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A NOTE ON GROUP RINGS WITH TRIVIAL UNITS

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Abstract

Let R be a ring with identity of characteristic two and G a nontrivial torsion group. We show that if the units in the group ring RG are all trivial, then G must be cyclic of order two or three. We also consider the case where R is a commutative ring with identity of odd prime characteristic and G is a nontrivial locally finite group. We show that in this case, if the units in RG are all trivial, then G must be cyclic of order two. These results improve on a result of Herman *et al.* ['Trivial units for group rings with G-adapted coefficient rings', *Canad. Math. Bull.* **48**(1) (2005), 80–89].

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1. Introduction

Let R be an associative ring with identity and let G be a group. A unit u in the group ring RG is said to be trivial if it is of the form rg for some unit $r \in R$ and some $g \in G$. Any unit of RG not of this form is said to be a nontrivial unit.

A well-known conjecture on units of group rings, first formulated by Higman in his doctoral thesis [5] and made popular by Kaplansky [6], states that if K is a field and G is a torsion-free group, then the group algebra KG has only trivial units. In an online talk on 22 February 2021, the mathematician Giles Gardam announced that this long-standing conjecture of more than 80 years is false. Gardam's counterexample, where the field has order two and the group is virtually abelian, is available in a preprint [2].

Conversely, questions have been asked on the conditions that the ring R and the group G would satisfy if the group ring RG has only trivial units. Among the results obtained on this question is the following result of Herman *et al.* [4].

Proposition 1.1. ([4], Proposition 8) *Let R be a commutative ring with identity of finite characteristic $l > 1$ and let G be a finite group such that RG has only trivial units. Then G is cyclic of order two or three.*

A question that comes to mind then is whether Proposition 1.1 holds true even when R is not commutative. It turns out that the answer to this is in the affirmative when R

has characteristic two and we present an elementary proof of this result here. We also consider commutative rings of odd prime characteristic and improve on Proposition 1.1 by showing that if G is a nontrivial locally finite group such that the units in RG are all trivial, then G is cyclic of order two.

It is worthwhile to mention here that in a continuation of the work in [4], Herman and Li [3] obtained necessary and sufficient conditions for the group ring RG to have only trivial units when R is the ring of integers of an algebraic number field and G is a nontrivial torsion group.

Throughout this paper, C_n is used to denote the cyclic group of order n and all rings considered are associative with identity.

2. Rings of characteristic two

We first give a constructive proof of the existence of nontrivial units.

PROPOSITION 2.1. *Let R be a ring of characteristic two and let G be a group. If G has an element of order n where $n \geq 4$, then RG has a nontrivial unit.*

PROOF. Suppose that $x \in G$ has finite order n where $n \geq 4$. Consider the case $n = 2m$ where $m \geq 2$. Then it is easy to see that $1 + x + x^2 + \cdots + x^{2m-2}$ is a unit in RG because $(1 + x + x^2 + \cdots + x^{2m-2})^{-1} = 1 + x^2 + x^3 + \cdots + x^{2m-1}$.

Suppose now that $n = 2m + 1$ where $m \geq 2$. Then $1 + x + x^2 + \cdots + x^{2m-2}$ is a unit in RG with

$$(1 + x + x^2 + \cdots + x^{2m-2})^{-1} = \begin{cases} x + x^2 + x^4 + \cdots + x^{2i} + \cdots + x^{2m} & \text{if } m \geq 2 \text{ is even,} \\ 1 + x^3 + x^5 + \cdots + x^{2i-1} + \cdots + x^{2m-1} & \text{if } m \geq 3 \text{ is odd.} \end{cases}$$

We have thus shown the existence of nontrivial units in RG. □

We now prove the main result in this section, which extends the characteristic-two case in Proposition 1.1 to rings that are not necessarily commutative.

THEOREM 2.2. *Let R be a ring of characteristic two and let G be a nontrivial torsion group. If the units in RG are all trivial, then G is cyclic of order two or three.*

PROOF. By Proposition 2.1, every nonidentity element of G has order two or three. Suppose that G is not cyclic. We consider the following cases.

Case 1: Every nonidentity element of G has order two. Then G is a 2-group with a subgroup H isomorphic to the direct product $C_2 \times C_2$. Let x, y be generators of $C_2 \times C_2$. Then $u = 1 + x + y \in RG$ and $u^2 = 1$. That is, RG has a nontrivial unit, which is a contradiction.

Case 2: Every nonidentity element of G has order three. Then G is a 3-group with a subgroup H isomorphic to $C_3 \times C_3$. Let x, y be generators of $C_3 \times C_3$. Then $v =$

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$1 + x + x^2 + y + xy \in RG$ is a unit with $v^{-1} = 1 + x + x^2 + y^2 + x^2y^2$. This tells us that RG has a nontrivial unit, which is a contradiction.

Case 3: G has at least one element of order two and at least one element of order three. Let P be a Sylow 2-subgroup and Q a Sylow 3-subgroup of G. Since every nonidentity element in P has order two, so P must be abelian. If $|P| \neq 2$, then P has a subgroup isomorphic to $C_2 \times C_2$ and, hence, by the same argument as in Case 1, there is a nontrivial unit in RG, which is a contradiction. It follows that $|P| = 2$, that is, $P \cong C_2$. Now if Q is not cyclic, then since every nonidentity element in Q has order three, so Q has a subgroup isomorphic to $C_3 \times C_3$ and, by the same argument as in Case 2, RG would have a nontrivial unit, which again is a contradiction. Hence, we have $Q \cong C_3$. It follows that G has six elements and, since G is assumed to be not cyclic, P and Q cannot commute with one another. Therefore, G must be the symmetric group S_3, also known as the dihedral group of order six, given by the presentation $\langle a, b \mid a^3 = b^2 = 1, ba = a^2b\rangle$. But then we have $u = 1 + a + a^2 + b + ab \in RG$ with $u^2 = 1$, that is, u is a nontrivial unit in RG, which gives us a contradiction.

In all the above cases, we have a contradiction. Therefore, G must be cyclic and has order two or three. □

EXAMPLE 2.3. Let D be a division ring of characteristic two (such division rings do exist; see, for example, [8]). Then DG has a nontrivial unit for any nontrivial torsion group G except when $G = C_2$ or C_3.

3. Rings of odd prime characteristic

By Proposition 1.1, we see that if $\mathbb{F}$ is a field of prime characteristic and G is a finite group such that $\mathbb{F}G$ has only trivial units, then G has order at most three. In fact, in the case where the field has odd prime characteristic, a sharper upper bound on the order of G is two (see [7]). For our purpose here, we prove the following version for fields of odd prime characteristic, which is somewhat analogous to Proposition 2.1.

THEOREM 3.1. *Let $\mathbb{F}$ be a field of characteristic p where p is an odd prime and let G be a group with an element of finite order $n \geq 3$. Then $\mathbb{F}G$ has a nontrivial unit.*

PROOF. Suppose that $x \in G$ has finite order $n \geq 3$. Assume first that p does not divide n and let $e = (n \cdot 1_{\mathbb{F}})^{-1} \sum_{i=0}^{n-1} x^i$. Then e is an idempotent of $\mathbb{F}G$ and $u = 1 + (p - 2)e$ is a unit of $\mathbb{F}G$ with $u^2 = 1$. Note that

$$u = 1 + (p - 2)(n \cdot 1_{\mathbb{F}})^{-1} + (p - 2)(n \cdot 1_{\mathbb{F}})^{-1} \sum_{i=1}^{n-1} x^i \neq kg$$

for any $k \in \mathbb{F} \setminus \{0\}$ and $g \in G$. Thus, u is a nontrivial unit of $\mathbb{F}G$.

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Now assume that p divides n. Then $n = pm$ for some integer m and it follows that x^m has order p. Let $\mathbb{Z}_p$ be the prime subfield of $\mathbb{F}$. Note that

$$(1 + x^m + x^{2m} + \cdots + x^{(p-1)m})(1 - x^m) = 1$$

in $\mathbb{Z}_p\langle x^m\rangle$. It follows that $\mathbb{Z}_p\langle x^m\rangle$ (and hence $\mathbb{F}G$) contains a nontrivial unit. □

Now let R be a commutative ring. We first observe that if $u \in R$ is a unit and $x \in R$ is nilpotent, then $u + x$ is a unit because

$$(1 + u^{-1}x)(1 - (u^{-1}x) + (u^{-1}x)^2 - \cdots + (-1)^{n-1}(u^{-1}x)^{n-1}) = 1,$$

where n is the smallest positive integer such that $x^n = 0$. In the following, we obtain some conditions on R and the group G so that RG has a nontrivial unit.

Proposition 3.2. *Let R be a commutative ring such that p is nilpotent in R for some odd prime p and let G be a nontrivial locally finite p-group. Then RG has a nontrivial unit.*

Proof. By [1, Proposition 16(ii)], the augmentation ideal Δ of RG is nil. Suppose that $g_1, g_2 \in G \setminus \{1\}$ are distinct elements in G. Then $1 - g_2 \in \Delta$ is nilpotent and, hence, $g_1 + (1 - g_2)$ is a nontrivial unit in RG. This completes the proof. □

Finally, we prove the following, which improves on Proposition 1.1 in the case where the ring R has odd prime characteristic.

Theorem 3.3. *Let R be a commutative ring of odd prime characteristic p and let G be a nontrivial locally finite group. If the units in RG are all trivial, then G is cyclic of order two.*

Proof. If the units in RG are all trivial, then it follows by Proposition 3.2 that G is not a p-group. Therefore, there exists $x \in G$ such that x has order q for some prime q with $q \neq p$. If q is odd, then $q \geq 3$ and, hence, it follows by Theorem 3.1 that $(\mathbb{Z}/p\mathbb{Z})\langle x\rangle$ (and, therefore, RG) has a nontrivial unit, which is a contradiction. Thus, $q = 2$. We also note that none of the nonidentity elements in G has as its order a power of either p or 2^k, where $k \geq 2$; otherwise, by Theorem 3.1 again, RG would contain a nontrivial unit, which is a contradiction. Thus, all the nonidentity elements in G have order two. If $|G| \neq 2$, then G has a subgroup H isomorphic to $C_2 \times C_2$. Let x, y be generators of H. Then $u = 1 + x + y - xy$ is a unit in $(\mathbb{Z}/p\mathbb{Z})H$ with $u^{-1} = \lambda(1 + x + y - xy)$, where $4\lambda \equiv 1 \pmod{p}$. Thus, RG has a nontrivial unit, which again is a contradiction. Therefore, $|G| = 2$ and G is cyclic, as asserted. □

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