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Random Star Processes

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A type of evolution of graphs with maximum vertex degree at most d is introduced. This evolution can start from any initial graph whose set of vertices of degree less than d is independent. The main concern is the regularity of graphs generated by this graph process when the initial graph has no edges. By analysis of the solutions of systems of differential equations it is shown that the final graph of this evolution is asymptotically almost surely a d-regular graph (subject to the usual parity condition).

1. Introduction

The motivation for the study in this paper comes from the need to analyse ways of generating random graphs with restricted vertex degrees. This is connected with modelling processes in chemistry and biology, and networks, since the natural sciences usually demand some restriction on the number of connections between objects (see Balińska and Quintas [1] for example).

The most interesting and well-studied type of degree-restricted graphs, owing to stability in the natural setting, are the regular graphs. To obtain random r-regular graphs with uniform distribution is difficult. Efficient algorithms are known for small r. The algorithm is fairly simple if r is bounded, and is given implicitly in the work of Bollobás [2] and explicitly by Wormald [14], with some results obtained using the algorithm given by Bollobás [3]. A more complicated algorithm suffices for r up to about $\sqrt{n}$ (McKay and Wormald [5]). The difficulty in finding an algorithm for the uniform distribution is related to the difficulty in counting r-regular graphs on n vertices. Even asymptotically, the numbers are not known unless r is $o(\sqrt{n})$ (McKay and Wormald [7]) or close to a constant times n (McKay and Wormald [6]). Approximately uniform generation is easier in theory (Jerrum and Sinclair [4]) but still does not seem to be practicable for large r.

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The process where edges are added randomly subject to maintaining a maximum degree at most d, is called a d-process by Ruciński [9]. Exact results on the probabilities in this distribution are hard to obtain in general but some features can be computed for small numbers of vertices (Szmanda [13]). From the point of view of the present study, the most significant result obtained so far, by Ruciński and Wormald [10], is that the final graph of this process is almost surely regular of degree d (when the necessary parity condition holds).

The subject considered here is an extension of the idea of degree-restricted random graph processes. If the object is to generate a random d-regular graph, then the most efficient way is to cause the vertices to have degree d one after the other. This suggests a method of generation of regular graphs by sequentially adding stars. Of many possible processes of this type, we concentrate on one in particular, in which we start with an initial empty graph, and, roughly speaking, each star added causes at least one more vertex of minimum degree to become of degree d, where such a vertex is chosen uniformly at random. We call this a *random star d-process*. Mahmoud, Smythe and Szymański [8] consider a similar type of process, but on trees.

We show that such a process on n vertices (with dn even) asymptotically almost surely gives a d-regular graph, where d is fixed. The main tool used is the differential equation method for discrete graph processes discussed by Wormald [15], which has already produced results for d-processes (Ruciński and Wormald [10]). As noted after the main theorem, the probability of failure is actually exponentially small in n. This contrasts strongly with the behaviour of the random d-process in [10], where the probability of failure was so high that experimental evidence for the truth has always been inconclusive and even misleading.

In Section 5 we mention some of the other implications of this study, such as lower bounds on the size of maximal independent sets in the graph generated by the random star d-process. A major question is still open, namely, what is the distribution of the final graphs in the star d-process? Related to this is the problem of the asymptotic equivalence, or contiguity, of this model and the two other heavily studied ones, namely the uniform model and the d-processes of [10]. By contiguity, we mean that any events occurring with probability $1-o(1)$ in one model do so in the other model. It is suspected but not proved that all three models are contiguous to each other.

Other results on random star d-processes not touched upon here were obtained in [11], in particular for processes beginning with a non-empty initial graph, and many results on features of random star 2-processes also appear in [12].

The notation $o()$, $O()$ and $\sim$ refers to the passage of n to infinity, where appropriate subject to the restriction that n is even. *Asymptotically almost surely* (abbreviated to a.a.s.) denotes that an event occurs with probability $1-o(1)$. The word 'uniformly' refers to the convergence implicit in the $o()$ terms being independent of all parameters other than n.

2. Definitions

Let us take a set $[n]=\{1,2,\ldots,n\}$ of vertices and some fixed positive number d smaller than n. We will study the evolution of 'greedily' generated d-graphs on $[n]$, where a *d-graph* is a graph such that the degree of each of its vertices is at most d.

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If d is given, we will say that a vertex of a d-graph is *saturated* if it is of degree d. For any graph G let $\delta(G)$ denote the minimum degree of vertices in G and let $K_{1,i}$ denote the star with centre of degree i. Let t be a process step counter; t can be regarded as discrete time. $G_n(t)$ will denote the graph on n vertices which occurs during a given process at step t, but the graphs considered below will mainly be of order n, so the indicator of the number of vertices will often be omitted, that is, $G_n(t) = G(t)$. Another simplification of notation that will often be used throughout the paper is neglecting the reference to G in denoting the value of a graph function. Thus, for example, instead of $\delta(G(t))$ we write $\delta(t)$.

Definition. Let $\mathbf{G}$ be the set of sequences of graphs $(G(t))_{t\geqslant 0}$ where:

(i) $G(0)$ is the empty graph on $[n]$;
(ii) for $t > 0$, $G(t)$ can be obtained from $G(t-1)$ as follows.
  (a) If possible, choose a vertex v of degree $\delta(t-1) < d$ and add the edges from v to $d - \delta(t-1)$ vertices of degree strictly less than d; these edges form a star $K_{1,d-\delta(t-1)}$ centred at v.
  (b) Otherwise, if $\delta(t) = d$ or the set of stars $K_{1,d-\delta(t-1)}$ described in (a) is empty, put $G(t) = G(t-1)$.

A sequence of graphs $(G(t))_{t\geqslant 0}$ generated by this procedure is called a *star d-process*.

Note that, once (b) is reached, it will be repeated *ad infinitum*. At this point, therefore, the evolution is finished and the graph occurring at this time is called the *final graph* of the star d-process $\tilde{G} = (G(t))_{t\geqslant 0}$. At most $dn/2$ edges and n stars can be added during the process, so it is obvious that (b) is reached in at most n steps. Hence, any final graph of $\tilde{G}$ is equal to $G(n)$.

Each vertex of a graph G chosen to be saturated becomes the centre of a star $K_{1,d}$ obtained by adding a star $K_{1,d-\delta(G)}$ at this vertex. For a given step t we can consider the uniform probabilistic space of all possible stars $K_{1,d-\delta(t-1)}$ with centre being a minimal degree vertex of $G(t-1)$, and which can be added to the d-graph $G(t-1)$ without violating the property of $G(t)$ being a d-graph. This induces a probability measure $\mathbf{P}$ on $\mathbf{G}$. The resulting probability space of star d-processes is called the *random star d-process*. For $d \geqslant 2$ and $n \geqslant 5$ it is easy to see that $\mathbf{P}$ is not uniform.

Let $V(G)$ be the set of vertices in a graph G. For $i = 0, \ldots, d$ let $\mathscr{N}_i$ be the set of vertices of degree i in G and for $k = 0, \ldots, d+1$ let $\mathscr{A}_k = \bigcup_{i=0}^{k-1} \mathscr{N}_i$ be the subset of $V(G)$ consisting of those vertices of degree strictly less than k. Obviously $\mathscr{A}_0 = \emptyset$ and, if G is a d-graph, $\mathscr{A}_{d+1} = V(G) = [n]$. The set $\mathscr{A}_d(t)$ is the set of all unsaturated vertices in a d-graph $G(t)$. Note that, for any $t \geqslant 0$, $\mathscr{A}_d(t)$ is an independent set of $G(t)$.

The stochastic processes $N_i(t) = |\mathscr{N}_i(t)|$ and $A_i(t) = |\mathscr{A}_i(t))|$ give the discrete time vector stochastic processes $\mathbf{N}(t) = (N_0(t), \ldots, N_d(t))$ and $\mathbf{A}(t) = (A_1(t), \ldots, A_d(t))$ on $\Omega = (\mathbf{G}, \mathbf{P})$ with states in $[n]^{d+1}$ and $[n]^d$, respectively. Obviously $\mathbf{N}(t)$ just exhibits the *degree vector* of $G(t)$. The last coordinate of $\mathbf{A}(t)$ counts the unsaturated vertices of $G(t)$. The position of the first nonzero coordinate of $\mathbf{A}(t)$ indicates $\delta(t) + 1$ and this coordinate shows the

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number of vertices of minimum degree. The statistics $\delta(t)$, $A_{\delta(t)+1}(t)$ and $A_d(t)$ determine the star d-process to a large extent, as will be seen.

It can be seen that these vector stochastic processes are Markovian, but the marginal stochastic processes $A_i(t)$ and $N_i(t)$ are not Markovian for $d \geqslant 2$. Note that for $d = 1$ both processes are deterministic, that is, $\mathbf{A}(t) = n - 2t$ and $\mathbf{N}(t) = (n - 2t, 2t)$. For $d = 2$ the vector process $\mathbf{A}(t)$ can be reduced to a 1-dimensional Markov process. The star 2-process is studied in depth in [12].

Let $\mathscr{G}$ be the σ-algebra on Ω defined by $\mathscr{G} = \bigcup_{t \geqslant 0} \mathscr{G}_t$ where $\mathscr{G}_t$ is the σ-algebra generated by the star processes with a given graph history to time t.

Definition. For each $i = -1, 0, \ldots, d-1$ let T_i be the random variable on Ω (with respect to $\mathscr{G}$) such that

$$T_{-1}(\tilde{G}) = 0, \quad \text{for all} \quad \tilde{G} \in \Omega, \qquad \text{and} \qquad T_i(\tilde{G}) = \inf\{t \geqslant 0 : (\forall s \geqslant t) A_{i+1}(s) = 0\}.$$

In other words, T_i is the first time when all vertices have degree greater than i. Moreover, the times T_i for $i = 0, \ldots, d-1$ divide the entire duration of the process into d *stages*. At each $t \in [T_{i-1}, T_i)$, one random $(d-i)$-star centred at a vertex of degree i is added to the evolving graph, which implies that $d - i$ edges are added to $G(t-1)$. A consequence of the non-decreasing degree of the vertices during the process is that $T_i = \inf\{t \geqslant 0 : A_{i+1}(t) = 0\}$. Moreover, for any $\tilde{G}$, and any $i = 0, \ldots, d-1$ and $k = 1, \ldots, d-i-1$, if $T_i(\tilde{G}) = \infty$ then $T_{i+k}(\tilde{G}) = \infty$.

We close this section with some easily verified remarks. The first is true because after time T_{d-2} the unsaturated vertices all have degree $d-1$ and form an independent set, so the process simply adds a set of independent edges.

Remark 2.1. $T_{d-2} < \infty$ is equivalent to the final graph having precisely $\lfloor \frac{dn}{2} \rfloor$ edges.

Remark 2.2. If $T_0 < \infty$ then

$$\frac{n}{d+1} \leqslant T_0 \leqslant \frac{n}{2}.$$

Proof. Obviously, the number of edges in $G(T_0)$ is equal to dT_0. $G(T_0)$ is a d-graph, so the number of edges in $G(T_0)$ cannot exceed $\frac{dn}{2}$. This implies $T_0 \leqslant \frac{n}{2}$. On the other hand, the number of vertices of degree d is at least T_0, so $T_0 \geqslant \frac{n}{d+1}$. □

Remark 2.3. For every $d \geqslant 0$ the star d-process generates a d-graph with at most d unsaturated vertices.

3. Probability on the star processes

We first give a formal recursive definition of the σ-algebras $\mathscr{G}_t$.

Definition. Let $W^v_{k_i, \ldots, k_{d-1}}(t)$ be the event of the appearance of a star $K_{1,d-i}$ whose centre v is an $i = \delta(t-1)$-degree vertex of $G(t-1)$, and such that k_j is the number of vertices in $\mathscr{N}_j(t-1)$ which become new neighbours of v in $G(t)$.

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Remark 3.1. For each t, $W^v_{k_i,\dots,k_{d-1}}(t)$ is an element of the σ-algebra $\mathscr{G}_t$. Let $\mathscr{W}_t$ be the σ-algebra generated by the set of events

$$\{W^v_{k_i,\dots,k_{d-1}}(t) : v \in \mathscr{N}_{\delta(t-1)}, k_{\delta(t-1)} + \cdots + k_{d-1} = d - \delta(t-1)\}.$$

Then, for any $t \geqslant 0$, the elements of the σ-algebra $\mathscr{G}_t$ are precisely the intersections of elements of $\mathscr{G}_{t-1}$ with elements of $\mathscr{W}_t$, and we write $\mathscr{G}_t = \mathscr{G}_{t-1} \cap \mathscr{W}_t$.

The probability of the events in $\mathscr{W}_t$ need to be determined, but for simpler expressions we consider $\mathscr{W}_{t+1}$. The number of all possible stars $K_{1,d-i}$ with centre in $\mathscr{N}_i(t)$ and neighbours of the centre in $\mathscr{A}_d(t)$ is

$$N_i(t)\binom{A_d(t)-1}{d-i}.$$

Let us consider the conditional probability $\mathbf{P}_t(\cdot) = \mathbf{P}(\cdot|\mathscr{G}_t)$ on $\mathscr{G}$. For each t such that $A_d(t) > d - \delta(t)$, since the step is chosen uniformly at random we have

$$\mathbf{P}_t\left(W_{k_{\delta(t)},\dots,k_{d-1}}(t+1)\right) = \begin{cases} \dfrac{\binom{N_{\delta(t)}(t)-1}{k_{\delta(t)}}\cdots\binom{N_{d-1}(t)}{k_{d-1}}}{\binom{A_d(t)-1}{d-\delta(t)}}, & \text{if } k_{\delta(t)} + \cdots + k_{d-1} = d - \delta(t), \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

It can be seen that, for all other t, the conditional probability of an empty star occurring is 1 and the probability of any other star is 0.

From (3.1) it is clear that the probability $\mathbf{P}_t$ is determined by $\mathbf{A}(t)$ or $\mathbf{N}(t)$. In other words it depends only on the degree sequence of $G(t)$.

In the case of a large population of points ($n \to \infty$) the following proposition makes (3.1) more useful. The proof is quite routine.

Proposition 3.2. *Let d be a fixed number smaller than n. Then, for every $0 \leqslant t < T_{d-1}$ such that $A_d(t) > d - \delta(t)$, and each $k_{\delta(t)}, \dots, k_{d-1}$ such that $k_j \leqslant N_j$ and $k_{\delta(t)} + \cdots + k_{d-1} = d - \delta(t)$,*

$$\mathbf{P}_t\left(W_{k_i,\dots,k_{d-1}}(t+1)\right) = \frac{(d-i)!}{k_i!\cdots k_{d-1}!} \times \frac{N_i(t)^{k_i}\cdots N_{d-1}(t)^{k_{d-1}}}{A_d^{d-i}(t)} + O(A_d(t)^{-1})$$

where $i = \delta(t)$. □

The conditional probabilities of the events $W_{k_i,\dots,k_{d-1}}(t)$, which describe the occurrence of stars at step t, determine the transition probabilities from any state $(n_0, \dots, n_d)$ of $\mathbf{N}$ to a state $(n_0, \dots, n_i - k_i - 1, n_{i+1} - k_{i+1} + k_i, \dots, n_{d-1} - k_{d-1} + k_{d-2}, n_d + k_{d-1} + 1)$. The processes $\mathbf{A}(t)$ and $\mathbf{N}(t)$ are isomorphic, and thus at once we can derive transition probabilities for $\mathbf{A}$.

Let $\Delta A_j(t) = A_j(t+1) - A_j(t)$ denote the increase in the number of vertices of degree less than j in step $t+1$ of the star process. For $j > \delta(t)$, $\Delta A_j(t) < 0$, whilst for $j \leqslant \delta(t)$, $\Delta A_j(t) = 0$. The asymptotic distribution of $\Delta A_j(t)$ is as follows.

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Proposition 3.3. *For each j and every $0 \leqslant t < T_{j-1}$, if $A_d(t) \to \infty$ then*

$$\mathbf{P}_t\left(\Delta A_j(t) = -k\right) \sim \binom{d-\delta(t)}{k-1} \left(\frac{N_{j-1}(t)}{A_d(t)}\right)^{k-1} \left(1 - \frac{N_{j-1}(t)}{A_d(t)}\right)^{d+1-k-\delta(t)}.$$

Proof. Let $i = \delta(t)$. For $1 \leqslant k \leqslant d-i$, from the above remarks we have

$$\mathbf{P}_t\left(\Delta A_j(t) = -k\right) = \sum_{k_i+\cdots+k_{j-2}+k_j+\cdots+k_{d-1}=d-i-k+1} \mathbf{P}_t\left(W(k_i,\ldots,k_{j-2},k-1,k_j,\ldots,k_{d-1})(t)\right).$$

By Proposition 3.2, and the fact that each asymptotic term in the sum contains the same factor $\left(N_{j-1}(t)/A_d(t)\right)^{k-1}$,

$$\mathbf{P}_t\left(\Delta A_j(t) = -k\right) \sim \frac{(d-i)!}{(k-1)!} \left(\frac{N_{j-1}(t)}{A_d(t)}\right)^{k-1} \sum_{\substack{k_i+\cdots+k_{d-1}=d-i-k+1 \\ k_j=0}} \frac{\left(\frac{N_i(t)}{A_d(t)}\right)^{k_i} \cdots \left(\frac{N_{d-1}(t)}{A_d(t)}\right)^{k_{d-1}}}{k_i! \cdots k_{d-1}!}.$$

The sum is equal to

$$\frac{1}{(d-i-k+1)!} \left(\frac{N_i(t)+\cdots+N_{j-2}(t)+N_j(t)+\cdots+N_{d-1}(t)}{A_d(t)}\right)^{d-i-k+1}.$$

Note that $N_i(t) + \cdots + N_{d-1}(t) = A_d(t)$, so the asymptotic distribution of $\Delta A_j(t)$ is as stated. □

Corollary 3.4. *For every $0 \leqslant t < T_{d-1}$ such that $A_d(t) \to \infty$,*

$$\mathbf{E}\left(\Delta A_j(t) | \mathscr{G}(t)\right) = \left(-1 - (d-\delta(t))\frac{N_{j-1}(t)}{A_d(t)} + o(1)\right) \mathbf{I}_{t<T_{j-1}},$$

where $\mathbf{I}_H$ denotes the indicator function of an event H. □

4. Main results

We describe the behaviour of the vector process $\mathbf{A}(t)$ asymptotically. Note that $\mathbf{A}(t)$ depends on the number n of vertices of the evolving graphs, but, as in previous sections, for convenience, we will usually neglect indices n for almost all variables which indeed depend on n.

Now we state the main result, that a scaling of the vector process $\mathbf{A}(t)$ is a.a.s. closely approximated by a fixed function.

Theorem 4.1. *For any fixed integer $d > 0$ such that dn is even and for any i, $1 \leqslant i \leqslant d$, there exists a function $\alpha_i : [0,1] \to [0,1]$ such that a.a.s.*

$$A_i(t) = \alpha_i(t/n)n + o(n) \tag{4.1}$$

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uniformly for all $t \geqslant 0$. *Moreover, there exists* $\mathbf{s} = (s_0, \ldots, s_{d-1})$ *with* $s_0 < s_1 < \cdots < s_{d-1} < 1$ *such that, for any* $i = 1, \ldots, d$ *and for* $k = 0, \ldots, i-1$, $\alpha_i(s)$ *satisfies the differential equation*

$$\frac{d\alpha_i(s)}{ds} = -1 - (d-k)\frac{\alpha_i(s) - \alpha_{i-1}(s)}{\alpha_d(s)} \tag{4.2}$$

for $s_{k-1} < s < s_k$, *where* $s_{-1} = 0$ *and* $\alpha_i(s) = 0$ *for* $s \geqslant s_{i-1}$. *In addition,* $T_i = s_i n + o(n)$ *for* $i = 0, \ldots, d-2$ *a.a.s.*

In the proof, given after a preliminary lemma, we will construct the function α_i by piecing together i nontrivial functions. The pieces correspond to the i stages of the star d-process during which $A_i > 0$. These functions arise from Corollary 3.4, which can be written as

$$\mathbf{E}\left(\Delta A_i(t) | \mathscr{G}(t)\right) = f_i\left(t/n, a_1(t/n), \ldots, a_d(t/n)\right) + o(1) \tag{4.3}$$

(provided that $t < T_{i-1}$ and $A_d(t) \to \infty$), where $f_i(s, \alpha_1, \ldots, \alpha_d)$ is the right-hand side of (4.2). The main theorem of [15] gives the required approximation of random variables by solutions of differential equations a.a.s. We apply this theorem separately to each stage of the process. This will establish the relationship between the systems of differential equations and $\mathbf{A}(t)$ given in the theorem. The s_i correspond to the dividing points between the d stages of the star d-process, scaled by a factor $1/n$. They and the functions α_j are defined as follows: $s_{-1} = 0$, and inductively for $0 \leqslant i \leqslant d-1$, s_i is the least zero of the function $\alpha_{i+1}(s)$ for $s \geqslant s_{i-1}$, where $\alpha_{i+1}(s), \ldots, \alpha_d(s)$ satisfy (4.2) for $s_{i-1} \leqslant s \leqslant s_i$ and the α_j are continuous with initial condition $\alpha_j = 1$ for $1 \leqslant j \leqslant d$. The difficult part is the analysis of the differential equations, to show that all the s_i exist and are distinct, and that $\alpha_d(s_{d-1}) > 0$. One consequence of this last statement is that stage $d-1$ a.a.s. endures for time at least cn for some $c > 0$. This implies the following by Remark 2.1.

Corollary 4.2. *The final graph of the random star d-process a.a.s. has exactly* $\lfloor dn/2 \rfloor$ *edges.*

In particular, if dn is even, this implies the final graph is regular.

We will be considering linear systems of ordinary differential equations with variables α_i, such as (4.2). In each case the derivatives, as functions of these variables, satisfy a Lipschitz condition in any bounded open domain excluding a neighbourhood of $\alpha_d = 0$. The existence of unique solutions of the system extending to the boundary of such a domain is a standard result in the theory of first-order differential equations. We will be concerned with showing properties of the solutions of the equations, but always in a bounded domain which excludes $\alpha_d \leqslant \epsilon_1$ for some sufficiently small $\epsilon_1 > 0$. Hence the systems involved have solutions which are unique. Thus, proving existence of the s_i requires showing that the unique functions $\alpha_i(s)$, $i \leqslant d-1$, become 0 at distinct values of s, increasing with i, and before α_d reaches 0.

The argument for the existence of s_0 is a little different from the others, so we give it first. Recall that s_0 is defined as the least positive zero of $\alpha_1(s)$.

Lemma 4.3. *Let* $d \geqslant 2$. *The zero* s_0 *of* $\alpha_1(s)$ *exists, and*

$$\alpha_1(s) < \cdots < \alpha_d(s) \quad \text{and} \quad \alpha_d(s) > \epsilon_1 \tag{4.4}$$

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for $0 < s \leqslant s_0$, *where*

$$\epsilon_1 = e^{-d-3}. \tag{4.5}$$

Proof. Note first that $\alpha_0(s) = 0$ for all $s \geqslant 0$. It can be checked by series solution of the equations (4.2) that (4.4) is valid for $0 < s < \epsilon$, for sufficiently small $\epsilon > 0$. Here are some details which suffice. In a small neighbourhood of the initial conditions, we have existence of $\alpha_i(s) = 1 - O(s)$ for each $i > 0$ as the derivatives in (4.2) are all bounded. This gives $\frac{d\alpha_1}{ds} = -1 - d + O(s)$, whence $\alpha_1 = 1 - (1+d)s + O(s^2)$. Then $\frac{d\alpha_2}{ds} = -1 - O(s)$, from which $\alpha_2 = 1 - s + O(s^2)$ and hence $\alpha_2(s) - \alpha_1(s) = ds + O(s^2)$, $\frac{d\alpha_2}{ds} = -1 - d^2 s + O(s^2)$, and so $\alpha_2 = 1 - (1+d^2)s + O(s^3)$. In this way we obtain in general $\alpha_i = 1 - s - d^i s^i + O(s^{i+1})$ for $1 \leqslant i \leqslant d$. Thus $\alpha_i(s) - \alpha_{i-1}(s) = d^{i-1}s^{i-1} + O(s^i)$ which is strictly positive for $s \in (0, \epsilon]$ for sufficiently small ϵ.

We proceed by contradiction. Assume that (4.4) is false for some $s = u > 0$ where

$$\alpha_1(s) > 0, \text{ for } 0 \leqslant s < u. \tag{4.6}$$

Necessarily, from the argument above, $u \geqslant \epsilon$ (for ϵ chosen sufficiently small). As the functions α_i are continuous, we can take u to be the infimum of the set containing all such s which violate (4.4). By continuity of the α_i we have

$$\alpha_i(u) = \alpha_{i+1}(u), \text{ for some } i,\ 1 \leqslant i < d, \text{ or } \alpha_d(u) = \epsilon_1. \tag{4.7}$$

Note that (4.4) holds for $0 < s < u$. So the differential equations (4.2) can be written as

$$\frac{d\alpha_i(s)}{ds} = -1 - \alpha_i(s)g(s) + \alpha_{i-1}(s)g(s)$$

for $i = 0, 1, \ldots, d$ with some function $g(s) = \frac{d}{\alpha_d(s)} \leqslant d/\epsilon_1$. Thus, for $i = 1, \ldots, d-1$,

$$\frac{d}{ds}(\alpha_{i+1}(s) - \alpha_i(s)) = -(\alpha_{i+1}(s) - \alpha_i(s))g(s) + f_i(s) \geqslant -(\alpha_{i+1}(s) - \alpha_i(s))d/\epsilon_1,$$

where $f_i(s)$ stands for a positive function when $s \in (0, u)$. Since $\alpha_{i+1}(\epsilon) - \alpha_i(\epsilon) > 0$, it follows that $\alpha_{i+1}(u) - \alpha_i(u) > 0$.

We deduce that the last option in (4.7) must hold, that is, $\alpha_d(u) = \epsilon_1$. This will lead to a contradiction. (This part of the proof seems to be the hardest to handle: it must be shown that $\alpha_d(s)$ gets close to 0 significantly later than $\alpha_1(s)$.) To save confusing d with the differential operator, we write r for d in the rest of this proof. From the expression for the derivatives of $\alpha_1(s)$ and $\alpha_r(s)$ in (4.2) we obtain for $s \in (0, u)$ (noting $d\alpha_r/ds < 0$ here)

$$\frac{d\alpha_1}{d\alpha_r} = \frac{1 + r\alpha_1/\alpha_r}{1 + r - r\alpha_{r-1}/\alpha_r} \geqslant \frac{1 + rv}{1 + r - rv},$$

where

$$v = \frac{\alpha_1}{\alpha_r},$$

and we have used $\alpha_{r-1} \geqslant \alpha_1$. (This is one point where we assume $d \geqslant 2$, *i.e.*, $r \geqslant 2$.) The usual solution of such a differential equation involves noting $\frac{d\alpha_1}{d\alpha_r} = \alpha_r \frac{dv}{d\alpha_r} + v$, and we

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proceed with this to obtain

$$\frac{dv}{d\alpha_r} \geqslant \frac{1 - v + rv^2}{(1 + r - rv)\alpha_r}. \tag{4.8}$$

Noting that $1 - v + rv^2 \geqslant 1 - 1/2r$ always, and $v > 0$ by (4.6), this implies

$$\frac{d\alpha_r}{dv} \leqslant \frac{(1 + r)\alpha_r}{1 - 1/2r} < (r + 3)\alpha_r \tag{4.9}$$

using $r \geqslant 2$. In order to clarify the generalization of this proof to later stages, we write $\bar{v} = 1$ and $\bar{\alpha} = 1$ for the (initial) values of v and α_r when $s = 0$. We know that α_r is a decreasing function of s, and thus by (4.8) $v(s)$ is also decreasing with s. So $v(s) < \bar{v}$ for $s > 0$. Hence, defining $y(v)$ to satisfy the differential equation $\frac{dy}{dv} = (r + 3)y$ with initial condition $y(\bar{v}) = \bar{\alpha}$, we have from (4.9)

$$\alpha_r(v(s)) > y(v(s)) \tag{4.10}$$

for $v < \bar{v}$ and hence $s > 0$. The solution of the equation defining y is $y(v) = Ce^{(r+3)v}$, whence

$$y(0) = e^{-(r+3)\bar{v}} y(\bar{v}) = e^{-(r+3)}\bar{\alpha} = e^{-(r+3)}. \tag{4.11}$$

Hence by (4.10), $\alpha_r > e^{-(r+3)} = \epsilon_1$ for $v \geqslant 0$, that is, for $\alpha_1(s) > 0$. This contradicts the last option in (4.7).

Since (4.4) cannot hold for all $s > 0$ (for, while it holds, the derivative of α_d is less than -1), there must exist some zero of α_1, which establishes the existence of s_0. Since $\alpha_1(s) > 0$ for $0 \leqslant s < s_0$, (4.4) holds for $0 < s \leqslant s_0$. □

Proof of Theorem 4.1. We first establish the sharp concentration of $\mathbf{A}(t)$ during stage 0. The case $d = 1$ is trivial so we take $d \geqslant 2$.

Let

$$D_0 = \{(x_0, x_1, \ldots, x_d) \in [0, 1]^{d+1} : 0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_d \leqslant 1\}$$

and

$$\tilde{D}_0 = \mathrm{Int}(D_0 \cap \{[0, 1]^d \times [\epsilon_1, 1]\}) \cup S(z_0, \epsilon_1)$$

where $S(z_0, \epsilon_1)$ denotes an open ball with centre $z_0 = (0, 1, \ldots, 1) \in D_0$ and radius ϵ_1, and $\mathrm{Int}(D)$ denotes the interior of any set D. Here ϵ_1 can be defined as in Lemma 4.3. By that lemma, the vector $(s, \alpha_1(s), \ldots, \alpha_d(s))$ reaches the boundary of $\tilde{D}_0$ at $s = s_0$. For $i = 1, \ldots, d$, let $a_i(t/n) = A_i(t)/n$, and let T be the first time t that $(t/n, a_1(t/n), \ldots, a_d(t/n)) \notin \tilde{D}_0$. Obviously $T \leqslant T_0$.

Since $A_d(t) \geqslant \epsilon_1 n$ for all $t < T$ and by (4.3), [15, Theorem 1, Note 4] applies to the vector process $\mathbf{A}(t)$ with $D = \tilde{D}_0$, and we obtain (4.1) a.a.s. uniformly for $t \leqslant T$. From this and by Lemma 4.3 we have $T = T_0$ a.a.s.

It now follows that $A_1(T_0) = \alpha_1(T_0/n)n + o(n)$ a.a.s., and so $\alpha_1(T_0/n) = o(1)$. Thus by boundedness of α_1', a.a.s. $T_0/n - s_0 = o(1)$, which gives the statement about T_0 in the theorem. Since each A_i changes by at most $d + 1$ in one step of the process, the approximations by the functions $\alpha_i(s)$ stated in the theorem hold for all $t \leqslant s_0 n$.

For the later stages, first note that by Lemma 4.3 the numbers $\alpha_1(s_0), \ldots, \alpha_d(s_0)$ are

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positive and strictly increasing in this order. Hence, the argument of that lemma on the interval $\epsilon < s < s_0$ can be repeated to show that s_1 exists and that $\alpha_2(s) < \cdots < \alpha_d(s) < 1$ for $s_0 \leqslant s \leqslant s_1$. (This time we work with α_2 in place of α_1, and in (4.11) we have $\bar{v} = \alpha_2(s_0)/\alpha_d(s_0)$ and $\bar{\alpha} = \alpha_d(s_0)$, and can choose $\epsilon_1 = e^{-(r+3)}\alpha_d(s_0)$, or even $e^{-(r+2)}\alpha_d(s_0)$ owing to the appearance of $d-k$ in (4.2).) Further repeating the argument on the intervals $s_{k-1} < s < s_k$, for $k = 2, \ldots, d-1$, shows that the s_i are distinct and that the functions α_i do not meet each other after $s = 0$. For $k = d-2$ we are left with $\alpha_d(s_{d-2}) > 0$, and we do not need to solve any differential equation corresponding to stage $d-1$ because its behaviour is deterministic.

The rest of the theorem now follows by inductively repeating the argument above about the approximation of a_i by α_i, for stages 1 up to $d-2$. The only difference between the initial step and the inductive step for stage k is that the initial values of the functions $A_i(t)$ at $t = \lfloor s_{k-1}n \rfloor$ are not deterministic but are randomly distributed, and a.a.s. have values $\alpha_i(t/n)n + o(n)$. The theorems of [15] and [16] quoted above permit such randomized initial values in the processes, and the conclusion is that, with the required probability, the random variables $A_i(t)$ are approximated within $o(n)$ by functions $\tilde{\alpha}_i(t/n)n$ where the $\tilde{\alpha}_i$ satisfy the same differential equations as the α_i but with the stochastic initial conditions at $s = s_{k-1}$. From this it follows that $\tilde{\alpha}_i(s) = \alpha_i(s) + o(1)$ for $s_{k-1} \leqslant s \leqslant s_k$, which permits the induction to go through. The conclusion from stage $d-2$ is that $A_d(T_{d-2}) = \alpha_d(s_{d-2})n + o(n)$ a.a.s., and so the next stage a.a.s. exists. In this way the full statement of the theorem follows. □

5. Further remarks

Theorem 4.1 implies that a.a.s. the process proceeds through all the various stages and they all last for times at least cn.

Various explicit bounds on the functions $\alpha_i(s)$ were obtained in [11]. For instance, it was shown in [11] that $s_0 \leqslant \frac{1}{d}\ln(1+d)$, by comparing the differential equation for α_1 with the equation $\frac{d\beta}{ds} = -1 - r\beta$. Moreover, this simple bound was improved, and it was shown that $s_0 \sim \frac{\ln d}{d}$ as $d \to \infty$. Since $T_0 \sim s_0 n$ a.a.s., this implies almost sure lower bounds on T_0. It is easy to see that the centres of the stars placed before time T_0 form an independent dominating set in the final graph of the process. We conclude that the final graph a.a.s. has an independent dominating set of size $s_0 n + o(n)$.

Finally, we note that by very minor modifications of the proof of Theorem 1 in [15], we obtain (4.1) uniformly over t with probability $1 - o(\exp(-\delta n))$ for some $\delta > 0$. (Such modifications are to be be found in [16, Theorem 5.1], but, owing to the generality of that theorem, its conclusion does not quite lend itself to the range of t used in the application here, although it is simple to deduce such an extension in this case.)

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References

[1] Balińska, K. T. and Quintas, L. V. (1987) Random graph models for physical systems. In *Graph Theory and Topology in Chemistry*, Vol. 51 of *Studies in Physical and Theoretical Chemistry*, Elsevier, Amsterdam, pp. 349–361.

[2] Bollobás, B. (1980) A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *Europ. J. Combinatorics* **1** 311–316.

[3] Bollobás, B. (1985) *Random Graphs*, Academic Press, London.

[4] Jerrum, M. and Sinclair, A. (1988) Fast uniform generation of regular graphs. Internal Report CSR-281-88, Department of Computer Science, University of Edinburgh.

[5] McKay, B. D. and Wormald, N. C. (1990) Uniform generation of random regular graphs of moderate degree. *J. Algorithms* **11** 52–67.

[6] McKay, B. D. and Wormald, N. C. (1990) Asymptotic enumeration by degree sequence of graphs of high degree. *Europ. J. Combinatorics* **11** 565–580.

[7] McKay, B. D. and Wormald, N. C. (1991) Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$. *Combinatorica* **11** 369–382.

[8] Mahmoud, H. M., Smythe, R. T. and Szymański, J. (1993) On the structure of random plane-oriented recursive trees and their branches. *Random Struct. Algorithms* **4** 151–176.

[9] Ruciński, A. (1990) Maximal graphs with bounded maximum degree: Structure, asymptotic enumeration, randomness. In *Proc. III of the 7th Fischland Colloquium*, Vol. 41 of *Rostock Math. Kolloq.*, pp. 47–58.

[10] Ruciński, A. and Wormald, N. C. (1992) Random graph processes with degree restrictions, *Combinatorics, Probability and Computing* **1** 169–180.

[11] Robalewska, H. (1996) Stochastic processes and random graphs. PhD thesis, University of Melbourne.

[12] Robalewska, H. (1997) Random graphs generated by the star 2-process. *Random Struct. Algorithms* **11** 125–149.

[13] Szmanda, P. (1990) The algorithm DOV for the random 2-graph process. Computer Science Centre Report No. 355, Technical University of Poznań.

[14] Wormald, N. C. (1984) Generating random regular graphs. *J. Algorithms* **5** 247–280.

[15] Wormald, N. C. (1995) Differential equations for random processes and random graphs. *Ann. Appl. Probab.* **5** 1217–1235.

[16] Wormald, N. C. (1999) The differential equation method for random graph processes and greedy algorithms. In *Lectures on Approximation and Randomized Algorithms* (M. Karoński and H. J. Prömel, eds), PWN, Warsaw, pp. 73–155.