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Abstract

We extend field patching to the setting of Berkovich analytic geometry and use it to prove a local–global principle over function fields of analytic curves with respect to completions. In the context of quadratic forms, we combine it with sufficient conditions for local isotropy over a Berkovich curve to obtain applications on the u-invariant. The patching method we adapt was introduced by Harbater and Hartmann [*Patching over fields*, Israel J. Math. **176** (2010), 61–107] and further developed by these two authors and Krashen [*Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), 231–263]. The results presented in this paper generalize those of Harbater, Hartmann, and Krashen [*Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), 231–263] on the local–global principle and quadratic forms.

Introduction

Patching techniques were introduced as one of the main approaches to inverse Galois theory. Originally of purely geometric nature, this method provided a way to obtain a global Galois cover from local ones; see, for example, [Har03]. Another example is [Poi10], where Poineau used patching on analytic curves in the Berkovich sense and consequently generalized results shown by Harbater in [Har87, Har88]. In [HH10], Harbater and Hartmann extended the technique to structures over fields, while constructing a setup of heavily algebraic flavor. Patching over fields has recently seen many applications to local–global principles and quadratic forms; see for example [HHK09, CPS12]. In particular, in [HHK09], Harbater, Hartmann, and Krashen (from now on referred to as HHK) obtained results on the u-invariant, generalizing those of Parimala and Suresh [PS10], which were proven through different methods. Another source for results on the u-invariant is Leep's article [Lee13].

In this paper, we use field patching in the setting of Berkovich analytic geometry. A convenience of this point of view is the clarity it provides into the overall strategy. By patching over analytic curves, we prove a local–global principle and provide applications to quadratic forms and the u-invariant. The results we obtain generalize those of [HHK09]. Because of the geometric nature of this approach, we believe it to be a nice framework for potential generalizations in different directions and in particular to higher dimensions.

Before presenting the main results of this paper, let us introduce some terminology.

Definition (HHK). Let K be a field. Let X be a K-variety and G a linear algebraic group over K. We say that G acts *strongly transitively* on X if G acts on X and, for any field extension L/K, either $X(L)=\emptyset$ or $G(L)$ acts transitively on $X(L)$.

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Our main results, the local–global principles we show, are as follows.

Theorem. *Let k be a complete non-trivially valued ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Let X be an F-variety and G a connected rational linear algebraic group over F acting strongly transitively on X.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations on F which either extend the valuation of k or are trivial when restricted to k.

Denote by C^{an} the Berkovich analytification of C, so that $F = \mathscr{M}(C^{\mathrm{an}})$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C^{an}. Then the following local–global principles hold.

- *(Theorem 3.11) $X(F) \neq \emptyset \Longleftrightarrow X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$.*
- *(Corollary 3.18) If F is a perfect field or X is a smooth variety, then*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \quad \text{for all } v \in V(F),$$

  where F_v denotes the completion of F with respect to v.

The statement above remains true for affinoid curves if $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. Being a local–global principle with respect to completions, the second equivalence evokes some resemblance to more classical versions of local–global principles. The statement can be made to include trivially valued base fields, even though in this case we obtain no new information (since one of the overfields will be equal to F).

We recall that for any finitely generated field extension F/k of transcendence degree 1, there exists a unique normal projective k-algebraic curve with function field F. Thus, the result of the theorem above is applicable to any such field F.

While HHK work over models of an algebraic curve, we work directly over analytic curves. We remark that we put no restrictions on the complete valued base field k. Apart from the framework, this is one of the fundamental differences with [HHK09, Theorem 3.7], where the base field needs to be complete with respect to a discrete valuation. Another difference lies in the nature of the overfields, which here are completions or fields of meromorphic functions. Section 4 shows that the latter contain the ones appearing in HHK's article and thus that [HHK09, Theorem 3.7] is a direct consequence of the local–global principle stated in Theorem 3.11. Moreover, we show that the converse is true as well provided we choose a 'fine' enough model. The proof of the theorem above is based on the patching method, but used in a different setting from the one of [HHK09].

As a consequence, in the context of quadratic forms we obtain the following theorem, which is a generalization of [HHK09, Theorem 4.2].

Theorem. *Let k be a complete non-trivially valued ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Suppose that* $\mathrm{char}(F) \neq 2$. *Let q be a quadratic form over F of dimension different from 2.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations on F which either extend the valuation of k or are trivial when restricted to k.

Let C^{an} be the Berkovich analytification of C, so that $F = \mathscr{M}(C^{\mathrm{an}})$, where $\mathscr{M}$ is the sheaf of meromorphic functions on C^{an}.

1. *(Theorem 3.12) The quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C^{\mathrm{an}}$.*
2. *(Corollary 3.19) The quadratic form q is isotropic over F if and only if it is isotropic over F_v for all $v \in V(F)$, where F_v is the completion of F with respect to v.*

As mentioned in the introduction of [HHK09], it is expected that for a 'nice enough' field K the u-invariant remains the same after taking finite field extensions and that it becomes $2^d u(K)$ after taking a finitely generated field extension of transcendence degree d. Since we work only in dimension one, this explains the motivation behind the following definition.

DEFINITION. Let K be a field.

(1) [Kaplansky] The *u-invariant* of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exist anisotropic quadratic forms over K of arbitrarily large dimension.

(2) [HHK] The strong u-invariant of K, denoted by $u_s(K)$, is the smallest real number m such that:
   - $u(E) \leqslant m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leqslant m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant.

The theorem above leads to applications on the u-invariant. Let k be a complete non-archimedean valued field with residue field $\widetilde{k}$ such that $\mathrm{char}(\widetilde{k}) \neq 2$. Suppose that either $|k^\times|$ is a free $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}}|k^\times| = n$ or more generally that $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = n$, where n is a non-negative integer and $\sqrt{|k^\times|}$ denotes the divisible closure of the value group $|k^\times|$. This is yet another difference with the corresponding results of HHK in [HHK09], where the requirement on the base field is that it be complete discretely valued, i.e. that its value group be a free $\mathbb{Z}$-module of rank 1. We obtain an upper bound on the u-invariant of a finitely generated field extension of k with transcendence degree at most 1, which depends only on $u_s(\widetilde{k})$ and n. More precisely, in terms of the strong u-invariant, we obtain the following.

COROLLARY (Corollary 6.2). *Let k be a complete valued non-archimedean field. Suppose that* $\mathrm{char}(\widetilde{k}) \neq 2$.

(1) *If $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = n$, then $u_s(k) \leqslant 2^{n+1}u_s(\widetilde{k})$.*

(2) *If $|k^\times|$ is a free $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}}|k^\times| = n$, then $u_s(k) \leqslant 2^n u_s(\widetilde{k})$.*

It is unknown to the author whether there is equality in the corollary above. This is true in the particular case of $n = 1$ by using [HHK09, Lemma 4.9], whose proof is independent of patching. In this way we recover [HHK09, Theorem 4.10], which is the main result of [HHK09] on quadratic forms. It also provides one more proof that $u(\mathbb{Q}_p(T)) = 8$, where p is a prime number different from 2, originally proven in [PS10].

COROLLARY (Corollary 6.4). *Let k be a complete discretely valued field such that* $\mathrm{char}(\widetilde{k}) \neq 2$. *Then $u_s(k) = 2u_s(\widetilde{k})$.*

The first section of this paper is devoted to proving that patching can be applied to an analytic curve. To do this, we follow along the lines of the proof of [HHK09, Theorem 2.5], making adjustments to render it suitable to our more general setup. Recall that an analytic curve is a graph (see [Duc, Théorème 3.5.1]). We work over any complete valued base field k such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. This condition is equivalent to asking the existence of type 3 points on k-analytic curves, which are characterized by simple topological and algebraic properties. More precisely, a point of type 3 has arity 2 in the graph associated to the curve and its local ring with respect to the sheaf of analytic functions is a field provided the curve is reduced. Type 3 points are crucial to the constructions we make in some of the following sections. Let C be an integral k-analytic curve. Let U, V be connected affinoid domains in C such that $W = U \cap V$ is a single type 3 point. We show that given two reasonable algebraic structures over $\mathscr{M}(U)$, $\mathscr{M}(V)$ and a suitable group action on them, they can be patched to give the same type of algebraic structure over $\mathscr{M}(U \cup V)$. The key step in proving this is a matrix decomposition result for certain linear algebraic groups.

In the second section, our aim is to show that any open cover of a projective k-analytic curve can be refined into a finite cover that satisfies conditions similar to those of the first section, i.e. one over which we can apply patching. The refinement $\mathcal{U}$ we construct is a finite cover of the curve such that for any $U \in \mathcal{U}$, U is a connected affinoid domain with only type 3 points in its boundary. Furthermore, for any distinct $U, V \in \mathcal{U}$, the intersection $U \cap V$ is a finite set of type 3 points. A cover with these properties will be called *nice* (cf. Definition 2.1). The existence of a refinement that is a nice cover will first be shown for the projective line $\mathbb{P}_k^{1,\mathrm{an}}$ and will then be generalized to a broader class of k-analytic curves. Our base field k will still be a complete valued non-archimedean field such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

The third section contains the main results of the paper. We show two local–global principles (Theorem 3.11 and Corollary 3.18) over fields of meromorphic functions of normal projective k-analytic curves and an application to quadratic forms (Theorem 3.12 and Corollary 3.19). In the simplest cases, the proofs use patching on nice covers and induction on the number of elements of said covers. We first prove these results over a complete ultrametric base field k such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. This is then generalized for projective curves to any complete ultrametric field using a descent argument that is based on results of model theory. We also prove similar results for affinoid curves.

In the fourth section, we interpret the overfields of HHK's [HHK09] in the Berkovich setting and show that [HHK09, Theorem 3.7] is a consequence of Theorem 3.11. We show that the converse is true as well provided one works over a 'fine' enough model.

The purpose of the fifth section is to find conditions under which there is local isotropy of a quadratic form q over analytic curves. The setup will be somewhat more general, which is partly why it is the most technical section of the paper. The idea is to find a nice enough representative of the isometry class of q to work with and then use Henselianity conditions. The hypotheses on the base field become stronger here. Namely, we require our complete valued non-archimedean base field k to be such that the dimension of the $\mathbb{Q}$-vector space $\sqrt{|k^\times|}$ is finite (a special case being when $|k^\times|$ is a free module of finite rank over $\mathbb{Z}$) and the residue characteristic unequal to 2. The restriction on the value group is not very strong: when working over a complete ultrametric field k satisfying this property, for every k-analytic space X and every point $x \in X$, the completed residue field $\mathcal{H}(x)$ of x satisfies it as well.

In the last section, we put together the local–global principle for quadratic forms and the local isotropy conditions of the previous section to give a condition for global isotropy of a quadratic form over an analytic curve. From there we deduce applications to the (strong) u-invariant of a complete valued field k with residue characteristic different from 2 and such that the dimension of the $\mathbb{Q}$-vector space $\sqrt{|k^\times|}$ is finite.

Conventions. Throughout this paper, we use the Berkovich approach to non-archimedean analytic geometry. A Berkovich analytic curve will be a separated analytic space of pure dimension 1.

A valued field is a field endowed with a non-archimedean absolute value. For any valued field l, we denote by $\widetilde{l}$ its residue field.

We call the topological boundary a *boundary* and denote it by $\partial(\cdot)$. We call the relative boundary (respectively boundary) introduced in [Ber90, Definition 2.5.7] and [Ber93, Definition 1.5.4] a *Berkovich relative boundary* (respectively a *Berkovich boundary*) and denote it by $\partial_B(\cdot/\cdot)$ (respectively $\partial_B(\cdot)$).

The empty set is considered to be connected.

A Berkovich analytic space which is reduced and irreducible is called *integral* in this text. Thus, an *integral affinoid space* is an affinoid space whose corresponding affinoid algebra is a domain.

Throughout the entire paper, we work over a complete valued base field k.

1. Patching over Berkovich curves

The purpose of this section is to prove a matrix decomposition result under conditions which generalize those of HHK's article [HHK09, § 3, Theorem 3.2]. As a consequence, we obtain a generalization of vector space patching on analytic curves. Let us start by fixing a somewhat more extensive framework, in which our proof works.

Setting 1. Let $R_i, i=1,2$, be an integral domain endowed with a non-archimedean submultiplicative norm $|\cdot|_{R_i}$, with respect to which it is complete. Set $F_i = \operatorname{Frac} R_i, i=1,2$. Let F be an infinite field embedded in both F_1 and F_2. Let F_0 be a complete ultrametric field with non-trivial valuation such that there exist bounded morphisms $R_i \hookrightarrow F_0$, $i=1,2$. Suppose that the image of F_1 is dense in F_0. Let A_i be an R_i-module such that $A_i \subseteq F_i$. Suppose that A_i is finitely generated as an R_i-module, i.e. that there exists a surjective R_i-linear morphism $\varphi_i : R_i^{n_i} \twoheadrightarrow A_i$ for some positive integer $n_i, i=1,2$. Let us endow A_i with the quotient semi-norm induced from φ_i. Assume that A_i is complete and the morphism $A_i \hookrightarrow F_0$ is bounded for $i=1,2$. We remark that this implies that the semi-norm on A_i is a norm. Suppose that the induced map $\pi : A_1 \oplus A_2 \to F_0$ is surjective. Finally, suppose that the norm of F_0 is equivalent to the quotient norm induced by the surjective morphism $\pi : A_1 \oplus A_2 \twoheadrightarrow F_0$, where $A_1 \oplus A_2$ is endowed with the usual max semi-norm $|\cdot|_{\max}$.

As in [Ber90], a morphism $f : A \to B$ of semi-normed rings is said to be *admissible* if the quotient semi-norm on $A/\ker(f)$ is equivalent to the restriction to $f(A)$ of the semi-norm on B. Thus, in the setting above, we suppose that the morphism π is admissible.

Before giving the motivating example for Setting 1, we need to recall Berkovich meromorphic functions.

DEFINITION 1.1. Let X be a reduced k-analytic space. Let $\mathcal{S}$ be the presheaf of functions on X which associates to any analytic domain U the set of analytic functions on U whose restriction to any affinoid domain in it is not a zero-divisor. Let $\mathscr{M}_-$ be the presheaf on X that associates to any analytic domain U the ring $\mathcal{S}(U)^{-1}\mathcal{O}(U)$. The sheafification $\mathscr{M}$ of the presheaf $\mathscr{M}_-$ is said to be the sheaf of meromorphic functions on X.

We notice that for any $x \in X$, $\mathscr{M}_x$ is the total ring of fractions of $\mathcal{O}_{X,x}$. In particular, if $\mathcal{O}_{X,x}$ is a domain, then $\mathscr{M}_x = \operatorname{Frac} \mathcal{O}_{X,x}$. We make note of the following, well-known, fact.

LEMMA 1.2. *Let X be an integral k-affinoid space. Then $\mathscr{M}(X) = \operatorname{Frac} \mathcal{O}(X)$.*

By replacing the fraction field of $\mathcal{O}(X)$ with its total ring of fractions, the statement remains true when removing the condition of integrality on X.

Proof. Since $\mathcal{O}(X)$ is an integral domain, $\operatorname{Frac}\mathcal{O}(X) \subseteq \mathscr{M}(X)$ by the definition of $\mathscr{M}$. Let $f \in \mathscr{M}(X)$. The sheaf $f\mathcal{O} \cap \mathcal{O} \subseteq \mathscr{M}$ is non-zero and coherent, so, by Kiehl's theorem, it has a non-zero global section x. Then there exists $y \in \mathcal{O}(X)\backslash\{0\}$ for which $f = x/y \in \operatorname{Frac}\mathcal{O}(X)$. $\square$

Another result that will be needed throughout this paper is the following.

Lemma 1.3. *Let C be a reduced k-analytic curve. Let U, V be affinoid domains of C such that $U \cap V = \{\eta\}$, where η is a point of type 3. Then the images of $\mathscr{M}(U)$ and $\mathscr{M}(V)$ in $\mathscr{M}(\{\eta\})$ are dense.*

Proof. Let us start by remarking that the set of poles of a meromorphic function is a divisor and as such consists of only rigid points. This implies that a meromorphic function cannot have a pole on any non-rigid point (including η), which is why it makes sense to evaluate it at η.

That $\{\eta\}$ is an affinoid domain of U (respectively V) can be checked directly from the definition of an affinoid domain. By the Gerritzen–Grauert theorem (see [Tem05]), we obtain that it is a rational domain in U (respectively V). Then, by the easy implication of [Ber90, Corollary 2.2.10], the meromorphic functions on U (respectively V) with no poles in $\{\eta\}$ are dense in $\mathcal{O}(\{\eta\})$. Seeing as η is a type 3 point, $\mathcal{O}(\{\eta\}) = \mathscr{M}(\{\eta\}) = \mathcal{H}(\eta)$: the completed residue field of η. Finally, this implies that the image with respect to the restriction morphism of $\mathscr{M}(U)$ (respectively $\mathscr{M}(V)$) in $\mathscr{M}(\{\eta\})$ is dense. $\square$

The example of Setting 1 we will be working with is the following.

Proposition 1.4. *Let C be an integral k-analytic curve. Set $F_C = \mathscr{M}(C)$. Let D be an effective divisor of degree n on C. Take two connected affinoid domains U, V in C such that $W = U \cap V = \{\eta\}$, where η is a type 3 point. Let $R_U = \mathcal{O}(U), F_U = \operatorname{Frac} R_U$, $R_V = \mathcal{O}(V), F_V = \operatorname{Frac} R_V$, and $F_W = \mathcal{O}(W)$. Set $A_U = \mathcal{O}(D)(U)$ and $A_V = \mathcal{O}(D)(V)$.*

For large enough n such that $H^1(C, \mathcal{O}(D)) = 0$, the conditions of Setting 1 are satisfied with $R_1 = R_U, R_2 = R_V$, $A_1 = A_U, A_2 = A_V$, $F = F_C$, and $F_0 = F_W$.

Proof. By construction, the rings R_U, R_V, F_W are all complete with respect to non-archimedean semi-norms. Since $\mathcal{O}(W) = \mathcal{H}(\eta)$, the normed ring F_W is a complete ultrametric non-trivially valued field. As U, V, and W are integral, by Lemma 1.2, $\mathscr{M}(U) = F_U$, $\mathscr{M}(V) = F_V$, and $\mathscr{M}(W) = F_W$. This shows the existence of embeddings of F_C into F_U, F_V, and F_W. The restriction morphisms $R_U, R_V \to F_W$ are bounded by construction. From Lemma 1.3, F_U, F_V have dense images in F_W.

Notice that for $Z \in \{U, V, W\}$, $\mathcal{O}(Z) \hookrightarrow \mathcal{O}(D)(Z) \hookrightarrow \mathscr{M}(Z)$. In particular, this means that $\mathcal{O}(D)(W) = \mathcal{O}(W) = \mathscr{M}(W)$. Since $\mathcal{O}(D)$ is a coherent sheaf, A_U (respectively A_V) is a finitely generated R_U-module (respectively R_V-module). The completeness of A_U (respectively A_V) follows from the fact that ideals of affinoid algebras are closed. The morphism $\mathcal{O}(D)(U) = A_U \hookrightarrow F_W = \mathcal{O}(D)(W)$ is the restriction morphism of the sheaf $\mathcal{O}(D)$, so it is bounded. The same is true for $A_V \hookrightarrow F_W$.

If $U \cup V$ is not the entire C, it is an affinoid domain thereof (see [Duc, Théorème 6.1.3]). By Tate's acyclicity theorem [Ber90, ch. 2, Proposition 2.2.5],

$$0 \to H^0(U \cup V, \mathcal{O}(D)) \to H^0(U, \mathcal{O}(D)) \oplus H^0(V, \mathcal{O}(D)) \to H^0(U \cap V, \mathcal{O}(D)) \to 0$$

is an exact admissible sequence, from which we obtain the surjective admissible morphism $A_U \oplus A_V \twoheadrightarrow \mathcal{O}(D)(W) = F_W$.

Suppose that $U\cup V=C$. Since C is then compact and integral, by [Duc, Théorèmes 6.1.3 and 3.7.2], it is either an affinoid domain (a case we dealt with in the paragraph above) or a projective curve. If C is projective, by [Liu02, § 7.5, Proposition 5.5] for large enough n, $H^1(U\cup V,\mathcal{O}(D))=0$. The Mayer–Vietoris exact sequence now produces a bounded surjective morphism $A_U\oplus A_V\twoheadrightarrow\mathcal{O}(D)(W)=F_W$. Admissibility follows from Banach's open mapping theorem if k is not non-trivially valued (for a proof, see [Bou53]) and by a change of basis followed by the open mapping theorem if it is (see [Ber90, ch. 2, Proposition 2.1.2(ii)]). $\square$

We make note of the fact that Proposition 1.4 assumes the existence of a point of type 3, which is equivalent to $\sqrt{|k^{\times}|}\neq\mathbb{R}_{>0}$.

Remark 1.5. Other examples of Setting 1 could be obtained by taking instead of $\mathcal{O}(D)$ any coherent sheaf $\mathcal{F}$ of $\mathcal{O}$-algebras that is a subsheaf of $\mathscr{M}$ for which $H^1(C,\mathcal{F})=0$.

DEFINITION 1.6. Let K be a field. A *rational* variety over K is a K-variety that has a Zariski open isomorphic to an open of some $\mathbb{A}^n_K$.

Using the same notation as in Setting 1, the main goal of this section is to prove the following matrix decomposition result.

THEOREM 1.7. *Let G be a connected linear algebraic group over F that is a rational variety over F. For any $g\in G(F_0)$, there exist $g_1\in G(F_1)$, $g_2\in G(A_2)$ such that $g=g_1\cdot g_2$.*

This was proven in a slightly different setting by HHK in [HHK09]. We follow along the lines of their proof, making adjustments to render it suitable for the hypotheses we want to work with.

Let K be an infinite field. Since a connected rational linear algebraic group G over some infinite field K has a non-empty open subset U' isomorphic to an open subset U of an affine space $\mathbb{A}^n_K$, by translation (since K is infinite) we may assume that the identity element of G is contained in U', that $0\in U$, and that the identity is sent to 0. Let us denote the isomorphism $U'\to U$ by ϕ.

Let m be the multiplication in G and set $\widetilde{U'}=m^{-1}(U')\cap(U'\times U')$, which is an open subset of $G\times G$. It is isomorphic to an open subset $\widetilde{U}$ of $\mathbb{A}^{2n}_K$ and $m_{|\widetilde{U'}}$ gives rise to a map $\widetilde{U}\to U$, i.e. to a rational function $f:\mathbb{A}^{2n}_K\dashrightarrow\mathbb{A}^n_K$. We remark that for any $(x,0),(0,x)\in\widetilde{U}$, this function sends them both to x.

$$\begin{tikzcd}
\widetilde{U'} \arrow[r, "m_{|\widetilde{U'}}"] \arrow[d, "(\phi\times\phi)_{|\widetilde{U'}}"'] & U' \arrow[d, "\phi"] \\
\widetilde{U} \arrow[r, "f"'] & U
\end{tikzcd}$$

The theorem we want to prove can be interpreted in terms of the map f. Lemma 1.9 below, formulated to fit a more general setup, shows that said theorem is true on some neighborhood of the origin of an affine space. It is the analogue of [HHK09, Theorem 2.5].

We proceed first with an auxiliary result. Since the morphisms $A_i\hookrightarrow F_0, i=1,2$, are bounded, there exists $C>0$ such that for any $x_i\in A_i$, $|x_i|_{F_0}\leqslant C\cdot|x_i|_{A_i}$. By changing to an equivalent norm on A_i if necessary, we may assume that $C=1$. Let us fix the quotient norm $|\cdot|_{F_0}$ on F_0, induced from the surjective morphism $\pi:A_1\oplus A_2\twoheadrightarrow F_0$.

LEMMA 1.8. (1) *For any $x_i\in A_i, i=1,2$, $|x_i|_{F_0}\leqslant|x_i|_{A_i}$.*

(2) *There exists a constant $d\in(0,1)$ such that for any $c\in F_0$, there exist $a\in A_1, b\in A_2$ for which $\pi(a+b)=c$ and $d\cdot\max(|a|_{A_1},|b|_{A_2})\leqslant|c|_{F_0}$.*

Proof. (1) See the paragraph above the statement.

(2) Suppose that $c \neq 0$. Let D be any real number such that $D > 1$. For any $c \in F_0$, there exist $a \in A_1, b \in A_2$ (depending on D) such that $\pi(a+b) = c$ and $\max(|a|_{A_1}, |b|_{A_2}) \leqslant D \cdot |c|_{F_0}$. Otherwise, for any $x \in A_1, y \in A_2$ for which $\pi(x+y) = c$, one would have $|x+y|_{\max} = \max(|x|_{A_1}, |y|_{A_2}) > D \cdot |c|_{F_0}$. Then

$$|c|_{F_0} = \inf_{\substack{x \in A_1, y \in A_2 \\ \pi(x+y)=c}} |x+y|_{\max} \geqslant D \cdot |c|_{F_0},$$

which is impossible if $c \neq 0$. Thus, there exist a and b as above and, for $d = D^{-1} \in (0,1)$, one obtains $d \cdot \max(|a|_{A_1}, |b|_{A_2}) \leqslant |c|_{F_0}$.

If $c = 0$, the statement is true regardless of the choice of d. $\square$

From now on, instead of writing $\pi(x+y) = c$ for $x \in A_1, y \in A_2, c \in F_0$, we will just put $x + y = c$ without risk of ambiguity.

In what follows, for any positive integer n, let us endow F_0^n with the max norm induced from the norm on F_0 and let us also denote it by $|\cdot|_{F_0}$.

LEMMA 1.9. *Let* $f : \mathbb{A}^n_{F_0} \times \mathbb{A}^n_{F_0} \dashrightarrow \mathbb{A}^n_{F_0}$ *be a rational map defined on a Zariski open* $\widetilde{S}$ *such that* $(0,0) \in \widetilde{S}$ *and* $f(x,0) = f(0,x) = x$ *whenever* $(x,0),(0,x) \in \widetilde{S}$. *Then there exists* $\varepsilon > 0$ *such that for any* $a \in \mathbb{A}^n(F_0)$ *with* $|a|_{F_0} \leqslant \varepsilon$, *there exist* $u \in A_1^n$ *and* $v \in A_2^n$ *for which* $(u,v) \in \widetilde{S}(F_0)$ *and* $f(u,v) = a$.

Proof. The rational function f can be written as $(f_1, \ldots, f_n)$, where the f_i are elements of $F_0[T_1, \ldots, T_n, S_1, \ldots, S_n]_{(T_1,\ldots,T_n,S_1,\ldots,S_n)}$. Furthermore, since $f_i(0,0) = 0$, they belong to the maximal ideal of this ring. Lemmas 2.1 and 2.3 of [HHK09] remain true in our setting without any changes to their proofs (this is where the condition $f(x,0) = f(0,x) = x$ is crucial). They tell us that:

(1) we can see these rational functions as elements of $F_0[[T_1, \ldots, T_n, S_1, \ldots, S_n]]$;

(2) there exists $M \geqslant 1$ such that

$$f_i = S_i + T_i + \sum_{|(l,m)| \geqslant 2} c^i_{l,m} \underline{T}^l \underline{S}^m \in F_0[[T_1, \ldots, T_n, S_1, \ldots, S_n]],$$

with $|c^i_{l,m}|_{F_0} \leqslant M^{|(l,m)|}$ for $i = 1, 2, \ldots, n$ and $(l,m) \in \mathbb{N}^{2n}$, where $|(l,m)|$ is the sum of the coordinates of (l,m). (We remark that since $f_i(x,0) = f_i(0,x) = x$ for any x for which $(0,x),(x,0) \in \widetilde{S}$, we can even assume that l, m are both non-zero.)

Since $\widetilde{S}$ is open, $\widetilde{S}(F_0)$ is Zariski open in $\mathbb{A}^{2n}(F_0)$ and so it is open in F_0^{2n} in the topology induced by the max norm (which is finer than the Zariski one). Seeing as $0 \in \widetilde{S}(F_0)$, there exists $\delta > 0$ such that for any $(x,y) \in F_0^{2n}$ with $|(x,y)|_{F_0} < \delta$, one has $(x,y) \in \widetilde{S}(F_0)$ and $f(x,y)$ is defined.

Let us fix the constant d given by Lemma 1.8. Let $0 < \varepsilon' \leqslant \min\{1/2M, d^2/M^4, \delta/2\}$. Set $\varepsilon = d\varepsilon'$. Since $\varepsilon < \varepsilon' < \min(1/M, \delta/2)$, by [HHK09, Lemma 2.1], for any $(x,y) \in \widetilde{S}(F_0)$ with $|(x,y)|_{F_0} \leqslant \varepsilon'$, $f(x,y)$ is well defined and the series by which $f_i(x,y)$ is given is convergent in F_0, $i = 1, 2, \ldots, n$.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n(F_0)$ be such that $|a|_{F_0} \leqslant \varepsilon$. Let $u_0 = 0 \in A_1^n$ and $v_0 = 0 \in A_2^n$. Using induction, one constructs sequences $(u_s)_s$ in A_1^n and $(v_s)_s$ in A_2^n such that the following conditions are satisfied:

(1) $|u_s|_{A_1}, |v_s|_{A_2} \leqslant \varepsilon'$ for all $s \geqslant 0$;

(2) $|u_s - u_{s-1}|_{A_1}, |v_s - v_{s-1}|_{A_2} \leqslant \varepsilon'^{(s+1)/2}$ for all $s \geqslant 1$;

(3) $|f(u_s, v_s) - a|_{F_0} \leqslant d\varepsilon'^{(s+2)/2}$ for all $s \geqslant 0$.

The first terms u_0 and v_0 satisfy conditions (1) and (3). We notice that the first condition implies that $|(u_s, v_s)|_{F_0} \leqslant \varepsilon'$, so $f(u_s, v_s)$ is well defined and $f_i(u_s, v_s)$ is convergent for $s \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. Suppose that for $j \geqslant 0$, we have constructed u_j and v_j satisfying all three conditions above. Then $d_j := a - f(u_j, v_j) \in F_0^n$ is well defined and $|d_j|_{F_0} \leqslant d\varepsilon'^{(j+2)/2}$. From Lemma 1.8, there exist $u'_j \in A_1^n$ and $v'_j \in A_2^n$ such that $d_j = u'_j + v'_j$ and $d \cdot \max(|u'_j|_{A_1}, |v'_j|_{A_2}) \leqslant |d_j|_{F_0} \leqslant d\varepsilon'^{(j+2)/2}$.

Set $u_{j+1} = u_j + u'_j$ and $v_{j+1} = v_j + v'_j$. Then $|u_{j+1}|_{A_1} \leqslant \max(\varepsilon', \varepsilon'^{(j+2)/2}) = \varepsilon'$ and the same is true for v_{j+1}. Also, $|u_{j+1} - u_j|_{A_1} = |u'_j|_{A_1} \leqslant \varepsilon'^{(j+2)/2}$ and, similarly, $|v_{j+1} - v_j|_{A_2} \leqslant \varepsilon'^{(j+2)/2}$.

For $r \in \mathbb{N}$, $i \in \{1, 2, \ldots, r\}$, and $\alpha \in F_0^r$, let α_i be the ith coordinate of α. For $p = (p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$, set $\alpha^p := \prod_{i=1}^r \alpha_i^{p_i}$. Then, for the third condition,

$$\begin{aligned}
|f_i(u_{j+1}, v_{j+1}) - a_i|_{F_0} &= \Bigg| u_{j+1,i} + v_{j+1,i} - a_i + \sum_{|(l,m)| \geqslant 2} c^i_{l,m} u^l_{j+1} v^m_{j+1} \Bigg|_{F_0} \\
&= \Bigg| u_{j,i} + v_{j,i} + u'_{j,i} + v'_{j,i} - a_i + \sum_{|(l,m)| \geqslant 2} c^i_{l,m} u^l_{j+1} v^m_{j+1} \Bigg|_{F_0} \\
&= \Bigg| f_i(u_j, v_j) - a_i + u'_{j,i} + v'_{j,i} + \sum_{|(l,m)| \geqslant 2} c^i_{l,m} (u^l_{j+1} v^m_{j+1} - u^l_j v^m_j) \Bigg|_{F_0} \\
&= \Bigg| -d_{j,i} + u'_{j,i} + v'_{j,i} + \sum_{|(l,m)| \geqslant 2} c^i_{l,m} (u^l_{j+1} v^m_{j+1} - u^l_j v^m_j) \Bigg|_{F_0} \\
&= \Bigg| \sum_{|(l,m)| \geqslant 2} c^i_{l,m} (u^l_{j+1} v^m_{j+1} - u^l_j v^m_j) \Bigg|_{F_0} \\
&\leqslant \max_{|(l,m)| \geqslant 2} |c^i_{l,m}|_{F_0} \cdot |u^l_{j+1} v^m_{j+1} - u^l_j v^m_j|_{F_0}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
u^l_{j+1} v^m_{j+1} - u^l_j v^m_j &= (u_j + u'_j)^l (v_j + v'_j)^m - u^l_j v^m_j \\
&= \sum_{\substack{0 \leqslant \beta \leqslant l \\ 0 \leqslant \gamma \leqslant m}} A_\beta B_\gamma u^\beta_j u'^{l-\beta}_j v^\gamma_j v'^{m-\gamma}_j - u^l_j v^m_j \\
&= \sum_{0 \leqslant \alpha \leqslant (l,m)} \sum_{\substack{\beta+\gamma=\alpha \\ 0 \leqslant \beta \leqslant l \\ 0 \leqslant \gamma \leqslant m}} A_\beta B_\gamma u^\beta_j u'^{l-\beta}_j v^\gamma_j v'^{m-\gamma}_j - u^l_j v^m_j,
\end{aligned}$$

where A_β, B_γ are integers (implying that they are of norm at most one on F_0). Thus, $u^l_{j+1} v^m_{j+1} - u^m_j v^l_j$ is a finite sum of monomials of degree $|(l,m)|$ in the variables $u_{j,i}, u'_{j,i}, v_{j,i}, v'_{j,i}$, $i = 1, 2, \ldots, n$, where the degree in $u_{j,i}, v_{j,i}$ is strictly smaller than $|(l,m)|$. Finally, since the norm is multiplicative and non-archimedean,

$$\begin{aligned}
|u^l_{j+1} v^m_{j+1} - u^l_j v^m_j|_{F_0} &\leqslant \max_{\substack{0 \leqslant \beta+\gamma < (l,m) \\ 0 \leqslant \beta \leqslant l, 0 \leqslant \gamma \leqslant m}} |u^\beta_j|_{F_0} |v^\gamma_j|_{F_0} |u'^{l-\beta}_j|_{F_0} |v'^{m-\gamma}_j|_{F_0} \\
&\leqslant \max_{\substack{0 \leqslant \beta+\gamma < (l,m) \\ 0 \leqslant \beta \leqslant l, 0 \leqslant \gamma \leqslant m}} \varepsilon'^{|(\beta,\gamma)|} \big(\varepsilon'^{(j+2)/2}\big)^{|(l,m)| - |(\beta,\gamma)|},
\end{aligned}$$

so $|u_{j+1}^l v_{j+1}^m - u_j^l v_j^m|_{F_0} \leqslant \max_{0\leqslant\theta<|(l,m)|} \varepsilon'^{\theta} \cdot (\varepsilon'^{(j+2)/2})^{|(l,m)|-\theta}$. This, combined with

$$|c_{l,m}^i|_{F_0} \leqslant M^{|(l,m)|},$$

implies that

$$\begin{aligned}|f_i(u_{j+1}, v_{j+1}) - a_i|_{F_0} &\leqslant \max_{\substack{|(l,m)|\geqslant 2\\ 0\leqslant\theta<|(l,m)|}} M^{|(l,m)|}\varepsilon'^{\theta} \cdot (\varepsilon'^{(j+2)/2})^{|(l,m)|-\theta}\\ &= \max_{\substack{|(l,m)|\geqslant 2\\ 0\leqslant\theta<|(l,m)|}} (M\varepsilon')^{\theta} \cdot (M\varepsilon'^{(j+2)/2})^{|(l,m)|-\theta}.\end{aligned}$$

Since $\varepsilon' \geqslant \varepsilon'^{(j+2)/2}$ and $M\varepsilon' < 1$, one obtains

$$|f_i(u_{j+1}, v_{j+1}) - a_i|_{F_0} \leqslant \max_{|(l,m)|\geqslant 2} (M\varepsilon')^{|(l,m)|-1} \cdot (M\varepsilon'^{(j+2)/2}) \leqslant M\varepsilon' \cdot M\varepsilon'^{(j+2)/2}.$$

At the same time, $M^2 \cdot \varepsilon'^{1+((j+2)/2)} = ((M^2/d)\varepsilon'^{1/2})\, d\varepsilon'^{(j+3)/2} \leqslant d\varepsilon'^{(j+3)/2}$, which concludes the induction argument.

The second property of the sequences $(u_s)_s, (v_s)_s$ tells us that they are Cauchy (hence convergent) in the complete spaces A_1^n, A_2^n, respectively. Let $u \in A_1^n$ and $v \in A_2^n$ be the corresponding limits. The first property implies that $|(u, v)|_{F_0} \leqslant \varepsilon' < \delta$, so $(u, v) \in \widetilde{S}(F_0)$ and $f(u, v)$ is well defined. Lastly, the third property implies that $f(u, v) = a$. □

From this point on, Theorem 1.7 can be proven in the same way as in [HHK09, Theorem 3.2]. Since $A_i \hookrightarrow F_i$, $i = 1, 2$, we immediately obtain the following corollary.

Corollary 1.10. *Let G be a connected linear algebraic group over F that is a rational variety over F. For any $g \in G(F_0)$, there exist $g_i \in G(F_i)$, $i = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(F_0)$.*

2. Retracting covers

In the first section, we mentioned that the most important example of Setting 1 was the one given by Proposition 1.4. This should serve as motivation for the following.

Definition 2.1. A finite cover $\mathcal{U}$ of a k-analytic curve will be called *nice* if:

(1) the elements of $\mathcal{U}$ are connected affinoid domains with only type 3 points in their topological boundaries;
(2) for any different $U, V \in \mathcal{U}$, $U \cap V = \partial U \cap \partial V$;
(3) for any two different elements of $\mathcal{U}$, neither is contained in the other.

We recall once again that we will use the term *boundary* for the topological boundary.

The purpose of this section is to prove that, under certain conditions, for any open cover of a k-analytic curve, there exists a *nice refinement*, i.e. a refinement that is a nice cover of the curve. The main goal is to be able to apply Corollary 1.10 to any open cover.

Definition 2.2. Let $P \in k[T]$ be any irreducible polynomial. We will denote by $\eta_{P,0}$ the only (type 1) point of $\mathbb{A}_k^{1,\mathrm{an}}$ for which $|P| = 0$. For $s \in \mathbb{R}_{>0}$, we will denote by $\eta_{P,s}$ the point of $\mathbb{A}_k^{1,\mathrm{an}}$ that is the Shilov boundary of the affinoid domain $\{|P| \leqslant s\} \subseteq \mathbb{A}_k^{1,\mathrm{an}}$.

PROPOSITION 2.3. *For any point $\eta \in \mathbb{A}_k^{1,\mathrm{an}}$ of type 2 or 3, there exist an irreducible polynomial $P \in k[T]$ and $r \in \mathbb{R}_{>0}$ such that $\eta = \eta_{P,r}$. Then $|P|_\eta = r$ and:*

1. *$r \in \sqrt{|k^\times|}$ if and only if η is a type 2 point;*
2. *$r \notin \sqrt{|k^\times|}$ if and only if η is a type 3 point, in which case η is the only element of $\mathbb{A}_k^{1,\mathrm{an}}$ for which $|P| = r$.*

Proof. We recall that the projective line $\mathbb{P}_k^{1,\mathrm{an}}$ is uniquely path connected and can be obtained by adding a rational point ∞ to $\mathbb{A}_k^{1,\mathrm{an}}$. For any two points $a, b \in \mathbb{P}_k^{1,\mathrm{an}}$, we denote by $[a, b]$ the unique path connecting them.

Let A be a connected component of $\mathbb{P}_k^{1,\mathrm{an}} \backslash \{\eta\}$ that does not contain ∞. In particular, $A \subseteq \mathbb{A}_k^{1,\mathrm{an}}$. Let η_0 be any rigid point of A. There exists a unique irreducible polynomial $P \in k[T]$ such that $\eta_0 = \eta_{P,0}$. Then $\eta \in [\eta_{P,0}, \infty]$.

Let φ be the finite morphism $\mathbb{P}_k^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}$ determined by the map $k[T] \to k[T]$, $T \mapsto P(T)$. Seeing as $\varphi(\eta_{P,0}) = \eta_{T,0}$ and $\varphi(\infty) = \infty$, $[\eta_{P,0}, \infty]$ is mapped by φ to $[\eta_{T,0}, \infty]$. Set $\eta' = \varphi(\eta)$. The path connecting $\eta_{T,0}$ to ∞ in $\mathbb{P}_k^{1,\mathrm{an}}$ is $\{\eta_{T,s} : s \in \mathbb{R}_{\geqslant 0}\} \cup \{\infty\}$. For any $s \geqslant 0$, $|T|_{\eta_{T,s}} = s$ and, if $\eta_{T,s}$ is a type 3 point, then it is the only one in $\mathbb{P}_k^{1,\mathrm{an}}$ for which $|T| = s$. Furthermore, $\eta_{T,s}$ is a type 2 (respectively type 3) point if and only if $s \in \sqrt{|k^\times|}$ (respectively $s \notin \sqrt{|k^\times|}$).

Thus, there exists $r > 0$ such that $\eta' = \eta_{T,r}$. Since $\varphi(\eta) = \eta_{T,r}$, by construction, $\eta = \eta_{P,r}$ and $|P|_{\eta_{P,r}} = r$. Seeing as a finite morphism preserves the type of the point (i.e. $\eta_{T,r}$ is a type 2 (respectively type 3) point if and only if $\eta_{P,r}$ is so), we obtain (1) and the first part of (2).

To prove the second part of (2), we need to show that if $r \notin \sqrt{|k^\times|}$, $\eta_{P,r}$ is the only point in $\mathbb{A}_k^{1,\mathrm{an}}$ for which $|P| = r$. Since P is irreducible, by [Duc, 3.4.24.3], $|P|$ is strictly increasing in $[\eta_{P,0}, \infty)$ and locally constant elsewhere. Hence, $\eta_{P,r}$ is the only point in $[\eta_{P,0}, \infty)$ for which $|P| = r$ and, since it is a type 3 point (i.e. $\mathbb{A}_k^{1,\mathrm{an}}$ has exactly two connected components), it is the only such point in $\mathbb{A}_k^{1,\mathrm{an}}$. □

Let us recall that we denote by $\partial_B(\cdot/\cdot)$ the Berkovich relative boundary and by $\partial_B(\cdot)$ the boundary relative to the base field k, i.e. $\partial_B(\cdot/\mathcal{M}(k))$ (see [Ber90, Definition 2.5.7] and [Ber93, Definition 1.5.4]).

LEMMA 2.4. *Let V be a k-affinoid curve. The following sets are equal:*

1. *the Berkovich boundary $\partial_B(V)$ of V;*
2. *the Shilov boundary $\Gamma(V)$ of V.*

Proof. If V is strictly affinoid, this is [Váz16, Lemma 2.3]. The proof can be extended to the general case by replacing classical reduction with Temkin's graded reduction (see [Tem04, Propositions 3.3 and 3.4]). □

PROPOSITION 2.5. *Let C be a k-analytic curve such that $\partial_B(C) = \emptyset$. Let V be an affinoid domain of C. The three following sets coincide:*

1. *the topological boundary ∂V of V in C;*
2. *the Berkovich relative boundary $\partial_B(V/C)$ of V in C;*
3. *the Shilov boundary $\Gamma(V)$ of V.*

Proof. By [Ber90, Corollary 2.5.13(ii)], $\partial_B(V/C) = \partial V$. By [Ber93, Proposition 1.5.5(ii)], since C is boundaryless, $\partial_B(V/C) = \partial_B(V)$. Finally, in view of Lemma 2.4, $\partial V = \partial_B(V/C) = \Gamma(V)$. □

Let us recall that analytification of an algebraic variety is boundaryless. In particular, projective k-analytic curves are boundaryless.

Until the end of this section, suppose that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, so that there exist type 3 points in $\mathbb{P}_k^{1,\mathrm{an}}$.

Theorem 2.6. *Let C be a k-analytic curve. The family of connected affinoid domains with only type 3 points in their topological boundaries forms a basis of neighborhoods of the Berkovich topology on C.*

Proof. Let $x \in C$. Seeing as any curve is a good Berkovich space (i.e. all points have a neighborhood that is an affinoid domain), we may assume that C is an affinoid domain. Let U be an open neighborhood of x in C. There exists an open neighborhood of x in U given by $\{|f_i| < r_i, |g_j| > s_j : i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$, where f_i, g_j are analytic functions on C and $r_i, s_j \in \mathbb{R}_{>0}$.

Let $r_i', s_j' \in \mathbb{R}_{>0} \backslash \sqrt{|k^\times|}$ be such that $r_i' < r_i, s_j' > s_j$ and $|f_i(x)| < r_i', |g_j(x)| > s_j'$ for all i and j. Set $V = \{|f_i| \leqslant r_i', |g_j| \geqslant s_j'\}$. It is an affinoid domain of C and a neighborhood of x contained in U.

As $\{|f_i| < r_i', |g_j'| > s_j\}$ is open, it is contained in $\mathrm{Int}(V)$, so $\partial V \subseteq \bigcup_{i=1}^n \{|f_i| = r_i'\} \cup \bigcup_{j=1}^m \{|g_j| = s_j'\}$. Let $y \in \bigcup_{i=1}^n \{|f_i| = r_i'\} \cup \bigcup_{j=1}^m \{|g_j| = s_j'\}$. Since there exists an analytic function f on C such that $|f(y)| \notin \sqrt{|k^\times|}$, the point y is of type 3, implying that the boundary of V contains only type 3 points. $\square$

2.1 The Case of $\mathbb{P}_k^{1,\mathrm{an}}$

Recall that $\mathbb{P}_k^{1,\mathrm{an}}$ is uniquely path-connected. For any $x, y \in \mathbb{P}_k^{1,\mathrm{an}}$, let us denote by $[x, y]$ the unique injective path connecting them. The next few properties of the projective line will be essential to the remainder of this section.

Lemma 2.7. *Let $A \subseteq \mathbb{P}_k^{1,\mathrm{an}}$. Then A is connected if and only if for any $x, y \in A$, $[x, y] \subseteq A$. Furthermore, the intersection of any two connected subsets of $\mathbb{P}_k^{1,\mathrm{an}}$ is connected.*

Lemma 2.8. *Let U, V be two non-disjoint connected affinoid domains of $\mathbb{P}_k^{1,\mathrm{an}}$ such that they have disjoint interiors. Then $U \cap V$ is a single point.*

Proof. Since $U \cap V = \partial U \cap \partial V$, it is a finite set of points. At the same time, by Lemma 2.7, $U \cap V$ is connected, so it must be a single point. $\square$

Lemma 2.9. *Let U be an affinoid domain of $\mathbb{P}_k^{1,\mathrm{an}}$ with only type 3 points in its boundary. If $\mathrm{Int}(U) \neq \emptyset$, then $(\mathrm{Int}\, U)^c$ is an affinoid domain of $\mathbb{P}_k^{1,\mathrm{an}}$ with only type 3 points in its boundary.*

Proof. Let us show that $\mathrm{Int}\, U$ has only finitely many connected components.

Since U is an affinoid domain, it has a finite number of connected components U_i and, by [Ber90, Corollary 2.2.7], they are all affinoid domains. Furthermore, U_i has only type 3 points in its boundary for all i.

Let $x, y \in \mathrm{Int}\, U_i$. Since U_i is connected, $[x, y] \subseteq U_i$. Let $z \in \partial U_i$. We aim to show that $z \notin [x, y]$, implying $[x, y] \subseteq \mathrm{Int}\, U_i$ and thus the connectedness of $\mathrm{Int}\, U_i$.

By [Duc, Théorème 4.5.4], there exists a neighborhood of z in U_i such that it is a closed virtual annulus and its Berkovich boundary is $\partial_B(V) = \{z, u\}$ for some $u \in U_i$. By shrinking this annulus if necessary, we may assume that $x, y \notin V$. Since V is an affinoid domain in U_i, by [Ber90, Corollary 2.5.13(ii)], the topological boundary $\partial_U V$ of V in U is a subset of $\partial_B(V) = \{z, u\}$. Since V is a neighborhood of z, $\partial_U V = \{u\}$.

Suppose that $z \in [x, y]$. Then we could decompose $[x, y] = [x, z] \cup [z, y]$. Since $x, y \notin V$ and $z \in V$, the sets $[x, z] \cap \partial_U V$, $[z, y] \cap \partial_U V$ are non-empty, thus implying that u is contained in both $[x, z]$ and $[z, y]$, which contradicts the injectivity of $[x, y]$.

We have shown that $\operatorname{Int} U$ has finitely many connected components, so, by [Duc, Proposition 4.2.14], $(\operatorname{Int} U)^c$ is a closed proper analytic domain of $\mathbb{P}_k^{1,\mathrm{an}}$. By [Duc, Théorème 6.1.3], it is an affinoid domain. □

We can now show a special case of the result we prove in this section.

LEMMA 2.10. *Let C, D be connected affinoid domains of $\mathbb{P}_k^{1,\mathrm{an}}$ with only type 3 points in their boundaries. There exists a nice refinement $\{C_1, \dots, C_n, D\}$ of the cover $\{C, D\}$ of $C \cup D$ such that $C_i \cap C_j = \emptyset$ for any $i \neq j$.*

Proof. If $C = D$, it is straightforward. Otherwise, suppose that $C \not\subseteq D$. By Lemma 2.9, $C \backslash \operatorname{Int} D$ is an affinoid domain. Let $C_1', C_2', \dots, C_m'$ be its connected components. They are mutually disjoint connected affinoid domains with only type 3 points in their boundaries. Furthermore, for any i, the intersection $C_i' \cap D$ is either empty or a single type 3 point, $C_i' \cap C_j' = \emptyset$ for all $i \neq j$, and $\{C_1', C_2', \dots, C_m', D\}$ is a refinement of $\{C, D\}$. For any i, if C_i' is a single point, i.e. $C_i' \subseteq D$, we remove it from $\{C_1', C_2', \dots, C_m'\}$ and, if not, we keep it there. Let $C_1, C_2, \dots, C_n$ be the remaining connected components of $C \backslash \operatorname{Int} D$. Then $\{C_1, C_2, \dots, C_n, D\}$ is a nice refinement of the cover $\{C, D\}$ of $C \cup D$. □

The main result of this section in the case of the projective line is the following generalization.

PROPOSITION 2.11. *For any $n \in \mathbb{N}$, let $\{U_i\}_{i=1}^n$ be a set of affinoid domains of $\mathbb{P}_k^{1,\mathrm{an}}$ with only type 3 points in their boundaries. Set $V_n = \bigcup_{i=1}^n U_i$. Then there exists a nice cover of V_n that refines $\{U_i\}_{i=1}^n$ satisfying the following properties:*

1. *the intersection of any two of its elements is either empty or a single type 3 point;*
2. *if two domains of the refinement intersect, there is no third one that intersects them both.*

Proof. We will use induction on the number of affinoid domains n. For $n = 1$, the statement is trivial. Suppose that the proposition is true for any positive integer smaller than or equal to some $n-1$. Let $\{U_i\}_{i=1}^n$ be affinoid domains of $\mathbb{P}_k^{1,\mathrm{an}}$ with only type 3 points in their boundaries. If they are all of empty interior, i.e. unions of points, then the statement is trivially true. Otherwise, let $i_0 \in \{1, 2, \dots, n\}$ be any index for which U_{i_0} has non-empty interior. To simplify the notation, suppose that $i_0 = n$. By removing the U_i contained in U_n if necessary, we may assume that for all i, $U_i \not\subseteq U_n$.

From Lemmas 2.9 and 2.10, $\mathcal{U} = \{U_n\} \cup \{U_i \cap (\operatorname{Int} U_n)^c\}_{i=1}^{n-1}$ is a refinement of $\{U_i\}_{i=1}^n$ containing affinoid domains with only type 3 points in their boundaries. Let $\{W_l\}_{l=1}^s$ be a nice refinement of $\{U_i \cap (\operatorname{Int} U_n)^c\}_{i=1}^{n-1}$. Then, for any l, $U_n \cap W_l \subseteq \partial U_n$. By removing those W_l for which $W_l \subseteq U_n$ if necessary, we obtain that $\{U_n\} \cup \{W_l\}_{l=1}^s$ is a nice refinement of $\{U_i\}_{i=1}^n$. The first condition of the statement is a direct consequence of Lemma 2.8.

We have proven that for any positive integer n, there exists a nice refinement of $\{U_i\}_{i=1}^n$ which satisfies the first property of the statement. Property 2 is immediate from the following.

LEMMA 2.12. *Let W_1, W_2, W_3 be three connected affinoid domains of $\mathbb{P}_k^{1,\mathrm{an}}$ with non-empty interiors and only type 3 points in their boundaries. Suppose that their interiors are mutually disjoint. Then at least one of $W_1 \cap W_2, W_2 \cap W_3, W_3 \cap W_1$ is empty.*

Proof. Suppose that $W_1 \cap W_2, W_2 \cap W_3$, and $W_3 \cap W_1$ are all non-empty. If $W_1 \cap W_2 \cap W_3 \neq \emptyset$, then it is a single type 3 point $\{z\}$. Since $\mathbb{P}_k^{1,\mathrm{an}} \backslash \{z\}$ has exactly two connected components and the interiors of W_1, W_2, W_3 are non-empty and mutually disjoint, this is impossible. Hence, $W_1 \cap W_2 \cap W_3 = \emptyset$ and so $W_1 \cap W_2, W_2 \cap W_3$, and $W_3 \cap W_1$ are all non-empty and different. Since $W_1 \cap W_2 \neq \emptyset$, $W_1 \cup W_2$ is a connected affinoid domain with only type 3 points in its boundary. Furthermore, $\mathrm{Int}(W_3) \cap \mathrm{Int}(W_1 \cup W_2) \subseteq (W_3 \cap W_1) \cup (W_3 \cap W_2)$ and, since this is a finite set of type 3 points, $\mathrm{Int}(W_3) \cap \mathrm{Int}(W_1 \cup W_2) = \emptyset$.

Thus, the interior of $W_1 \cup W_2$ is disjoint to the interior of W_3. By Lemma 2.8, $(W_1 \cup W_2) \cap W_3$ is a single type 3 point. But $W_1 \cap W_3$ and $W_2 \cap W_3$ were both assumed to be non-empty and shown to be different, implying that $(W_1 \cap W_3) \cup (W_2 \cap W_3) = (W_1 \cup W_2) \cap W_3$ contains at least two different points, which is a contradiction.

Thus, at least one of $W_1 \cap W_2, W_2 \cap W_3$, $W_3 \cap W_1$ must be empty. □

This completes the proof of the proposition. □

In view of Theorem 2.6, we obtain the following result.

Theorem 2.13. *Any open cover of a compact subset of* $\mathbb{P}_k^{1,\mathrm{an}}$ *has a nice refinement.*

The following will be needed later.

Lemma 2.14. *Let A be a connected affinoid domain of $\mathbb{P}_k^{1,\mathrm{an}}$. Let S be a finite subset of* $\mathrm{Int}(A)$ *containing only type 3 points. There exists a nice cover $\mathcal{A}$ of A such that the set of points of intersection of different elements of $\mathcal{A}$ is S.*

Proof. Seeing as S consists of type 3 points, they are all contained in a copy of $\mathbb{A}_k^{1,\mathrm{an}}$ in $\mathbb{P}_k^{1,\mathrm{an}}$. Thus, for any element $\eta \in S$, there exist an irreducible polynomial P over k and a real number $r \notin \sqrt{|k^\times|}$ such that $\eta = \eta_{P,r}$.

Let us prove the statement using induction on the cardinality of S. If S is empty, then the statement is trivially true. Suppose we know that the statement is true if the cardinality of S is equal to some $n-1$.

Let us assume that S contains n points. Fix some element $\eta_{P,r} \in S$. Let $\mathcal{U}$ be a nice cover of A that satisfies the properties of the statement for $S' = S \backslash \{\eta_{P,r}\}$. There exists a unique $U \in \mathcal{U}$ such that $\eta_{P,r} \in U$, in which case $\eta_{P,r} \in \mathrm{Int}(U)$. Then $\{U \cap \{|P| \leqslant r\}, U \cap \{|P| \geqslant r\}\} \cup \{V \in \mathcal{U} : V \neq U\}$ is a nice cover that fulfills our requirements. □

2.2 Nice covers of a Berkovich curve

Lemma 2.15. *Let C be a projective irreducible generically quasi-smooth k-analytic curve. There exists a type 3 point η in C such that $C \backslash \{\eta\}$ has exactly two connected components E_1, E_2. Furthermore, $E_1 \cup \{\eta\}, E_2 \cup \{\eta\}$ are affinoid domains of C.*

Proof. Let η_0 be any smooth rigid point in C. By [Duc, Théorème 4.5.4], there exists a virtual disc D ($\subsetneq C$) centered at η_0 such that its boundary is a type 3 point η. Then $C \backslash \{\eta\}$ has two connected components. One of them is $D \backslash \{\eta\}$ and, since $D \neq C$, there has to be at least another one. There cannot be more than one, seeing as the topological boundary ∂D of D consists of a single point.

Let E_1, E_2 be the two connected components of $C \backslash \{\eta\}$ such that $E_1 \cup \{\eta\} = D$. It remains to show that the connected subset $E_2 \cup \{\eta\}$ is an affinoid domain. This follows from [Duc, Proposition 4.2.14 and Théorème 6.1.3]. □

Remark 2.16. In general, $C\backslash\{\eta\}$ has at most two connected components and it might happen that it has exactly one (for example in a Tate curve); see also the remarks made after Lemma 3.7.

Proposition 2.17. *Let C be an integral projective k-algebraic curve. Then there exists a nice cover $\{U_1, U_2\}$ of C^{an} such that $U_1 \cap U_2$ is a single type 3 point.*

Proof. Let $C \to \mathbb{P}^1_k$ be a finite morphism. It induces an embedding of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$. Let K be the separable closure of $k(\mathbb{P}^1_k)$ in $k(C)$. There exists a connected normal projective algebraic curve Y over k such that $k(Y) = K$. Since the field extension $K/k(\mathbb{P}^1_k)$ is separable, the induced morphism $Y \to \mathbb{P}^1_k$ is generically étale, so Y is a generically smooth curve. In particular, this implies that the k-analytic curve Y^{an} is generically quasi-smooth. At the same time, since the extension $k(C)/K$ is purely inseparable, the finite type morphism $C \to Y$ is a homeomorphism. Consequently, by [Ber90, Proposition 3.4.6], its analytification $f : C^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a finite morphism that is a homeomorphism.

By Lemma 2.15, there exists a nice cover $\{U'_1, U'_2\}$ of Y^{an} such that $U'_1 \cap U'_2$ is a single type 3 point. Seeing as f is finite and a homeomorphism, $U_i = f^{-1}(U'_i), i = 1, 2$, is a connected affinoid domain and $U_1 \cap U_2$ is a single type 3 point. □

Notation 1. For a nice cover $\mathcal{U}$ of a k-analytic curve, let us denote by $S_{\mathcal{U}}$ the finite set of type 3 points that are in the intersections of different elements of $\mathcal{U}$.

We remark that for a nice cover $\mathcal{U}$ of a k-analytic curve C, if $s \in S_{\mathcal{U}}$, the set $\{s\}$ is an affinoid domain of C. This is because $\{s\}$ is a connected component of the intersection of two affinoid domains.

The following notion will be needed in the next section.

Definition 2.18. Let C be a k-analytic curve. Let $\mathcal{U}$ be a nice cover of C. A function $T_{\mathcal{U}} : \mathcal{U} \to \{0, 1\}$ will be called *a parity function for* $\mathcal{U}$ if for any different $U', U'' \in \mathcal{U}$ that intersect, $T_{\mathcal{U}}(U') \neq T_{\mathcal{U}}(U'')$.

Lemma 2.19. *For any $n \in \mathbb{N}$, let $U_1, U_2, \ldots, U_n$ be affinoid domains in $\mathbb{P}^{1,\mathrm{an}}_k$ such that $\mathcal{U}_n := \{U_i\}_{i=1}^n$ is a nice cover of $K_n := \bigcup_{i=1}^n U_i$. Then there exists a parity function $T_{\mathcal{U}_n}$ for $\mathcal{U}_n$.*

Proof. It suffices to prove the result under the assumption that K_n is connected. We will use induction on the cardinality n of $\mathcal{U}_n$. If $n = 1$, the statement is trivially true. Suppose it to be true for some $n - 1$.

Lemma 2.20. *Let Z be a topological space. For any positive integer m, let $\{W_i\}_{i=1}^m$ be a set of closed connected subsets of Z. Suppose that $\bigcup_{i=1}^m W_i$ is connected. Then there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $\bigcup_{i \neq i_0} W_i$ is connected.*

Proof. Let $l < m$ be the largest positive integer such that there exist $W_{i_1}, W_{i_2}, \ldots, W_{i_l}$ with $\bigcup_{j=1}^l W_{i_j}$ connected. Set $J = \{1, 2, \ldots, m\}\backslash\{i_1, i_2, \ldots, i_l\}$. If $l < n-1$, then, for any W_p such that $p \in J$, the intersection $W_p \cap \bigcup_{j=1}^l W_{i_j}$ is empty. But this implies that $(\bigcup_{p\in J} W_p) \cap (\bigcup_{j=1}^l W_{i_j}) = \emptyset$, which contradicts the connectedness of $\bigcup_{i=1}^m W_i$. Thus, $l = n - 1$. □

Seeing as $\bigcup_{i=1}^n U_i$ is connected, from Lemma 2.20, there exist $n - 1$ elements of $\mathcal{U}_n$ whose union remains connected. For simplicity of notation, assume them to be the elements of $\mathcal{U}_{n-1} := \{U_1, U_2, \ldots, U_{n-1}\}$. Then $\mathcal{U}_{n-1}$ is a nice cover of $K_{n-1} := \bigcup_{i=1}^{n-1} U_i$. Let $T_{\mathcal{U}_{n-1}}$ be a parity function for $\mathcal{U}_{n-1}$. By Lemma 2.8, $U_n \cap \bigcup_{i=1}^{n-1} U_i$ is a single type 3 point, so U_n intersects exactly one of the elements of $\mathcal{U}_{n-1}$. Suppose it to be U_{n-1}. Define $T_{\mathcal{U}_n}$ as follows:

(1) for any $U \in \mathcal{U}_{n-1}$, $T_{\mathcal{U}_n}(U) = T_{\mathcal{U}_{n-1}}(U)$;

(2) $T_{\mathcal{U}_n}(U_n) = 1 - T_{\mathcal{U}_{n-1}}(U_{n-1})$.

The function $T_{\mathcal{U}_n}$ is a parity function for $\mathcal{U}_n$. □

Proposition 2.21. *Let Y, Z be k-affinoid curves. Let $f : Z \to Y$ be a finite surjective morphism. Suppose that $\mathcal{V}$ is a nice cover of Y. Then the connected components of $f^{-1}(V), V \in \mathcal{V}$ form a nice cover $\mathcal{U}$ of Z such that $f^{-1}(S_{\mathcal{V}}) = S_{\mathcal{U}}$.*

Furthermore, if $T_{\mathcal{V}}$ is a parity function for $\mathcal{V}$, then the function $T_{\mathcal{U}}$ that to an element $U \in \mathcal{U}$ associates $T_{\mathcal{V}}(f(U))$ is a parity function for $\mathcal{U}$.

Proof. The connected components of $f^{-1}(V)$ for all $V \in \mathcal{V}$ form a finite cover $\mathcal{U}$ of Z consisting of affinoid domains. By [Ber90, Proposition 2.5.8(iii) and Corollary 2.5.13(i)], for any $V \in \mathcal{V}$, $\partial(f^{-1}(V)) = \partial_B(f^{-1}(V)/Y) \subseteq \partial_B(f^{-1}(V)) = f^{-1}(\partial_B(V))$, so the elements of $\mathcal{U}$ are affinoid domains with only type 3 points in their boundaries. Let $U_1, U_2 \in \mathcal{U}$ be such that $U_1 \cap U_2 \neq \emptyset$. Set $V_i = f(U_i), i = 1, 2$. Then $V_1, V_2 \in \mathcal{V}$ and $V_1 \neq V_2$. To see the second part, if $V_1 = V_2$, then U_1, U_2 would be connected components of $f^{-1}(V_1)$ and thus disjoint, which contradicts the assumption $U_1 \cap U_2 \neq \emptyset$. Seeing as $U_1 \cap U_2 \subseteq f^{-1}(V_1 \cap V_2)$, we obtain that $U_1 \cap U_2$ is a finite set of type 3 points. Hence, $U_1 \cap U_2 = \partial U_1 \cap \partial U_2$. The third condition of a nice cover is trivially satisfied. Since $f^{-1}(\partial V) = \partial f^{-1}(V)$ for all $V \in \mathcal{V}$, it follows that $f^{-1}(S_{\mathcal{V}}) = S_{\mathcal{U}}$. Finally, $T_{\mathcal{U}}(U_1) = T_{\mathcal{V}}(V_1) \neq T_{\mathcal{V}}(V_2) = T_{\mathcal{U}}(U_2)$, so $T_{\mathcal{U}}$ is a parity function for $\mathcal{U}$. □

Corollary 2.22. *Let C be a projective k-analytic curve or a strict k-affinoid curve. Any open cover of C has a nice refinement.*

Proof. By Theorem 2.6, we may assume that the open cover only contains elements with finite boundary consisting of type 3 points. Since C is compact, there is a finite subcover $\mathcal{U}$ of the starting open cover. Set $S = \bigcup_{U \in \mathcal{U}} \partial U$. Suppose that C is a projective k-analytic curve. Then there exists a finite surjective morphism $C \to \mathbb{P}_k^{1,\mathrm{an}}$. Set $S' = f(S)$. By Lemma 2.14, there exists a nice cover $\mathcal{D}$ of $\mathbb{P}_k^{1,\mathrm{an}}$ such that $S_{\mathcal{D}} = S'$. We conclude by applying Proposition 2.21.

If C is a strict k-affinoid curve, by Noether's normalization lemma there exists a finite surjective morphism $C \to \mathbb{D}$, where $\mathbb{D}$ is the closed unit disc in $\mathbb{P}_k^{1,\mathrm{an}}$. We conclude as above. □

3. A local–global principle over Berkovich curves

Unless mentioned otherwise, we assume that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. Following [HHK09], we have the next definition.

Definition 3.1. Let F be a field. A linear algebraic group G over F acts *strongly transitively* on an F-variety X if G acts on X and, for any field extension E/F, either $X(E) = \emptyset$ or the action of $G(E)$ on $X(E)$ is transitive.

We start by showing some patching results over nice covers.

Proposition 3.2. *Let D be $\mathbb{P}_k^{1,\mathrm{an}}$ or a connected affinoid domain of $\mathbb{P}_k^{1,\mathrm{an}}$. Let $\mathcal{D}$ be a nice cover of D and $T_{\mathcal{D}}$ a parity function for $\mathcal{D}$. Let $G/\mathscr{M}(D)$ be a connected rational linear algebraic group. Then, for any $(g_s)_{s \in S_{\mathcal{D}}} \in \prod_{s \in S_{\mathcal{D}}} G(\mathscr{M}(\{s\}))$, there exists $(g_U)_{U \in \mathcal{D}} \in \prod_{U \in \mathcal{D}} G(\mathscr{M}(U))$ satisfying: for any $s \in S_{\mathcal{D}}$, if $U_0, U_1 \in \mathcal{D}$ contain s and $T_{\mathcal{D}}(U_0) = 0$, then $g_s = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(\{s\}))$.*

Proof. Let Z be any effective Cartier divisor on $\mathbb{P}_k^{1,\mathrm{an}}$ so that $H^1(\mathbb{P}_k^{1,\mathrm{an}}, \mathcal{O}(Z)) = 0$. We will use induction on the cardinality n of a nice cover. If $n = 2$, then this is Corollary 1.10. Suppose that the result is true for some $n-1$. If $\mathcal{D} = \{U_1, U_2, \dots, U_n\}$, since $\bigcup_{i=1}^{n} U_i$ is connected, from Lemma 2.20, there exist $n-1$ elements of $\mathcal{U}$ whose union remains connected. For simplicity of notation, suppose them to be the elements of $\mathcal{D}' = \{U_1, U_2, \dots, U_{n-1}\}$. By Lemma 2.8, $\bigcup_{i=1}^{n-1} U_i \cap U_n$ is single type 3 point, so U_n intersects exactly one of the elements of $\mathcal{D}'$. To simplify the notation, suppose it to be U_{n-1}. Set $\{\eta\} = U_{n-1} \cap U_n$, so that $S_{\mathcal{D}} = S_{\mathcal{D}'} \cup \{\eta\}$.

Let $(g_s)_{s \in S_{\mathcal{D}}}$ be any element of $\prod_{s \in S_{\mathcal{D}}} G(\mathscr{M}(\{s\}))$. By the induction hypothesis, for $(g_s)_{s \in S_{\mathcal{D}'}} \in \prod_{s \in S_{\mathcal{D}'}} G(\mathscr{M}(\{s\}))$, there exists $(g_U)_{U \in \mathcal{D}'} \in \prod_{U \in \mathcal{D}'} G(\mathscr{M}(U))$ satisfying the conditions of the statement.

- Suppose that $T_{\mathcal{D}}(U_n) = 0$. By Corollary 1.10, there exist $a \in G(\mathscr{M}(U_n))$ and $b \in G(\mathscr{M}(\bigcup_{i=1}^{n-1} U_i))$ such that $g_\eta \cdot g_{U_{n-1}} = a \cdot b$ in $G(\mathscr{M}(\{\eta\}))$. For any $i \neq n$, set $g'_{U_i} = g_{U_i} \cdot b^{-1}$ in $G(\mathscr{M}(U_i))$. Also, set $g'_{U_n} = a$ in $G(\mathscr{M}(U_n))$.
- Suppose that $T_{\mathcal{D}}(U_n) = 1$. By Corollary 1.10, there exist $c \in G(\mathscr{M}(\bigcup_{i=1}^{n-1} U_i))$ and $d \in G(\mathscr{M}(U_n))$ such that $g_{U_{n-1}}^{-1} \cdot g_\eta = c \cdot d$ in $G(\mathscr{M}(\{\eta\}))$. For any $i \neq n$, set $g'_{U_i} = g_{U_i} \cdot c$ in $G(\mathscr{M}(U_i))$. Also, set $g'_{U_n} = d^{-1}$ in $G(\mathscr{M}(U_n))$.

The family $(g'_{U_i})_{i=1}^{n} \in \prod_{i=1}^{n} G(\mathscr{M}(U_i))$ satisfies the conditions of the statement for $(g_s)_{s \in S_{\mathcal{D}}}$. ☐

Proposition 3.3. *Let Y be an integral strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let G/K be a connected rational linear algebraic group. For any open cover $\mathcal{V}$ of Y, there exists a nice refinement $\mathcal{U}$ of $\mathcal{V}$ with a parity function $T_{\mathcal{U}}$ such that for any given $(g_y)_{y \in S_{\mathcal{U}}} \in \prod_{y \in S_{\mathcal{U}}} G(\mathscr{M}\{y\})$, there exists $(g_U)_{U \in \mathcal{U}} \in \prod_{U \in \mathcal{U}} G(\mathscr{M}(U))$ satisfying: for any $y \in S_{\mathcal{U}}$, if U', U'' are the elements of $\mathcal{U}$ containing y and $T_{\mathcal{U}}(U') = 0$, then $g_y = g_{U'} \cdot g_{U''}^{-1}$ in $G(\mathscr{M}\{y\})$.*

Proof. By Theorem 2.6, we may assume that the cover $\mathcal{V}$ only contains elements with finite boundary consisting of only type 3 points. Since Y is compact, we may also assume that $\mathcal{V}$ is finite.

Let $f : Y \to \mathbb{D}$ be a finite surjective morphism we obtain from Noether's normalization lemma, where $\mathbb{D}$ is the closed unit disc in $\mathbb{P}_k^{1,\mathrm{an}}$. Set $S = f(\bigcup_{V \in \mathcal{V}} \partial V)$. It is a finite set of type 3 points. By Lemma 2.14, there exists a nice cover $\mathcal{D}$ of $\mathbb{D}$ such that $S_{\mathcal{D}} = S$. Let $T_{\mathcal{D}}$ be a parity function for $\mathcal{D}$ (it exists by Lemma 2.19). From Proposition 2.21, the connected components of $f^{-1}(Z')$ for all $Z' \in \mathcal{D}$ form a nice cover $\mathcal{U}$ of Y such that $f^{-1}(S_{\mathcal{D}}) = S_{\mathcal{U}}$ and $T_{\mathcal{D}}$ induces a parity function $T_{\mathcal{U}}$ for $\mathcal{U}$.

Let us show that $\mathcal{U}$ refines $\mathcal{V}$. Suppose, by contradiction, that $Z \in \mathcal{U}$ is such that there does not exist an element of $\mathcal{V}$ containing it. Then there must exist some $a \in \bigcup_{V \in \mathcal{V}} \partial V \subseteq S_{\mathcal{U}}$ such that $a \in \mathrm{Int}(Z)$. Since $a \in S_{\mathcal{U}}$, there exists $U \in \mathcal{U}$ such that $a \in \partial U$. But, then, $Z \cap U \neq \partial Z \cap \partial U$, which contradicts the fact that $\mathcal{U}$ is a nice cover of Y. Consequently, $\mathcal{U}$ must refine $\mathcal{V}$.

Set $G' = \mathcal{R}_{K/\mathscr{M}(\mathbb{D})}(G)$: the restriction of scalars from K to $\mathscr{M}(\mathbb{D})$ of G. It is still a connected rational linear algebraic group (see [Mil72, § 1]).

Lemma 3.4. *For any point s of type 3 in $\mathbb{D}$, $\mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y) = \prod_{x \in f^{-1}(s)} \mathscr{M}(\{x\})$.*

Proof. Seeing as s is a type 3 point, the set $f^{-1}(s)$ is finite consisting of only type 3 points. Hence, $\mathcal{O}(\{s\}) = \mathscr{M}(\{s\})$ and $\mathcal{O}(\{x\}) = \mathscr{M}(\{x\})$ for all $x \in f^{-1}(s)$.

Set $A = \mathcal{O}(\mathbb{D}), B = \mathcal{O}(Y)$, and $C = \mathcal{O}(\{s\})$. Let us denote by S the set of non-zero elements of A. We know that $C \otimes_A B = \prod_{x \in f^{-1}(s)} \mathcal{O}(\{x\}) = \prod_{x \in f^{-1}(s)} \mathscr{M}(\{x\})$. Then, localizing on both sides, we obtain $S^{-1}(C \otimes_A B) = C \otimes_{S^{-1}A} S^{-1}B$ and $S^{-1}(\prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})) = \prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})$. Since B is a finite A-module, $S^{-1}B$ is a domain that is a finite-dimensional $S^{-1}A$-vector space. Then, for any $b \in B\backslash\{0\}$, the map $S^{-1}B \to S^{-1}B, \alpha \mapsto b\alpha$ is injective and so surjective, so there exists $b' \in S^{-1}B$ such that $bb' = 1$ and thus $S^{-1}B = \operatorname{Frac} B$. Consequently, $S^{-1}(C \otimes_A B) = \mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y)$. □

Recall that for any $s \in S_{\mathcal{D}}$, one obtains $G'(\mathscr{M}(\{s\})) = G(\mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y))$. By the lemma above, $G'(\mathscr{M}(\{s\})) = \prod_{x\in f^{-1}(s)} G(\mathscr{M}(\{x\}))$.

Consequently, $(g_y)_{y\in S_{\mathcal{U}}} \in \prod_{y\in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$ determines uniquely an element $(h_s)_{s\in S_{\mathcal{D}}}$ of $\prod_{s\in S_{\mathcal{D}}} G'(\mathscr{M}(\{s\}))$. By Proposition 3.2, there exists $(h_Z)_{Z\in\mathcal{D}} \in \prod_{Z\in\mathcal{D}} G'(\mathscr{M}(Z))$ such that if for two different $Z_0, Z_1 \in \mathcal{D}$ with $T_{\mathcal{D}}(Z_0) = 0$, $s \in Z_0 \cap Z_1$, then $h_s = h_{Z_0} \cdot h_{Z_1}^{-1}$ in $G'(\mathscr{M}(\{s\}))$.

For any $Z \in \mathcal{D}$, let $Z_1, Z_2, \ldots, Z_r$ be the connected components of $f^{-1}(Z)$. The application $\mathscr{M}(Z) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y) \to \prod_{i=1}^r \mathscr{M}(Z_i)$ maps h_Z to an element $(g_{Z_1}, g_{Z_2}, \ldots, g_{Z_r})$ of $\prod_{i=1}^r G(\mathscr{M}(Z_i))$. Thus, for any $U \in \mathcal{U}$, we have an element $g_U \in G(\mathscr{M}(U))$. It remains to show that given different $U_0, U_1 \in \mathcal{U}$ with $T_{\mathcal{U}}(U_0) = 0$ such that $y \in U_0 \cap U_1$ for some $y \in S_{\mathcal{U}}$, we have $g_y = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(\{y\}))$. This is a consequence of the relation between $T_{\mathcal{D}}$ and $T_{\mathcal{U}}$ and of the commutativity of the following diagram for any $Z \in \mathcal{D}$ and any $s \in Z$ of type 3.

$$\begin{tikzcd}
\mathscr{M}(Z) \arrow[r] \arrow[d] & \mathscr{M}_{\mathbb{D}}(\{s\}) \arrow[d] \\
\prod_{i=1}^{r} \mathscr{M}(Z_i) \arrow[r] & \prod_{y\in f^{-1}(s)} \mathscr{M}_Y(\{y\})
\end{tikzcd}$$

□

Proposition 3.5. *Let Y be a normal irreducible strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local–global principles hold:*

- *$X(K) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

Proof. Since Y is irreducible and normal, $\mathcal{O}_x$ is a domain for all $x \in Y$ and $\mathscr{M}_x = \operatorname{Frac} \mathcal{O}_x$.

Seeing as $K \hookrightarrow \mathscr{M}_x$ for all $x \in Y$, the implication '⇒' is true.

Suppose that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$. Then there exists an open cover $\mathcal{V}$ of C such that for any $V \in \mathcal{V}$, $X(\mathscr{M}(V)) \neq \emptyset$. Let $\mathcal{U}$ be a nice refinement of $\mathcal{V}$ given by Proposition 3.3 and $T_{\mathcal{U}}$ its associated parity function. We remark that for any $U \in \mathcal{U}$, we have $X(\mathscr{M}(U)) \neq \emptyset$.

For $U \in \mathcal{U}$, let $x_U \in X(\mathscr{M}(U))$. For any $y \in S_{\mathcal{U}}$, there exists exactly one element $U_i \in \mathcal{U}$ with $T_{\mathcal{U}}(U_i) = i$, $i = 0, 1$, containing it. From the transitivity of the action of G, there exists $g_y \in G(\mathscr{M}(\{y\}))$ such that $x_{U_0} = g_y \cdot x_{U_1}$ in $G(\mathscr{M}(\{y\}))$. This gives us an element $(g_y)_{y\in S_{\mathcal{U}}} \in \prod_{y\in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$. By Proposition 3.3, there exists $(g_U)_{U\in\mathcal{U}} \in \prod_{U\in\mathcal{U}} G(\mathscr{M}(U))$ satisfying: for any different $U', U'' \in \mathcal{U}$ containing some point $y \in S_{\mathcal{U}}$ such that $T_{\mathcal{U}}(U') = 0$ (implying that $T_{\mathcal{U}}(U'') = 1$), $g_y = g_{U'} \cdot g_{U''}^{-1}$ in $G(\mathscr{M}\{y\})$.

For any $U \in \mathcal{U}$, set $x'_U = g_U^{-1} \cdot x_U \in X(\mathscr{M}(U))$. We have constructed a meromorphic function over U for any $U \in \mathcal{U}$. Let us show that they are compatible, i.e. that they coincide on the intersections of the elements of $\mathcal{U}$. Let $D, E \in \mathcal{U}$ be such that $D \cap E \neq \emptyset$. Suppose that $T_{\mathcal{U}}(D) = 0$. For any $s \in D \cap E$, $x'_E = g_E^{-1} \cdot x_E = g_D^{-1}(g_D g_E^{-1}) \cdot x_E = g_D^{-1} g_s \cdot x_E = g_D^{-1} x_D = x'_D$ in $X(\mathscr{M}(\{s\}))$. Consequently, $x'_E = x'_D$ in $X(\mathscr{M}(E \cap D))$.

Compatibility of these meromorphic functions implies that they can be patched to give a meromorphic function on the entire Y. Thus, $X(K) = X(\mathscr{M}(Y)) \neq \emptyset$.

The second version of this local–global principle is a direct consequence of the first one. $\square$

Let us show the same result (Theorem 3.9) for any k-affinoid space.

Lemma 3.6. *Let E be a k-affinoid space. Let e be any point of E. Then the following statements are equivalent:*

1. *there exists an affinoid neighborhood N_0 of e in E such that $e \in \Gamma(N_0)$;*
2. *for any affinoid neighborhood N of e in E, $e \in \Gamma(N)$;*
3. *$e \in \Gamma(E)$.*

Proof. Suppose that there exists an affinoid neighborhood N_0 of e in E such that $e \in \Gamma(N_0)$. By [Ber90, Proposition 2.5.20], $\Gamma(N_0) \subseteq \partial_B(N_0/E) \cup (\Gamma(E) \cap N_0)$. Since $\partial_B(N/E)$ is the topological boundary of N_0 in E (see [Ber90, Corollary 2.5.13(ii)]), we obtain that $e \notin \partial_B(N_0/E)$, implying $e \in \Gamma(E) \cap N_0 \subseteq \Gamma(E)$.

On the other hand, if $e \in \Gamma(E)$ for any affinoid neighborhood N of e in E, since $\Gamma(E) \cap N \subseteq \Gamma(N)$ (see [Ber90, Proposition 2.5.20]), we obtain $e \in \Gamma(N)$. $\square$

Lemma 3.7. *Let Y be an integral k-affinoid curve. Let $y \in Y$ be any point of type 3 and Z a connected affinoid neighborhood of y in Y. Then:*

1. *the subspace $Y \backslash \{y\}$ has at most two connected components at the neighborhood of y; it is connected at the neighborhood of y if and only if $y \in \Gamma(Y)$;*
2. *if $y \in \Gamma(Y)$, then there exist connected affinoid domains A, B of Y such that A is a neighborhood of y in Z, $\Gamma(Y) \cap A = \{y\}$, $A \cup B = Y$, and $A \cap B$ is a single type 3 point;*
3. *if k is non-trivially valued and $y \notin \Gamma(Y)$, there exists a strict affinoid neighborhood of y in Y.*

Proof. Let p denote the characteristic exponent of k. Then, by [Duc09, Theorem 6.10], there exists n such that $Y' := (Y \times k^{1/p^n})_{\mathrm{red}}$ is geometrically reduced. Since $k^{1/p^n}/k$ is a purely inseparable field extension, the map $f : Y' \to Y$ is a homeomorphism. As Y' is geometrically reduced, the set of its smooth points is a non-empty Zariski-open subset, i.e. the complement of a set of rigid points. Consequently, since $y' := f^{-1}(y)$ is non-rigid, it is smooth in Y'. We remark also that by [Duc, Theorem 4.2.14], the image (respectively preimage) of a connected affinoid domain is a connected analytic domain and thus, by [Duc, Théorème 6.1.3], a connected affinoid domain. Finally, for any affinoid domain U of Y', we have that $\Gamma(U) = f^{-1}(\Gamma(f(U)))$: by [Ber90, Proposition 2.5.8(iii) and Corollary 2.5.13(i)], this is true for finite morphisms and taking the reduction of an affinoid space does not change its Shilov boundary. Set $Z' = f^{-1}(Z)$. It suffices to prove the statement for Y', y', Z'.

(1) By [Duc, Théorème 4.5.4], y' has an affinoid neighborhood A' in Y' with only type 3 points in its boundary that is a virtual annulus. Thus, A' has at most two connected components at the neighborhood of y' and it is connected there if and only if $y' \in \Gamma(A')$.

Finally, Y' has at most two connected components at the neighborhood of y' and, by Lemma 3.6, it is connected there if and only if $y' \in \Gamma(Y')$.

(2) Suppose furthermore that $y' \in \Gamma(Y')$, implying $y' \in \Gamma(A')$. Set $\Gamma(A') = \{y', z'\}$, where z' is a type 3 point. Then $\partial A' = \{z'\}$ and, by [Duc, Proposition 4.3.14 and Théorème 6.1.3], $B' := (Y' \backslash A') \cup \{z'\}$ is an affinoid domain. We have $A' \cup B' = Y'$, $A' \cap B' = \{z'\}$ (which implies that B' is connected). Finally, by shrinking A' if necessary, we can always assume that $z' \notin \Gamma(Y')$ and, since $\Gamma(Y') \cap A' \subseteq \Gamma(A')$, this implies that $\Gamma(Y') \cap A' = \{y'\}$.

(3) If $y' \notin \Gamma(Y')$, then $y' \notin \Gamma(A')$ and, for the non-trivially valued field k^{1/p^n}, the statement follows from the fact that A' is a closed virtual annulus. □

By the terminology introduced in [Duc, § 1.7 and Théorème 3.5.1], the first part of Lemma 3.7 shows that points of type 3 of certain k-analytic curves have at most two branches. Furthermore, in view of Lemma 2.4 and [Ber93, Proposition 1.5.5(ii)], a type 3 point has one branch if and only if it is in the Berkovich boundary of the curve.

The following argument will be used often in what is to come.

Lemma 3.8. *Let C be a normal irreducible k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X.*

(1) Suppose that $X(\mathscr{M}_x) \neq \emptyset$ for any $x \in C$. Let Z be any affinoid domain of C. Then $G_Z := G \times_F \mathscr{M}(Z)$ is a connected rational linear algebraic group over $\mathscr{M}(Z)$ acting strongly transitively on the $\mathscr{M}(Z)$-variety $X_Z := X \times_F \mathscr{M}(Z)$. Furthermore, $X_Z(\mathscr{M}_{Z,x}) \neq \emptyset$ for any $x \in Z$, where $\mathscr{M}_Z$ is the sheaf of meromorphic functions over Z.

(2) Let U_1, U_2 be connected affinoid domains of C such that $U_1 \cap U_2 = \{s\}$, where s is a type 3 point. If $X(\mathscr{M}(U_i)) \neq \emptyset, i = 1, 2$, then $X(\mathscr{M}(U_1 \cup U_2)) \neq \emptyset$.

Proof. (1) That $G_Z = G \times_F \mathscr{M}(Z)$ is still a connected rational linear algebraic group acting strongly transitively on the variety $X_Z = X \times_F \mathscr{M}(Z)$ is immediate. Also, $\mathscr{M}_x \hookrightarrow \mathscr{M}_{Z,x}$ for any $x \in Z$. Thus, $X(\mathscr{M}_x) \neq \emptyset$ implies that $X(\mathscr{M}_{Z,x}) = X_Z(\mathscr{M}_{Z,x}) \neq \emptyset$ for any $x \in Z$.

(2) Let $x_i \in X(\mathscr{M}(U_i)), i = 1, 2$. By the transitivity of the action of G, there exists $g \in G(\mathscr{M}(\{s\}))$ such that $x_1 = g \cdot x_2$ in $X(\mathscr{M}(\{s\}))$. By Corollary 1.10, there exist $g_i \in G(\mathscr{M}(U_i))$ such that $g = g_1 \cdot g_2$ in $G(\mathscr{M}(\{s\}))$. Thus, $g_1^{-1} \cdot x_1 = g_2 \cdot x_2$ in $X(\mathscr{M}(\{s\}))$. Set $x_1' = g_1^{-1} \cdot x_1$ and $x_2' = g_2 \cdot x_2$. They represent meromorphic functions over U_1 and U_2, respectively, whose restrictions to $U_1 \cap U_2$ are compatible. Thus, they can be patched to give a meromorphic function x over $\mathscr{M}(U_1 \cup U_2)$, where $x \in X(\mathscr{M}(U_1 \cup U_2))$, implying that $X(\mathscr{M}(U_1 \cup U_2)) \neq \emptyset$. □

Theorem 3.9. *Suppose that k is non-trivially valued. Let Y be a normal irreducible k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local–global principles hold:*

- *$X(K) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

Proof. Seeing as $K \hookrightarrow \mathscr{M}_x$ for any $x \in Y$, the direction '$\Rightarrow$' is true.

For the other one, let us use induction on the number n of type 3 points in the Shilov boundary of Y. If $n = 0$, then, by [Ber90, Corollary 2.1.6], Y is a strict k-affinoid curve, in which case the statement has already been proven in Proposition 3.5. Assume that we know the statement for any positive integer not larger than $n - 1, n > 0$.

Suppose that $\Gamma(Y)$ contains n type 3 points. Let $u \in \Gamma(Y)$. Since $X(\mathscr{M}_u) \neq \emptyset$, there exists a connected affinoid neighborhood U_1' of u in Y such that $X(\mathscr{M}(U_1')) \neq \emptyset$. By Lemma 3.7, there exist two connected affinoid domains U_1, U_2 of Y such that U_1 is a neighborhood of u in U_1',

$\Gamma(Y)\cap U_1=\{u\}$, $U_1\cup U_2=Y$, and $U_1\cap U_2=\{s\}$, where s is a type 3 point. Since $U_1\subseteq U_1'$, we obtain $X(\mathscr{M}(U_1'))\subseteq X(\mathscr{M}(U_1))$, so $X(\mathscr{M}(U_1))\neq\emptyset$. Let U_s be a connected strict affinoid neighborhood of s in Y (see Lemma 3.7). Set $Z_i=U_i\cup U_s, i=1,2$. It is an integral affinoid domain. Let us show that $\Gamma(Z_2)$ contains at most $n-1$ type 3 points.

For any $y\in U_s$ of type 3, seeing as $\Gamma(U_s)$ does not contain any type 3 points, $y\notin\Gamma(U_s)$. Taking into account that $\Gamma(Z_i)\cap U_s\subseteq\Gamma(U_s)$, we obtain $y\notin\Gamma(Z_i)$. Similarly, for any $y\in U_i\backslash\Gamma(U_i)$, we have $y\notin\Gamma(Z_i)$. Thus, if z is a type 3 point in the Shilov boundary of Z_i, then $z\in\Gamma(U_i)$. For a subset S of Y, let us denote by S_3 the set of type 3 points contained in S. We have just shown that $\Gamma(Z_i)_3=\Gamma(U_i)_3\backslash\{s\}, i=1,2$. At the same time, $\Gamma(Y)_3$ is a disjoint union of $\Gamma(U_i)_3\backslash\{s\}, i=1,2$. By construction, $u\in\Gamma(U_1)_3\backslash\{s\}$, so the cardinality of $\Gamma(Z_2)_3$ is at most $n-1$.

By the first part of Lemma 3.8, $X_{Z_2}(\mathscr{M}_{Z_2,x})\neq\emptyset$ for any $x\in Z_2$. In view of the paragraph above and the induction hypothesis, $X(\mathscr{M}(Z_2))=X_{Z_2}(\mathscr{M}(Z_2))\neq\emptyset$. Seeing as $\mathscr{M}(Z_2)\subseteq\mathscr{M}(U_2)$, we obtain $X(\mathscr{M}(U_2))\neq\emptyset$. Considering we also have $X(\mathscr{M}(U_1))\neq\emptyset$, we can conclude by applying the second part of Lemma 3.8.

The second version of this local–global principle is a direct consequence of the first one. □

We are now able to prove the following theorem.

THEOREM 3.10. *Let k be a complete valued non-archimedean field such that $\sqrt{|k^\times|}\neq\mathbb{R}_{>0}$. Let C be a normal irreducible projective k-analytic curve. Set $F=\mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local–global principles hold:*

- *$X(F)\neq\emptyset\iff X(\mathscr{M}_x)\neq\emptyset$ for all $x\in C$;*
- *for any open cover $\mathcal{P}$ of C, $X(F)\neq\emptyset\iff X(\mathscr{M}(U))\neq\emptyset$ for all $U\in\mathcal{P}$.*

Proof. Since $F\hookrightarrow\mathscr{M}_x$ for any $x\in C$, the direction '$\Rightarrow$' is true.

Suppose that k is non-trivially valued. By Proposition 2.17, there exists a nice cover $\{Z_1,Z_2\}$ of C such that $Z_1\cap Z_2$ is a single type 3 point. Set $\{\eta\}=Z_1\cap Z_2$. By the first part of Lemma 3.8, G_{Z_i} is a connected rational linear algebraic group acting strongly transitively on the variety X_{Z_i} and $X_{Z_i}(\mathscr{M}_{Z_i,x})\neq\emptyset$ for any $x\in Z_i, i=1,2$. Thus, by Theorem 3.9, $X(\mathscr{M}(Z_i))=X_{Z_i}(\mathscr{M}(Z_i))\neq\emptyset$. We now conclude by the second part of Lemma 3.8.

Suppose that k is trivially valued. Being a projective analytic curve over a trivially valued field, the curve C has exactly one type 2 point x. In that case, $\mathscr{M}_x=F$, so the statement is trivially satisfied.

The second version of this local–global principle is a direct consequence of the first one. □

The condition on the value group of k can be removed using model-theoretic arguments. We are very grateful to Antoine Ducros for bringing this to our attention.

THEOREM 3.11. *Let k be a complete ultrametric field. Let C be an irreducible normal projective k-analytic curve. Set $F=\mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local–global principles hold:*

- *$X(F)\neq\emptyset\iff X(\mathscr{M}_x)\neq\emptyset$ for all $x\in C$;*
- *for any open cover $\mathcal{P}$ of C, $X(F)\neq\emptyset\iff X(\mathscr{M}(U))\neq\emptyset$ for all $U\in\mathcal{P}$.*

Proof. If $\sqrt{|k^\times|}\neq\mathbb{R}_{>0}$, then the statement was already proven in Theorem 3.10. Let us show that we can always reduce to this case.

Since $F \hookrightarrow \mathscr{M}_x$ for all $x \in C$, the direction '$\Rightarrow$' is clear. Assume that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Since C is compact, there exists a finite cover $\mathcal{V}$ of C containing only affinoid domains such that $\{\mathrm{Int}(V) : V \in \mathcal{V}\}$ is also a cover of C and $X(\mathscr{M}(V)) \neq \emptyset$ for all $V \in \mathcal{V}$. Let $x_V \in X(\mathscr{M}(V))$.

Recall that for any V, $\mathscr{M}(V)$ is the fraction field of an algebra of convergent series over k. Hence, C, X, G, the action of G on X, the isomorphism of a Zariski open of G to an open of some $\mathbb{A}^n_F$, and x_V, $V \in \mathcal{V}$, are determined by countably many elements of k. Let $S \subseteq k$ denote a countable subset containing all these elements.

Let k_0 be the prime subfield of k. Let k_1 be the field extension of k_0 generated by S. We remark that k_1 is countable. By [Mar02, Theorem 2.3.7], there exists a subfield k_2 of k that is a countable extension of k_1 such that $k_2 \subseteq k$ is an elementary embedding in the language of valued fields.

Then, by [Mar02, Theorem 2.5.36], there exists a field extension K of k such that $K = k_2^I/D$, where I is an index set and D is a non-principal ultra-filter on I. Furthermore, by [Mar02, Exercise 2.5.22], it is an elementary extension.

Since k_2 is a countable subfield of k, the value group of k_2 with respect to the valuation induced by that of k satisfies $\sqrt{|k_2^\times|} \neq \mathbb{R}_{>0}$. Let k' be the completion of k_2 with respect to this valuation. Then $\sqrt{|k'^\times|} \neq \mathbb{R}_{>0}$.

Since C is defined over k', there exists a compact integral k'-analytic curve C' such that $C' \times_{k'} k = C$. Set $F' = \mathscr{M}(C')$. By construction, there exist an F'-variety X' and a connected rational linear algebraic group G'/F' acting on X' such that $X = X' \times_{F'} F$, $G = G' \times_{F'} F$, and the action of G induced on X is the one given in the statement. Let us show that G' acts strongly transitively on X'. Let L/F' be any field extension such that $X'(L) \neq \emptyset$. Set $L_1 = L^I/D$. This is a field containing F' and k (since $k \subseteq k'^I/D \subseteq L_1$), so it is a field extension of F. Consequently, $G'(L_1) = G(L_1)$ acts transitively on $X'(L_1) = X(L_1)$ and, since, by [Mar02, Exercise 2.5.22], $L \subseteq L_1$ is an elementary embedding, $G'(L)$ acts transitively on $X'(L)$.

For any $V \in \mathcal{V}$, let V' denote the image of V with respect to the projection morphism $C \to C'$. By construction, $X'(\mathscr{M}(V')) \neq \emptyset$. Hence, $X'(\mathscr{M}_x) \neq \emptyset$ for all $x \in C'$, implying that $X'(F') \neq \emptyset$ and thus in particular $X'(F') = X(F') \subseteq X(F) \neq \emptyset$.

The second part of the statement is a direct consequence of the first one. $\square$

We can apply Theorem 3.11 to the projective variety X defined by a quadratic form q over F. In [HHK09, Theorem 4.2], HHK show that for a regular quadratic form q over F, if $\mathrm{char}(F) \neq 2$, $\mathrm{SO}(q)$, the special orthogonal group of q, acts strongly transitively on X when $\dim q \neq 2$, so in that case we can take $G = \mathrm{SO}(q)$. If $\dim q = 2$, then X may not be connected and consequently the group $\mathrm{SO}(q)$ does not necessarily act strongly transitively on X (see [HHK09, Example 4.4] and the proof of [HHK09, Theorem 4.2]).

THEOREM 3.12. *Let k be a complete ultrametric field. Let C be a compact irreducible normal k-analytic curve. If $\sqrt{|k^\times|} = \mathbb{R}_{>0}$ (respectively $|k^\times| = \{1\}$), assume that C is projective (respectively strict). Set $F = \mathscr{M}(C)$. Suppose that $\mathrm{char}(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2.*

(1) *The quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C$.*

(2) *Let $\mathcal{U}$ be an open cover of C. Then q is isotropic over F if and only if it is isotropic over $\mathscr{M}(U)$ for all $U \in \mathcal{U}$.*

Proof. By Witt decomposition, $q = q_t \perp q_r$, where q_r is regular and q_t is totally isotropic. If $q_t \neq 0$, then q is isotropic, so we may assume that q is regular. Consequently, Proposition 3.5 and Theorems 3.9 and 3.11 are applicable according to the paragraph above the statement. $\square$

Because of the relation of Berkovich points to valuations of the function field of a curve, as a result of Theorem 3.11 we will obtain a local–global principle with respect to completions.

DEFINITION 3.13. Let k be a complete ultrametric field. Let F be a field extension of k. For any valuation v on F, we denote by R_v the valuation ring of F with respect to v and m_v its maximal ideal. We denote by F_v the completion of F with respect to v. We use the following notation:

- $V_k(F)$ is the set of all rank 1 valuations v on F that extend the valuation of k;
- $V_0(F)$ is the set of all non-trivial rank 1 valuations on F that when restricted to k are trivial;
- for a k-subalgebra R of F, $R \neq k$, $V'_R(F)$ is the set of valuations $v \in V_0(F)$ such that $R \subseteq R_v$;
- $V(F) := V_k(F) \cup V_0(F)$;
- for a k-subalgebra R of F, $R \neq k$, $V_R(F) := V_k(F) \cup V'_R(F)$.

We remark that if k is trivially valued, then $V(F)$ and $V_R(F)$ contain the trivial valuation on F for any k-subalgebra R of F, $R \neq k$.

Remark 3.14. Let C be a normal irreducible k-analytic curve. Then, for any point $x \in C$, $\mathcal{O}_x$ is either a field or a discrete valuation ring. If $\mathcal{O}_x$ is a field, then $\mathscr{M}_x = \mathcal{O}_x \hookrightarrow \mathcal{H}(x)$, so we endow $\mathscr{M}_x$ with the valuation induced from $\mathcal{H}(x)$. If $\mathcal{O}_x$ is a discrete valuation ring, then we endow $\mathscr{M}_x$ with the corresponding discrete valuation.

PROPOSITION 3.15. *Let k be a non-trivially valued complete ultrametric field. Let C be a normal irreducible k-analytic curve.*

1. *Suppose that there exists an affine curve S over k such that $S^{\mathrm{an}} = C$. Let F denote the function field of S. Then there exists a bijective correspondence $C \longleftrightarrow V_{\mathcal{O}(S)}(F)$.*
2. *If C is projective, set $F = \mathscr{M}(C)$. Then there exists a bijective correspondence $C \longleftrightarrow V(F)$.*

In either case, if to $x \in C$ is associated the valuation v of F, then $\widehat{\mathscr{M}_x} = F_v$, where the completion of $\mathscr{M}_x$ is taken with respect to the valuation introduced in Remark 3.14.

Proof. (1) Let $x \in C$. If x is a non-rigid point, then by definition it is a norm on $A := \mathcal{O}(S)$ extending that of k, so it defines a valuation v_x on F extending that of k. If x is a rigid point, $\mathcal{O}_{C,x}$ is a discrete valuation ring, and $k^\times \subseteq \mathcal{O}_{C,x}^\times$, so the embedding $A \hookrightarrow \mathcal{O}_{C,x}$ induces a discrete valuation on A whose restriction to k is trivial, i.e. a discrete valuation v_x on F whose restriction to k is trivial.

Let us look at the function $C \longrightarrow V_A(F)$, $x \mapsto v_x$. It is injective by the paragraph above. It is also surjective: if $v \in V_k(F)$, then it determines a norm on A that extends that of k, so it corresponds to a non-rigid point of C; if $v \in V'_A(F)$, then $A \subseteq R_v$ and $P := A \cap m_v$ is a prime ideal of A, so it corresponds to a rigid point of C.

If $x \in C$ is non-rigid, then $\widehat{\mathscr{M}_x}$ is $\mathcal{H}(x)$, which is the completion of F with respect to v_x. If x is a rigid point of C and P its corresponding prime ideal in A, then, by [Ber90, Theorem 3.4.1(ii)], $\widehat{\mathcal{O}_{C,x}} = \widehat{A_P} = \widehat{A}$, where $\widehat{A}$ denotes the completion of A with respect to the ideal P. Consequently, $\widehat{\mathscr{M}_x} = \operatorname{Frac} \widehat{A} = F_{v_x}$.

(2) Suppose that C is projective. Let $x \in C$. Let C' be an affine Zariski open of C containing x. Since C is irreducible, the function field of C' is F. Thus, by (1), there exists an injective map $C \longrightarrow V(F)$, $x \mapsto v_x$.

Let us show that it is also surjective. Let $v \in V(F)$ be such that $v_{|k}$ is the valuation on k. Then, by taking any affine Zariski open subset C' of C, we obtain that v corresponds to some non-rigid point of $C' \subseteq C$.

Suppose that $v \in V(F)$ is such that $v_{|k}$ is trivial. Let C^{alg} be the normal irreducible projective k-algebraic curve such that its Berkovich analytification is C. Let us consider an embedding $C^{\mathrm{alg}} \to \mathbb{P}_k^n = \operatorname{Proj} k[x_0, x_1, \ldots, x_n]$. Let $\{U_i := \operatorname{Spec} k[x_j/x_i]_{j\neq i}/I_i\}_{i=1}^n$ be a cover of C^{alg} by standard open sets. Let i_0 be such that $|x_{i_0}|_v \geqslant |x_i|_v$ for all i. Since $|x_i/x_{i_0}|_v \leqslant 1$, $\mathcal{O}(U_{i_0}) \subseteq R_v$, so, by (1), v corresponds to a rigid point of $U_{i_0}^{\mathrm{an}} \subseteq C$.

That $\widehat{\mathscr{M}_x} = F_{v_x}$ for all $x \in C$ follows from part (1) by taking an affine Zariski open containing the point x. $\square$

Let us now show a local–global principle with respect to all such completions of the field F. We are very grateful to the referee for bringing to our attention the following lemma.

Lemma 3.16. *Let K be a complete valued field and K_0 a dense Henselian (called* quasicomplete *in [Ber93, Definition 2.3.1]) subfield. Let F be a subfield of K_0 and X an F-variety. Then, if F is perfect or X is smooth,*

$$X(K_0) \neq \emptyset \iff X(K) \neq \emptyset.$$

Proof. Since K_0 is a subfield of K, the implication '$\Rightarrow$' is clear. Suppose that $X(K) \neq \emptyset$.

Suppose that F is perfect. By taking the reduction of X if necessary, we may assume that X is reduced. Let $a \in X(K)$. Denote by X' the (reduced) Zariski closure of $\{a\}$ in X. Since F is perfect, the smooth locus X'' of X' is a dense Zariski open subset of X' containing a. Thus, X'' is a smooth F-variety such that $X''(K) \neq \emptyset$, implying that it suffices to prove the statement in the case X is smooth.

Suppose that X is smooth. Let $a \in X(K)$. Since X is smooth, there exists a neighborhood U of a in X such that there exists an étale morphism $\varphi : U \to \mathbb{A}_F^d$ for some $d \in \mathbb{N}$. Let $\varphi_K : U_K \to \mathbb{A}_K^d$ be the tensorization by K and let us look at its analytification φ_K^{an}. Since a is a rational point, φ_K^{an} induces an isomorphism between a neighborhood V of x in U_K^{an} and an open V' of $\mathbb{A}_K^{d,\mathrm{an}}$. Since K_0 is dense in K, there exists b in V' such that $b \in \mathbb{A}^d(K) = K^d$ has coordinates over K_0. Let c be the only preimage of b in V. Then c is a K-rational point over b.

$$\begin{tikzcd}
U_K \arrow[r] \arrow[d, "\varphi_K"'] & U_{K_0} \arrow[d, "\varphi_{K_0}"] \\
\mathbb{A}_K^d \arrow[r, "g"'] & \mathbb{A}_{K_0}^d
\end{tikzcd}$$

Set $b' = g(b) \in \mathbb{A}_{K_0}^d$. By commutativity of the diagram, b' is a closed point of $\mathbb{A}_{K_0}^d$ which is in the image of φ_{K_0}.

Since φ is étale, $\varphi_{K_0}^{-1}(b')$ is a disjoint union $\bigsqcup_i \operatorname{Spec} F_i$, where F_i are separable finite field extensions of $\kappa(b') = K_0$. At the same time, $\varphi_K^{-1}(b) = \bigsqcup_i F_i \otimes_{K_0} K$. Set $\widehat{F_i} = F_i \otimes_{K_0} K$.

We know that $\varphi_K^{-1}(b)(K) \neq \emptyset$. Then there exists i such that $(\operatorname{Spec} \widehat{F_i})(K) \neq \emptyset$, so $\widehat{F_i} = K$. By [Ber93, Proposition 2.4.1], this implies that $F_i = K_0$ and so $\varphi_{K_0}^{-1}(b')(K_0) \neq \emptyset$, implying $X(K_0) \neq \emptyset$. $\square$

COROLLARY 3.17. *Let k be a complete ultrametric field. Let C be a normal irreducible k-analytic curve. Set $F = \mathscr{M}(C)$. Let X be an F-variety. Then, if F is perfect or X is smooth,*

$$X(\mathscr{M}_x) \neq \emptyset \iff X(\widehat{\mathscr{M}_x}) \neq \emptyset$$

for all $x \in C$, where the completion $\widehat{\mathscr{M}_x}$ of $\mathscr{M}_x$ is taken with respect to the valuations introduced in Remark 3.14.

Proof. If $\mathcal{O}_x$ is a field, then $\mathscr{M}_x$ is Henselian by [Ber93, Theorem 2.3.3]. If $\mathcal{O}_x$ is not a field, then it is a discrete valuation ring that is Henselian (see [Ber93, Theorem 2.1.5]), so $\mathscr{M}_x$ is Henselian by [Ber93, Proposition 2.4.3]. We conclude by Lemma 3.16. □

Recall once again that an irreducible compact analytic curve is either projective or affinoid (see [Duc, Théorèmes 6.1.3 and 3.7.2]).

COROLLARY 3.18. *Let k be a complete ultrametric valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local–global principles hold if F is perfect or X is smooth:*

(1) *if C is affinoid and $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$,*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ for all } v \in V_{\mathcal{O}(C)}(F);$$

(2) *if C is projective,*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ for all } v \in V(F).$$

Proof. If k is trivially valued, then the trivial valuation v_0 of F is in $V_{\mathcal{O}(C)}(F)$ (respectively $V(F)$) and, since $F_{v_0} = F$, the statement is clear in this case.

Otherwise, it is a consequence of Proposition 3.5 and Theorems 3.9 and 3.11 in view of Proposition 3.15 and Corollary 3.17. □

COROLLARY 3.19. *Let k be a complete non-archimedean valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Suppose that $\operatorname{char}(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2. The following local–global principles hold.*

(1) *If C is affinoid and $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, q is isotropic over F if and only if it is isotropic over all completions $F_v, v \in V_{\mathcal{O}(C)}(F)$, of F.*
(2) *If C is projective, q is isotropic over F if and only if it is isotropic over all completions $F_v, v \in V(F)$, of F.*

Proof. If k is trivially valued, then the trivial valuation v_0 of F is in $V_{\mathcal{O}(C)}(F)$ (respectively $V(F)$) and, since $F_{v_0} = F$, the statement is clear in this case.

Otherwise, by Witt decomposition, $q = q_t \perp q_r$, where q_r is regular and q_t is totally isotropic. If $q_t \neq 0$, then q is isotropic. Otherwise, q is regular, so smooth, and we conclude by Corollary 3.18. □

Remark 3.20. Recall that for any finitely generated field extension F/k of transcendence degree 1, there exists a unique normal projective k-algebraic curve C^{alg} with function field F. Let C be the analytification of C^{alg}. Then $\mathscr{M}(C) = F$ (see [Ber90, Proposition 3.6.2]), so the local–global principles above are applicable to any such field F.

By [HHK09, Corollary 3.8], if G_1 and G_2 are linear algebraic groups such that $G_1 \times G_2$ is a connected rational linear algebraic group, then all the results proven in this section remain true for G_1 and G_2.

4. Comparison of overfields

The purpose of this section is to draw a comparison between the local–global principle we proved (Theorem 3.11) and the one proven in [HHK09, Theorem 3.7]. More precisely, we will interpret what the overfields appearing in [HHK09] represent in the Berkovich setting and show that [HHK09, Theorem 3.7] can be obtained as a consequence of Theorem 3.11. For a 'fine' enough model, we show that the converse is also true.

Throughout this section, for a non-archimedean valued field E, we will denote by E° the ring of integers of E, by $E^{\circ\circ}$ the maximal ideal of E°, and by $\widetilde{E}$ the residue field of E.

Until the end of this section, we assume k to be a complete discretely valued field.

4.1 Analytic generic fiber and the reduction map

We will be using the notion of generic fiber in the sense of Berkovich. To see the construction in more detail and under less constrictive conditions, we refer the reader to [Ber94, § 1] and [Ber96, § 1].

Let $\mathscr{X} = \operatorname{Spec} A$ be a flat finite type scheme over k°. Then the formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along its special fiber is $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is a topologically finitely presented ring over k° (i.e. isomorphic to some $k^\circ\{T_1, \dots, T_n\}/I$, where I is a finitely generated ideal). We remark that $\widehat{A} \otimes_{k^\circ} k$ is a k-affinoid algebra.

The *analytic generic fiber* of $\widehat{\mathscr{X}}$, denoted by $\widehat{\mathscr{X}_\eta}$, is defined to be $\mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. There exists a *reduction map* $\pi : \widehat{\mathscr{X}_\eta} \to \widehat{\mathscr{X}_s}$, where $\widehat{\mathscr{X}_s}$ is the special fiber of $\widehat{\mathscr{X}}$, which is anti-continuous, meaning that the preimage of a closed subset is open. We remark that $\widehat{\mathscr{X}_s} = \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fiber of $\mathscr{X}$. Let us describe π more explicitly.

There are embeddings $A \hookrightarrow \widehat{A} \hookrightarrow (\widehat{A} \otimes_{k^\circ} k)^\circ$, where $(\widehat{A} \otimes_{k^\circ} k)^\circ$ is the set of all elements f of $\widehat{A} \otimes_{k^\circ} k$ for which $|f(x)| \leqslant 1$ for all $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. Let $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. This point then determines a bounded morphism $A \to \mathcal{H}(x)^\circ$, which induces an application $\varphi_x : A \otimes_{k^\circ} \widetilde{k} \to \widetilde{\mathcal{H}(x)}$. The reduction map π sends x to $\ker \varphi_x$.

The following commutative diagram, where $\phi : \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \to \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$ is the canonical map, gives the relation between this reduction map and the one from [Ber90, § 2.4]. The morphism ϕ is finite and dominant (see [BGR84, 6.1.2 and 6.4.3] and [Thu05, p. 17]).

$$\begin{tikzcd}
\mathcal{M}(\widehat{A} \otimes_{k^\circ} k) \arrow[r, "r"] \arrow[dr, "\pi"'] & \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \arrow[d, "\phi"] \\
 & \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})
\end{tikzcd}$$

The construction above has nice gluing properties. Let $\mathscr{X}$ be a finite type scheme over k° and $\widehat{\mathscr{X}}$ its formal completion along the special fiber. Then the *analytic generic fiber* $\widehat{\mathscr{X}_\eta}$ of $\widehat{\mathscr{X}}$ is the k-analytic space we obtain by gluing the analytic generic fibers of an open affine cover of the formal scheme $\widehat{\mathscr{X}}$. In general, $\widehat{\mathscr{X}_\eta}$ is a compact analytic domain of the Berkovich analytification $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$. If $\mathscr{X}$ is proper, then $\mathscr{X}^{\mathrm{an}} = \widehat{\mathscr{X}_\eta}$ (see [MN15, 2.2.2]). Similarly, there exists an anti-continuous *reduction map* $\pi : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fiber of $\mathscr{X}$.

A property we will need is the following.

Proposition 4.1. *With the same notation as above, suppose that A is a normal domain. Then $\widehat{A} = (\widehat{A} \otimes_{k^\circ} k)^\circ$ and the finite morphism ϕ from the diagram above is a bijection.*

Proof. Let us denote by t a uniformizer of k° and by I the ideal $t\widehat{A}$. Set $B = (\widehat{A} \otimes_{k^\circ} k)^\circ$ and $J = (\widehat{A} \otimes_{k^\circ} k)^{\circ\circ}$.

We remark that for any maximal ideal m of A, $t \in m$ (i.e. the closed points of $\operatorname{Spec} A$ are in the special fiber). This means that tA is contained in the Jacobson radical of A. Considering this and the fact that A is excellent and normal, by [EGAIV, 7.8.3.1], $\widehat{A}$ is also normal. At the same time, by [BGR84, 6.1.2 and 6.3.4], B is the integral closure of $\widehat{A}$ in $\widehat{A} \otimes_{k^o} k$. Since $\operatorname{Frac} \widehat{A} = \operatorname{Frac} B$, we obtain $\widehat{A} = B$.

Let us look at the canonical map $A/p = \widehat{A}/I \to B/J$ inducing ϕ. Let $|\cdot|_{\widehat{A}}$ be the norm on the affinoid algebra $\widehat{A}$.

We remark that $\sqrt{I} = J$: let $x \in J$, so that $\lim_{n\to\infty} |x^n|_{\widehat{A}}^{1/n} < 1$, implying that $|x^n|_{\widehat{A}} \to 0$, $n \to +\infty$. Thus, for large enough n, $x^n \in I$, so $J \subseteq \sqrt{I}$. The other containment is clear. This means that any prime ideal of $\widehat{A}$ contains I if and only if it contains J and thus that ϕ is a bijection. □

4.2 The setup

Let us start by recalling HHK's framework (see [HHK09, Notation 3.3]).

Notation 2. Let $T = k^\circ$ be a complete discrete valuation ring with uniformizer t, fraction field k, and residue field $\widetilde{k}$. Let $\mathscr{C}$ be a flat normal irreducible projective T-curve with function field F. Let us denote by $\mathscr{C}_s$ the special fiber of $\mathscr{C}$.

For any point $P \in \mathscr{C}_s$, set $R_P = \mathcal{O}_{\mathscr{C},P}$. Since T is complete discretely valued, R_P is an excellent ring. Let us denote by $\widehat{R_P}$ the completion of R_P with respect to its maximal ideal. Since R_P is normal and excellent, $\widehat{R_P}$ is also a domain. Set $F_P = \operatorname{Frac} \widehat{R_P}$.

Let U be a proper subset of one of the irreducible components of $\mathscr{C}_s$. Set $R_U = \bigcap_{P\in U} \mathcal{O}_{\mathscr{C},P}$. Let us denote by $\widehat{R_U}$ the t-adic completion of R_U. By [HHK09, Notation 3.3], for any $Q \in U$, $\widehat{R_U} \subseteq \widehat{R_Q}$. Thus, $\widehat{R_U}$ is an integral domain. Set $F_U = \operatorname{Frac} \widehat{R_U}$.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{C}_s$ containing all points at which distinct irreducible components of $\mathscr{C}_s$ meet. Let $\mathscr{U}$ be the set of all irreducible components of $\mathscr{C}_s \backslash \mathscr{P}$ (which here are also its connected components).

The following is the local–global principle proven by HHK in [HHK09, HHK15].

THEOREM 4.2 ([HHK09, Theorem 3.7], [HHK15, Theorem 9.1]). *Let G be a connected rational linear algebraic group over F that acts strongly transitively on an F-variety X. The following statements are equivalent:*

(1) $X(F) \neq \emptyset$;
(2) $X(F_P) \neq \emptyset$ *for all* $P \in \mathscr{P}$ *and* $X(F_U) \neq \emptyset$ *for all* $U \in \mathscr{U}$;
(3) $X(F_Q) \neq \emptyset$ *for all* $Q \in \mathscr{C}_s$.

The implication (1) ⇒ (2) is immediate seeing as F is embedded into F_P and F_U for all $P \in \mathscr{P}$ and $U \in \mathscr{U}$. Considering for any $U \in \mathscr{U}$ and any $Q \in U$, $F_U \subseteq F_Q$, we obtain that (2) ⇒ (3).

We now proceed to show that the remaining implication (3) ⇒ (1) is a consequence of Theorem 3.11. To do this, a comparison will be drawn between the fields $F_Q, Q \in \mathscr{C}_s$, and the ones appearing in Theorem 3.11.

4.3 The comparison

Let us denote by C the Berkovich analytification of the generic fiber of $\mathscr{C}$. It is a normal irreducible projective k-analytic curve. By [Ber90, Proposition 3.6.2], $\mathscr{M}(C) = F$, where $\mathscr{M}$ is the sheaf of meromorphic functions on C. Since $\mathscr{C}$ is projective, $C = \widehat{\mathscr{C}_\eta}$. Let $\pi : C \to \mathscr{C}_s$ be the reduction map.

Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Then $\mathcal{O}_{\mathscr{C},\mu}$ is a discrete valuation ring with fraction field F whose valuation extends that of k. Consequently, μ determines a unique type 2 point x_μ on the Berkovich curve C.

Lemma 4.3. *Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Then $\pi^{-1}(\mu) = \{x_\mu\}$.*

Proof. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of μ in $\mathscr{C}$. Since $\mathscr{C}$ is irreducible, we obtain that $\operatorname{Frac} A = F$. The restriction of π on $\widehat{U}_\eta$ is the reduction map $\widehat{U}_\eta \to U_s$. Explicitly, we have $\pi : \mathcal{M}(\widehat{A} \otimes_{k^\circ} k) \to \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$, where $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$ is sent to the kernel of the map $A \otimes_{k^\circ} \widetilde{k} = A/k^{\circ\circ}A \to \widetilde{\mathcal{H}(x)}$.

By construction, for any $x \in \pi^{-1}(\mu)$ and any $f \in A$, $f(\mu) = 0$ if and only if $|f|_x < 1$, and $f(\mu) \neq 0$ if and only if $|f|_x = 1$. As a consequence, $|f|_{x_\mu} < 1$ if and only if $|f|_x < 1$, and $|f|_{x_\mu} = 1$ if and only if $|f|_x = 1$. This implies that x and x_μ define the same norm on A (and hence on F), so $x_\mu = x$ in C and $\pi^{-1}(\mu) = \{x_\mu\}$. $\square$

Proposition 4.4. *Let X be an F-variety. Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Set $\{x_\mu\} = \pi^{-1}(\mu)$. If $X(F_\mu) \neq \emptyset$, then $X(\mathscr{M}_{x_\mu}) \neq \emptyset$.*

Proof. From Lemma 4.3, $F_\mu = \operatorname{Frac} \widehat{\mathcal{O}_{\mathscr{C},\mu}} = \mathcal{H}(x_\mu)$. If X is smooth or F is perfect, we can conclude by Corollary 3.17.

Otherwise, since x_μ is a type 2 point, we have $\mathcal{O}_{C,x_\mu} = \mathscr{M}_{C,x_\mu}$. The restriction morphism of the sheaf of meromorphic functions gives us $\operatorname{Frac} \mathcal{O}_{\mathscr{C},\mu} = F \hookrightarrow \mathcal{O}_{C,x_\mu}$, so there exist embeddings $\mathcal{O}_{\mathscr{C},\mu} \subseteq \mathcal{O}_{C,x_\mu} \subseteq \mathcal{H}(x_\mu)$. Seeing as all elements of $\mathcal{O}_{\mathscr{C},\mu}$ have norm at most one, $R_\mu = \mathcal{O}_{\mathscr{C},\mu} \subseteq \mathcal{O}_{C,x_\mu}^\circ$.

By the proof of [HHK15, Proposition 5.8], $X(F_\mu) \neq \emptyset$ implies that $X(\widehat{R_\mu}) \neq \emptyset$. The ring $R_\mu = \mathcal{O}_{\mathscr{C},\mu}$ is excellent, so, by Artin's approximation theorem [Art69, Theorem 1.10], $X(R_\mu^h) \neq \emptyset$, where R_μ^h denotes the Henselization of the local ring R_μ. Seeing as $\mathcal{O}_{x_\mu}^o$ is Henselian [Ber93, § 2.1], $R_\mu \subseteq R_\mu^h \subseteq \mathcal{O}_{C,x_\mu}^\circ \subseteq \mathscr{M}_{x_\mu}$. Consequently, $X(\mathscr{M}_{x_\mu}) \neq \emptyset$. $\square$

We recall that the reduction map is anti-continuous.

Proposition 4.5. *Let P be a closed point of $\mathscr{C}_s$. Then $\widehat{R_P} = \mathcal{O}_C^\circ(\pi^{-1}(P))$, where $\mathcal{O}^\circ$ is the sheaf of analytic functions f such that $|f|_{\text{sup}} \leqslant 1$. Consequently, if $X(F_P) \neq \emptyset$, then $X(\mathscr{M}(\pi^{-1}(P))) \neq \emptyset$.*

Proof. Let $V = \operatorname{Spec} A$ be an open integral affine neighborhood of P in $\mathscr{C}$. Then $P \in V_s$, where V_s is the special fiber of V. The restriction of π on $\widehat{V}_\eta$ is the reduction map $\widehat{V}_\eta \to V_s$ and $\pi^{-1}(V_s) = \widehat{V}_\eta$ (cf. [Ber94, p. 541]). Thus, $\pi^{-1}(P) \subseteq \widehat{V}_\eta$. Let us come back to the commutative diagram

$$\begin{tikzcd}
\widehat{V}_\eta = \mathcal{M}(\widehat{A}\otimes_{k^\circ} k) \arrow[r, "r"] \arrow[dr, "\pi"] & \mathrm{Spec}(\widetilde{\widehat{A}\otimes_{k^\circ} k}) \arrow[d, "\phi"] \\
 & \mathrm{Spec}(A\otimes_{k^\circ} \widetilde{k}) = V_s
\end{tikzcd}$$

By Proposition 4.1, ϕ is a bijection. Let m_P be the maximal ideal of A corresponding to the point P on the special fiber and $\widehat{m_P}$ the corresponding ideal in $\widehat{A}$, i.e. the completion of m_P along the special fiber. Then $\phi^{-1}(B)$ is a closed point of $\mathrm{Spec}(\widetilde{\widehat{A}\otimes_{k^\circ} k})$ corresponding to the maximal ideal $\widehat{m_P}$ of $(\widehat{A}\otimes_{k^\circ} k)^\circ = \widehat{A}$.

Since $k^{\circ\circ}A \subseteq m_P$, $\widehat{A}^{m_P} = \widehat{\widehat{A}}^{\widehat{m_P}} = \widehat{B}^{\widehat{m_P}}$, where the notation $\widehat{R}^S$ is used for the completion of a ring R with respect to the topology induced by an ideal S.

Since $\widehat{V}_\eta$ is an analytic domain of the analytification V^{an} of V, and V^{an} is reduced, so is $\widehat{V}_\eta$ (see [Duc09, Théorème 3.4]). By [Mar17, Theorem 3.1] (see also [Bos77, Theorem 5.8]),

$$\widehat{B}^{\widehat{m_P}} = \mathcal{O}^\circ_{\widehat{V_\eta}}(r^{-1}(\phi^{-1}(P))) = \mathcal{O}^\circ_C(\pi^{-1}(P)).$$

As a consequence,

$$\widehat{R_P} = \widehat{\mathcal{O}_{\mathscr{C},P}}^{m_P} = \widehat{A}^{m_P} = \widehat{B}^{\widehat{m_P}} = \mathcal{O}^\circ_C(\pi^{-1}(P)).$$

This implies that $F_P = \mathrm{Frac}\,\mathcal{O}^\circ(\pi^{-1}(P)) \hookrightarrow \mathscr{M}(\pi^{-1}(P))$. ☐

We are now able to state and prove the following argument, thus concluding the proof that HHK's local–global principle (Theorem 4.2) can be obtained as a consequence of Theorem 3.11.

Proposition 4.6. *Using the same notation as in Theorem 4.2,* (3) $\Rightarrow$ (1).

Proof. Let x be any point of C.

(1) If $\pi(x) = \mu \in \mathscr{C}_s$ is the generic point of one of the irreducible components of $\mathscr{C}_s$, then, by Proposition 4.4, $X(F_\mu) \neq \emptyset$ implies that $X(\mathscr{M}_x) \neq \emptyset$.

(2) If $\pi(x) = P \in \mathscr{C}_s$ is a closed point, by Proposition 4.5, $F_P \subseteq \mathscr{M}(\pi^{-1}(P))$. Since $x \in \pi^{-1}(P)$, we obtain $\mathscr{M}(\pi^{-1}(P)) \subseteq \mathscr{M}_x$. Hence, $X(F_P) \neq \emptyset$ implies that $X(\mathscr{M}_x) \neq \emptyset$.

Finally, seeing as $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$, by Theorem 3.11, $X(F) \neq \emptyset$. ☐

Lastly, using Ducros' work on semi-stable reduction in the analytic setting (see [Duc], in particular ch. 6), we can say something in the other direction as well.

Proposition 4.7. *Let F be a finitely generated field extension of k of transcendence degree 1. Let C be the normal irreducible projective Berkovich k-analytic curve for which $F = \mathscr{M}(C)$. Let X/F be a variety. Then there exists a flat normal irreducible projective model $\mathscr{C}'$ over T of F such that*

$$X(\mathscr{M}_x) \neq \emptyset \textit{ for all } x \in C \;\Rightarrow\; X(F_P) \neq \emptyset \textit{ for all } P \in \mathscr{C}'_s,$$

where $F_P = \widehat{\mathcal{O}_{\mathscr{C}',P}}$ and $\mathscr{C}'_s$ is the special fiber of $\mathscr{C}'$.

Consequently, a local–global principle with respect to the overfields $F_P, P \in \mathscr{C}'_s$, implies a local–global principle with respect to the $\mathscr{M}_x, x \in C$.

Proof. Suppose that $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Let $\mathcal{U}$ be a finite cover of C such that:

(1) for any $U \in \mathcal{U}$, U is a connected strict affinoid domain in C;
(2) $\bigcup_{U\in\mathcal{U}} \mathrm{Int}(U) = C$;
(3) for any $U \in \mathcal{U}, X(\mathscr{M}(U)) \neq \emptyset$.

Let S be the set of all boundary points of the elements of $\mathcal{U}$. By construction, S is a finite set of type 2 points.

Let us show that S is a *vertex set* of C using [Duc, Théorème 6.3.15] (see [Duc, 6.3.17] for the definition of a vertex set, which is called *ensemble sommital* there). Since C is projective (implying boundaryless) and irreducible, conditions α), β), and γ) of [Duc, Théorème 6.3.15(ii)] are satisfied. Finally, condition δ) is a consequence of the fact that S contains only type 2 points (see [Duc, Commentaire 6.3.16]).

By [Duc, 6.3.23], this implies the existence of an irreducible projective model $\mathscr{C}'$ of F over T with special fiber $\mathscr{C}'_s$ and specialization map $\pi : C \to \mathscr{C}'_s$ such that π induces a bijection between S and the generic points of the irreducible components of $\mathscr{C}'_s$. Furthermore, by [Duc, 6.3.9.1], since k is discretely valued and C reduced, $\mathscr{C}'$ is locally topologically finitely presented. Finally, by [Duc, 6.3.10], since C is normal, the model $\mathscr{C}'$ is flat and normal.

By Proposition 4.5, for any closed point $P \in \mathscr{C}'$, $\widehat{\mathcal{O}_{\mathscr{C}',P}} = \mathcal{O}^o(\pi^{-1}(P))$, where $\mathcal{O}^o$ is the sheaf of holomorphic functions f such that $|f|_{\sup} \leqslant 1$. In particular, we remark that if V is an affinoid domain of C, since all holomorphic functions are bounded on V, we have $\mathcal{O}^o(V) \subseteq \mathcal{O}(V)$. This implies that $\mathrm{Frac}\,\mathcal{O}^o(V) \subseteq \mathscr{M}(V)$. Let $f/g \in \mathscr{M}(V)$ with $f, g \in \mathcal{O}(V)$. Let $\alpha \in k$ be such that $|\alpha f|_{\sup}, |\alpha g|_{\sup} \leqslant 1$ (it suffices to choose α so that $|f|_{\sup}, |g|_{\sup} \leqslant |\alpha^{-1}|$, which is possible seeing as k is non-trivially valued). Then $f/g = \alpha f/\alpha g \in \mathrm{Frac}\,\mathcal{O}^o(V)$, implying that $\mathscr{M}(V) = \mathrm{Frac}\,\mathcal{O}^o(V)$. By construction, there exists $U \in \mathcal{U}$ such that $\pi^{-1}(P) \subseteq U$. In particular, $\mathscr{M}(U) = \mathrm{Frac}\,\mathcal{O}^o(U) \subseteq \mathrm{Frac}(\mathcal{O}^o(\pi^{-1}(P))) = F_P$, so $X(F_P) \neq \emptyset$.

If P is a generic point of $\mathscr{C}'_s$, then $\pi^{-1}(P)$ is a single type 2 point x_P and $\mathscr{M}_{x_P} \subseteq \mathcal{H}(x_P) = F_P$. Thus, $X(F_P) \neq \emptyset$.

Since π is surjective, this implies that $X(F_P) \neq \emptyset$ for all $P \in \mathscr{C}'_s$. □

5. The local part for quadratic forms

In view of the local–global principle we proved for quadratic forms (Theorem 3.12), we now want to find sufficient conditions under which there is local isotropy. To do this, we will need to put further restrictions on the base field. Throughout this section, we will suppose that the dimension of $\sqrt{|k^\times|}$ as a $\mathbb{Q}$-vector space (i.e. the rational rank of $|k^\times|$) is $n \in \mathbb{Z}$. In the special case that $|k^\times|$ is a free $\mathbb{Z}$-module (e.g. if k is a discretely valued field), the sufficient conditions for local isotropy can be refined. The class of such fields is quite broad, especially when it comes to arithmetic questions: if we work over a complete ultrametric base field k satisfying these conditions, then for any k-analytic space and any of its points x, the field $\mathcal{H}(x)$ also satisfies them.

For any valued field E, we denote by E° its ring of integers, by $E^{\circ\circ}$ the corresponding maximal ideal, and by $\widetilde{E}$ its residue field.

For the following two propositions, the case of characteristic 2 can be treated uniformly with the general one. Afterwards, we will restrict to residual characteristic different from 2.

Proposition 5.1. *Let l be a valued field. Suppose that $|l^\times|$ is a free $\mathbb{Z}$-module of finite rank n. Let L be a valued field extension of l. Let q be a non-zero diagonal quadratic form over L. Suppose that for any non-zero coefficient a of q, $|a| \in |l^\times|$. There exists a family Q of at most*

2^n *quadratic forms with coefficients in* $(L^\circ)^\times$ *such that* q *is* L*-isometric to* $\perp_{\sigma\in Q} C_\sigma\cdot\sigma$, *where* $C_\sigma\in L^\times$ *for any* $\sigma\in Q$.

Proof. Let us fix $\pi_1,\pi_2,\ldots,\pi_n\in l^\times$ such that their norms form a basis of the $\mathbb{Z}$-module $|l^\times|$. Set $\mathcal{A}=\{\prod_{i=1}^n \pi_i^{\delta_i}|\delta_i\in\{0,1\}\}$. For any coefficient a of q, let $p_1,p_2,\ldots,p_n\in\mathbb{Z}$ be such that $|a|=\prod_{i=1}^n|\pi_i|^{p_i}$. Then there exist $v_a\in(L^o)^\times$ and $s_a\in\mathcal{A}$ such that $a\equiv v_a s_a \bmod (L^\times)^2$. Consequently, for any $A\in\mathcal{A}$, there exists a diagonal quadratic form σ_A with coefficients in $(L^\circ)^\times$ such that q is L-isometric to $\perp_{A\in\mathcal{A}} A\cdot\sigma_A$. $\square$

The following is the analogue of Proposition 5.1 in a more general case.

Proposition 5.2. *Let* l *be a valued field such that* $\dim_{\mathbb{Q}}\sqrt{|l^\times|}$ *equals an integer* n. *Let* L *be a valued field extension of* l. *Let* q *be a non-zero diagonal quadratic form over* L. *Suppose that for any non-zero coefficient* a *of* q, $|a|\in\sqrt{|l^\times|}$. *Then there exists a family* Q *of at most* 2^{n+1} *quadratic forms with coefficients in* $(L^\circ)^\times$ *such that* q *is* L*-isometric to* $\perp_{\sigma\in Q} C_\sigma\cdot\sigma$, *where* $C_\sigma\in L^\times$ *for any* $\sigma\in Q$.

Proof. To ease the notation, let us start by introducing the following.

Notation 3. Let M be a multiplicative $\mathbb{Z}$-module such that the divisible closure $\sqrt{M}$ of M as a group is a finite-dimensional $\mathbb{Q}$-vector space. Set $n=\dim_{\mathbb{Q}}\sqrt{M}$. Set $M^2=\{m^2 : m\in M\}$.

There exist $t_1,t_2,\ldots,t_n\in M$ such that for any $t\in M$, there exist unique $p_1,p_2,\ldots,p_n\in\mathbb{Q}$ for which $t=\prod_{i=1}^n t_i^{p_i}$. Let us fix such elements $t_1,t_2,\ldots,t_n$.

In the particular situation that is of interest to us, $M=|l^\times|$ and there exist $\pi_1,\pi_2,\ldots,\pi_n\in l$ with $|\pi_i|=t_i$ such that for any $\epsilon\in\sqrt{|l^\times|}$, there exist unique $p_1,p_2,\ldots,p_n\in\mathbb{Q}$ for which $\epsilon=\prod_{i=1}^n|\pi_i|^{p_i}$. Let us fix such elements $\pi_1,\pi_2,\ldots,\pi_n$.

Definition 5.3. Let $\epsilon\in M$. Suppose that $\epsilon=\prod_{i=1}^n t_i^{s_i/r_i}$ for $s_i/r_i\in\mathbb{Q}$ with s_i,r_i coprime, $i=1,2,\ldots,n$.

(1) Let r be the least common multiple of $r_i, i=1,2,\ldots,n$. We will say that r is *the order of* ϵ.

(2) Let $s_i/r_i=s_i'/r$, $i=1,2,\ldots,n$. If there exists i_0 such that $s_{i_0}'=1$, then t_{i_0} will be said to be *a base of* ϵ.

Let $\epsilon\in M$ and suppose that $\epsilon=\prod_{i=1}^n t_i^{p_i}$ for $p_i\in\mathbb{Q}$, $i=1,2,\ldots,n$. Let α be the order of ϵ.

Lemma 5.4. *If* α *is odd, then, for any* $i=1,2,\ldots,n$, *there exist* $\delta_i\in\{0,1\}$ *such that* $\epsilon\equiv\prod_{i=1}^n t_i^{\delta_i} \bmod M^2$.

Proof. We remark that since α is odd, $\epsilon\equiv\epsilon^\alpha \bmod M^2$ and $\epsilon^\alpha=\prod_{i=1}^n t_i^{s_i}$ with $s_i\in\mathbb{Z}$ for all i. Let $s_i=2s_i'+\delta_i$, where $s_i'\in\mathbb{Z}$ and $\delta_i\in\{0,1\}$. Then $\epsilon\equiv\prod_{i=1}^n t_i^{\delta_i} \bmod M^2$. $\square$

Lemma 5.5. *If* α *is even, then there exist* $m\in M$, $x_i,y\in\mathbb{Z}$, $i=1,2,\ldots,n$, *with* $y>0$ *satisfying:*

(1) $\epsilon\equiv m \bmod M^2$;

(2) $m=\prod_{i=1}^n t_i^{x_i/2^y}$;

(3) *there exists* $i_0\in\{1,2,\ldots,n\}$ *such that* $x_{i_0}=1$.

We remark that t_{i_0} is a base of m and its order is 2^y.

Proof. Let $\alpha = 2^y \cdot z$ with z odd and $y > 0$. Then $\epsilon \equiv \epsilon^z \bmod M^2$ and $(\epsilon^z)^{2^y} = \prod_{i=1}^n t_i^{e_i}$ with $e_i \in \mathbb{Z}, i = 1, 2, \ldots, n$. Furthermore, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that e_{i_0} is odd.

Seeing as $(2^y, e_{i_0}) = 1$, there exist $A, B \in \mathbb{Z}$ with A odd such that $Ae_{i_0} + 2^y B = 1$. Then $\epsilon^z \equiv \epsilon^z \cdot (\epsilon^z)^{A-1} \bmod M^2$ and $\epsilon^{zA} = t_{i_0}^{1/2^y - B} \cdot \prod_{i \neq i_0} t_i^{Ae_i/2^y}$. Hence, there exists $m'_B \in M$ such that $\epsilon^{zA} \equiv m'_B \bmod M^2$ and:

- $m'_B = t_{i_0}^{1/2^y} \prod_{i \neq i_0} t_i^{Ae_i/2^y}$ if B is even;
- $m'_B = t_{i_0}^{1/2^y+1} \prod_{i \neq i_0} t_i^{Ae_i/2^y}$ if B is odd.

If B is odd, $m''_B := m'_B \cdot {m'_B}^{2^y} t_{i_0}^{-2-2^y} \equiv m'_B \bmod M^2$ and $m''_B = t_{i_0}^{1/2^y} \prod_{i \neq i_0} t_i^{(Ae_i/2^y)(2^y+1)}$.

Consequently, in either case, there exist $m \in M$ and $x_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$ with $x_{i_0} = 1$ such that $\epsilon \equiv m \bmod M^2$ and $m = \prod_{i=1}^n t_i^{x_i/2^y}$. □

For $\varepsilon \in L$ such that $|\varepsilon| \in \sqrt{|l^\times|}$, we will say that the order of $|\varepsilon|$ is *the order of* ε. If $|\pi_{i_0}|$ is a base of $|\varepsilon|$, we will say that π_{i_0} is *a base of* ε. By applying the last two lemmas to the valued field L, we obtain the following corollary.

Corollary 5.6. *Let $\varepsilon \in L^\times$. Suppose that $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{p_i}$ for $p_i \in \mathbb{Q}, i = 1, 2, \ldots, n$.*

1. *If the order of $|\varepsilon|$ is odd, then, for any $i = 1, 2, \ldots, n$, there exists $\delta_i \in \{0, 1\}$ such that $\varepsilon \equiv \prod_{i=1}^n \pi_i^{\delta_i} \bmod (L^\times)^2 (L^o)^\times$.*
2. *If the order of $|\varepsilon|$ is even, then there exist $\varepsilon' \in L^\times$, $x_i, y \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, with $y > 0$ satisfying:*
   - (a) *$\varepsilon \equiv \varepsilon' \bmod (L^\times)^2 (L^o)^\times$;*
   - (b) *$|\varepsilon'| = \prod_{i=1}^n |\pi_i|^{x_i/2^y}$;*
   - (c) *there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $x_{i_0} = 1$.*

We immediately obtain as a by-product of the proof the following corollary.

Corollary 5.7. *Let $\varepsilon \in L^\times$ be such that $|\varepsilon| \in \sqrt{|l^\times|}$. Suppose the order of $|\varepsilon|$ is 2^ν, so that there exist $\nu_i \in \mathbb{Z}, i = 1, 2, \ldots, n$, such that $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{\nu_i/2^\nu}$. If $\nu_{i'}$ is odd for some i', then there exists $\varepsilon' \in L^\times$ such that $\varepsilon \equiv \varepsilon' \bmod (L^\times)^2 (L^\circ)^\times$ and $|\pi_{i'}|$ is a base of $|\varepsilon'|$.*

Let q_1 (respectively q_2) be the part of q whose coefficients have odd (respectively even) order. We remark that q_1, q_2 are diagonal quadratic forms over L and that $q = q_1 \perp q_2$.

Decomposition of q_1: Set $\mathcal{A} = \{\prod_{i=1}^n \pi_i^{\delta_i} | \delta_i \in \{0, 1\}\}$. Let e be any coefficient of q_1. By Corollary 5.6(1), there exist $u_e \in (L^\circ)^\times$ and $A_e \in \mathcal{A}$ such that $e \equiv u_e \cdot A_e \bmod (L^\times)^2$. Consequently, for any $A \in \mathcal{A}$, there exists a diagonal quadratic form σ_A with coefficients in $(L^\circ)^\times$ such that q_1 is L-isometric to $\perp_{A \in \mathcal{A}} A \cdot \sigma_A$.

Decomposition of q_2: We first need an auxiliary result, which requires the following definition.

Definition 5.8. Let $\varepsilon \in L^\times$ be such that there exist $p_i \in \mathbb{Q}$, $i = 1, 2, \ldots, n$, for which $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{p_i}$. Let $I \subseteq \{0, 1, \ldots, n\}$ be such that $\{i : p_i \neq 0\} \subseteq I$. We will say that ε is *given in* $|I|$ *parameters*, where $|I|$ is the cardinality of I, or that ε is *given in parameters over* I.

Notice that $a \in L$ is given in zero parameters if and only if $a \in (L^\circ)^\times$.

LEMMA 5.9. *Let τ be a diagonal quadratic form over L with coefficients of order either 1 or an even number. Let $I \subseteq \{1, 2, \ldots, n\}$ with $1 \leqslant |I| = m \leqslant n$ be such that the coefficients of τ are given in parameters over I. Then there exist*

- *$J \subseteq I$ with $|J| = m - 1$,*
- *$x_1, x_2 \in L^\times$,*
- *diagonal quadratic forms τ_1, τ_2 over L with coefficients of order either 1 or an even number and in parameters over J*

such that τ is L-isometric to $x_1\tau_1 \perp x_2\tau_2$.

Proof. Roughly, the idea is to find some i_0 and a partition $A_j, j = 1, 2$, of the set of coefficients for which there exist $x_j \in L^\times$ satisfying: if $a \in A_j$, there exists $B_a \in L^\times$ such that, modulo squares, $a = x_j \cdot B_a$ and $|B_a| = \prod_{i \neq i_0} |\pi_i|^{p_{i,a}}, p_{i,a} \in \mathbb{Q}$. In what follows, we find suitable representatives of the coefficients modulo squares, from which we can read the factorization $x_j \cdot B_a$.

Without loss of generality, let us assume that $I = \{1, 2, \ldots, m\}$. Suppose that the coefficients of τ are all of order 1. If they are given in zero parameters, the statement is clear. Otherwise, suppose that there is a coefficient given over a set of parameters containing t_1.

Let d be any coefficient of the quadratic form. There exist $s_i \in \mathbb{Z}, i = 1, 2, \ldots, n$, such that $|d| = \prod_{i=1}^n |\pi_i|^{s_i}$. As a consequence, there exist $d' \in L^\times$ and $s_i' \in \mathbb{Z}, i = 2, \ldots, n$, for which $d \equiv d' \bmod (L^\times)^2(L^\circ)^\times$ and either $|d'| = \prod_{i=2}^n |\pi_i|^{s_i'}$ or $|d'| = |\pi_1| \cdot \prod_{i=2}^n |\pi_i|^{s_i'}$. Hence, there exist diagonal quadratic forms τ_1, τ_2 whose coefficients are all of order 1, in parameters over $\{2, 3, \ldots, m\}$, such that τ is L-isometric to $\pi_1\tau_1 \perp \tau_2$.

Suppose that there exists at least one coefficient of τ of even order. Let τ' be the quadratic form obtained from τ by:

(1) leaving the coefficients of order 1 intact;

(2) applying Corollary 5.6(2) to the coefficients of even order to substitute them by elements of $L^\times$ that satisfy the second and third properties of Corollary 5.6(2).

We remark that due to the proof of Corollary 5.6(2) (i.e. Lemma 5.5), the set of parameters over which the coefficients of τ' are given does not change. The quadratic form τ' is L-isometric to τ. Let us fix a', one of the coefficients of τ' with largest order. Suppose that the order of a' is $2^{\alpha'}$. Without loss of generality, we may assume that π_1 is a base of a'. For $i = 2, \ldots, m$, let $\alpha_i \in \mathbb{Z}$ be such that $|a'| = |\pi_1|^{1/2^{\alpha'}} \cdot \prod_{i=2}^m |\pi_i|^{\alpha_i/2^{\alpha'}}$.

Let c be any other coefficient of τ'. Let π_{i_0} be a base of c and $2^\gamma, \gamma \geqslant 0$, its order. For $i = 1, 2, \ldots, m$, let $\gamma_i \in \mathbb{Z}$ be such that $|c| = \prod_{i=1}^m |\pi_i|^{\gamma_i/2^\gamma}$.

- Suppose that $\alpha' > \gamma$. Set $c' = c \cdot a'^{(2^\gamma - \gamma_1) \cdot 2^{\alpha' - \gamma}}$. Then $c' \equiv c \bmod (L^\times)^2(L^\circ)^\times$ and $|c'| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{(\gamma_i + \alpha_i(2^\gamma - \gamma_1))/2^\gamma}$.
- Suppose that $\alpha' = \gamma$ and γ_1 is odd. By Corollary 5.7, there exist $\alpha_i' \in \mathbb{Z}, i = 2, 3, \ldots, n$, and $c'' \in L^\times$ of order $2^{\alpha'}$, having π_1 as a base, such that $c'' \equiv c \bmod (L^\times)^2(L^\circ)^\times$ and $|c''| = |\pi_1|^{1/2^{\alpha'}} \cdot \prod_{i=2}^m |\pi_i|^{\alpha_i'/2^{\alpha'}}$.
- Suppose that $\alpha' = \gamma$ and γ_1 is even. Let $\gamma_1'/2^\delta$ be the reduced form of $\gamma_1/2^\gamma$, meaning that γ_1' is odd. Set $c''' = c \cdot a'^{(2^\delta - \gamma_1') \cdot 2^{\alpha' - \delta}}$. Then $c''' \equiv c \bmod (L^\times)^2(L^\circ)^\times$ and $|c'''| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{(\gamma_i + \alpha_i(2^\gamma - \gamma_1))/2^\gamma}$.

To summarize, there exist $\bar{c} \in L^\times$ and $\epsilon_2, \ldots, \epsilon_m \in \mathbb{Z}$ such that $c \equiv \bar{c} \bmod (L^\times)^2(L^\circ)^\times$ and either $|\bar{c}| = |\pi_1|^{1/2^\alpha} \cdot \prod_{i=2}^m |\pi_i|^{\epsilon_i/2^{\alpha'}} = |a'| \cdot \prod_{i=2}^m |\pi_i|^{(\epsilon_i - \alpha_i)/2^{\alpha'}}$ or $|\bar{c}| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{\epsilon_i/2^{\alpha'}}$.

Therefore, there exist diagonal quadratic forms τ_1, τ_2 over L such that $\tau \cong \pi_1\tau_1 \perp a'\tau_2$ and, for any coefficient h of τ_1 or τ_2, the order of h is either 1 or an even integer. Furthermore, h is with parameters over $\{2, 3, \dots, m\}$. $\square$

Using induction, an immediate consequence of Lemma 5.9 is that there exists a family T of 2^n quadratic forms with coefficients in $(L^\circ)^\times$ such that τ is L-isometric to $\perp_{\sigma \in T} B_\sigma \cdot \sigma$, where $B_\sigma \in L^\times$ for any $\sigma \in T$.

Finally, by combining the decomposition results of q_1 and q_2, we obtain the statement of Proposition 5.2. $\square$

The following framework corresponds to Berkovich curves.

Setting 2. Let k be a complete ultrametric field. Let $k \subseteq R$ be a Henselian valuation ring with maximal ideal m_R and fraction field $F_R = \operatorname{Frac} R$. Set $L' = R/m_R$ and suppose that it is endowed with a valuation making it a Henselian (called *quasicomplete* in [Ber93]) valued field extension of k. Let L/L' be an immediate Henselian extension. Set $t = \operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}}(|L'^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q})$ and $s = \deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} = \deg \operatorname{tr}_{\widetilde{k}} \widetilde{L'}$. Suppose that $s + t \leqslant 1$.

The motivation behind this setup is the following example.

Example 5.10. Let C be any k-analytic curve and $x \in C$ any point. The hypotheses of the setting above are satisfied for $R = \mathcal{O}_x$, $F_R = \mathscr{M}_x$, $L' = \kappa(x)$, and $L = \mathcal{H}(x)$.

For any quadratic form σ with coefficients in R, let us denote by σ_L (respectively $\sigma_{L'}$) its image over L (respectively L').

We recall the following definition.

Definition 5.11. Let K be a field.

1. [Kaplansky] The *u-invariant* of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exist anisotropic quadratic forms over K of arbitrarily large dimension.
2. [HHK] The strong u-invariant of K, denoted by $u_s(K)$, is the smallest real number m such that:
   - $u(E) \leqslant m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leqslant m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant.

Notation 4. From now on, let k be a complete ultrametric field such that $\dim_{\mathbb{Q}} \sqrt{|k^\times|}$ equals an integer n. Also, suppose that $\operatorname{char} \widetilde{k} \neq 2$.

Proposition 5.12. *Let L/k be a valued field extension such that $\operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ and $\deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} = 0$. Let τ be a quadratic form over L with $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.*

1. *Suppose that L is Henselian. Then τ is isotropic.*
2. *Under the same hypotheses as in Setting 2, let q be a diagonal quadratic form over R such that $q_L = \tau$. Then q is isotropic over F_R.*

Proof. Since $\mathrm{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Seeing as $\dim_{\mathbb{Q}} \sqrt{|L^\times|} = n$, by Proposition 5.2, there exists a set Q of at most 2^{n+1} quadratic forms with coefficients in $(L^\circ)^\times$ such that τ is L-isometric to $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$ with $C_\sigma \in L^\times$ for every $\sigma \in Q$.

Since $\dim \tau > 2^{n+1} u_s(\widetilde{k})$, there exists $\tau' \in Q$ such that $\dim \tau' > u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' over $\widetilde{L}$. Seeing as the coefficients of τ' are all in $(L^\circ)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > u_s(\widetilde{k})$. Since $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 0$, the extension $\widetilde{L}/\widetilde{k}$ is algebraic. Let E be the finite field extension of $\widetilde{k}$ generated by the coefficients of $\widetilde{\tau'}$. Then $u(E) \leqslant u_s(\widetilde{k}) < \dim \widetilde{\tau'}$, implying that $\widetilde{\tau'}$ is isotropic over E and hence over $\widetilde{L}$. Since L is Henselian, τ' is isotropic over L and thus so is τ.

For the second part, if $\tau = q_L$ for some diagonal R-quadratic form q, seeing as $\widetilde{\tau'}$ is isotropic over $\widetilde{L} = \widetilde{L'}$, the image of q in $\widetilde{L'}$ is so as well. From Henselianity of L', we obtain that the image of q in L' is isotropic there. Finally, from Henselianity of R, the quadratic form q is isotropic over F_R. □

The bound $2^{n+1} u_s(\widetilde{k})$ in Proposition 5.12 will remain the same regardless of whether we demand $|k^\times|$ to be a free $\mathbb{Z}$-module or not. The reason behind this is that in any case, the hypotheses of said proposition tell us only that $\dim_{\mathbb{Q}} \sqrt{|L^\times|} = n$, but not necessarily that $|L^\times|$ is a free $\mathbb{Z}$-module.

PROPOSITION 5.13. *Let L/k be a valued field extension such that $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ and $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 1$. Let τ be a quadratic form over L with $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.*

1. *Suppose that L is Henselian. Then τ is isotropic.*
2. *Under the same hypotheses as in Setting 2, let q be a diagonal quadratic form over R such that $q_L = \tau$. Then q is isotropic over F_R.*

If $|L^\times|$ is a free $\mathbb{Z}$-module of dimension n, the statement is true for $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

Proof. Since $\mathrm{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Again, let $\perp_{\sigma\in Q} C_\sigma \sigma$ be the L-quadratic form isometric to τ obtained from Proposition 5.2 (respectively Proposition 5.1), where Q has cardinality at most 2^{n+1} (respectively 2^n). Then there exists $\tau' \in Q$ such that $\dim \tau' > 2u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' over $\widetilde{L}$. Since the coefficients of τ' are all in $(L^o)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > 2u_s(\widetilde{k})$.

As the extension $\widetilde{L}/\widetilde{k}$ is finitely generated of transcendence degree 1, one obtains $u(\widetilde{L}) \leqslant 2u_s(\widetilde{k}) < \dim \tau'$. This implies that τ' is isotropic over $\widetilde{L}$. Since L is Henselian, the quadratic form τ' is isotropic over L and thus so is τ.

For the second part, if $\tau = q_L$ for some diagonal quadratic form q over R, we conclude by using the same argument as in Proposition 5.12, seeing as $\widetilde{\tau'}$ is isotropic over $\widetilde{L'}$. □

PROPOSITION 5.14. *Let L/k be a valued field extension such that $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$ and $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 0$. Let τ be a quadratic form over L with $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.*

1. *Suppose that L is Henselian. Then τ is isotropic.*
2. *Under the same hypotheses as in Setting 2, let q be a diagonal quadratic form over R such that $q_L = \tau$. Then q is isotropic over F_R.*

If $|k^\times|$ is a free $\mathbb{Z}$-module, the statement is true for $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

Proof. Since $\mathrm{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Since $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$, there exists $\rho \in \mathbb{R}_{>0}\backslash\sqrt{|k^\times|}$ such that the group $|L^\times|$ is generated by $|k^\times|$ and ρ. Let T be an element of L with $|T| = \rho$. Then, for any $a \in L^\times$, there exist $m \in \mathbb{Z}$, $p_i \in \mathbb{Q}$ (respectively $p_i \in \mathbb{Z}$), $i = 1, 2, \ldots, n$, such that $|a| = |T|^m \cdot \prod_{i=1}^n |\pi_i|^{p_i}$. Consequently, there exist diagonal quadratic forms q_1, q_2 over L for which τ is isometric to $q_1 \perp Tq_2$, where the coefficients of q_1, q_2 have norms in $|k^\times|$.

By applying Proposition 5.2 (respectively Proposition 5.1) to q_1 and q_2, we obtain a family S of at most 2^{n+2} (respectively 2^{n+1}) diagonal quadratic forms with coefficients in $(L^\circ)^\times$ such that τ is isometric to $\perp_{\sigma\in S} C_\sigma \cdot \sigma$, where $C_\sigma \in L^\times$ for every $\sigma \in S$. Thus, there exists $\tau' \in S$ such that $\dim \tau' > u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' in $\widetilde{L}$. Seeing as the coefficients of τ' are all in $(L^\circ)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > u_s(\widetilde{k})$.

The extension $\widetilde{L}/\widetilde{k}$ is finite algebraic, so $u(\widetilde{L}) \leqslant u_s(\widetilde{k}) < \dim \widetilde{\tau'}$, implying that $\widetilde{\tau'}$ is isotropic over $\widetilde{L}$. Since L is Henselian, τ' is isotropic over L and thus so is τ.

For the second part, if $\tau = q_L$ for some q, as $\widetilde{\tau'}$ is isotropic over $\widetilde{L'}$, we conclude as in Proposition 5.12. □

Keeping the same notation, the three propositions above can be summarized as follows.

THEOREM 5.15. *Let L/k be a valued field extension. Suppose that the inequality $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) + \deg \mathrm{tr}_{\widetilde{k}}\widetilde{L} \leqslant 1$ holds. Let τ be a quadratic form over L with $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.*

1. *Suppose that L is Henselian. Then τ is isotropic.*
2. *Under the same hypotheses as in Setting 2, let q be a diagonal quadratic form over R such that $q_L = \tau$. Then q is isotropic over F_R.*

If $|k^\times|$ is a free $\mathbb{Z}$-module and $|L^\times|$ is a free $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}}|L^\times| = n$ if $\deg \mathrm{tr}_{\widetilde{k}}\widetilde{L} = 1$ and $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, then the statement is true for $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

The next result will be used often in what follows.

LEMMA 5.16. *Suppose that $|k^\times|$ is a free $\mathbb{Z}$-module of dimension n. Let k'/k be a valued field extension such that $|k'^\times|$ is finitely generated over $|k^\times|$ and $|k'^\times|/|k^\times|$ is a torsion group. Then $|k'^\times|$ is also a free $\mathbb{Z}$-module of dimension n.*

Suppose that k'/k is a finite field extension. Let τ be a diagonal quadratic form over k' with $\dim \tau > 2^n u_s(\widetilde{k})$. Then q is k'-isotropic.

Proof. Seeing as $|k'^\times|/|k^\times|$ is a torsion group, its rank as a $\mathbb{Z}$-module is zero. Considering $\mathrm{rank}_{\mathbb{Z}}|k'^\times| = \mathrm{rank}_{\mathbb{Z}}|k'^\times|/|k^\times| + \mathrm{rank}_{\mathbb{Z}}|k^\times|$, we obtain $\mathrm{rank}_{\mathbb{Z}}|k'^\times| = n$. Furthermore, being a finitely generated torsion-free module over $\mathbb{Z}$, it is free.

Let $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$ be the quadratic form k'-isometric to τ obtained by applying Proposition 5.1. There exists $\sigma_0 \in Q$ with coefficients in $(k'^\circ)^\times$ such that $\dim \widetilde{\sigma_0} = \dim \sigma_0 > u_s(\widetilde{k})$, where $\widetilde{\sigma_0}$ is the image of σ_0 over $\widetilde{k'}$. Suppose that k'/k is a finite field extension. Seeing as then $\widetilde{k'}/\widetilde{k}$ is also finite, $\widetilde{\sigma_0}$ is $\widetilde{k'}$-isotropic. From Henselianity of k', we obtain that σ_0 is k'-isotropic and thus so is τ. □

The following shows that if $|k^\times|$ is a free finitely generated $\mathbb{Z}$-module of dimension n, the last conditions of Theorem 5.15 are satisfied in the Berkovich setting.

COROLLARY 5.17. *Suppose that* $|k^\times|$ *is a free* $\mathbb{Z}$*-module with* $\mathrm{rank}_{\mathbb{Z}}|k^\times| = n$*. Let* C *be a* k*-analytic curve. If* $x \in C$ *is a type 2 point, then* $|\mathcal{H}(x)^\times|$ *is a free* $\mathbb{Z}$*-module and* $\mathrm{rank}_{\mathbb{Z}}(|\mathcal{H}(x)^\times|) = n$*.*

Proof. Since x is an Abhyankar point, $|\mathcal{H}(x)^\times|$ is finitely generated over $|k^\times|$ and, since it is of type 2, $|\mathcal{H}(x)^\times|/|k^\times|$ is a torsion group, so this follows from Lemma 5.16. □

The following is another result we will be needing in what is to come.

LEMMA 5.18. *Under the same hypotheses as in Setting 2, suppose that* R *is a discrete valuation ring. Let* q *be a diagonal quadratic form over* F_R*. Then there exist diagonal* F_R*-quadratic forms* q_1, q_2 *with coefficients in* R *and* $a \in F_R^\times$ *such that:*

- q *is isometric to* $q_1 \perp aq_2$*;*
- $q_{i,L}$ *has coefficients in* $(L^\circ)^\times, i = 1, 2$*;*
- *there exists* $i_0 \in \{1, 2\}$ *such that* $\dim q_{i_0,L} \geqslant \frac{1}{2}\dim q$*.*

In particular, if either of q_1, q_2 *is isotropic over* F_R*, then so is* q*.*

Proof. Let π be a uniformizer of R. For any coefficient b of q, either $b \equiv 1 \mod (F_R^\times)^2(F_R^\circ)^\times$ or $b \equiv \pi \mod (F_R^\times)^2(F_R^\circ)^\times$. Hence, there exist diagonal F_R-quadratic forms q_1, q_2 with coefficients in $(F_R^\circ)^\times = R^\times$ such that q is F_R-isometric to $q' = q_1 \perp \pi q_2$. Then $\dim q = \dim q'$ and there exists i_0 such that $\dim q_{i_0} \geqslant \frac{1}{2}\dim q$. Since the coefficients of q_1, q_2 are in $R^\times$, their images over L are of the same dimension, so $\dim q_{i_0,L} \geqslant \frac{1}{2}\dim q$. Finally, the last sentence of the statement is obvious. □

The following theorem gives the motivation behind the hypotheses we put upon R, L', and L.

THEOREM 5.19. *Suppose that* $\mathrm{char}(\widetilde{k}) \neq 2$*. Let* C *be a normal irreducible* k*-analytic curve. Set* $F = \mathscr{M}(C)$*. Let* q *be a quadratic form over* F *of dimension* d*, with* $d > 2^{n+2}u_s(\widetilde{k})$*. Then, for any* $x \in C$*, the quadratic form* q *is isotropic over* $\mathscr{M}_x$ *for all* $x \in C$*.*

If $|k^\times|$ *is a free* $\mathbb{Z}$*-module, the statement is true for* $d > 2^{n+1}u_s(\widetilde{k})$*.*

Proof. Seeing as $\mathrm{char}(\widetilde{k}) \neq 2$, neither of the overfields of k has characteristic 2. In particular, $\mathrm{char}(F) \neq 2$, so there exists a diagonal quadratic form q' over F isometric to q. By replacing q with q' if necessary, we may directly assume that q is a diagonal quadratic form.

Recall that $\mathcal{O}_x$ and $\kappa(x)$ are Henselian [Ber93, §§ 2.1 and 2.3]. Furthermore, $\mathcal{H}(x)$ is the completion of $\kappa(x)$, so it is a Henselian immediate extension. We know that for any $x \in C$, the field $\mathcal{H}(x)$ is either a finite extension of k or a completion of F with respect to some valuation extending that of k. Abhyankar's inequality tells us that $\mathrm{rank}_{\mathbb{Q}}(|\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) + \deg\mathrm{tr}_{\widetilde{k}}\widetilde{\mathcal{H}(x)} \leqslant 1$. We will apply part (2) of Theorem 5.15 by taking $R = \mathcal{O}_x$, $F_R = \mathscr{M}_x$, $L' = \kappa(x)$, and $L = \mathcal{H}(x)$.

If $\mathcal{H}(x)/k$ is finite, i.e. if x is a rigid point, then $\mathcal{H}(x) = \kappa(x) = \mathcal{O}_x/m_x$. Being a normal Noetherian local ring with Krull dimension one, $\mathcal{O}_x$ is a discrete valuation ring. By Lemma 5.18, there exists a diagonal $\mathscr{M}_x$-quadratic form τ with coefficients in $\mathcal{O}_x$ such that $\dim \tau_L \geqslant \frac{1}{2}\dim q > 2^{n+1}u_s(\widetilde{k})$ (respectively $\dim \tau_L \geqslant \frac{1}{2}\dim q > 2^n u_s(\widetilde{k})$) and the isotropy of τ implies that of q. Seeing as $\mathrm{rank}_{\mathbb{Q}}(|\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = \deg\mathrm{tr}_{\widetilde{k}}\widetilde{\mathcal{H}(x)} = 0$, we can apply Proposition 5.12 (respectively Lemma 5.16) to τ.

Otherwise, $\mathcal{O}_x = \kappa(x)$ is a field and $\mathcal{H}(x)$ is its completion. In the general case, we conclude by a direct application of Theorem 5.15. In particular, if $|k^\times|$ is a free $\mathbb{Z}$-module, then this is an application of Theorem 5.15 in view of Corollary 5.17. □

We also obtain the following corollary.

Corollary 5.20. *Suppose that* $\operatorname{char}(\widetilde{k}) \neq 2$. *Let* C *be a normal irreducible* k*-analytic curve. Let* x *be any point of* C. *Let* q *be a quadratic form over* $\mathcal{H}(x)$ *such that* $\dim q > 2^{n+2}u_s(\widetilde{k})$. *Then* q *is isotropic.*

If $|k^\times|$ *is a free* $\mathbb{Z}$*-module, then the statement is true for* $\dim q > 2^{n+1}u_s(\widetilde{k})$.

Proof. This is a direct consequence of part (1) of Theorem 5.15 (in view of Corollary 5.17 for the special case). □

6. Applications

We will now apply the results obtained in the previous section to the (strong) u-invariant. Let k be a complete ultrametric field.

Theorem 6.1. *Suppose that* $\operatorname{char}(\widetilde{k}) \neq 2$. *Let* F *be a finitely generated field extension of* k *of transcendence degree 1. Let* q *be a quadratic form over* F *with dimension* d.

1. *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = n$ *and* $d > 2^{n+2}u_s(\widetilde{k})$, *then* q *is isotropic.*
2. *If* $|k^\times|$ *is a free* $\mathbb{Z}$*-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| = n$ *and* $d > 2^{n+1}u_s(\widetilde{k})$, *then* q *is isotropic.*

Proof. There exists a connected normal projective k-analytic curve C such that $F = \mathscr{M}(C)$. By Theorem 3.12, the quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C$. The statement now follows in view of Theorem 5.19. □

Corollary 6.2. *Suppose that* $\operatorname{char}(\widetilde{k}) \neq 2$.

1. *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = n$, *then* $u_s(k) \leqslant 2^{n+1}u_s(\widetilde{k})$.
2. *If* $|k^\times|$ *is a free* $\mathbb{Z}$*-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| = n$, *then* $u_s(k) \leqslant 2^{n}u_s(\widetilde{k})$.

Proof. Let l/k be a finite field extension. Let q be an l-quadratic form of dimension $d > 2^{n+1}u_s(\widetilde{k})$ (respectively $d > 2^{n}u_s(\widetilde{k})$). Since $\operatorname{char}(\widetilde{k}) \neq 2$, we may assume q to be diagonal. In view of part (1) of Proposition 5.12 (respectively Lemma 5.16), q is l-isotropic, so $u(l) \leqslant 2^{n+1}u_s(\widetilde{k})$ (respectively $u(l) \leqslant 2^{n}u_s(\widetilde{k})$). In combination with Theorem 6.1, this completes the proof of the statement. □

Corollary 6.3. *Suppose that* $\operatorname{char}(\widetilde{k}) \neq 2$. *Let* C *be a normal irreducible* k*-analytic curve. Let* x *be any point of* C.

1. *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = n$, *then* $u(\mathcal{H}(x)) \leqslant 2^{n+2}u_s(\widetilde{k})$.
2. *If* $|k^\times|$ *is a free* $\mathbb{Z}$*-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| = n$, *then* $u(\mathcal{H}(x)) \leqslant 2^{n+1}u_s(\widetilde{k})$.

Proof. See Corollary 5.20. □

In particular, when k is discretely valued we obtain the upcoming corollary. It is the most important result on quadratic forms of HHK in [HHK09] and from it we obtain that $u(\mathbb{Q}_p(T)) = 8$ when $p \neq 2$, originally shown in [PS10].

Corollary 6.4. *Let* k *be a complete discretely valued field such that* $\operatorname{char}(\widetilde{k}) \neq 2$. *Then* $u_s(k) = 2u_s(\widetilde{k})$.

Proof. The inequality $u_s(k) \leqslant 2u_s(\widetilde{k})$ is a special case of Corollary 6.2. For the other direction, a proof that is independent of the patching method and relies on the theory of quadratic forms is given in [HHK09, Lemma 4.9]. □

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