* (respectively, $\Sigma^\top$) the free monoidal category (respectively, groupoid) generated by Σ.*

Explicitly, the 0*-cells are given by Σ_0^* (the free monoid on Σ_0), while the* 1*-cells of Σ^* are given by composable sequences of arrows of the form ufv, with $u, v \in \Sigma_0^*$ and $f \in \Sigma_1$, up to the following relation, where $x, x', y, y', u_1, u_2, u_3$ are elements of Σ_0^* and f, f' of Σ_1.*

$$u_1 f u_2 x' y_3 \star u_1 y u_2 f' u_3 = u_1 x u_2 f' y_3 \star u_1 f u_2 y' u_3.$$

Definition 3.3. *A monoidal cubical* 2*-polygraph (respectively,* (2, 0)*-polygraph) is given by three sets Σ_0, Σ_1 and Σ_2, together with maps $\partial^\alpha : \Sigma_1 \to \Sigma_0^*$ and $\partial_i^\alpha : \Sigma_2 \to \Sigma_1^*$ (respectively, $\partial_i^\alpha : \Sigma_2 \to \Sigma_1^\top$) for $i = 1, 2$ and $\alpha = \pm$.*

We denote by Σ^* (respectively, $\Sigma^\top$) the free monoidal cubical (2, 1)*-category (respectively,* 2*-groupoid) generated by Σ.*

Example 3.4. If Σ is a monoidal cubical 2-polygraph, the cells of Σ and Σ^* together with the face operations can be visualized as follows (a similar diagram could be drawn for $\Sigma^\top$):

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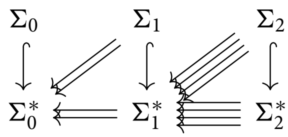

In what follows, we will use the following monoidal cubical 2-polygraph as our running example:

$$\Sigma_0 = \{s, t, a\}$$
$$\Sigma_1 = \{\alpha : ta \to as, \beta : st \to a, \gamma : sas \to aa, \delta : saa \to aat\}$$

And finally Σ_2 consists of the following cells:

$$\begin{array}{ccc} sta & \xrightarrow{\beta a} & aa \\ {\scriptstyle s\alpha}\downarrow & A & \| \\ sas & \xrightarrow[\gamma]{} & aa \end{array} \qquad \begin{array}{ccc} sast & \xrightarrow{\gamma t} & aat \\ {\scriptstyle sa\beta}\downarrow & B & \| \\ saa & \xrightarrow[\delta]{} & aat \end{array}$$

$$\begin{array}{ccccc} sasas & & \xrightarrow{\gamma as} & & aaas \\ {\scriptstyle sa\gamma}\downarrow & & C & & \| \\ saaa & \xrightarrow[\delta a]{} & aata & \xrightarrow[aa\alpha]{} & aaas \end{array} \qquad \begin{array}{ccccccc} sasaa & & & \xrightarrow{\gamma aa} & & & aaaa \\ {\scriptstyle sa\delta}\downarrow & & & D & & & \| \\ saaat & \xrightarrow[\delta at]{} & aatat & \xrightarrow[aa\alpha t]{} & aaast & \xrightarrow[aaa\beta]{} & aaaa. \end{array}$$

The main point about these generating 2-cells is that the pairs formed of the top and left faces correspond precisely to the critical pairs of Σ_1.

Remark 3.5. The presentation in this article is slightly different from that of Guiraud and Malbos (2018). The monoidal cubical (n, p)-categories defined in this article correspond precisely to one-object $(n + 1, p + 1)$-categories in Guiraud and Malbos (2018). Up to the equivalence between cubical and globular categories, this is analogous to the fact that monoids are one-object categories.

In particular, here generators are 0-cells and generating relations are 1-cells, while in Guiraud and Malbos (2018), generators are 1-cells and generating relations are 2-cells.

Definition 3.6. *Let Σ be a monoidal* 1*-polygraph. A* rewriting step *in Σ_1^* is a* 1*-cell of the form ufv, where f is in Σ_1, and u and v are elements of Σ_0^*.*

Definition 3.7. *Let Σ be a monoidal* 1*-polygraph. A* branching *is a pair of* 1*-cells $f, g \in \Sigma_1^*$ with the same source. It is said to be* local *if f and g are rewriting steps.*

Up to permutation of f and g, there are three distinct types of local branchings:

- *If $f = g$, (f, g) is said to be an* aspherical branching.
- *If there exists $f', g' \in \Sigma_1^*$ and $u, v \in \Sigma_0^*$ such that $f = f'v$ and $g = ug'$ with $\partial^- f' = u$ and $\partial^- g' = v$, (f, g) is said to be a* Peiffer branching.
- *Otherwise, (f, g) is said to be an* overlapping branching.

Finally a critical branching *is a minimal overlapping branching, where overlapping branchings are ordered by the (well-founded) relation: $(f, g) \leqslant (ufv, ugv)$ for $u, v \in \Sigma_0^*$*

Example 3.8. Using our example, $sa\gamma : sasas \to saaa$ and $\delta a : saa \to saaat$ are rewriting steps, but not $sa\gamma \star \delta a$ (since it is a composite of rewriting steps) or $\beta a\beta$ (since it is a product of rewriting steps).

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Let us now look at local branchings. Peiffer branchings are local branchings that rewrite disjoint parts of a word. For example, $(\beta st, st\beta) : stst \to (ast, sta)$ is a Peiffer branching. On the other hand, $(a\beta a, as\alpha)$, of source *asta* is an overlapping branching, but not a critical one since it is not minimal: $(a\beta a, as\alpha) > (\beta a, s\alpha)$. In this example, one can check that there are four critical branchings: $(\beta a, s\alpha)$, $(\gamma t, sa\beta)$, $(\gamma as, sa\gamma)$, and $(\gamma aa, sa\gamma)$.

Definition 3.9. *Let Σ be a monoidal 1-polygraph. A branching (f, g) is* confluent *if there exist 1-cells f' and g' in Σ_1^* with the same target and such that $\partial^+ f = \partial^- f'$ and $\partial^+ g = \partial^- g'$:*

$$\begin{array}{ccc} & \xrightarrow{\;f\;} & \\ g\big\downarrow & & \big\downarrow f' \\ & \xrightarrow[\;g'\;]{} & \end{array}$$

We say that Σ is locally confluent *if any local branching is confluent, and Σ is* confluent *if any branching is confluent.*

It is terminating *if there is no infinite sequence of rewriting steps $f_1, \ldots, f_n, \ldots$ satisfying that $\partial^+ f_i = \partial^- f_{i+1}$ for all i.*

$$\xrightarrow{f_1}\;\xrightarrow{f_2}\;\ldots\;\xrightarrow{f_i}\;\xrightarrow{f_{i+1}}\;\ldots$$

It is convergent *if it is both terminating and confluent.*

Example 3.10. The faces of the cells of Σ_2 show that all the critical branching of Σ are confluent. In addition, aspherical branchings are always confluent (using identities for f' and g'), and so are the Peiffer branchings. Indeed, given a Peiffer branching $(f, g) = (f'v, ug')$ with $f' : u \to u'$ and $g' : v \to v'$, one can form the following diagram:

$$\begin{array}{ccc} & \xrightarrow{\;f'v\;} & \\ ug'\big\downarrow & & \big\downarrow u'g' \\ & \xrightarrow[\;f'v'\;]{} & \end{array}$$

In the end, we have just shown that Σ is locally confluent.

Moreover, Σ is terminating. To show this, we consider the order $t > a$ and $s > a$ on Σ_0. We extend inductively this order to Σ_0^* as follows:

- If u is shorter than v, then $u < v$.
- If $u = xu'$ and $v = xv'$, with $x \in \Sigma_0$, and $u' < v'$, then $u < v$.
- If $u = xu'$ and $v = yv'$, with $x, y \in \Sigma_0$ and $x < y$, then $u < v$.

This is a well-founded ordering of Σ_0^* compatible with multiplication, and we can check that for any cell f of Σ_1, $s(f) > t(f)$. As a result, Σ is terminating.

Finally, by Newman's Lemma, a terminating locally confluent rewriting system is confluent, and so Σ is actually convergent.

4. Squier's Theorem

Before stating Squier's theorem, we need to define the cubical analogue to the notions of globe and of coherence.

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Definition 4.1. *Let $\mathcal{C}$ be a cubical 2-category. A* shell *over $\mathcal{C}_1$ is a family of cells f_i^α in $\mathcal{C}_1$, ($i = 1, 2$ and $\alpha = +, -$) satisfying $\partial^\alpha f_2^\beta = \partial^\beta f_1^\alpha$ for every α and β.*

$$\begin{array}{ccc} & \xrightarrow{f_1^-} & \\ {\scriptstyle f_2^-}\downarrow & & \downarrow{\scriptstyle f_2^+} \\ & \xrightarrow[f_1^+]{} & \end{array}$$

A filler *in $\mathcal{C}_2$ of a shell $S = (f_i^\alpha)$ over $\mathcal{C}_1$ is a 2-cell $A \in \mathcal{C}_2$ satisfying $\partial_i^\alpha A = f_i^\alpha$ for every i and α.*

If Σ is a monoidal cubical $(2, 0)$-polygraph, we say that Σ is coherent if any shell over $\Sigma_1^\top$ admits a filler in $\Sigma_2^\top$.

The main result of this paper is the following:

Theorem 4.2. Cubical Squier's theorem. *Let Σ be a convergent monoidal cubical 2-polygraph. Suppose that for every critical pair (f_1, f_2) of Σ, there exists a 2-cell in Σ_2^* whose shell is of the form:*

$$\begin{array}{ccc} & \xrightarrow{f_1} & \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ & \longrightarrow & \end{array}$$

Then Σ is coherent.

The proof of this result occupies the rest of this article and loosely follows the proof of the globular case from Guiraud and Malbos (2018). Before that though, we show that this result applies to our example.

Example 4.3. We have already proven that Σ is convergent. We have also made the list of all possible critical branchings, and we can check that each of them corresponds to a cell in Σ_2. Thus by Theorem 4.2, every shell over $\Sigma_1^\top$ admits a filler in $\Sigma_2^\top$.

Lemma 4.4. *For every local branching (f_1, f_2), there exists a cell A in $\Sigma_2^\top$ such that $\partial_1^- A = f_1$ and $\partial_2^- A = f_2$. So A is of the following shape:*

$$\begin{array}{ccc} & \xrightarrow{f_1} & \\ {\scriptstyle f_2}\downarrow & A & \downarrow \\ & \longrightarrow & \end{array}$$

Proof. We distinguish cases depending on the form of the branching (f_1, f_2). Note first that if A is a suitable cell for the branching (f_1, f_2), then TA satisfies the conditions for the branching (f_2, f_1), and uAv for the branching (uf_1v, uf_2v). Since by hypothesis on Σ_2 the property holds for critical branchings, it remains to show that the property holds for aspherical and Peiffer branchings.

If $(f_1, f_2) = (f, f)$ is an aspherical branching, then the 2-cell $\begin{array}{ccc} & \xrightarrow{f} & \\ {\scriptstyle f}\downarrow & \Gamma^- f & \Vert \\ & = & \end{array}$ satisfies the condition.

If $(f_1, f_2) = (fv, ug)$ is a Peiffer branching, then the 2-cell $(\epsilon_1 f)(\epsilon_2 g)$ satisfies the condition:

$$\begin{array}{ccc} & \xrightarrow{f} & \\ {\scriptstyle u}\Vert & \epsilon_1 f & \Vert{\scriptstyle u'} \\ & \xrightarrow[f]{} & \end{array} \cdot \begin{array}{ccc} & \overset{v}{=} & \\ {\scriptstyle g}\downarrow & \epsilon_2 g & \downarrow{\scriptstyle g} \\ & \underset{v'}{=} & \end{array} = \begin{array}{ccc} & \xrightarrow{fv} & \\ {\scriptstyle ug}\downarrow & (\epsilon_1 f)(\epsilon_2 g) & \downarrow{\scriptstyle u'g} \\ & \xrightarrow[fv']{} & \end{array}$$

□

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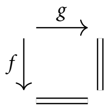
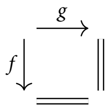
Lemma 4.5. *For every $f, g \in \Sigma_1^*$ of same source and whose target is a normal form, the shell*



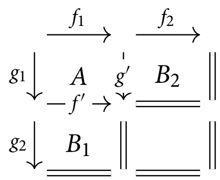
admits a filler in Σ_2^*.

Proof. Define the *origin* of a shell (f_i^α) as $\partial^- f_1^- \in \Sigma_0^*$. Let us prove that for any $u \in \Sigma_0^*$, any shell over Σ_1^* of origin u and of the form



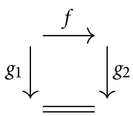
admits a filler. We reason by induction on u. If u is a normal form, then $f = g = \epsilon u$ and $\epsilon_1 \epsilon u$ is a filler of the shell.

If u is not a normal form, then we can write $f = f_1 \star f_2$ and $g = g_1 \star g_2$ in Σ_1^*, where f_1 and g_1 are rewriting steps. Let A be a 2-cell in Σ_2^* such that $\partial_1^- A = f_1$ and $\partial_2^- A = g_1$ (which exists thanks to the previous lemma). Denote $f' = \partial_1^+ A$ and $g' = \partial_2^+ A$. Then, we can apply the induction hypothesis to both (f', g_2) and (f_2, g') defining 2-cells B_1 and B_2, and we conclude using the following composite:



□

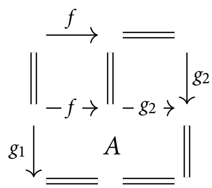
Lemma 4.6. *For every $f \in \Sigma_1^\top$, and every $g_1, g_2 \in \Sigma_1^*$ whose target is a normal form, the shell*



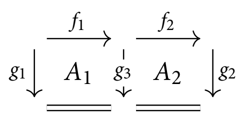
admits a filler in $\Sigma_2^\top$.

Proof. To prove that the set E of 1-cells f satisfying the lemma is $\Sigma_1^\top$, we show that it contains Σ_1^*, and that it is closed under composition and inverses.

- It contains Σ_1^*. Indeed, let $f \in \Sigma_1^*$ and g_1 and g_2 as in the lemma. We can form the following composite, where the cell A is obtained by the previous lemma:



- It is stable under composition. Indeed, let $f_1, f_2 \in E$ be two composable 1-cells, and $g_1, g_2 \in \Sigma_1^*$ as in the lemma. Let $g_3 \in \Sigma_1^*$ be a 1-cell such that $\partial^- g_3 = \partial^+ f_1$, and whose target is a normal form. Then, the following composite shows that $f_1 \star f_2$ is in E, where A_1 and A_2 exist since f_1 and f_2 are in E:



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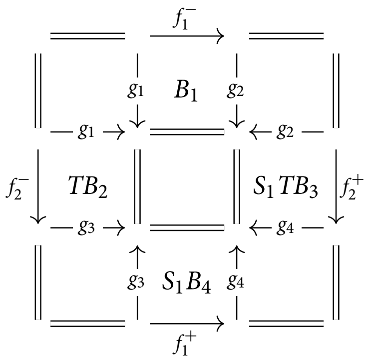
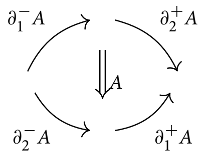
- It is stable under inverses. Indeed, let $f \in E$, and let $g_1, g_2 \in \Sigma_1^*$ as in the lemma. We can construct the following cell, where A comes from the fact that f is in E, applied to the pair (g_2, g_1): $\begin{array}{ccc} & \xleftarrow{f} & \\ g_1 \downarrow & S_2 A & \downarrow g_2 \\ & = & \end{array}$ ☐

Proof of Theorem 4.2. Let us fix a shell (f_i^α) over $\Sigma_1^\top$. The following cell is a filler of (f_i^α). The 1-cells g_1, g_2, g_3, and g_4 are arbitrary 1-cells in Σ_1^*, with the appropriate source, and a normal form as target. Applying the previous lemma to f_1^-, f_2^-, f_1^+, and f_2^+ (with a suitable choice of cells g_i), we get cells B_1, B_2, B_3, and B_4. We can now form the following diagram, which forms a filler



☐

Remark 4.7. It is possible to give an alternate proof of Theorem 4.2 using Squier's homotopical theorem. Let Σ be a cubical monoidal 2-polygraph. The equivalence between globular and cubical ω-categories associates to Σ a globular monoidal 2-polygraph $G\Sigma$. The underlying monoidal 1-polygraph is unchanged, while $G\Sigma$ contains a generating 2-cell of the following shape for every generating 2-cell $A \in \Sigma_2$:



Additionally, the 2-cells of $G\Sigma^*$ (respectively, $G\Sigma^\top$) correspond precisely with the 2-cells of Σ^* (respectively, $\Sigma^\top$) of the form $\begin{array}{ccc} & \xrightarrow{f} & \\ \| & A & \| \\ & \xrightarrow[g]{} & \end{array}$. Suppose now that Σ satisfies the hypothesis of Theorem 4.2. Then for any critical pair (f_1, f_2), there exists a 2-cell in Σ^* of shape $\begin{array}{ccc} & \xrightarrow{f_1} \xrightarrow{g_1} & \\ \| & & \| \\ & \xrightarrow[f_2]{} \xrightarrow[g_2]{} & \end{array}$. As a result, the coherence of $G\Sigma$ satisfies the hypothesis of Squier's homotopical theorem, and is therefore coherent. This means that any shell in $\Sigma^\top$ of shape $\begin{array}{ccc} & \xrightarrow{f} & \\ \| & A & \| \\ & \xrightarrow[g]{} & \end{array}$ admits a filler, which implies the coherence of Σ.

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This proves that Squier's homotopical theorem implies Theorem 4.2, and a similar argument shows the other implication. As a result, those two results are indeed equivalent.

5. Conclusion and Future Work

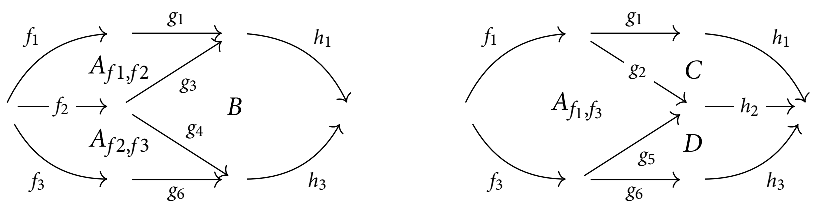
We end this article by discussing why the cubical setting would be helpful in generalizing Squier's homotopical theorem to higher dimensions. The general idea is that we want to be able to specify the shape of the source and target of some higher dimensional cells.

In globular categories, there is no structure on the source and target of a cell. Therefore, imposing a specific shape requires a lot of book-keeping, and growingly so in higher dimensions. In cubical categories however, we are able to inscribe relevant cells as faces of cubes, so that the book-keeping is hidden within the geometry of the cell.

Fillers for critical triples. One hypothesis of Squier's theorem is that any critical pair (f_1, f_2) admits a filler A_{f_1,f_2}. The first advantage of the cubical setting is the ease in how to generalise this hypothesis to critical triples (and above).

First note that the data of the A_{f_1,f_2}'s for (f_1, f_2) a critical branching induces the existence of cells of similar shape for any local branching (f_1, f_2). If (f_1, f_2) is an aspherical or a Peiffer branching, then A_{f_1,f_2} is the identity. Otherwise, then we can write $(f_1, f_2) = (uf_1'v, uf_2'v)$ for (f_1', f_2') a critical pair, and define A_{f_1,f_2} as $uA_{f_1',f_2'}v$.

A critical local branchings admit a similar classification as critical pairs, yielding a notion of critical triple. The hypothesis on critical triples would then say something like this: for any critical triple (f_1, f_2, f_3), there exists a 3-cell A_{f_1,f_2,f_3} whose source and target are, respectively, of the following shapes:



Which syntactically amounts to the (unpleasant) formulas:

$$s(A_{f_1,f_2,f_3}) = (A_{f_1,f_2} \star_0 h_1) \star_1 (f_2 \star_0 B) \star_1 (A_{f_2,f_3} \star_0 h_3)$$
$$t(A_{f_1,f_2,f_3}) = (f_0 \star_0 C) \star_1 (A_{f_1,f_3} \star_0 h_2) \star_1 (f_3 \star_0 D).$$

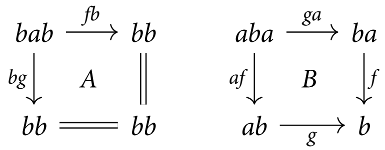
This condition is, for example, used in Lucas (2017), Definition 1.4.8.

In the cubical setting, 3-cells have the shape of cubes, and so it is possible to isolate A_{f_1,f_2}, A_{f_1,f_3} and A_{f_2,f_3} on the faces of the cube closest to the origin. The condition could therefore be rephrased more simply as:

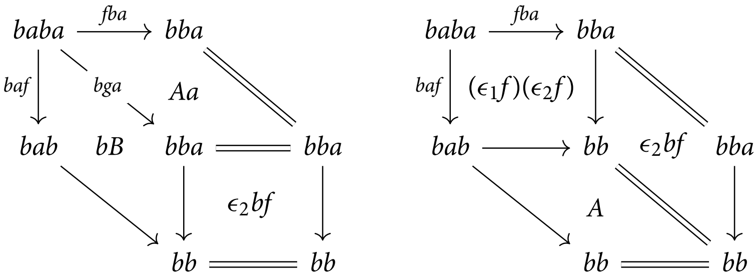
$$\partial_1^- A_{f_1,f_2,f_3} = A_{f_1,f_2} \qquad \partial_2^- A_{f_1,f_2,f_3} = A_{f_1,f_3} \qquad \partial_3^- A_{f_1,f_2,f_3} = A_{f_2,f_3}$$

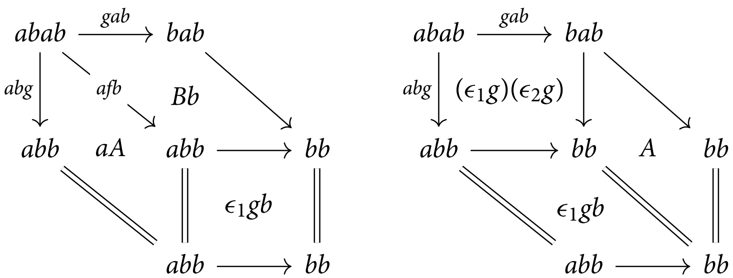
More specifically, notice that 1-cells appear when expressing the condition in globular terms, but not in cubical ones.

An example of monoidal cubical 3-polygraph. As an example, let us consider the (convergent) presentation $\Sigma = \langle a, b | f : ba \to b, g : ab \to b \rangle$. Using Squier's theorem, we can extend Σ to a monoidal cubical 2-polygraph, where the 2-cells correspond to the two critical pairs (fb, bg) and (ga, af):



There are then two critical triples: (fba, bga, baf) and (gab, afb, abg). Following the idea detailed in the previous section, we extend Σ to a cubical monoidal 3-polygraph using two 3-cells whose shapes are, respectively, given by the following diagrams.





As previously, the faces of those generating 3-cells can be easily described in cubical terms, while in the globular setting, the geometry of the source and target of the generating 3-cells would involve complex compositions.

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